On the Construction of Complete and Partial Nearest Neighbour Balanced Designs

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Abstract

In this paper, methods for constructing two dimensional nearest neighbour balanced (NNB) designs are considered. The methods given by Afsarinejad and Seeger (1988) are extended to give a new family of nearest neighbour balanced designs. Both nearest neighbour balanced designs with and without borders are constructed. A method of construction of a class of partial nearest neighbour balanced (PNNB) designs is also given.

1 Introduction

Since the introduction of nearest neighbour methods for analysing statistical experimental designs (see for example, Bartlett (1978), Kiefer and Wynn (1981), Wilkinson et al (1983)) there has been considerable interest in spatial analysis and optimal designs when the observations are dependent. Cullis and Gleeson (1991) extended their earlier one-dimensional analysis to two dimensions by modelling the covariance structure as a separable ARIMA processes and showed that it gave a significantly better fit than the one dimensional model, which in turn has been shown to be more efficient than the incomplete block model in their earlier paper (1989). They also stress that the proportion of trial data sets that require a two-dimensional model is surprisingly high.

Neighbour balance is important if it is known or thought that the effect of a plot is influenced by its neighbouring plots, in such cases nearest neighbour analysis is considered to be more efficient than classical analysis methods, see Wilkinson et al (1983). The construction of nearest neighbour balanced designs in one-dimension has received much attention from several authors, examples are Kiefer and Wynn (1981), Cheng (1983). Various types of two-dimensional nearest neighbour balanced designs are introduced and studied in Street and Street (1985), Street (1986), Freeman (1979, 1988), Afsarinejad and Seeger (1988), Morgan (1990), Morgan and Uddin (1991). A one dimensional (block) design is defined to be nearest neighbour balanced (NNB) if each treatment has every other treatment as its neighbour on an adjacent plot an equal number of times. A two dimensional (row-column) design is said to be row neighbour balanced if each treatment has every other treatment as its nearest neighbour in rows an equal number of times, and similarly defined for column neighbour balance. Thus a two dimensional design is called NNB if it is row neighbour balanced and column neighbour balanced. The designs considered in this paper are in two dimensions set out in a rectangular array of r rows and c columns.

Section 2 of this paper gives a method to construct NNB designs of size $\theta_1 a \times \theta_2 b$ given an initial NNB design of size $a \times b$ that is equireplicated in each row and column. In section 3, bordered NNB designs for v treatments with $\theta_1(v-1)$ rows and $\theta_2 v$ columns are considered. Bordered PNNB designs are considered in section 4, where a method is presented for constructing NNB designs with p^2 treatments in p rows and $p(p-1)$ columns, p being a prime.

2 Unbordered Nearest Neighbour Balanced Designs

It is easy to see that if an unbordered NNB design exists then there are positive integers m_1 and m_2 satisfying (necessary condition),

$$\binom{v}{2} = \frac{r(c-1)}{m_1} = \frac{c(r-1)}{m_2} \tag{1}$$

Then m_1 and m_2 represent the number of times each treatment is a row and column neighbour of every other treatment respectively. If the number of rows equals the number of treatments, that is $r = v$, then the above equation becomes,

$$v = \frac{2(c-1)}{m_1} + 1 = \frac{2c}{m_2} \tag{2}$$

Afsarinejad and Seeger (1988) give a method of constructing NNB designs for v treatments in v rows and v^2 columns with no borders. This method is extended for any number of columns c, provided condition (2) is satisfied.

The method of construction is based on complete and quasi-complete Latin squares. A Latin square in which every ordered pair of distinct treatments appears in adjacent positions precisely once in rows and in columns is said to be complete. A Latin square in which each unordered pair of treatments appears twice in rows and in columns is said to be quasi-complete.

Complete Latin squares exist for even number of treatments and similarly quasi-complete Latin squares exist if the number of treatments is odd. A method of constructing such Latin squares is given in Kiefer and Wynn (1981) by letting the (j,k)-th element of the square be the treatment number

$$\sum_{s=1}^{j}(-1)^s(s-1) + \sum_{s=1}^{k}(-1)^s(s-1)$$

reduced (mod v). For convenience all elements are increased by one so that the treatments are labelled 1 to v.

The NNB designs considered in this paper are composed of the union of a number of (quasi) complete Latin squares and since within each square there exists NNB, it remains to balance the pairs of treatments that occur at the joins and within any incomplete squares. The methods described below give alternative ways for deriving designs with parameter sets given in Street (1986). The designs constructed here are not necessarily the same as those in Street, however, they can be made so by taking appropriate permutations of treatments, rows and columns, not necessarily the same permutation for all three. By adopting the conventional definition for Latin squares, they are said to be *isotopic*.

Consider v odd. A NNB design with v rows and θv columns, where $\theta = \frac{m_2}{2}$ can be constructed as follows:

$$L \quad L(1) \quad L(2) \quad \ldots \quad L(\theta - 1)$$

where L is a quasi-complete Latin square constructed using method by Kiefer and Wynn. $L(d)$, $d = 1, 2, \ldots, v-1$, means add d to the last column of the previous square, then use this as the first column of the new square and complete by using the same permutation.

Consider now that v is even and treatments are arranged in v rows and $\frac{m_2}{2}v$ columns. If m_2 is even then the method of construction is an analogy to the case when v is odd with the quasi-complete Latin square replaced by complete Latin square (L) and $d = 1, 2, \ldots, \frac{v}{2} - 1$. If m_2 is odd, a NNB design with $\frac{m_2}{2}v = (\theta + \frac{1}{2})v$ columns is:

$$L \quad L(1) \quad L(2) \quad \ldots \quad L(\theta - 1) \quad H(\theta)$$

where $H(\theta)$ is the first $\frac{v}{2}$ columns of $L(\theta)$. For details of the above methods see Chan and Eccleston (1996).

The key idea in the construction methods of NNB designs discussed above is to make use of (quasi) complete Latin squares as building blocks which are themselves NNB and each treatment appears exactly once in each row and each column, and then balance the pairs of treatments at the joins. Hence the same principles would be expected to apply if the Latin squares are replaced by NNB designs with each treatment occurring equally frequent in each row and each column.

Theorem 1 *Let the building block A be an $a \times b$ NNB design such that every treatment appears k_1 times in each row and k_2 times in each column. Then a $\theta_1 a \times \theta_2 b$ NNB design exists if the following conditions are satisfied*

$$\binom{v}{2} \;\Big|\; (\theta_1 - 1)\theta_2 b \qquad\qquad \binom{v}{2} \;\Big|\; \theta_1(\theta_2 - 1)a \tag{3}$$

where θ_1, θ_2 are positive integers.

Proof

Case I. v is odd. Let $v = 2n + 1$. Our NNB resulting design is constructed by appending permutations of A horizontally and vertically to our building block, arranged in a way such that NNB is retained at the joins.

Let A be A_{11}, then the NNB design is

$$\begin{array}{cccc} A_{11} & A_{12} & \ldots & A_{1\theta_2} \\ A_{21} & A_{22} & \ldots & A_{2\theta_2} \\ \vdots & \vdots & & \vdots \\ A_{\theta_1 1} & A_{\theta_1 2} & \ldots & A_{\theta_1\theta_2} \end{array}$$

The difference between the two edges of A_{xy-1} and A_{xy} is $((x+y-3)$ mod n)+1 and similarly the difference between the two edges of A_{x-1y} and A_{xy} is $((x+y-3)$ mod n)+1.

Since each unordered pair of treatments has a difference of d, $d \in \{1, 2, \ldots, n\}$ and each difference d appear an equal number of times at joins when taking unions of the designs (equation 3), it can easily be seen that the pairwise treatments at the joins are balanced, and hence the resulting design is NNB.

Case II. v is even. Let $v = 2n$. Construct a design as above, but now let the difference between the two edges of A_{xy-1} and A_{xy} be $((x+y-3)$ mod 2n-1)+1 and similarly the difference between the two edges of A_{x-1y} and A_{xy} is $((x+y-3)$ mod 2n-1)+1.

The resulting design is NNB as each ordered pair has a difference of d, $d \in \{1, 2, \ldots, 2n-1\}$ and each difference d appear an equal number of times at joins. □

Examples of the above theorem are given in the appendix.

3 Bordered Nearest Neighbour Balanced Designs

NNB designs are known to exist for only limited combinations of numbers of treatments, row, and column sizes. It is therefore of interest to consider designs which are close to being NNB and can be made so if a border is placed around the design. In bordered designs only the neighbours of treatments in interior plots are of interest. Thus a border plot has no neighbours but can be a neighbour of an interior plot and all interior plots have four nearest neighbours. A bordered design is said to be NNB if it is row neighbour balanced and column neighbour balanced when considering the interior plots only.

The following theorem is an extension of the work by Afsarinejad and Seeger (1988).

Theorem 2 *A bordered NNB designs for v treatments in $\theta_1(v-1)$ rows and $\theta_2 v$ columns exists, where θ_1 and θ_2 are positive integers.*

Proof

Construct a (quasi) complete Latin square (depending on whether v is odd or even). Let the first $v-1$ rows be the $(v-1) \times v$ design, B_{11} and the last row be $B_{11}(v)$. Construct the second (quasi) complete Latin square by letting the first row be $B_{11}(v)$. Complete this Latin square given the first row by using the fact that the (quasi) complete Latin square is symmetric and that the treatments in rows have the same ordering as the rows of the first Latin square. Call the first $v-1$ rows of this square B_{21} and the last row $B_{21}(v)$. Similarly construct the t_1-th (quasi) complete Latin square by letting the first row be $B_{t_1-1,1}(v)$, $t_1 = 3, \ldots, \theta_1$. Then $B_{11}, B_{21}, \ldots, B_{\theta_1 1}$ together form a design with $\theta_1(v-1)$ rows and v columns. Let the right neighbours of the elements in the last column of B_{11} be the right neighbours of these elements in $B_{11}(v)$, complete the rows by adopting the same ordering of treatments as in the previous (quasi) complete Latin squares, call this B_{12}. Similarly with the others, the right neighbours of the elements in the last column of $B_{t_1 t_2}$, $t_1 = 1, \ldots, \theta_1, t_2 = 1, \ldots, (\theta_2 - 1)$ are the right neighbours of these treatments in $B_{t_1 1}(v)$. Complete the elements in each of the $B_{t_1(t_2+1)}$ until the desired $\theta_1(v-1) \times \theta_2 v$ interior design is obtained, namely,

$$\theta_1(v-1)\left\{\underbrace{\begin{array}{cccc} B_{11} & B_{12} & \ldots & B_{1\theta_2} \\ B_{21} & B_{22} & \ldots & B_{2\theta_2} \\ \vdots & \vdots & & \vdots \\ B_{\theta_1 1} & B_{\theta_1 2} & \ldots & B_{\theta_1 \theta_2} \end{array}}_{\theta_2 v}\right.$$

The Western borders of each of the $B_{t_1 1}$, $t_1 = 1, \ldots, \theta_1$ are the left neighbours of the elements in the first column of $B_{t_1 1}$ in $B_{t_1 1}(v)$, $t_1 = 1, \ldots, \theta_1$. The Eastern borders of each of the $B_{t_1 \theta_2}$, $t_1 = 1, \ldots, \theta_1$ are given by the right neighbours of the elements in the last column of $B_{t_1 \theta_2}$ in $B_{t_1 1}(v)$.

The Northern border is obtained from adding one (mod v) to each of the elements in the first row of the interior design. The Southern border is obtained from subtracting one (mod v) from each of the elements in the last row of the interior design. □

Example 1. Let $v = 4$, $\theta_1 = 2$ and $\theta_2 = 2$. Then the bordered NNB design is given below.

$$\begin{array}{c|cccccccc|c} & 2 & 3 & 1 & 4 & 1 & 2 & 4 & 3 & \\ \hline 2 & 1 & 2 & 4 & 3 & 4 & 1 & 3 & 2 & 1 \\ 4 & 2 & 3 & 1 & 4 & 2 & 3 & 1 & 4 & 2 \\ 3 & 4 & 1 & 3 & 2 & 1 & 2 & 4 & 3 & 4 \\ 4 & 3 & 4 & 2 & 1 & 2 & 3 & 1 & 4 & 3 \\ 2 & 4 & 1 & 3 & 2 & 4 & 1 & 3 & 2 & 4 \\ 1 & 2 & 3 & 1 & 4 & 3 & 4 & 2 & 1 & 2 \\ \hline & 1 & 2 & 4 & 3 & 2 & 3 & 1 & 4 & \end{array}$$

Example 2. The bordered NNB design for $v = 5$, $\theta_1 = 2$ and $\theta_2 = 3$ is as follows.

	2	3	1	4	5	1	2	5	3	4	2	3	1	4	5	
3	1	2	5	3	4	5	1	4	2	3	1	2	5	3	4	5
1	2	3	1	4	5	3	4	2	5	1	2	3	1	4	5	3
4	5	1	4	2	3	1	2	5	3	4	5	1	4	2	3	1
5	3	4	2	5	1	2	3	1	4	5	3	4	2	5	1	2
1	4	5	3	1	2	3	4	2	5	1	4	5	3	1	2	3
4	5	1	4	2	3	1	2	5	3	4	5	1	4	2	3	1
2	3	4	2	5	1	4	5	3	1	2	3	4	2	5	1	4
3	1	2	5	3	4	5	1	4	2	3	1	2	5	3	4	5
	5	1	4	2	3	4	5	3	1	2	4	1	4	2	3	

4 Bordered Partial Nearest Neighbour Balanced Designs

To achieve nearest neighbour balance may require a relatively large number of replicates and may often be impractical. An obvious compromise to achieve efficient and 'smaller' designs would be to consider designs which have partial nearest neighbour balance (PNNB), in which each treatment has every other treatment as its nearest neighbour m_1 or $m_1 + 1$ times in rows and m_2 or $m_2 + 1$ times in columns. There has been little written on partially neighbour balanced designs in comparison with complete neighbour balanced designs, for example, see Street and Street (1985). In the one dimensional case NNB designs have been looked at by several authors, for example Kiefer and Wynn (1981) and Cheng (1983), whereas designs that are partially balanced have not received much attention. One method to construct such designs would be to use as a starting point a balanced or resolvable incomplete block design, then attempt to rearrange treatments within a block to achieve a higher 'degree' of neighbour balance. In this section, a method of construction is given for a special class of two dimensional PNNB row column designs where $m_1 = m_2 = 0$ with p^2 treatments in p rows and $p(p-1)$ columns, where p must be a prime number.

The method of construction is based on standard complete or quasi-complete Latin squares of side p with equal back diagonal elements. A method of obtaining these Latin squares will also be described. In the resulting design, the columns will be grouped together so that each treatment is replicated exactly once in each group, and there will be $p - 1$ of these groups. Such designs are said to be resolvable.

Construction Method

Initially construct a (quasi) complete Latin square of side p and then rearrange the rows of the Latin square so that treatment p appears on the back diagonal. Now rename the treatment numbers so that the Latin square is in its standard form, that is, the elements in the first row (or column) are $1, 2, \ldots, p$.

Construct a $p \times p$ reference square for p^2 treatments by assigning the first p treatments to the first row, the next p treatments to the second row, etc. Call this

L_0.

Now construct a square design as follows. Consider the first row of our (quasi) complete Latin square. Suppose the treatments in this row are $t_{11}, t_{12}, ..., t_{1p}$. Then the elements in the first row of our design are given by the t_{11}-th element in the first row of L_0, the t_{12}-th element in the second row of L_0, and in general, the t_{1p}-th element in the p-th row of L_0. Consider now the second row of our (quasi) complete Latin square and let the elements in this row be $t_{21}, t_{22}, \ldots, t_{2p}$. The elements in the second row of our design are given by the t_{21}-th element in the first row of L_0, the t_{22}-th element in the second row of L_0, etc. In general, let t_{ij} be the treatment on the (i,j)-th element of our neighbour balanced Latin square. Then the (i,j)-th element of the design is given by the t_{ij}-th element in the j-th row of L_0. Repeat this process until the desired p rows are obtained, call this design L_1. By adopting the similar procedure, obtain the design L_2 using L_1 as the reference square. Similarly obtain $L_3, \ldots, L_{p-1}$, the union of these designs form a partial neighbour balanced design with p rows and $p^2 - p$ columns, and due to the way they are constructed, the resulting design is in fact resolvable with $p-1$ groups of complete replicates. However, treatments occurring more frequently on the edges or corners of this design are going to have fewer number of row or column neighbours as other treatments which always appear on an interior plot. If a set of border plots is added around our main plot then every treatment in the main plot would have four adjacent neighbours.

Now a border needs to be added so that the PNNB property is not destroyed. This requires the use of the initial reference square L_0. For every treatment t on the upper or lower edges of our design, suppose it appears as the (i,j)-th element in L_0, the border plot is taken as the treatment on the $(i+g \pmod p, j)$-th element in L_0, where g is the group number which t is in, $g = 1, \ldots, p-1$. For every treatment on the left or right edges of our design, let the corresponding treatment appear as the (i,j)-th element in L_0, then the border plot to the left or right can be taken as the $(i, j+g \pmod p)$-th element in L_0, $g = 1$ or $(p-1)$.

Example 5. Let $p = 5$. Construct the quasi-complete Latin square using the method given by Kiefer and Wynn. Now rearrange the rows so that treatment 5 appears on the back diagonal of the Latin square and finally standardize the Latin square by using the permutation (132). Note that the resultant Latin square remains quasi-complete. The standardized neighbour balanced Latin square and the first reference square L_0 are, respectively,

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{array} \qquad \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{array}$$

The bordered PNNB design constructed using the method described above is as follows.

	6	12	18	24	5	11	19	25	3	7	16	10	14	2	23	21	16	11	1	6	
2	1	7	13	19	25	1	9	15	18	22	1	20	24	12	8	1	21	16	6	11	15
3	2	9	11	20	23	7	20	3	21	14	9	21	15	5	17	20	5	15	25	10	9
4	3	6	15	17	24	13	2	24	10	16	15	7	16	23	4	24	9	4	14	19	18
5	4	10	12	18	21	19	23	6	12	5	18	14	2	6	25	12	17	7	2	22	21
1	5	8	14	16	22	25	11	17	4	8	22	3	10	19	11	8	13	23	18	3	2
	10	13	19	21	2	10	21	2	14	18	12	18	25	9	1	3	8	18	13	23	

5 Conclusion

The methods described here generate classes of unbordered and bordered NNB designs, and a class of bordered PNNB designs. Further development along these lines, that is through direct combinatorial procedures appears to be quite difficult and limited. A computational approach should be possible and produce NNB and PNNB designs for wide classes of parameter sets, including those that can be generated by direct combinatorial methods. Such an approach is part of ongoing research.

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Appendix : Examples

The following 21 $\times$ 21 NNB design is constructed using results from theorem 1 with a building block of size 7 $\times$ 7.

7	1	6	2	5	3	4	5	6	4	7	3	1	2	4	5	3	6	2	7	1
1	2	7	3	6	4	5	6	7	5	1	4	2	3	5	6	4	7	3	1	2
6	7	5	1	4	2	3	4	5	3	6	2	7	1	3	4	2	5	1	6	7
2	3	1	4	7	5	6	7	1	6	2	5	3	4	6	7	5	1	4	2	3
5	6	4	7	3	1	2	3	4	2	5	1	6	7	2	3	1	4	7	5	6
3	4	2	5	1	6	7	1	2	7	3	6	4	5	7	1	6	2	5	3	4
4	5	3	6	2	7	1	2	3	1	4	7	5	6	1	2	7	3	6	4	5
5	6	4	7	3	1	2	4	5	3	6	2	7	1	4	5	3	6	2	7	1
6	7	5	1	4	2	3	5	6	4	7	3	1	2	5	6	4	7	3	1	2
4	5	3	6	2	7	1	3	4	2	5	1	6	7	3	4	2	5	1	6	7
7	1	6	2	5	3	4	6	7	5	1	4	2	3	6	7	5	1	4	2	3
3	4	2	5	1	6	7	2	3	1	4	7	5	6	2	3	1	4	7	5	6
1	2	7	3	6	4	5	7	1	6	2	5	3	4	7	1	6	2	5	3	4
2	3	1	4	7	5	6	1	2	7	3	6	4	5	1	2	7	3	6	4	5
4	5	3	6	2	7	1	4	5	3	6	2	7	1	2	3	1	4	7	5	6
5	6	4	7	3	1	2	5	6	4	7	3	1	2	3	4	2	5	1	6	7
3	4	2	5	1	6	7	3	4	2	5	1	6	7	1	2	7	3	6	4	5
6	7	5	1	4	2	3	6	7	5	1	4	2	3	4	5	3	6	2	7	1
2	3	1	4	7	5	6	2	3	1	4	7	5	6	7	1	6	2	5	3	4
7	1	6	2	5	3	4	7	1	6	2	5	3	4	5	6	4	7	3	1	2
1	2	7	3	6	4	5	1	2	7	3	6	4	5	6	7	5	1	4	2	3

The following 36 $\times$ 6 NNB design is constructed using results from theorem 1 with a building block of size 6 $\times$ 6.

6	1	5	2	4	3
1	2	6	3	5	4
5	6	4	1	3	2
2	3	1	4	6	5
4	5	3	6	2	1
3	4	2	5	1	6
4	5	3	6	2	1
5	6	4	1	3	2
3	4	2	5	1	6
6	1	5	2	4	3
2	3	1	4	6	5
1	2	6	3	5	4
3	4	2	5	1	6
4	5	3	6	2	1
2	3	1	4	6	5
5	6	4	1	3	2
1	2	6	3	5	4
6	1	5	2	4	3
3	4	2	5	1	6
4	5	3	6	2	1
2	3	1	4	6	5
5	6	4	1	3	2
1	2	6	3	5	4
6	1	5	2	4	3
4	5	3	6	2	1
5	6	4	1	3	2
3	4	2	5	1	6
6	1	5	2	4	3
2	3	1	4	6	5
1	2	6	3	5	4
6	1	5	2	4	3
1	2	6	3	5	4
5	6	4	1	3	2
2	3	1	4	6	5
4	5	3	6	2	1
3	4	2	5	1	6

The following 20×30 NNB design is constructed using results from theorem 1 with a building block of size 5×15.

1	2	5	3	4	5	1	4	2	3	5	1	4	2	3	4	5	3	1	2	3	4	2	5	1	3	4	2	5	1
2	3	1	4	5	1	2	5	3	4	1	2	5	3	4	5	1	4	2	3	4	5	3	1	2	4	5	3	1	2
5	1	4	2	3	4	5	3	1	2	4	5	3	1	2	3	4	2	5	1	2	3	1	4	5	2	3	1	4	5
3	4	2	5	1	2	3	1	4	5	2	3	1	4	5	1	2	5	3	4	5	1	4	2	3	5	1	4	2	3
4	5	3	1	2	3	4	2	5	1	3	4	2	5	1	2	3	1	4	5	1	2	5	3	4	1	2	5	3	4
5	1	4	2	3	4	5	3	1	2	4	5	3	1	2	4	5	3	1	2	3	4	2	5	1	3	4	2	5	1
1	2	5	3	4	5	1	4	2	3	5	1	4	2	3	5	1	4	2	3	4	5	3	1	2	4	5	3	1	2
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