Flow decompositions and directed graph minors

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Flow graphs

Given is an s-t (multigraph) DAG $G = (V, E)$ and a flow $f : E \to \mathbb{N}$.
Conservation of flow: $\sum_{e \in \delta^+(v)} f(e) = \sum_{e \in \delta^-(v)} f(e) \quad \forall v \in V \setminus \{s, t\}$.

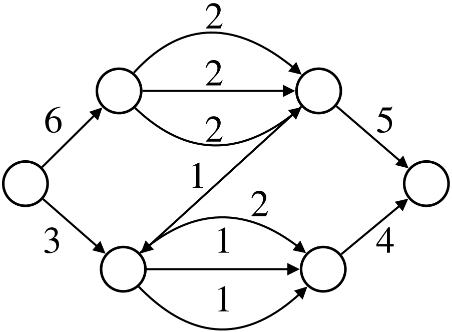

Figure: Simple flow graph.

Minimum Flow Decomposition (MFD)

Minimum Flow Decomposition (MFD) of (G, f): **minimum sized** set of s-t **paths and weights** $\{(P_1, w_1), \ldots, (P_k, w_k)\}$ $(w_i \in \mathbb{N})$ with

$$f = \sum_{i=1}^{k} w_i P_i.$$

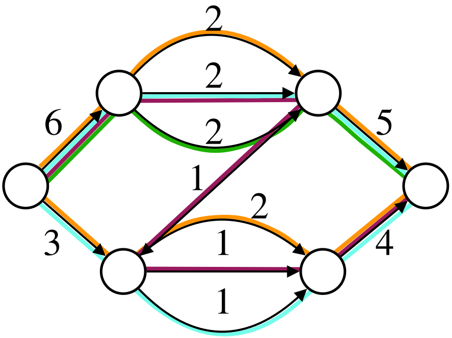

Figure: Flow decomposition in 6 s-t paths (weights omitted).

Solving MFD, state of the art

- MFD is NP-hard, even if the flow values come from $\{1, 2, 4\}$ [Hartman et al., 2012]
- MFD is APX-hard [Hartman et al., 2012]
- FPT in time $O(2^{k^2} \cdot (n + \log \|f\|))$ [Kloster et al., 2018] and ILP [Dias et al., 2022] solvers exist
- Greedy approximation factor $\Omega(m/\log m)$ [Cáceres et al., 2024]
- Greedy-weight commonly used in applications [Baaijens et al., 2020, Tomescu et al., 2013]

Outline

- Explore DAG structure with (minimal) flows
- $O(\log \|f\|)$-approximation for graphs with forbidden minors
- Implication: Quasi-polynomial algorithm for some class of instances
- Bridges gap between application and theory

Graph structure impact: digraph width

Definition

The *width* of an s-t DAG G is the **minimum sized set of s-t paths** to cover all edges.

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We have $\text{width}(G) \leq \text{mfd}(G, f)$ for all $f > 0$.

Width hinders greedy

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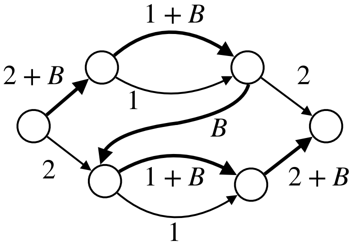
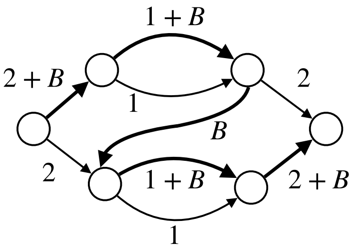


(a) The base case $(G_1, X_{1,B})$. Bold edges carry flow at least B.

Width hinders greedy



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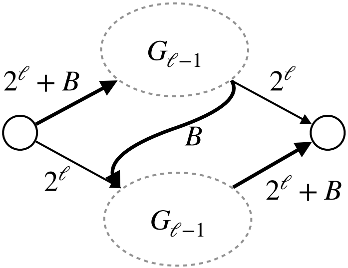

(b) Building $(G_\ell, X_{\ell,B})$ from two copies of $(G_{\ell-1}, X_{\ell-1,B})$ $(\ell > 1)$.

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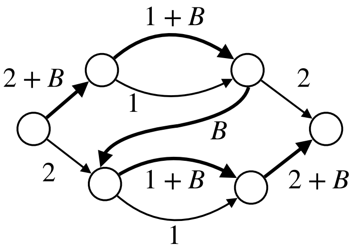

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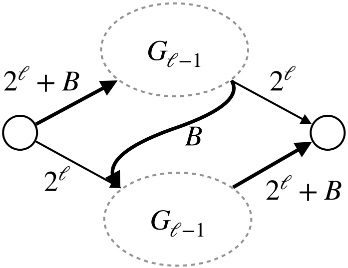

(b) Building $(G_\ell, X_{\ell,B})$ from two copies of $(G_{\ell-1}, X_{\ell-1,B})$ $(\ell > 1)$.

$(G_\ell, X_{\ell,B})$ can be decomposed into $\Theta(\ell)$ paths. Greedy-weight uses $\Theta(2^\ell)$ paths.

$\rightarrow$ Approximation ratio for greedy-weight on MFD is $\Omega(m/\log m)$.

Flow derived graph minors

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Definition [Grigorjew et al., 2024]

We call H a *flow-minor* of an *s*-*t* DAG G, if there exists H' such that

- $E(H') = \text{supp}(f)$ for some flow $f : E(G) \to \mathbb{N}$ (write $H' = H_f$),
- H is a butterfly contraction of H'

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From [Deligkas and Meir, 2017]: Delete edges (u, v), where $\deg^-(u) \geq 2$ and $\deg^+(v) \geq 2$, contract via butterfly contractions.

Flow derived graph minors

Definition [Deligkas and Meir, 2017]

The *parallel width* pw of an s-t DAG is the **largest minimal** s-t **cut-set**.

- Equivalently, the maximum width(G_f) throughout all $f \geq 0$.
- Equivalently, the maximum possible value of a minimal flow.

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Let $GP(c)$ be the s-t DAG of c parallel edges (s, t).

Lemma [Deligkas and Meir, 2017]

$\mathrm{pw}(G) < c$ iff G is $GP(c)$-f-minor free.

Ch_k DAG

We call the following DAGs Ch_k:

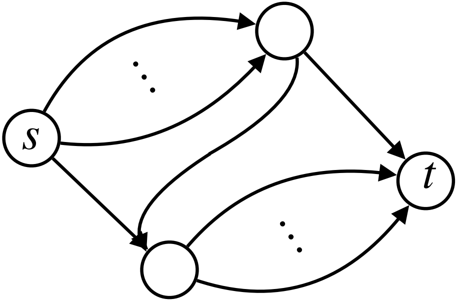

Figure: On each side, k parallel edges.

Lemma [Deligkas and Meir, 2017]

A DAG is series-parallel if and only if it is Ch_1-f-minor free.

Width-stable graphs

Definition ([Cáceres et al., 2024])

G is called *width-stable* if $\text{width}(G_f) \leq \text{width}(G_g)$ for all flows $f \leq g$ on G.

The following are equivalent [Cáceres et al., 2024]:

- G is width-stable,
- G is Ch_2-f-minor free.

$\rightarrow$ Greedy is a $O(\log \text{Val}(f))$-approximation on width-stable graphs [Cáceres et al., 2024].

Approximating MFD

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Power of two–approach

1. "Remove" the odd part: Given flow f, define flow g of the **same parity** as f,
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2. Choose a small g and decompose it to $\text{Val}(g)$ paths,
3. Divide the remaining flow by 2.

$\rightarrow$ We obtain a decomposition of size $\text{Val}(g_1) + \text{Val}(g_2) + \cdots + \text{Val}(g_{\log \|f\|})$.

How to choose g?

How to choose a small, parity fixing flow g?

$$\begin{aligned} &\min \mathrm{Val}(g), \text{s.t.} \\ &g \text{ is a flow on } G, \\ &0 \leq g \leq f, \\ &f - g \equiv_2 0. \end{aligned} \tag{1}$$

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Easier problem:

$$\begin{aligned} &\min \operatorname{Val}(h), \text{s.t.} \\ &h \text{ is a flow on } G, \\ &0 \leq h \leq f, \\ &0 < h(e) \quad \forall e \in E(G)\colon\ f(e) \text{ is odd.} \end{aligned} \tag{2}$$

How to choose g?

Lemma [Cáceres et al., 2024]

For every flow $f : E \to \mathbb{N}$ we can find a flow $\text{Unitary}(f) : E \to \{-1, 0, 1\}$ such that $f \equiv_2 \text{Unitary}(f)$, in time $O(m)$.

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Lemma [Grigorjew et al., 2024]

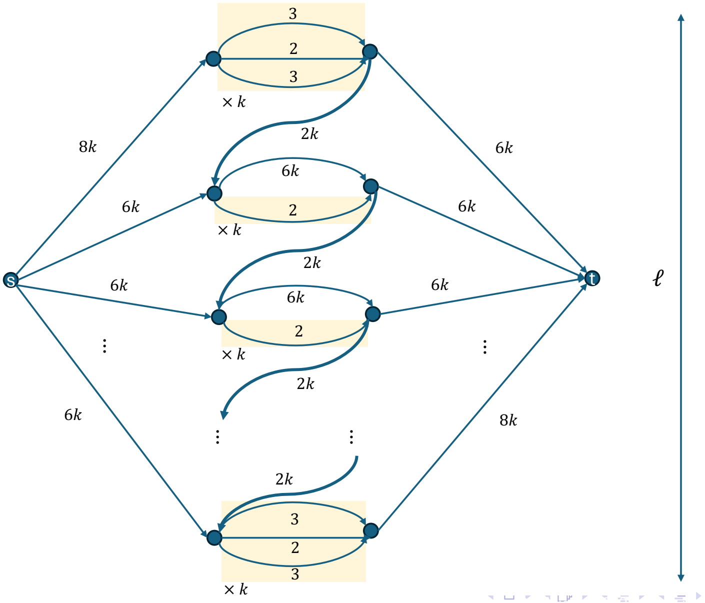
h is an optimal solution to (2)
$\iff g = h + \text{Unitary}(f + h)$ is an optimal solution to (1).

How large can the g get?

Lemma ([Grigorjew et al., 2024])

For all $c > 1$, there are instances with $pw(G)/mfd(G, f) > c$.

How large can the g get?



$O(\log \|f\|)$-approximation

Theorem [Grigorjew et al., 2024]

For all $c > 1$, MFD can be approximated with a factor of $O(\log \|f\|)$ in runtime $O(m \log ||f|| \cdot (\text{mfd}(G, f) + n))$ on Ch_2-f-minor free DAGs and on $GP(c)$-f-minor free DAGs.

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Theorem

For all $c > 1$, MFD can be approxmated with a factor of $O(\log \|f\|)$ on instances with $\text{pw}(G)/\text{mfd}(G, f) < c$.

Complexity result

Corollary [Grigorjew et al., 2024]

MFD can be solved in quasi-polynomial time on $GP(c)$-f-minor free DAGs, when the flow is coded in unary.

Proof:

- Approx. algorithm: $\mathrm{mfd}(G, f) \leq \mathrm{pw}(G) \cdot \log \|f\|$.
- MFD is in FPT [Kloster et al., 2018]: $O(2^{k^2} \cdot (n + \log \|f\|)) \leq O(\|f\|^{\log \|f\|} \cdot (n + \log \|f\|))$.

Further problems

- Good approximation for MFD, when $\mathrm{pw}(G)/\mathrm{mfd}(G,f) > c$.
- Is MFD in APX? What about $GP(c)$-f-minor free graphs?
- PTAS on $GP(c)$-f-minor free graphs?
- Parameterized algorithms for (generalized) max flow or other flow problems?
- Structural graph theorems using flows?

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Thank you!

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