Unstable classes for discriminant complements

Ishan Banerjee

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What's a discriminant anyway?

- $$V_{d,n} := \mathbb{C}[x_0, \ldots, x_n]_{(d)}$$
- $$\Sigma_{d,n} = \{f \in V_{d,n} | \exists p \in \mathbb{P}^n, f(p) = 0, df(p) = 0\}$$
- $$U_{d,n} = V_{d,n} - \Sigma_{d,n}$$

  Polynomials with smooth zero locus in $\mathbb{P}^n$
- More generally have discriminant varieties $\Sigma(L) \subseteq H^0(X, L)$, where X - smooth proj. variety, L ample line bundle,

$$U(L) = H^0(X, L) - \Sigma(L)$$

Stability properties

- Tommasi(2014)

$$H^*(U_{d,n}, \mathbb{Q}) \cong H^*(GL_{n+1}(\mathbb{C}), \mathbb{Q})$$

  for $* < \frac{d}{2}$.
- (B -2020) X- Riemann surface, $\deg(L) = d$,

$$H^*(U(L), \mathbb{Z}) \cong H^*(G_d(X); \mathbb{Z}) \text{ for } * << d$$

  $G_d(X)$ - group associated to braid group on X.
- Aumonier(2021) - for all (X, L), spaces $U(L)$ satisfy homological stability.

Limits of stability (Work in Progress)

1. Thm (B, WIP)

$$H^*(U_{d,n}, \mathbb{Q}) = H^*(GL_{n+1}(\mathbb{C}), \mathbb{Q}) \text{ for } * < 2d - O(1)$$

$$H^*(U_{d,2}, \mathbb{Q}) = H^*(GL_3(\mathbb{C}), \mathbb{Q}) \text{ for } * < 4d - O(1)$$

   , this is sharp.
2. Trying to improve bound for $\mathbb{P}^n$ to $* < 2nd - O(1)$ (Conj. optimal).
3. Need to understand stratification of $\mathrm{Conf}_k(\mathbb{P}^n)$.
4. Arithmetic consequences?

References

1. A. Aumonier An h-principle for complements of discriminants https://arxiv.org/abs/2112.00326
2. O. Tommasi, Stable cohomology of spaces of non-singular hypersurfaces. Adv. Math. 265 (2014), 428–440.
3. I. Banerjee Stable Cohomology of Discriminant Complements for an algebraic curve https://arxiv.org/abs/2010.14644

Monday, March 21, 2022 6:45 PM

Infinitely many planar exact Lagrangian fillings & symplectic Milnor fibers

Orsola Capovilla-Searle

UC Davis

Our setting:

$$(S^3, \ker(dz - y\,dx))$$

standard contact structure on S^3

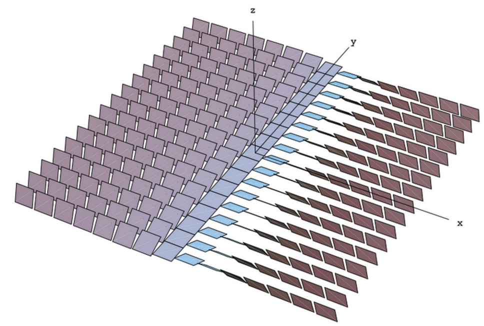
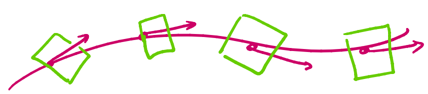


FIGURE 1. The standard contact structure on $\mathbb{R}^3$.
Image borrowed from Starkstons thesis.

Protagonists:

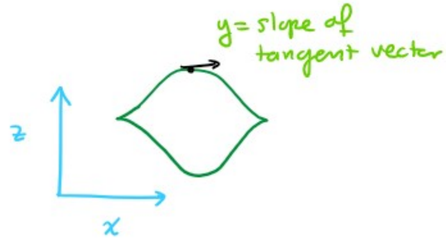
Legendrian links.

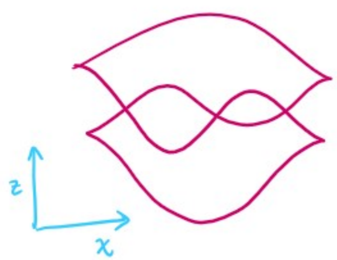


$\Lambda : S^1 \longrightarrow (S^3, \xi)$ is Legendrian if $T_p\Lambda \subset \xi$ for all $p \in \Lambda$

The front projection of a Legendrian

$F(x,y,z) = (x,z)$





Classification of legendrian links up to Legendrian isotopy

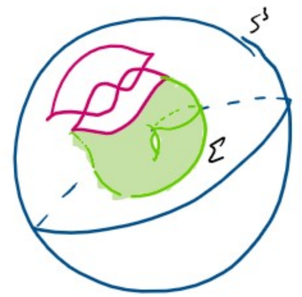
Classification of exact Lagrangian surfaces $\Sigma \subseteq (\mathbb{B}^4, \omega_{std})$

such that

$$\partial \Sigma = \Lambda \subseteq (S^3, \xi_{std})$$

(fillings of Λ)

up to Hamiltonian isotopy



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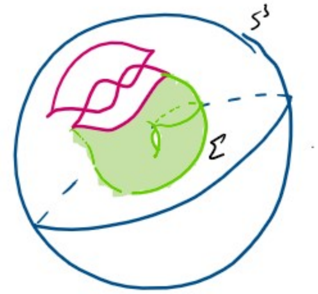
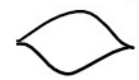


- for such fillings Σ the genus realizes the smooth 4-ball genus of $\partial\Sigma = \Lambda$
- The only Legendrian Λ whose exact fillings are classified is the max tb unknot  [Eliashberg Polterovich]

There exist Legendrian links with ∞-many distinct Lagrangian fillings

[Casals - Gao]

[Gao-Shen-Weng], [Casals-Zaslow]

[Casals - Ng]

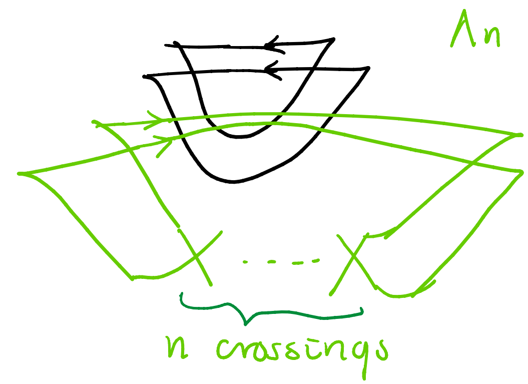
COR [- Legout - Limouzineau - Murphy - Pan - Traynor]

If Σ is an exact Lagrangian cobordism from Λ_- to Λ_+ and Λ_- has infinitely many distinct exact Lagrangian fillings distinguished by augmentations of the Legendrian contact homology dga $\Rightarrow$ so does Λ_+

THM [C-S '2021]

The Legendrian links $\Lambda_n \subseteq (\mathbb{R}^3, \xi_{std})$ for $n \geq 1$ have ∞-many distinct exact Lagrangian fillings.

Λ_1 and Λ_2 have ∞-many distinct genus 0 Lagrangian fillings.



Applications of the constructions of exact Lagrangian fillings

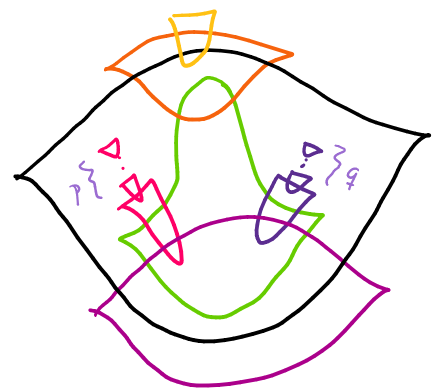
Weinstein 4-dim manifolds

Legendrian links in $(\#^k(S^1 \times S^2), \xi_{std})$ or (S^3, ξ_{std})

We find Weinstein handlebodies of

$T_{p,q,r} = \{(x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r + xyz = 1\}$

with ∞-many distinct exact Lagrangian tori built from fillings.
& spheres built from fillings of Λ_2.

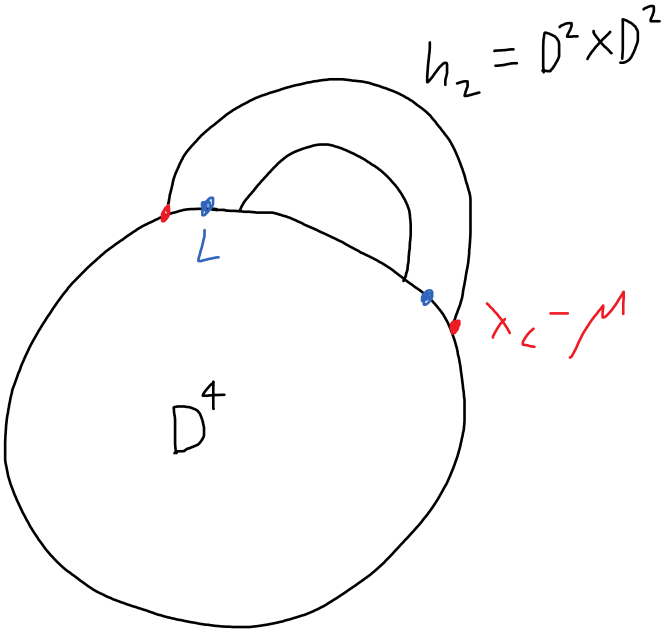


Thanks!

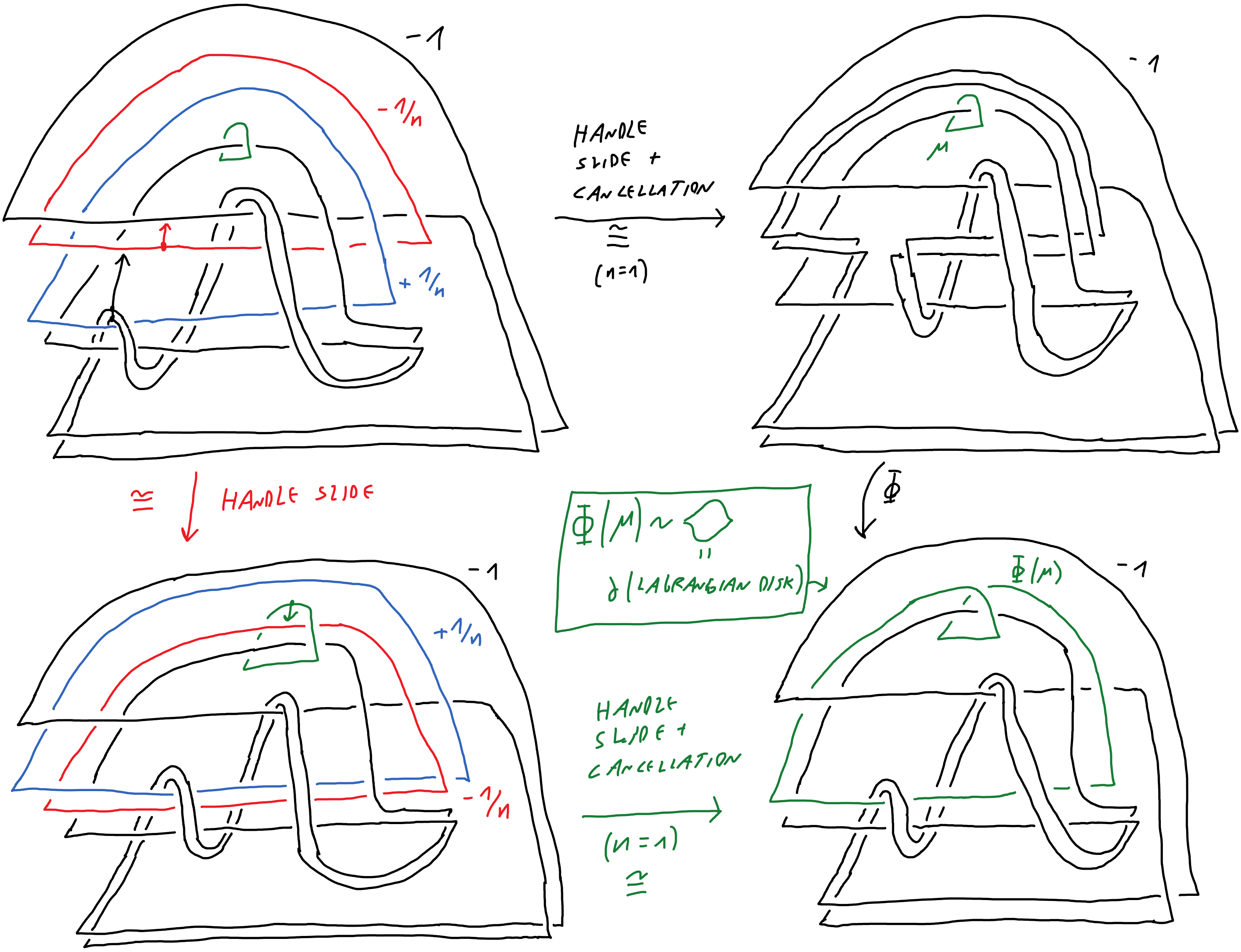
<u>Stein traces</u>

(joint with John Etnyre and Roger Casals)

arXiv:2111:00265



Proof of Theorem 2:



A winding number filtration on the Morse Complex

Francesco Morabito - École Polytechnique, CMLS

Setting

$\varphi \in Ham_c(\mathbb{R}^2)$

Laudenbach - Sikorav: existence of a generating function quadratic at infty for φ.

$$\begin{array}{ccc} E & \xrightarrow{h} & \mathbb{R} \\ \downarrow & & \\ \mathbb{R}^2 & & \end{array}$$

Contractibility of $Ham_c(\mathbb{R}^2)$: if $x, y \in Fix(\varphi)$, $x \neq y \Rightarrow WN(x,y)$ is well defined (i.e. it does not depend on the Hamiltonian isotopy).

Milinković - Oh: $HM(h,g) \simeq HF(H,J)$ as filtered complexes ($\varphi = \phi_H^1$)

Result

Proposition Let h be a generating function for φ. The function

$$I : CM(h;\mathbb{F}) \times CM(h;\mathbb{F}) \longrightarrow \mathbb{Z} \quad \text{defined on the generators via}$$

$$(x,y) \mapsto \begin{cases} WN(x,y) & x \neq y \\ -\left\lfloor \frac{CZ(x)}{2} \right\rfloor & x = y \end{cases}$$

and extended to $I\left(\sum_i \lambda_i x_i , \sum_j \mu_j x_j\right) = \min\{ I(x_i, x_j) \mid \lambda_i \neq 0 \neq \mu_j \}$

gives a filtration on $CM(h+h;\mathbb{F})$, $h+h : E \times E \to \mathbb{R}$.

$\Rightarrow H\left(CM_\bullet^{I \geq k}(h+h;\mathbb{F})\right)$ is well defined.

Proof

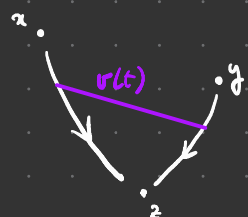
P. Le Calvez ('91, '99): winding numbers increase on pairs of orbits of $-\nabla h$, $h: E \to \mathbb{R}$

+

existence of an analogue for linearised dynamics: "dominated distribution"

<u>To do</u>: give a consistent definition of $WN(x,x)$.

Idea:



- Show that $\lim_{t\to\infty} \sigma(t) \in T_z W^s_{-\nabla h}(z)$

- Use properties of the dominated distribution to show that $I(x,y) \leq -\left\lfloor \frac{CZ(z)}{2} \right\rfloor$

Thank you for your attention.

Artin Braids from Infinitesimal Loops

Minh-Tâm Quang Trinh

Massachusetts Institute of Technology

$\Delta = \operatorname{Spec} \mathbf{C}[\![x]\!]$ = infinitesimal disk

$\eta = \operatorname{Spec} \mathbf{C}(\!(x)\!)$ = infinitesimal loop

$\pi_1^{\text{ét}}(\eta) \simeq \hat{\mathbf{Z}} = \varprojlim_n \mathbf{Z}/n\mathbf{Z}$

Suppose $W \curvearrowright V$ by real reflections.

$$Br_W := \pi_1^{top}(V^\circ/W)$$

where $V^\circ \subseteq V$ is the free locus.

Ex $S_n \curvearrowright \mathbf{C}^n$ gives $Br_W = Br_n$.

$$\begin{array}{ccc} \mathrm{Maps}(\eta, V^\circ/W) & \xrightarrow{\pi_1^{\text{ét}}} & \dfrac{\widehat{Br}_W}{\text{conjugacy}} \\ f & \mapsto & [\beta_f] \end{array}$$

Df $[\beta]$ *algebraic* iff $\exists f$ st $[\beta] = [\beta_f]$.

Prob Classify algebraic braids.

$\pi_W \in Z(Br_W)$ full twist

Thm If $[\beta] = [\beta_f]$ is algebraic, then

$$\beta^n \sim \pi_{W_1}^{e_1} \cdots \pi_{W_k}^{e_k}$$

for some n and $W = W_1 \supseteq \cdots \supseteq W_k$.

If f extends to $\Delta \to V /\!\!/ W$, then can take $n, e_1, \ldots, e_k \geq 0$.

Ex For $S_n \curvearrowright \mathbf{C}^n$, recover that links of plane curves are *iterated torus* type.

Knot of $y = x^{\frac{3}{2}} + x^{\frac{6}{7}}$ is closure of

$$\beta = (\sigma_2\sigma_1\sigma_3\sigma_2)^3\sigma_1^7 \in Br_4.$$

Have $\beta^4 \sim \pi_{S_4}^6 \pi_{S_2 \times S_2}$, where:

$$\pi_{S_4} = (\sigma_1\sigma_2\sigma_3)^4 \in Br_4$$

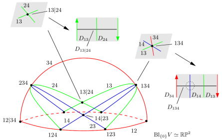
$$\pi_{S_2 \times S_2} = (\sigma_1\sigma_3)^2 \in Br_2 \times Br_2$$

Cor If $\Sigma(\beta)$ is the set of Burau eigenvalues of β, then

$$\max_{\lambda_q \in \Sigma(\beta_f)} \max_{|q|=1} \lambda_q = 1.$$

For $S_n \curvearrowright \mathbf{C}^n$, stronger: reducible with periodic components.

Proof of Thm Reduce to f lifting to $\tilde{f} : \Delta \to V$. Then lift to *wonderful compactification* $\tilde{V} \supseteq V^\circ$:



$D = \tilde{V} - V^\circ$ has normal crossings. *Winding* numbers e_i are *intersection* numbers with components of D.

Image: Feichtner, MSRI **52** (2005), 333-360