Thurston Theory: A tale of two theorems

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MSRI, College of the Holy Cross

joint with Jim Belk, Justin Lanier and Dan Margalit

  

Motto

Branched covers $S^2 \to S^2$ = higher degree braids

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A central goal in complex dynamics is to determine which branched covers are equivalent to rational maps

Thurston's Theorem

* f branched cover $(S^2, P) \to (S^2, P)$ $|P| < \infty$ is either:

1. Rational

2. Topologically obstructed

* Outside of a class of well-understood examples called Lattés maps

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Belk–Lanier–Margalit–W: Algorithm for polynomials

Main Result

f post-critically finite branched cover $\mathbb{C} \to \mathbb{C}$

Algorithm (Belk–Lanier–Margalit–W)

1. If polynomial

   $\rightsquigarrow$ determines the polynomial

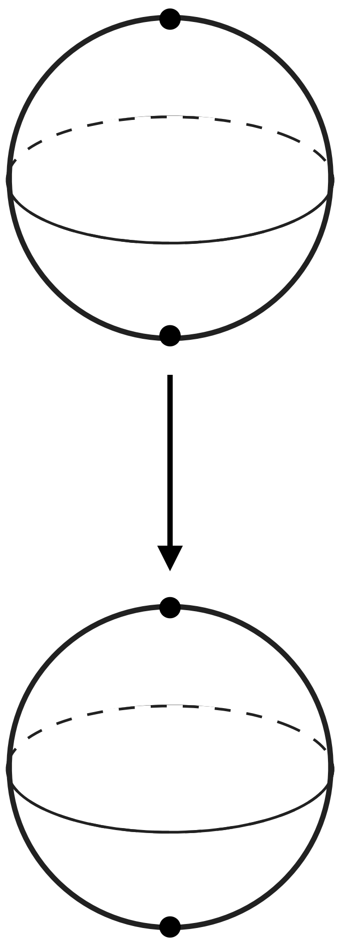
2. Otherwise, finds an obstruction

   canonical obstruction

Branched covers

Branched covers

Examples



$f(z) = z^d$

$0 \circlearrowleft$

$\infty \circlearrowleft$

Branched covers

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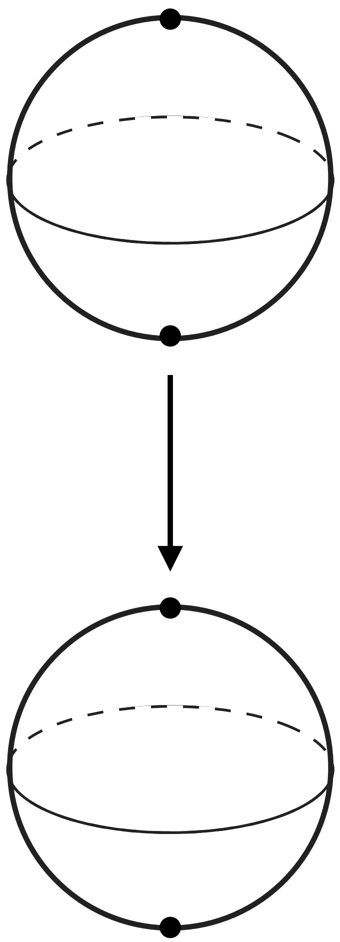

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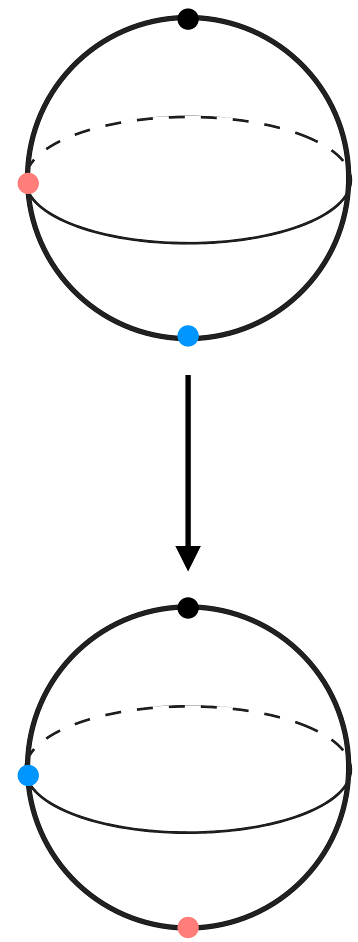

$$f(z) = z^2 - 1$$

$$0 \mapsto -1 \mapsto 0$$

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Branched covers

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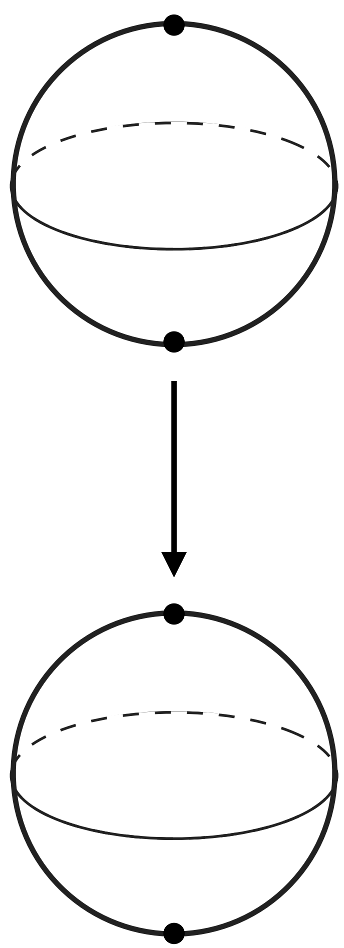

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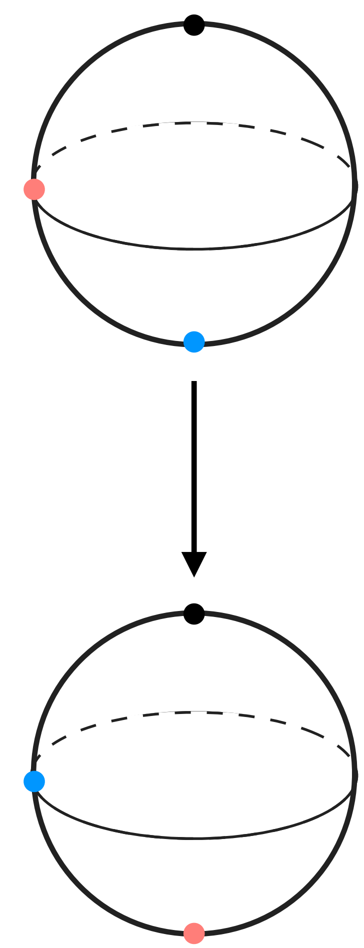

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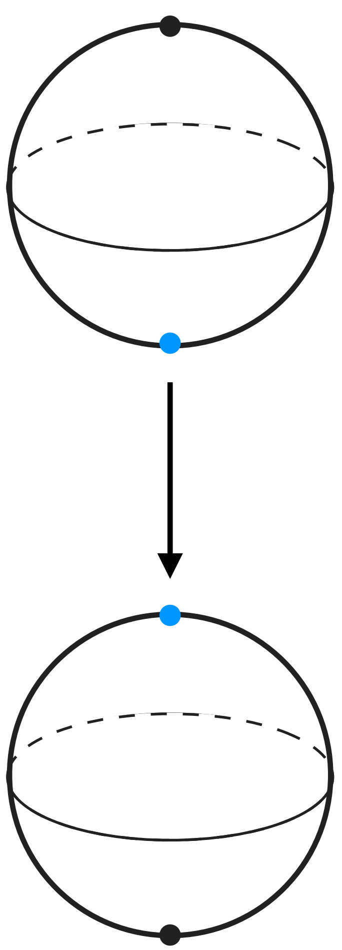

$$f(z) = \frac{1}{z^2}$$

$0 \mapsto \infty$

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Branched covers

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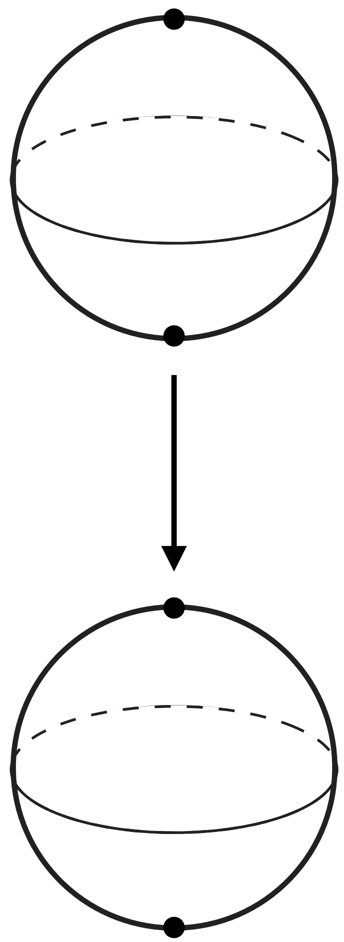

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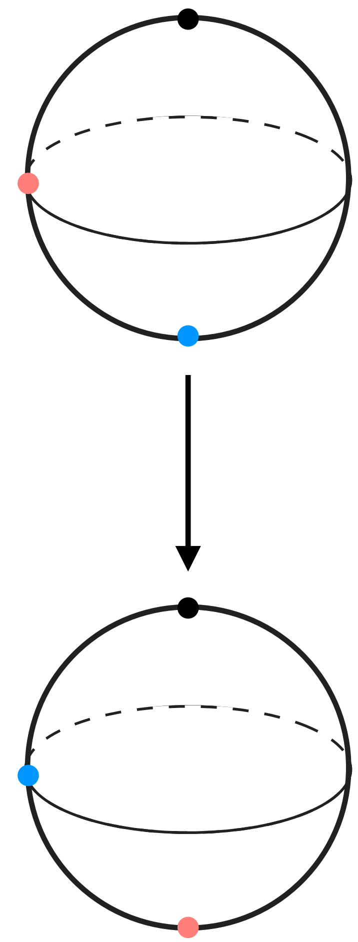

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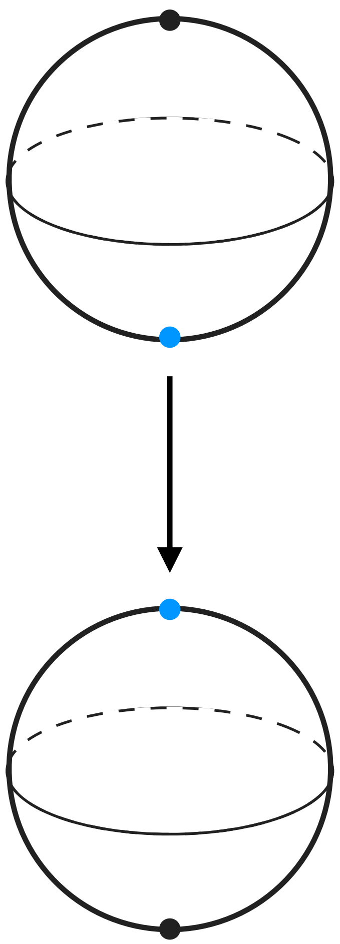

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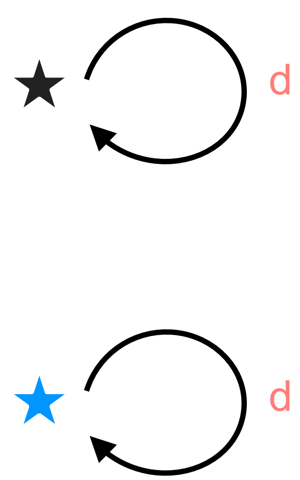
Portraits



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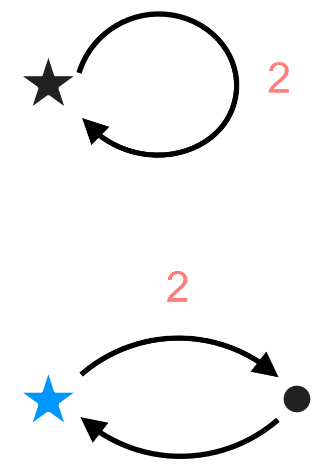

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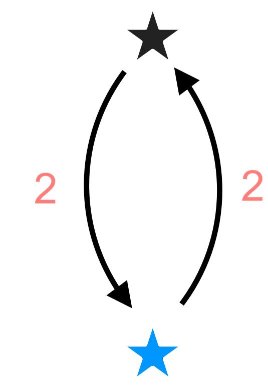

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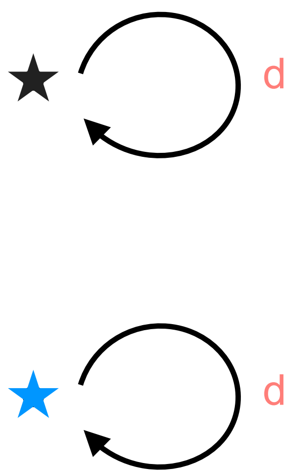

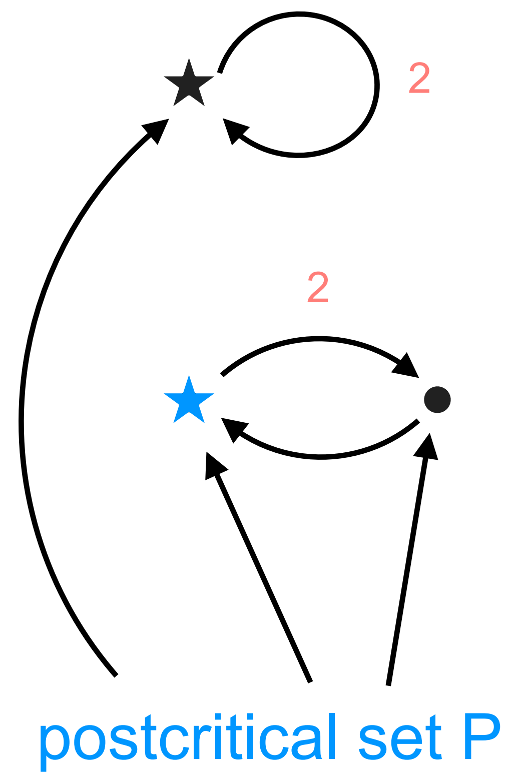

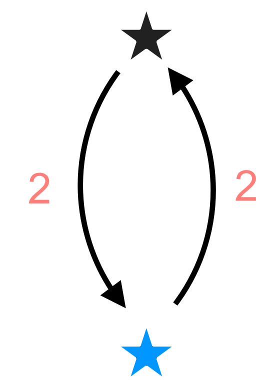

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Equivalence

$$f : (S^2, P_f) \to (S^2, P_f)$$

$$g : (S^2, P_g) \to (S^2, P_g)$$

branched covers

$$|P_f| = |P_g|$$

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$$\begin{array}{ccc} (S^2, P_f) & \xrightarrow{h_2} & (S^2, P_g) \\ \downarrow f & & \downarrow g \\ (S^2, P_f) & \xrightarrow{h_1} & (S^2, P_g) \end{array}$$

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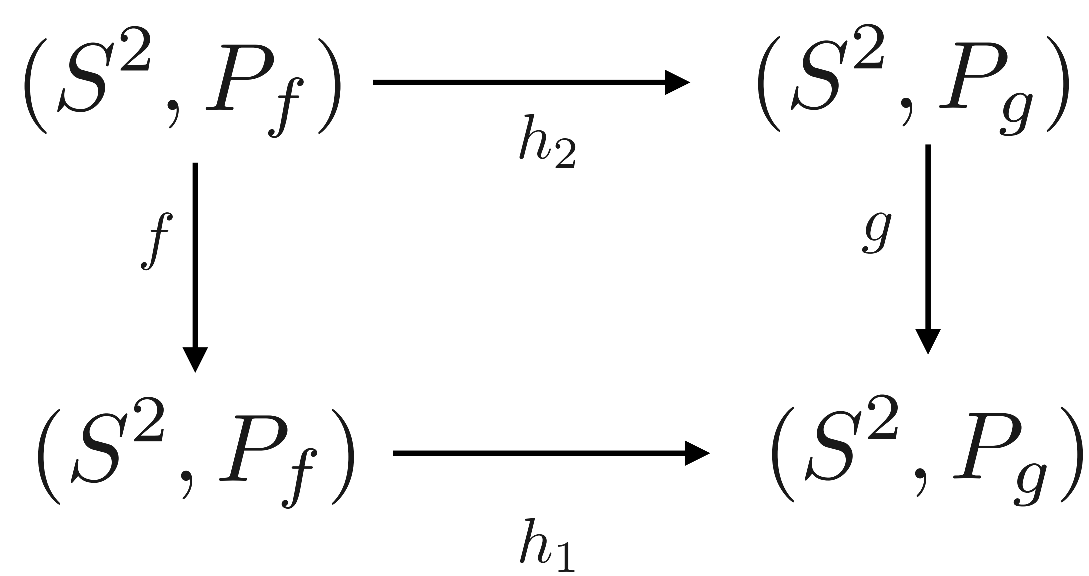

$$h_1 \underset{\text{rel. } P_f}{\simeq} h_2$$

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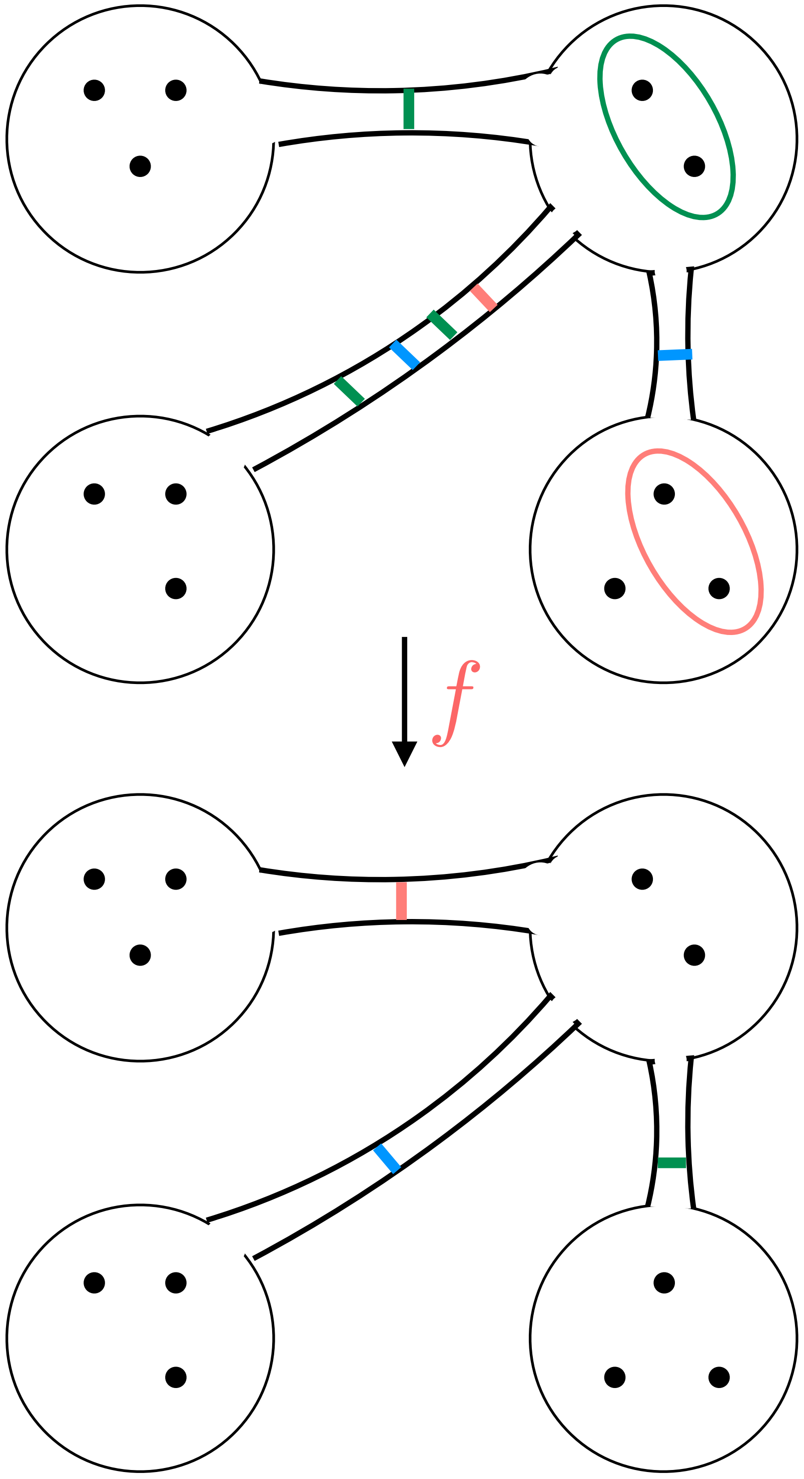

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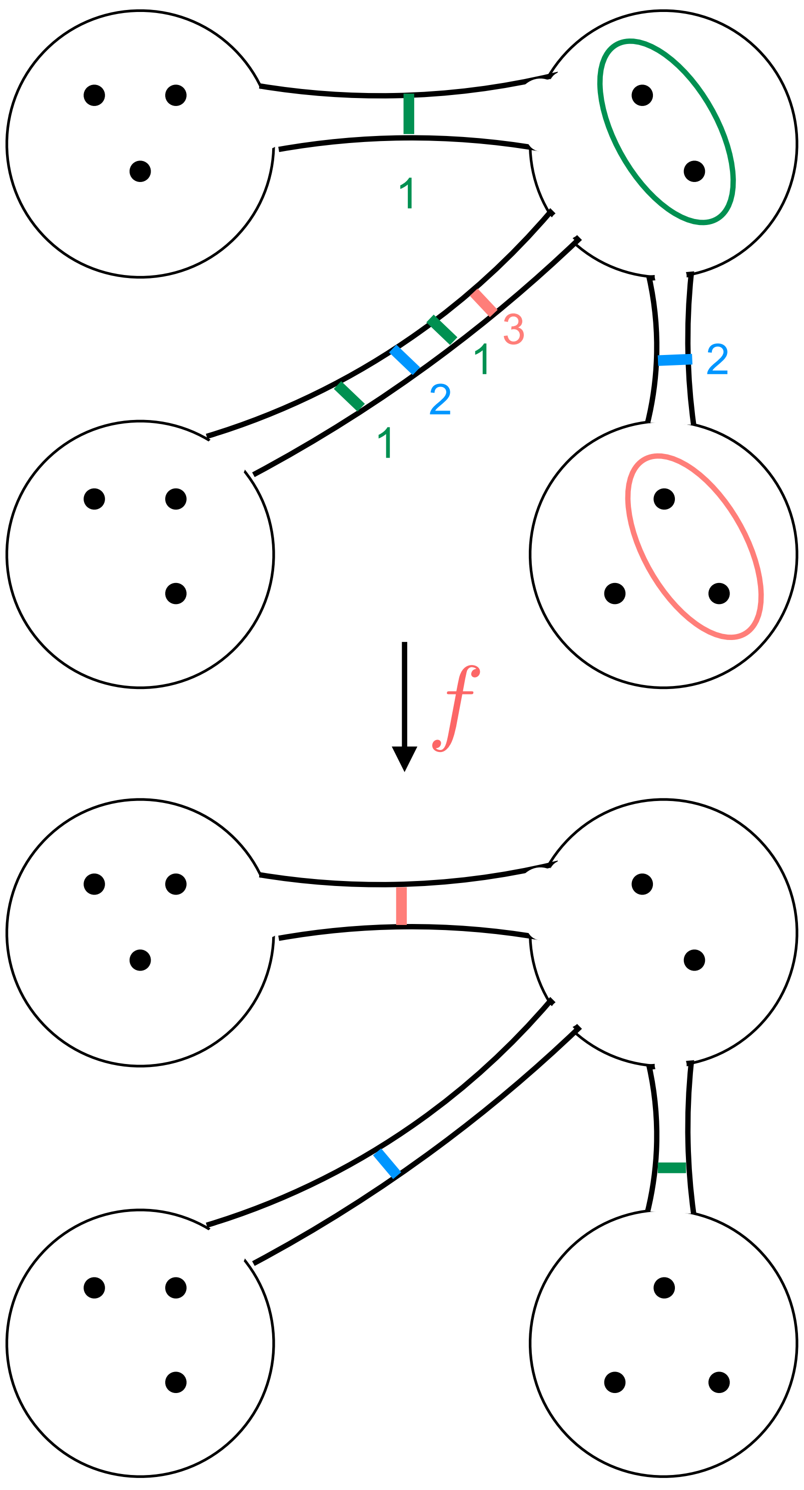

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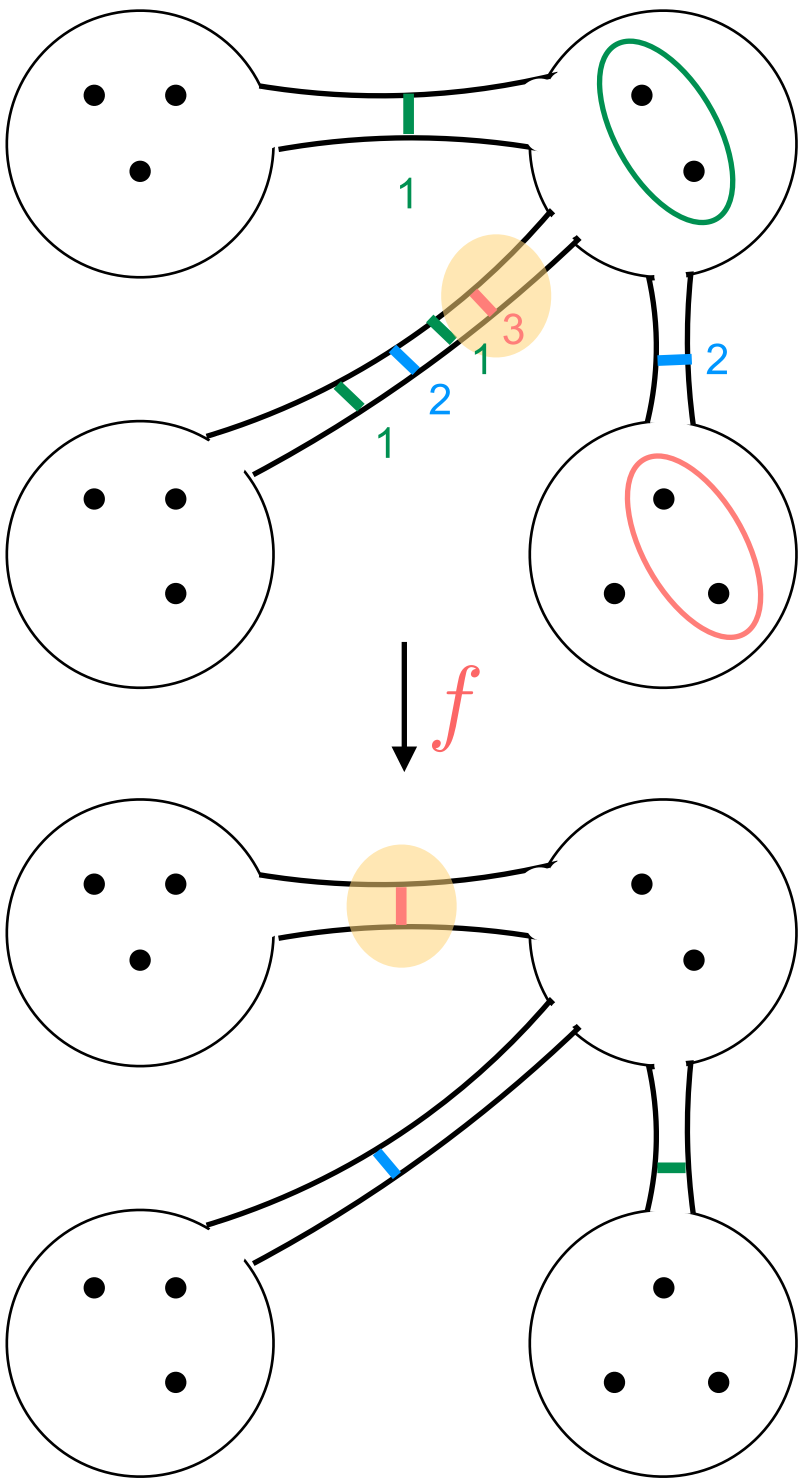

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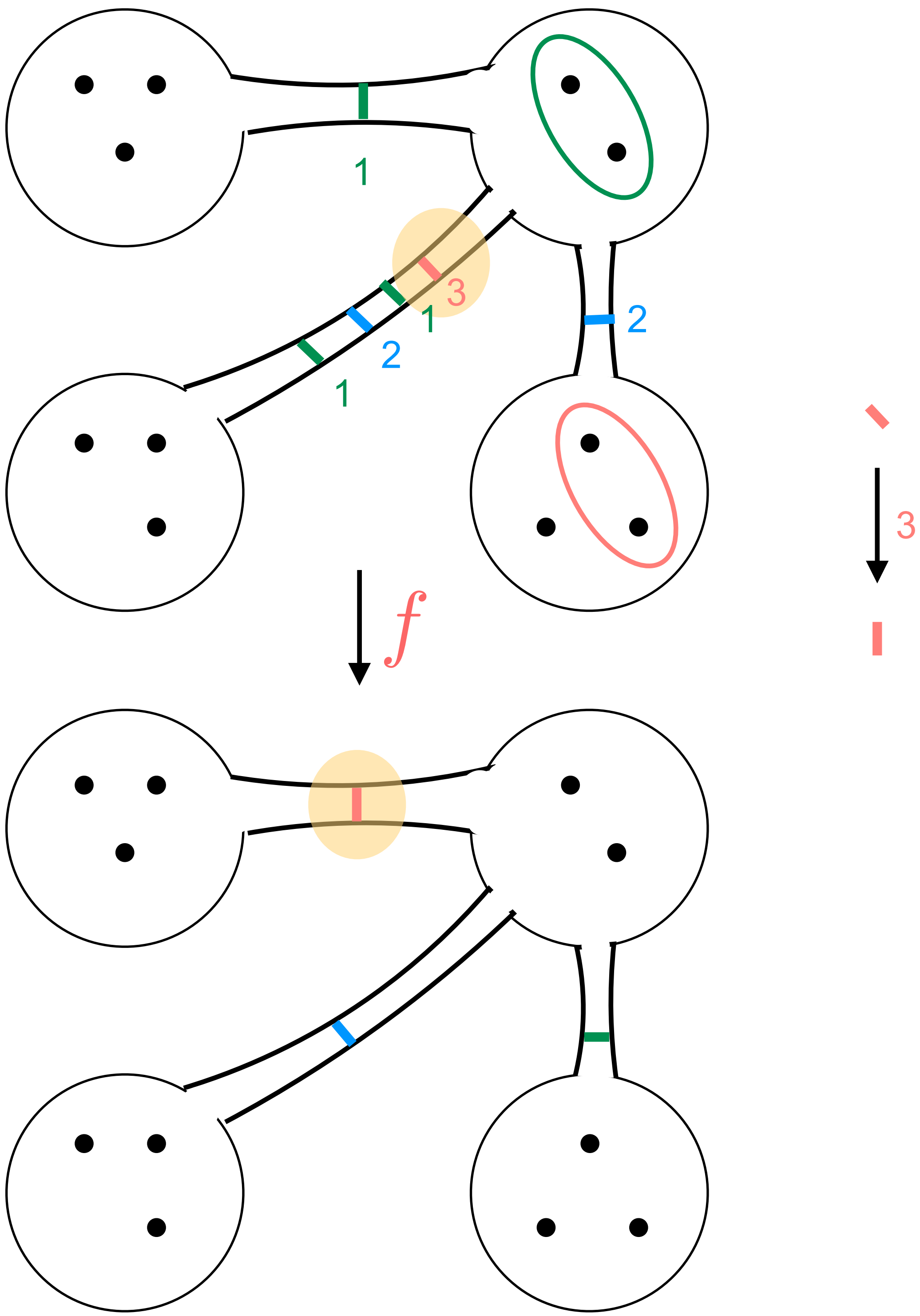

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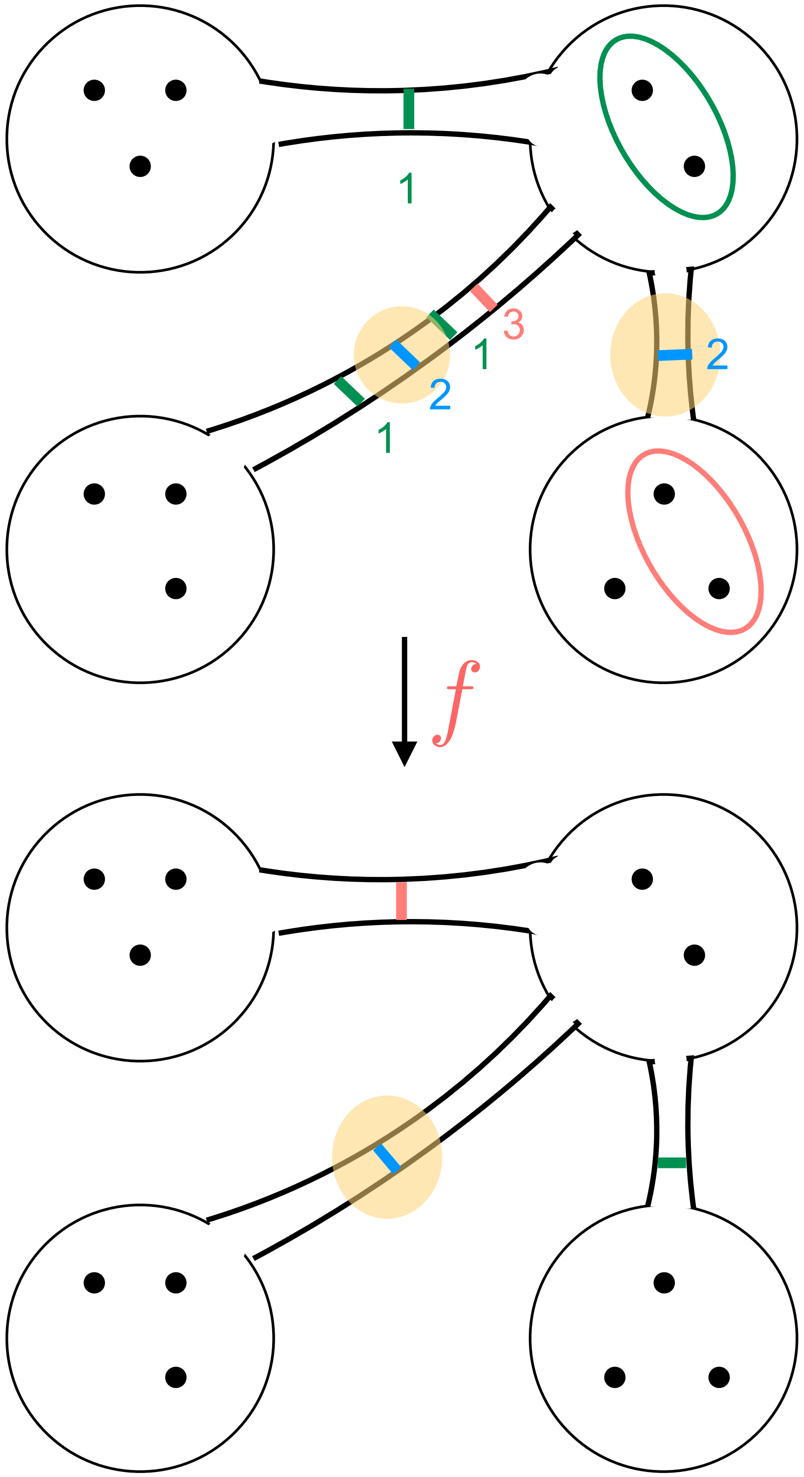

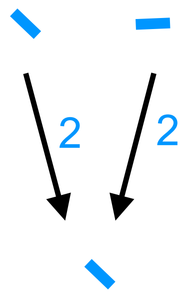

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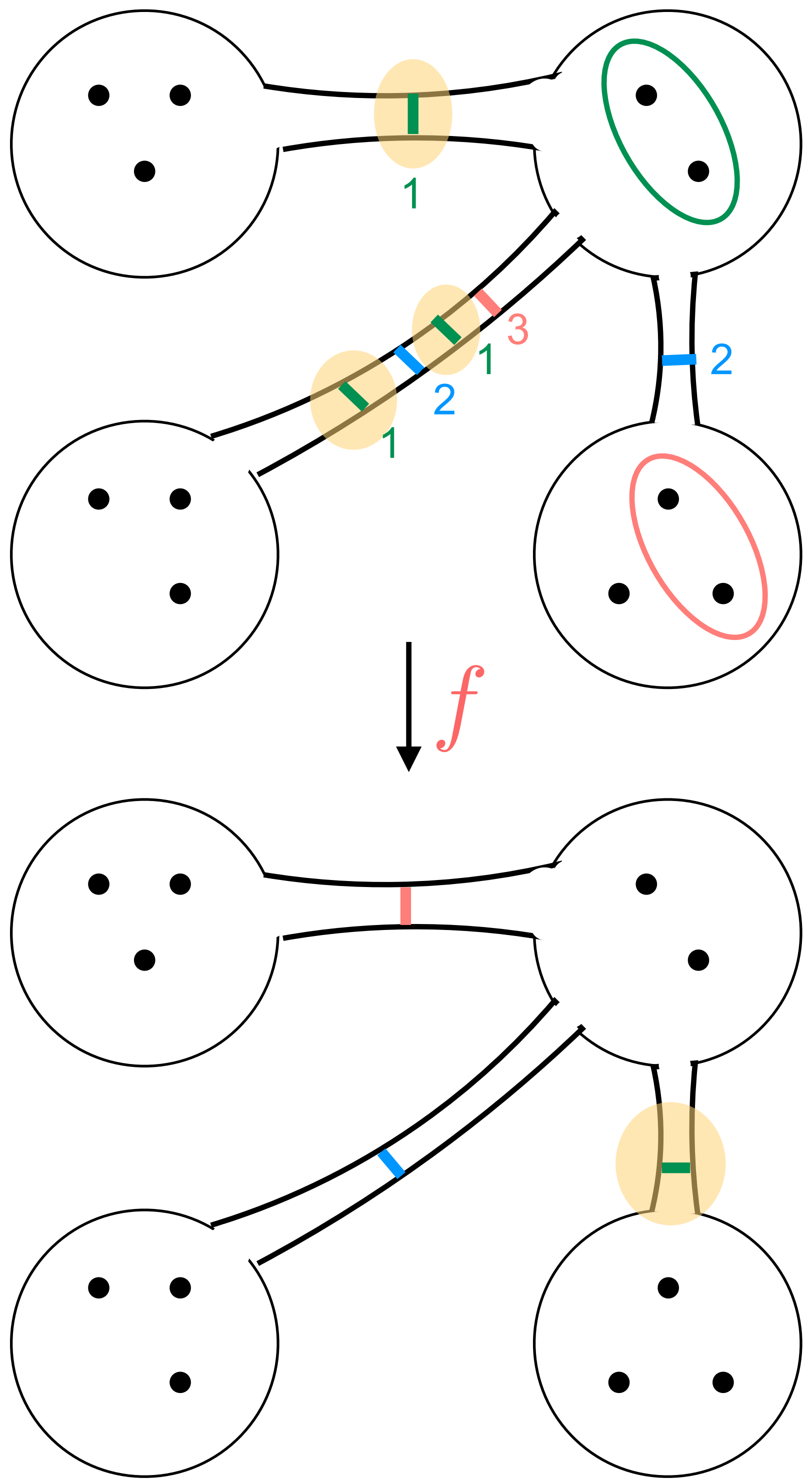

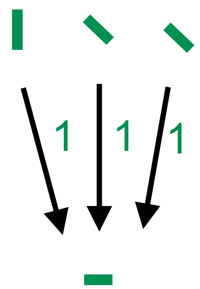

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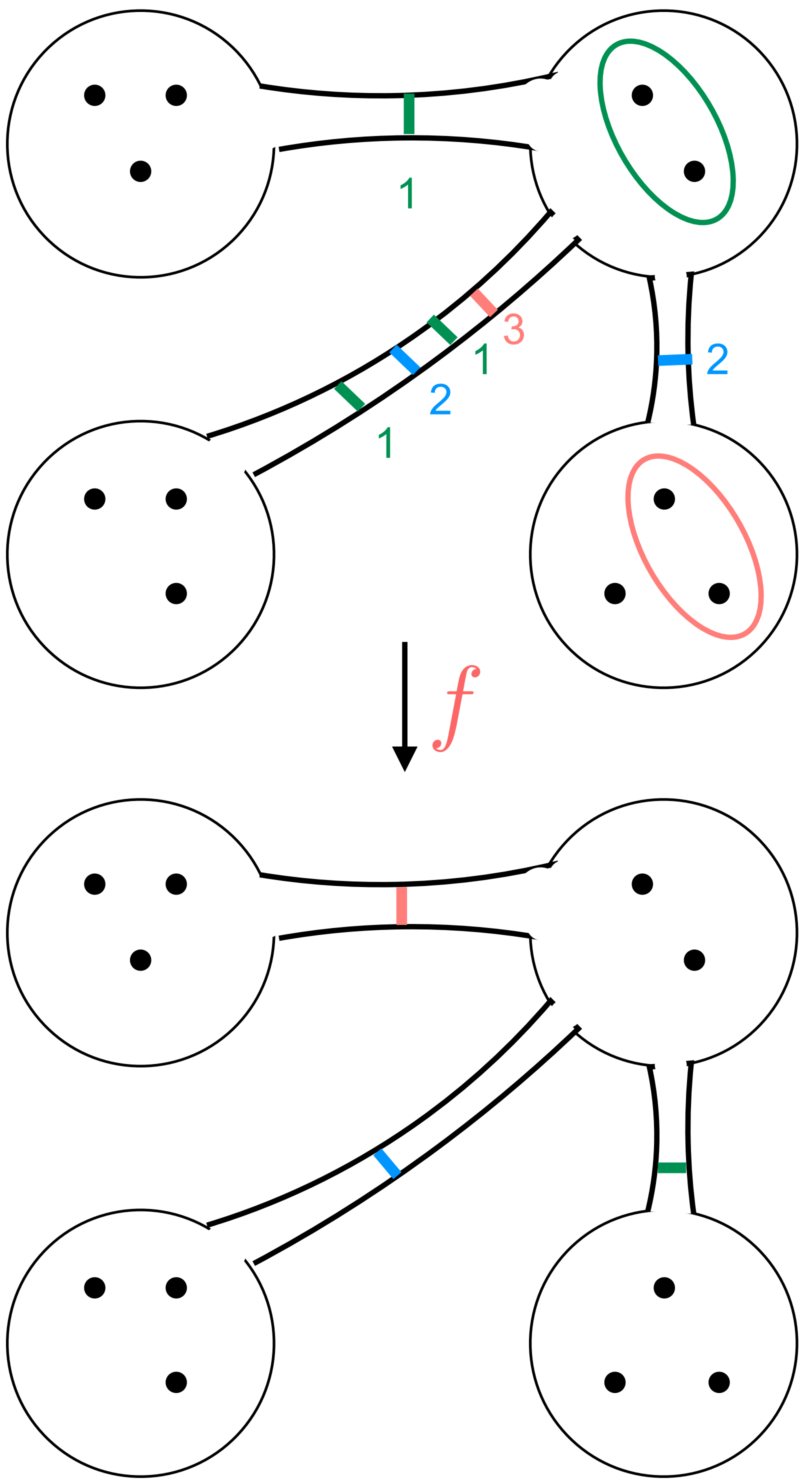

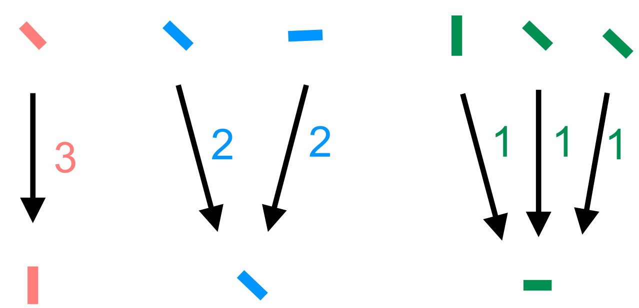

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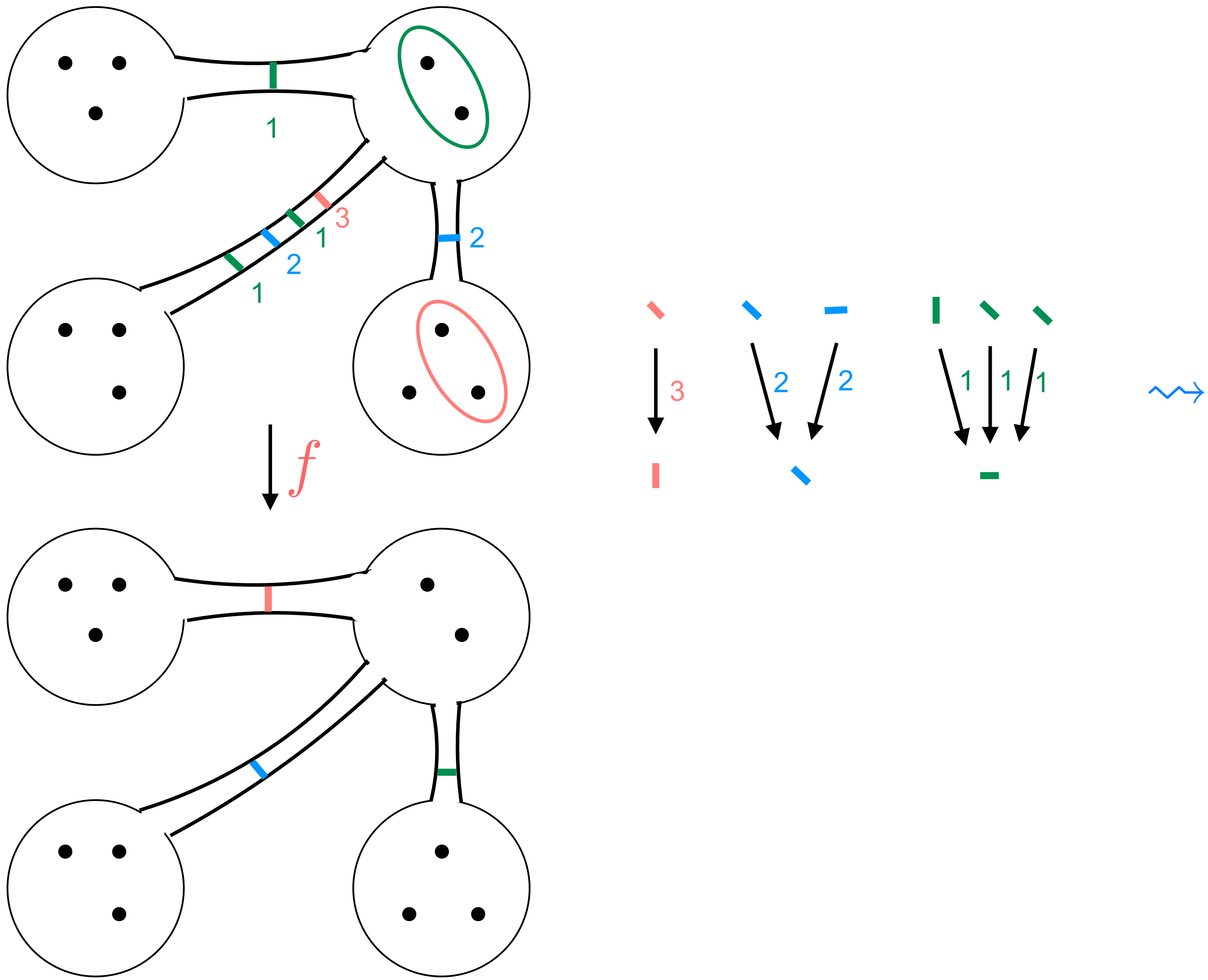

$$\begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{3} & \frac{1}{2} & 2 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

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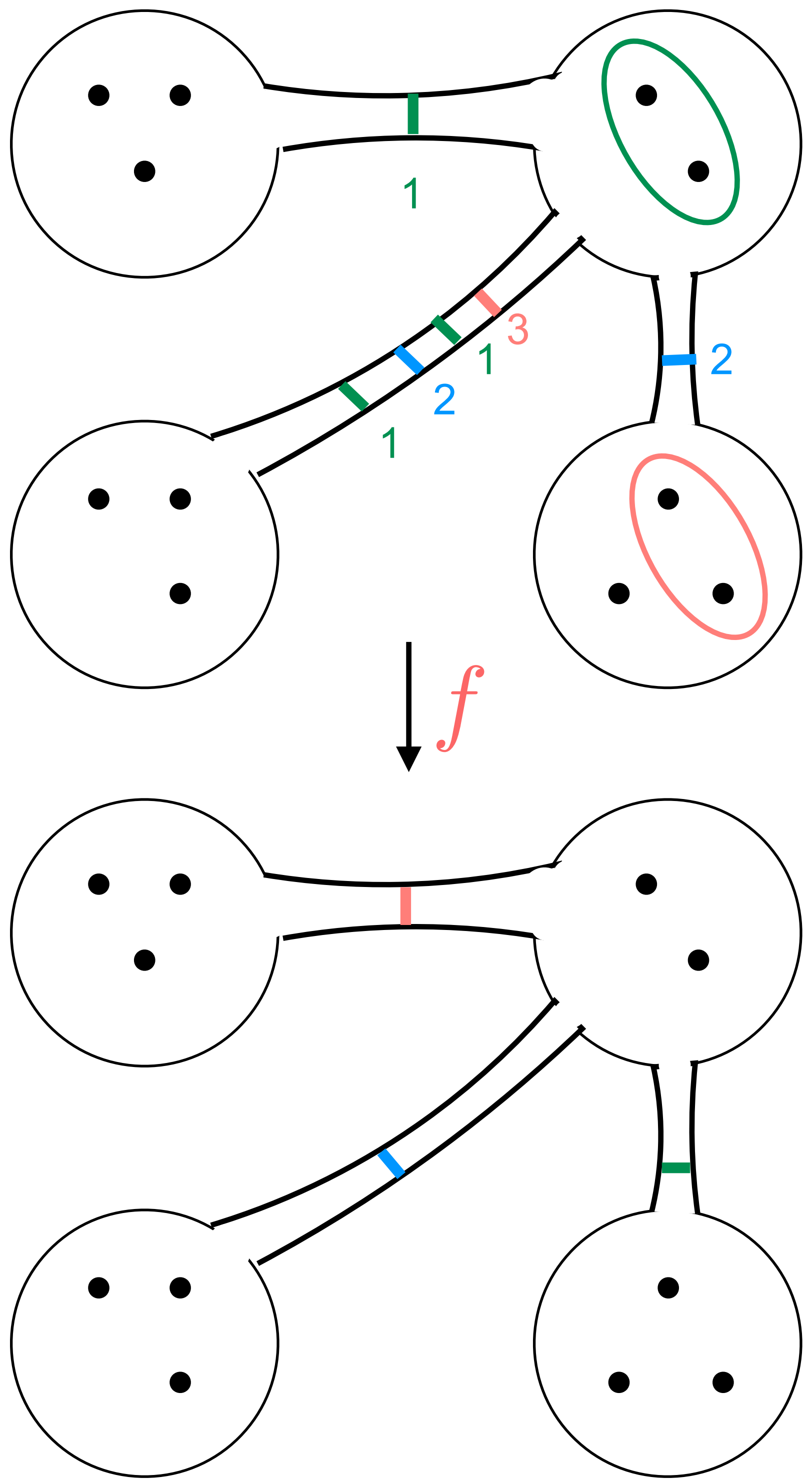

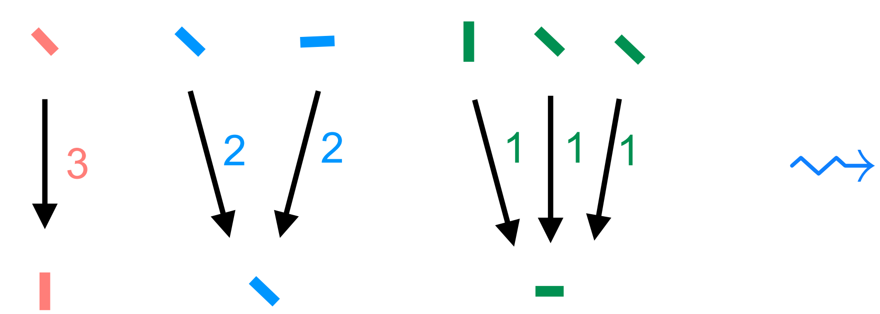

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Obstruction if eigenvalue ≥ 1

Characterization Theorem(s)

Nielsen–Thurston classification

$f:(S,P)\to(S,P)$ homeomorphism is homotopic to one:

1. Periodic

2. Reducible

3. pseudo-Anosov

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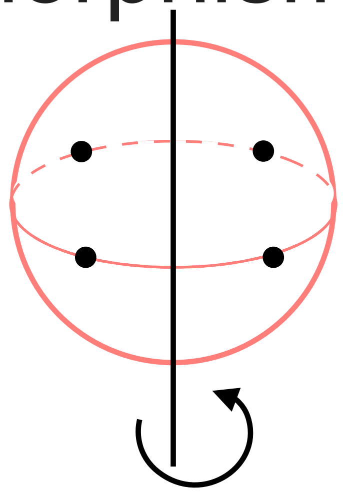

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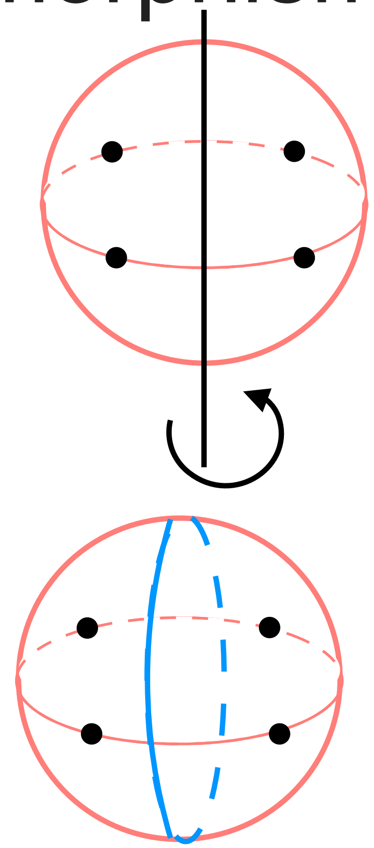

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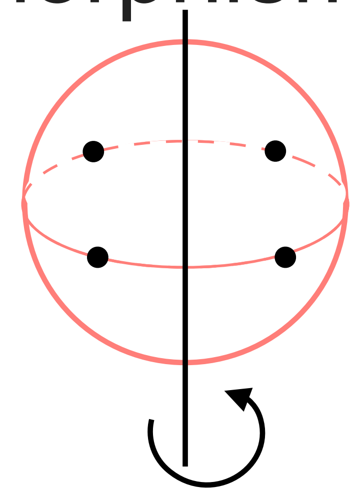

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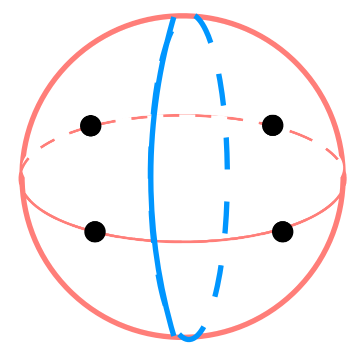
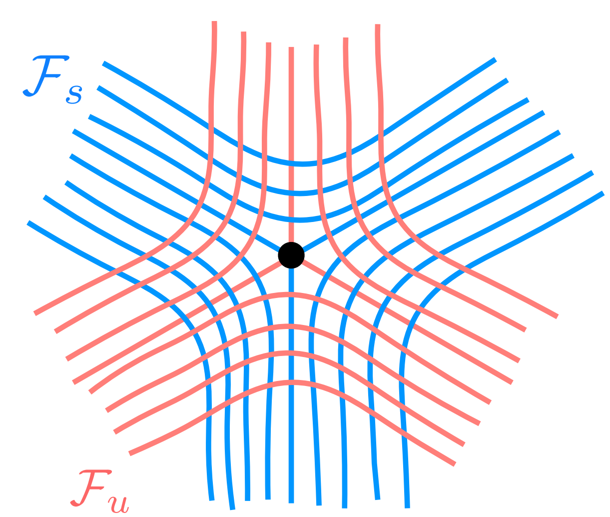


3. pseudo-Anosov

   $f(\mathcal{F}_s)=\lambda\mathcal{F}_s$

   $f(\mathcal{F}_u)=\frac{1}{\lambda}\mathcal{F}_u$



übertheorem

Theorem (Thurston)+epsilon

* f branched cover $(S^2, P) \to (S^2, P)$ is one of:

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* Also true for self-covers of tori, but those aren't braids

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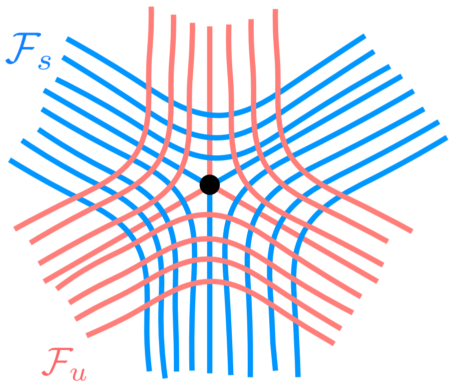

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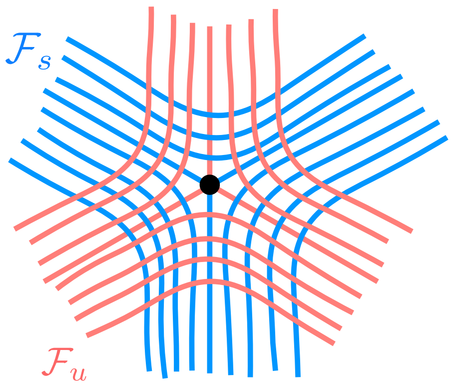
3. Pseudo-Anosov		

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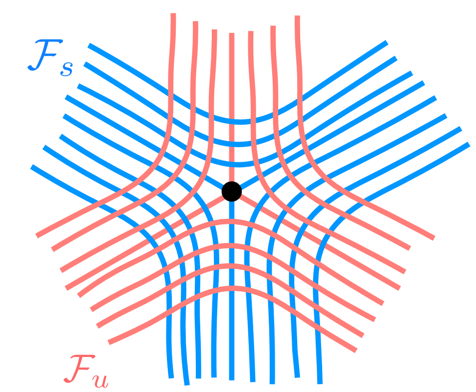
3. Pseudo-Anosov		

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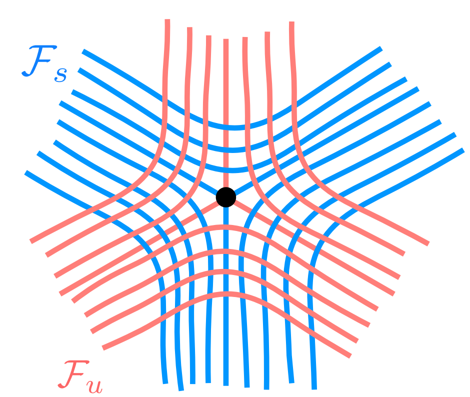
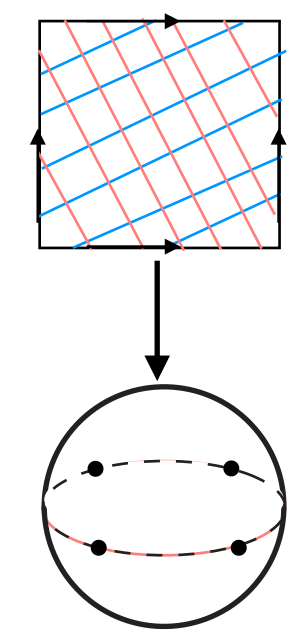
3. Pseudo-Anosov		

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Bers' Proof of Nielsen–Thurston (see Farb–Margalit)

$$\mathrm{Teich}(S^2, P) = \{\text{complex structures } (S^2, P)\}/\mathrm{Diff}_0(S^2, P)$$

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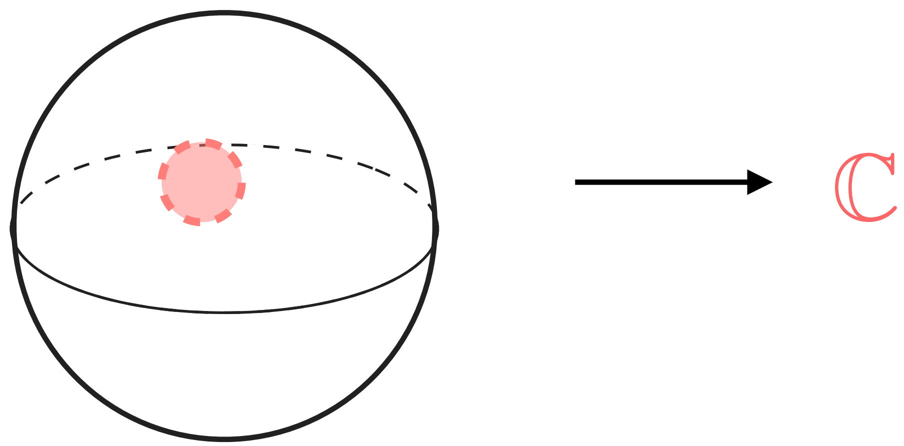

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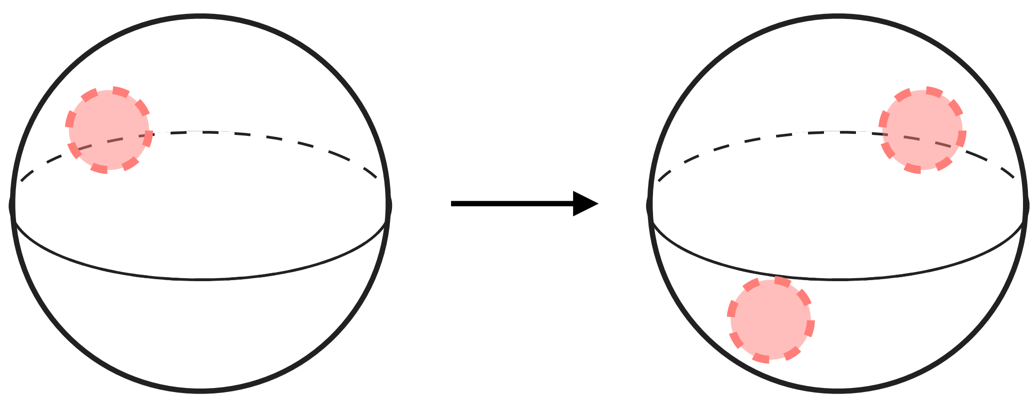

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σ_f^2 contraction unless special (parabolic orbifold with 4 branch points)

Teichmüller maps $\mapsto$ "Teichmüller maps" $\leq$ stretch factor

$\inf(d(X,\sigma_f(X))$

= 0 realized $\rightsquigarrow$ holomorphic

not realized $\rightsquigarrow$ invariant multi-curve

Proof Summary

Bers' Proof of Nielsen–Thurston (see Farb–Margalit)

$\mathrm{Teich}(S^2, P) = \{\text{complex structures } (S^2, P)\}/\mathrm{Diff}_0(S^2, P)$

$\sigma_f : \mathrm{Teich}(S^2, P) \to \mathrm{Teich}(S^2, P)$

σ_f weak contraction: $d(\sigma(X), \sigma(Y)) \leq d(X, Y)$

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Teichmüller maps $\mapsto$ "Teichmüller maps" $\leq$ stretch factor

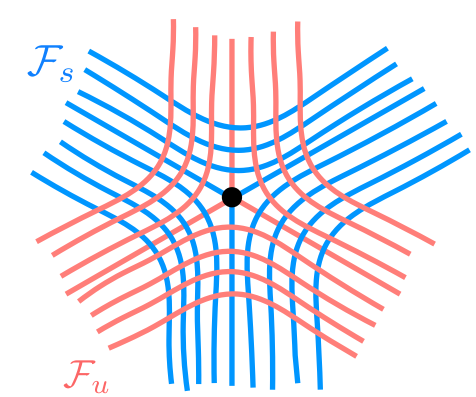
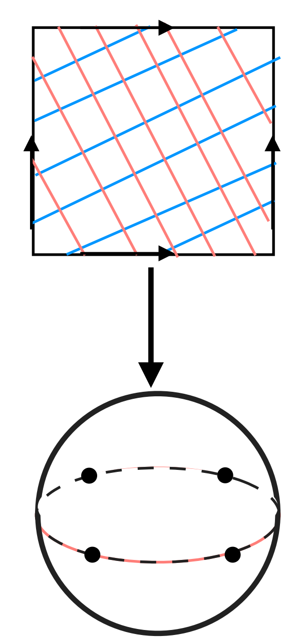
$\inf(d(X, \sigma_f(X))$

= 0 realized $\rightsquigarrow$ holomorphic
not realized $\rightsquigarrow$ invariant multi-curve
> 0 realized $\rightsquigarrow$ pseudo-Anosov

übertheorem

Theorem (Thurston)+epsilon

* f branched cover $(S^2, P) \to (S^2, P)$ is one of:

	d=1	d>1
1. Holomorphic	periodic	rational
2. Fixes multicurve	reducible	obstructed
3. Pseudo-Anosov		

* Also true for self-covers of tori, but those aren’t braids

Topological polynomials

Topological polynomials

Topological polynomials: branched self-covers $(\mathbb{C}, P)$

post-critical set $P \subset \mathbb{C}$

Topological polynomials

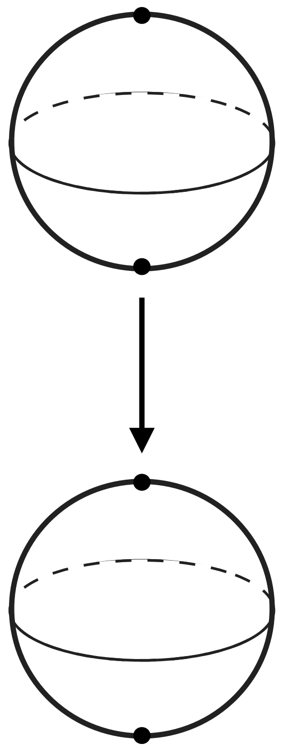
Topological polynomials: branched self-covers $(\mathbb{C}, P)$

post-critical set $P \subset \mathbb{C}$

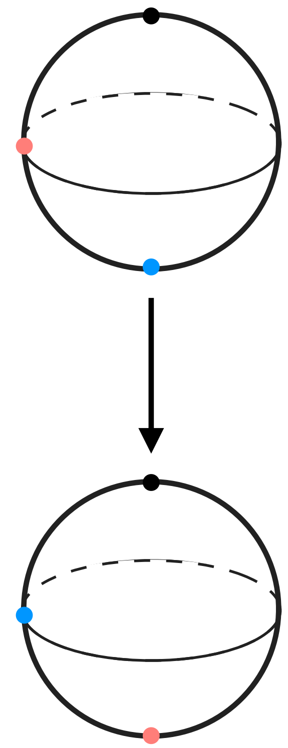
$\Leftrightarrow$

branched covers $f : (S^2, P \cup \infty) \to (S^2, P \cup \infty)$

such that $f^{-1}(\infty) = \{\infty\}$



$f(z) = z^d$



$f(z) = z^2 - 1$

Thurston's Theorem

Theorem (W. Thurston)

f post-critically finite topological polynomial, either

1. f is equivalent to a polynomial

2. f obstructed

Thurston's Theorem

Theorem (W. Thurston)

f post-critically finite topological polynomial, either

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Characterization problem:

Given a topological polynomial, determine whether or not it is equivalent to a polynomial. If so, which one?

Thurston's Theorem

Theorem (W. Thurston)
f post-critically finite topological polynomial, either

1. f is equivalent to a polynomial = has a Hubbard tree

2. f obstructed

Characterization problem:

Given a topological polynomial, determine whether or not it is equivalent to a polynomial. If so, which one?

Strategy

(Degree 1) braids

Maps (homeomorphisms) ⟷ Curves/Arcs

Alexander method

Strategy

(Degree 1) braids

Maps (homeomorphisms) ⟷ Curves/Arcs

Alexander method

Higher degree/Dynamical

Postcriticially finite

Polynomial ⟷ Hubbard tree

Strategy

(Degree 1) braids

Ma

Curves/Arcs

Alexander method

Hi

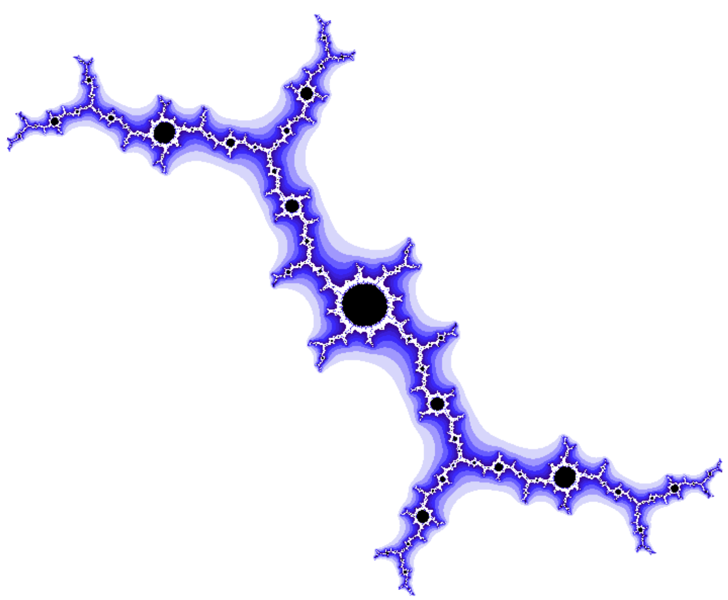
Po

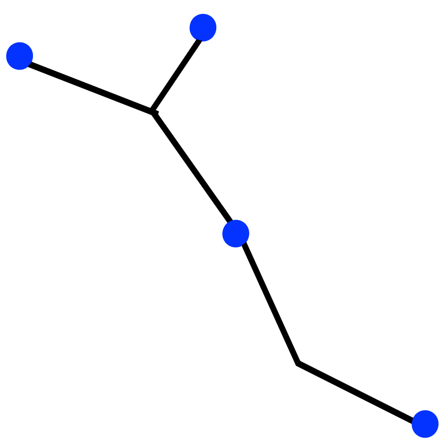
P

Hubbard tree

Tree + action

Invariant under preimage





Strategy

(Degree 1) braids

Ma ... ;urves/Arcs

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Po

P ... ɔbard tree

+ action

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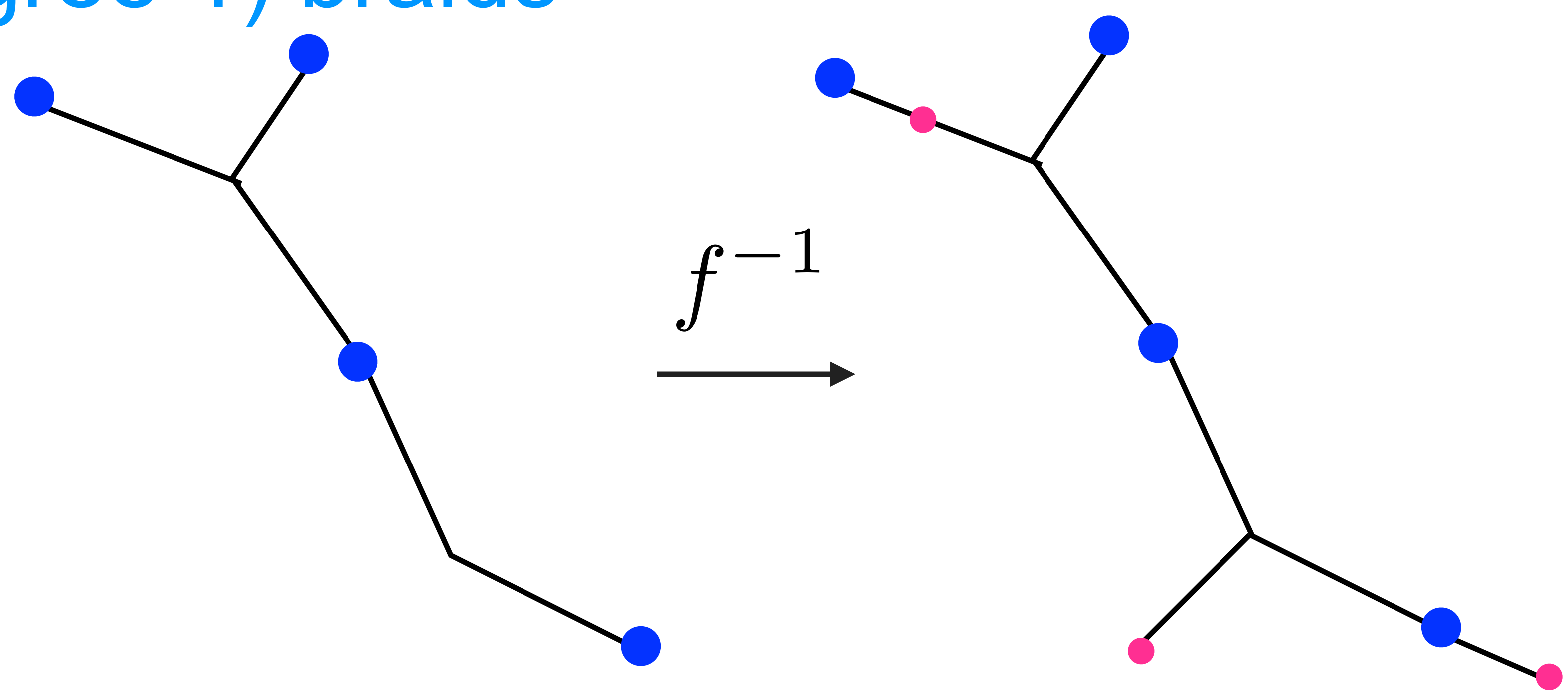

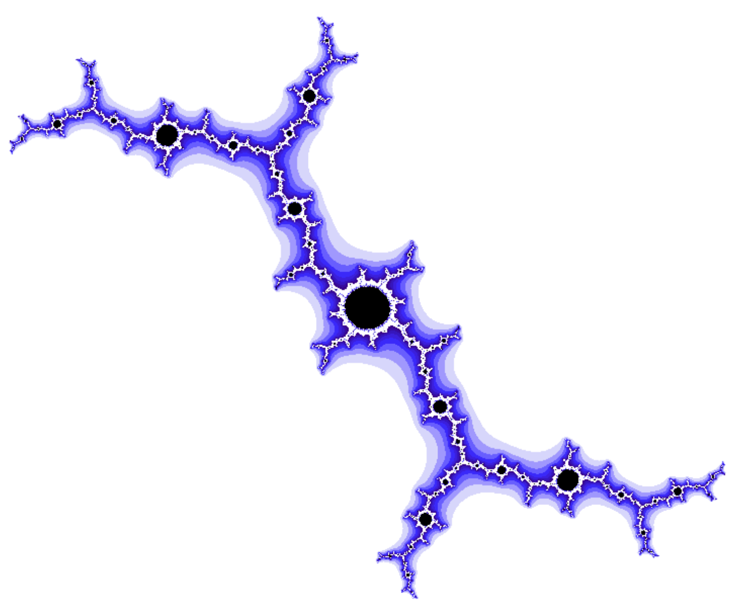

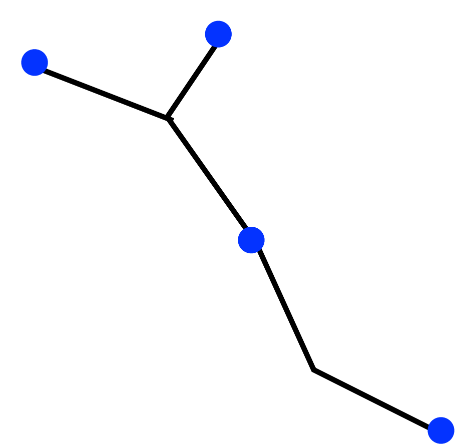

Strategy

(Degree 1) braids

Ma

Hig

Po

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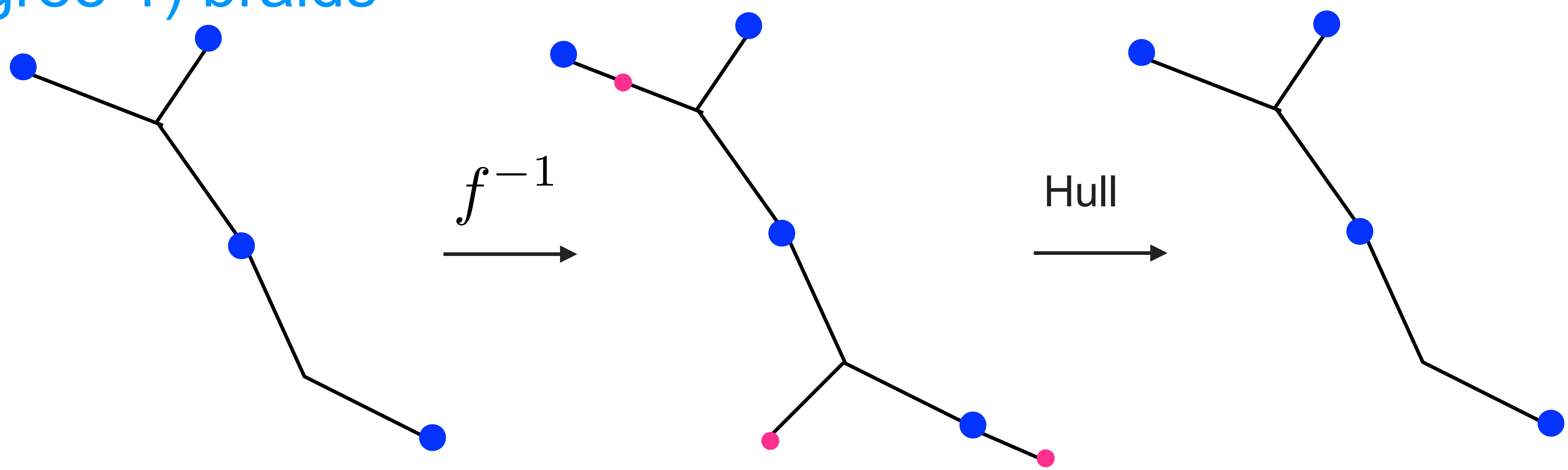
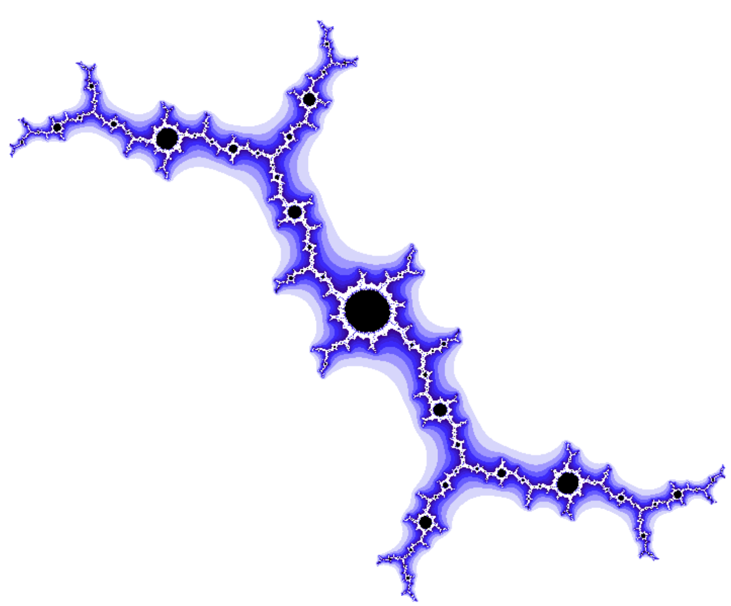
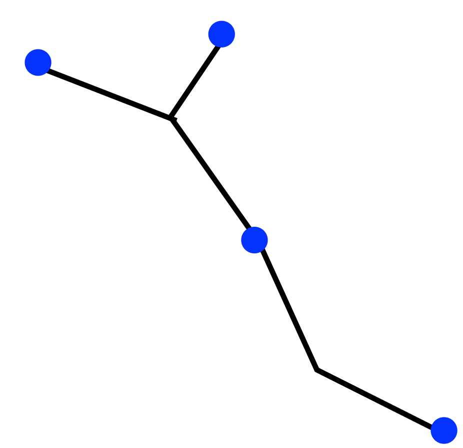






Strategy

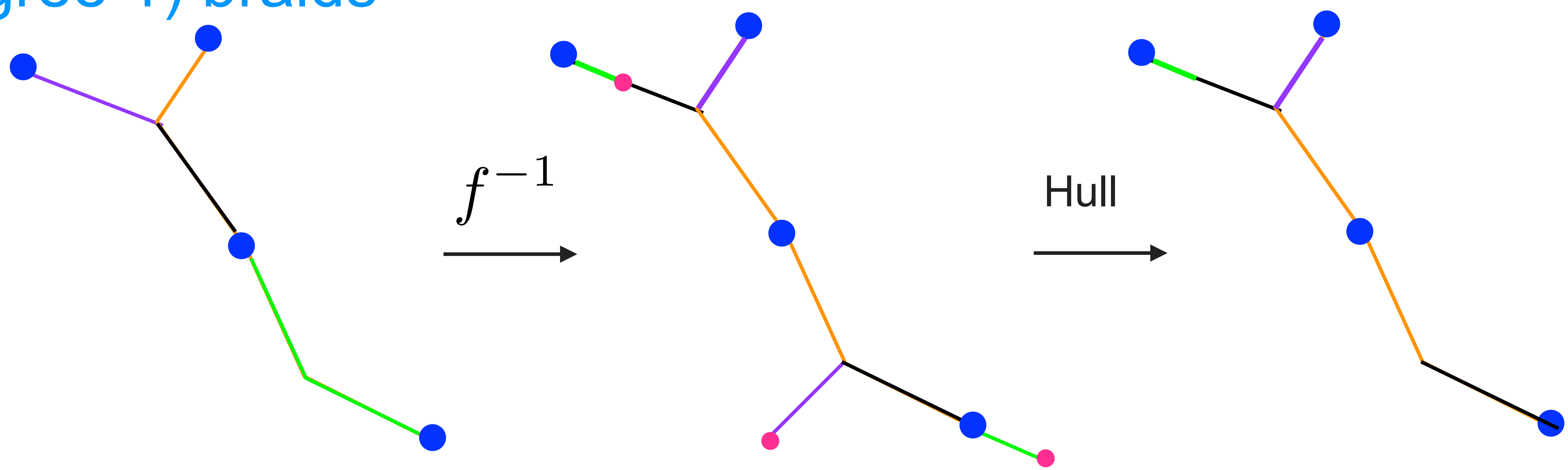
(Degree 1) braids

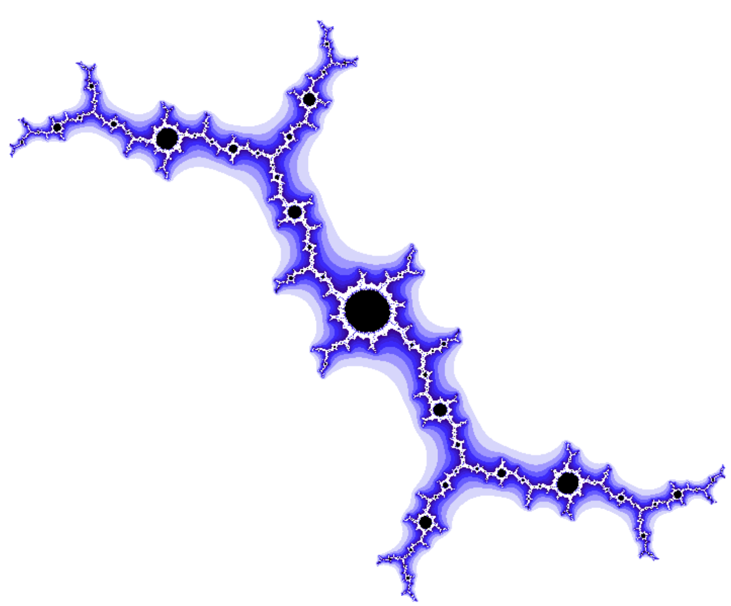
Ma

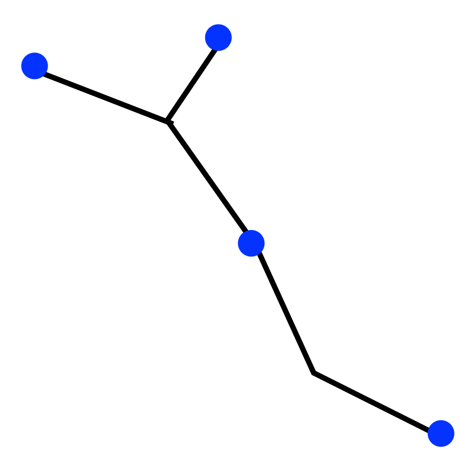
Hig

Po

P







Strategy

(Degree 1) braids

Maps (homeomorphisms) ⟷ Curves/Arcs

Alexander method

Higher degree/Dynamical

Postcriticially finite

Polynomial

Douady–Hubbard

⇄

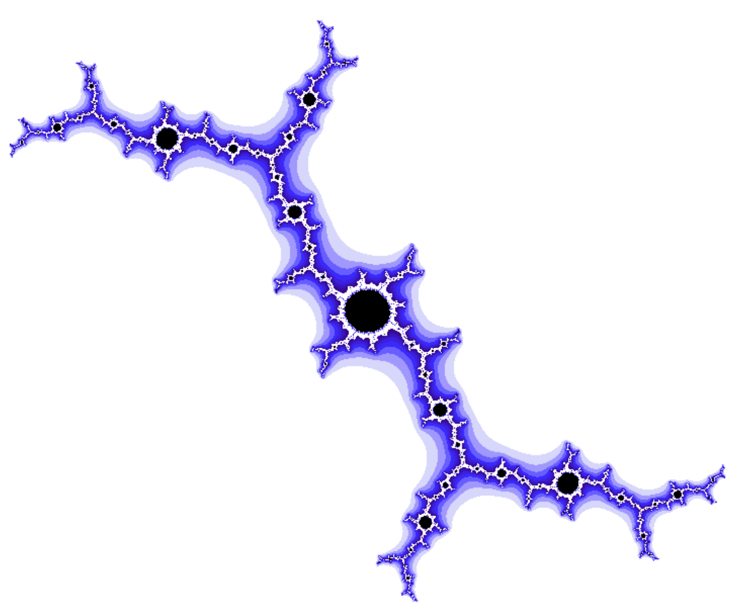
Poirier

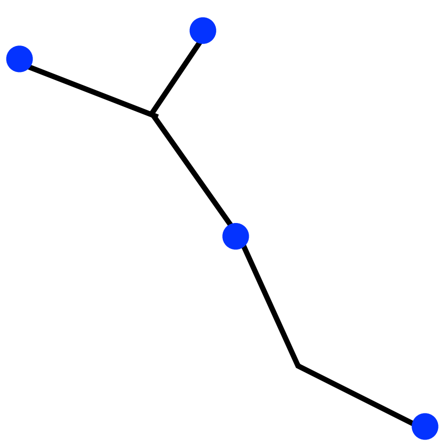
Alexander method

Hubbard tree

Tree + action

Invariant under preimage





Strategy

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Alexander method

Higher degree/Dynamical

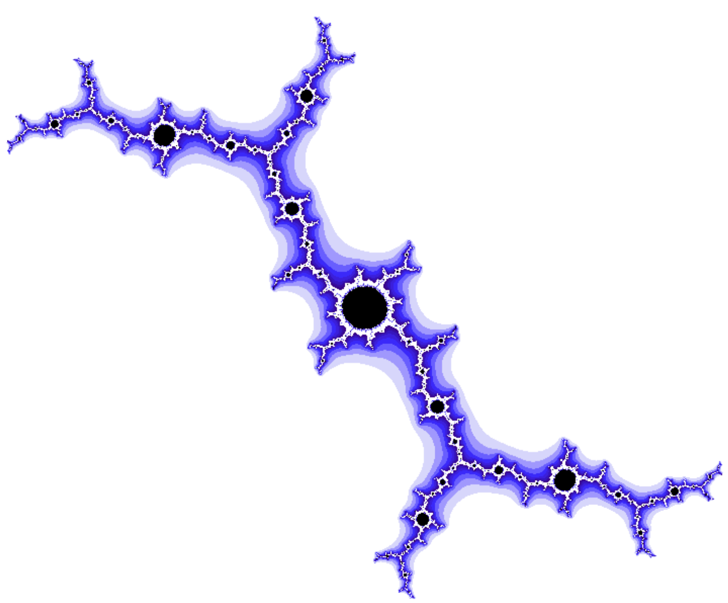
Postcritically finite [Polynomial] ⇄ [Hubbard tree]

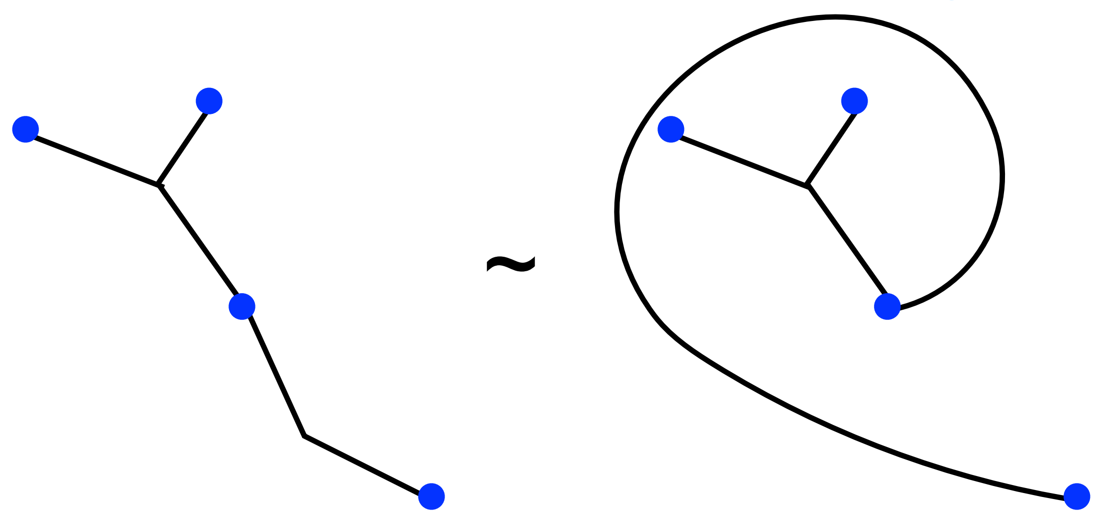
Douady–Hubbard (→)

Poirier Alexander method (←)

Tree + action

Invariant under preimage





Main Result

f post-critically finite branched cover $\mathbb{C} \to \mathbb{C}$

Algorithm (Belk–Lanier–Margalit–W)

1. Finds the Hubbard tree if equivalent to a polynomial

   $\rightsquigarrow$ determines the polynomial

2. Otherwise, finds an obstruction

   canonical obstruction

Strategy

1. Build a simplicial complex
2. Define simplicial map λ_f
3. Iterating λ_f converges to a finite set or horocycle
4. Check a neighborhood $\rightsquigarrow$ Hubbard tree or obstruction

Main Result

f post-critically finite branched cover $\mathbb{C} \to \mathbb{C}$

Algorithm (Belk–Lanier–Margalit–W)

1. Finds the Hubbard tree if equivalent to a polynomial

$\rightsquigarrow$ determines the polynomial

2. Otherwise, finds an obstruction

canonical obstruction

Strategy

1. Build a simplicial complex
2. Define simplicial map λ_f
3.
4.

f unobstructed $\Rightarrow \lambda_f$ converges to finite subcomplex

The Simplicial Complex

Tree Complex

Fixed set P

Tree Complex

Fixed set P

$\mathcal{T}_P$ = simplicial complex

Tree Complex

Fixed set P

$\mathcal{T}_P$ = simplicial complex

vertices: isotopy classes of trees

simplices: subforest collapses/expansions

Tree Complex

Fixed set P

$\mathcal{T}_P$ = simplicial complex

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Tree Complex

Fixed set P

$\mathcal{T}_P$ = simplicial complex

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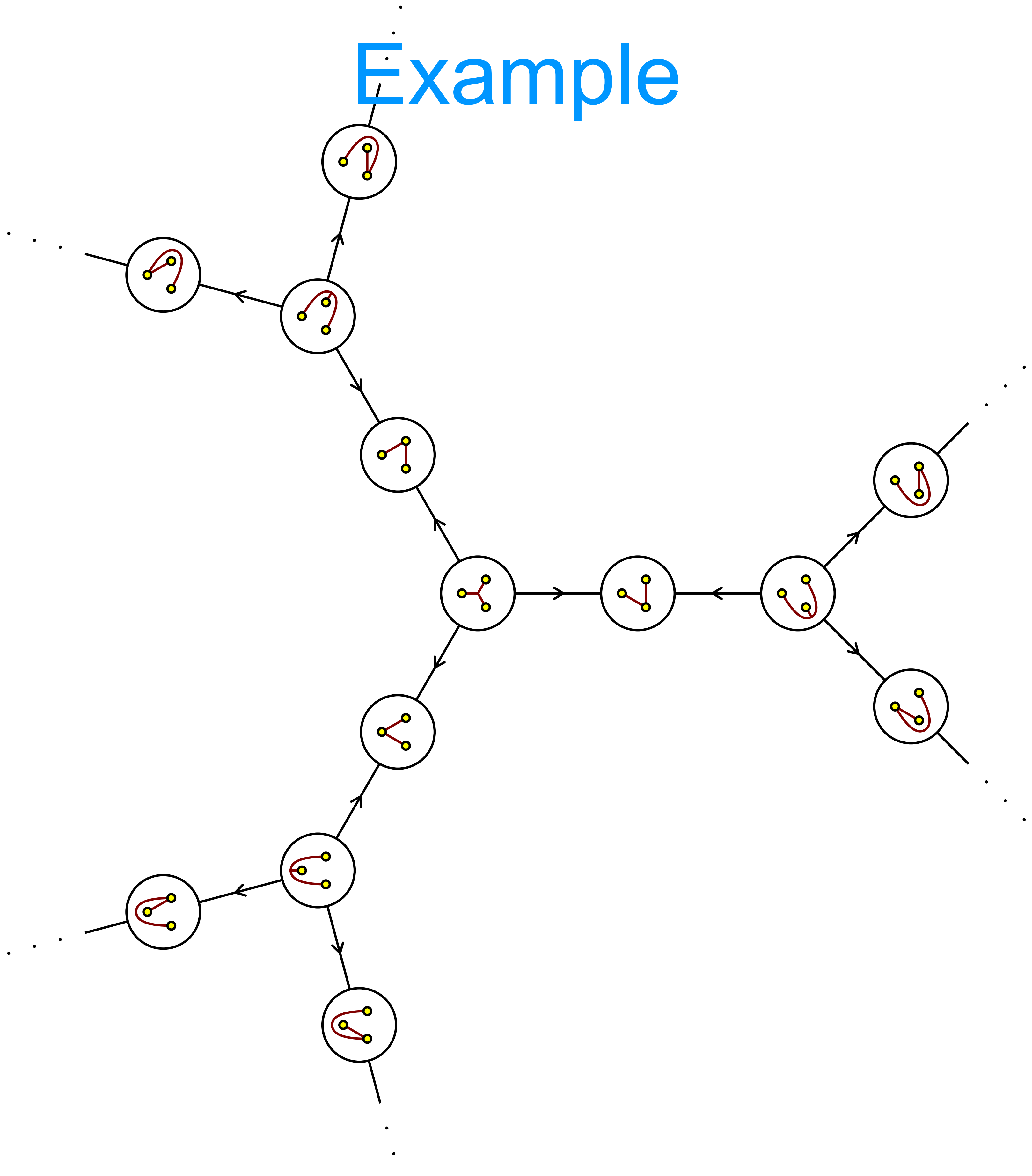
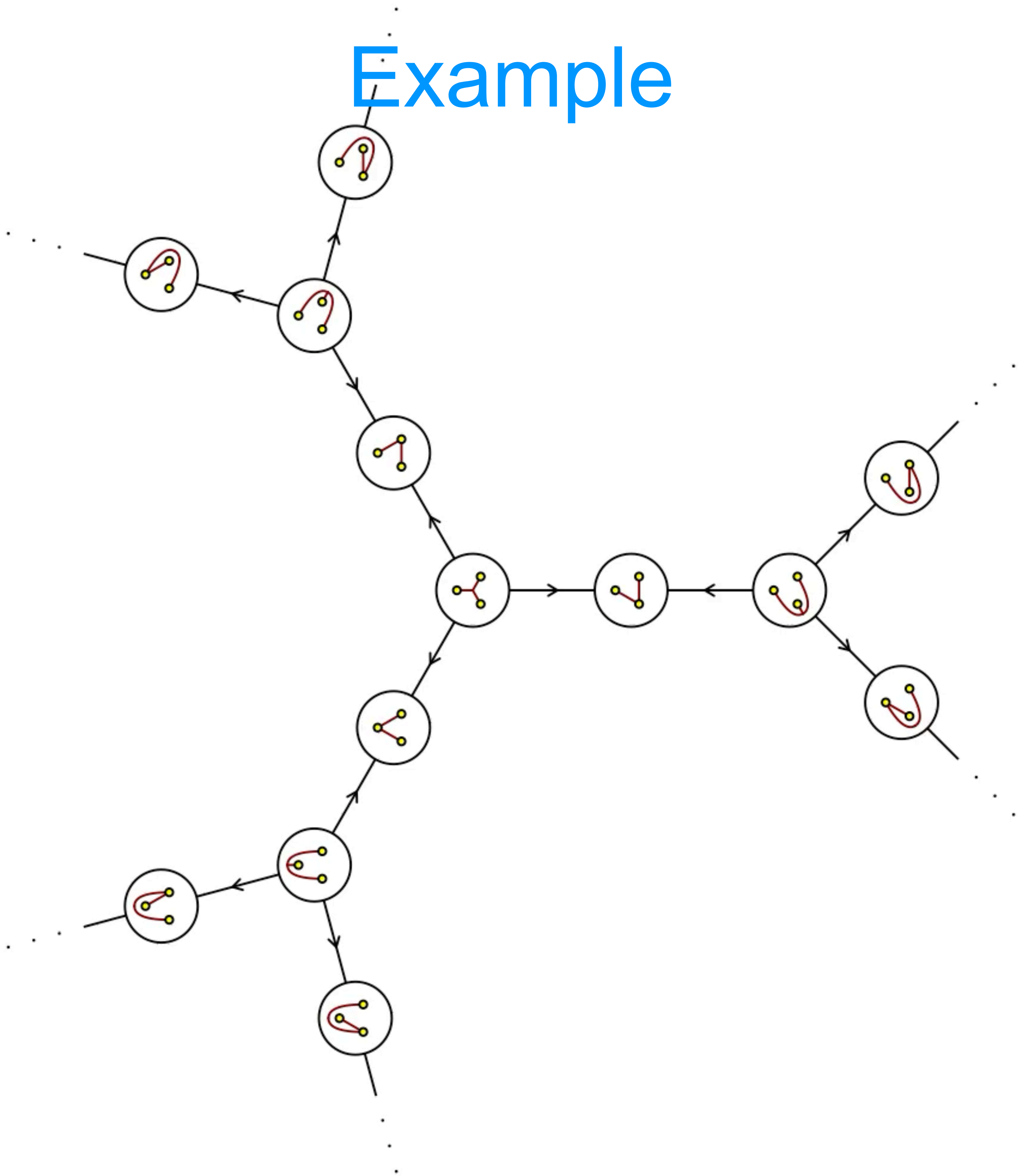
simplices: subforest collapses/expansions

Example

$\mathcal{T}_P$



Example



Tree Complex

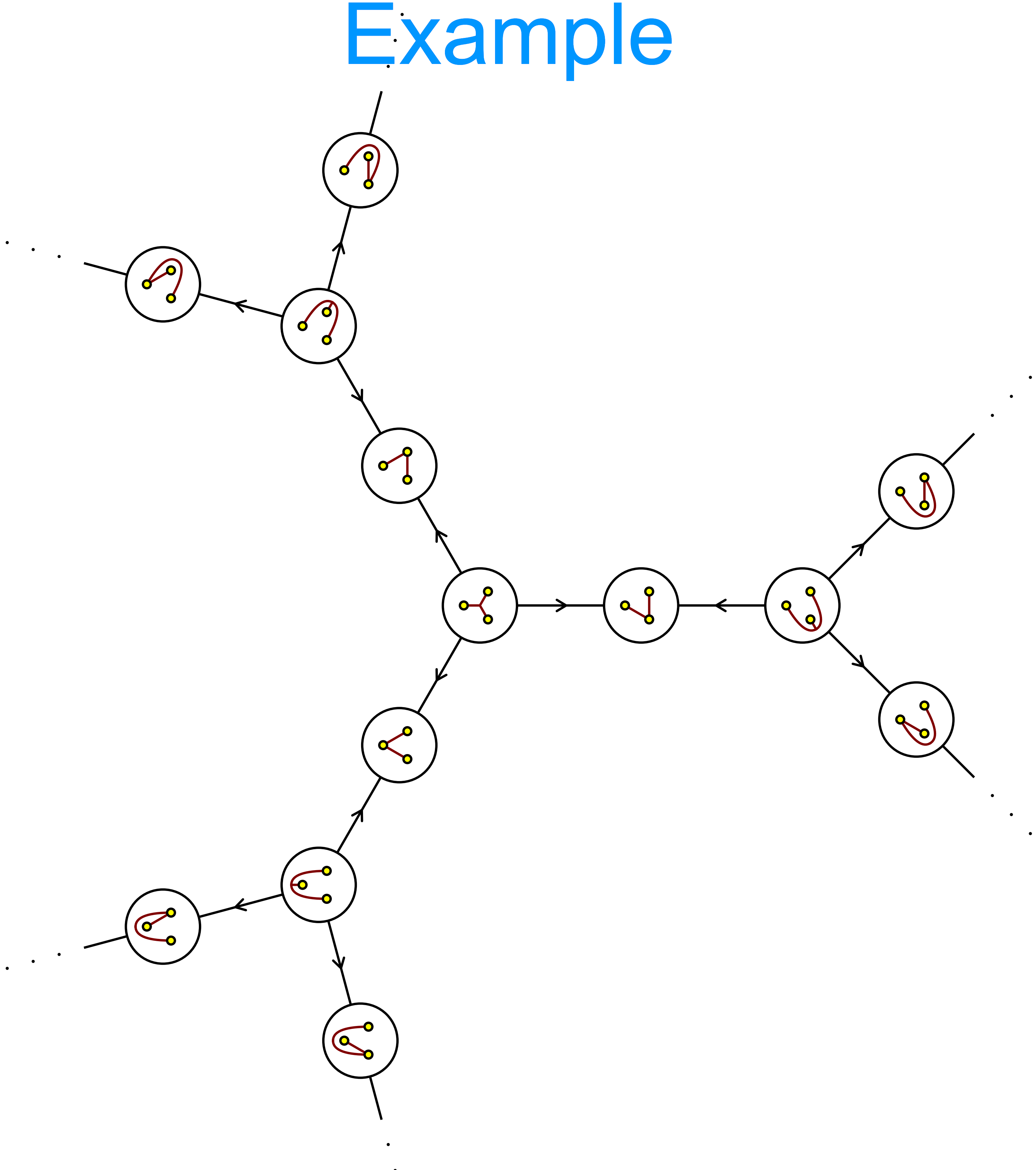
Proposition: $\mathcal{T}_P$ is connected (actually, simply connected)

Proof: (Hubbard–Masur, Penner)

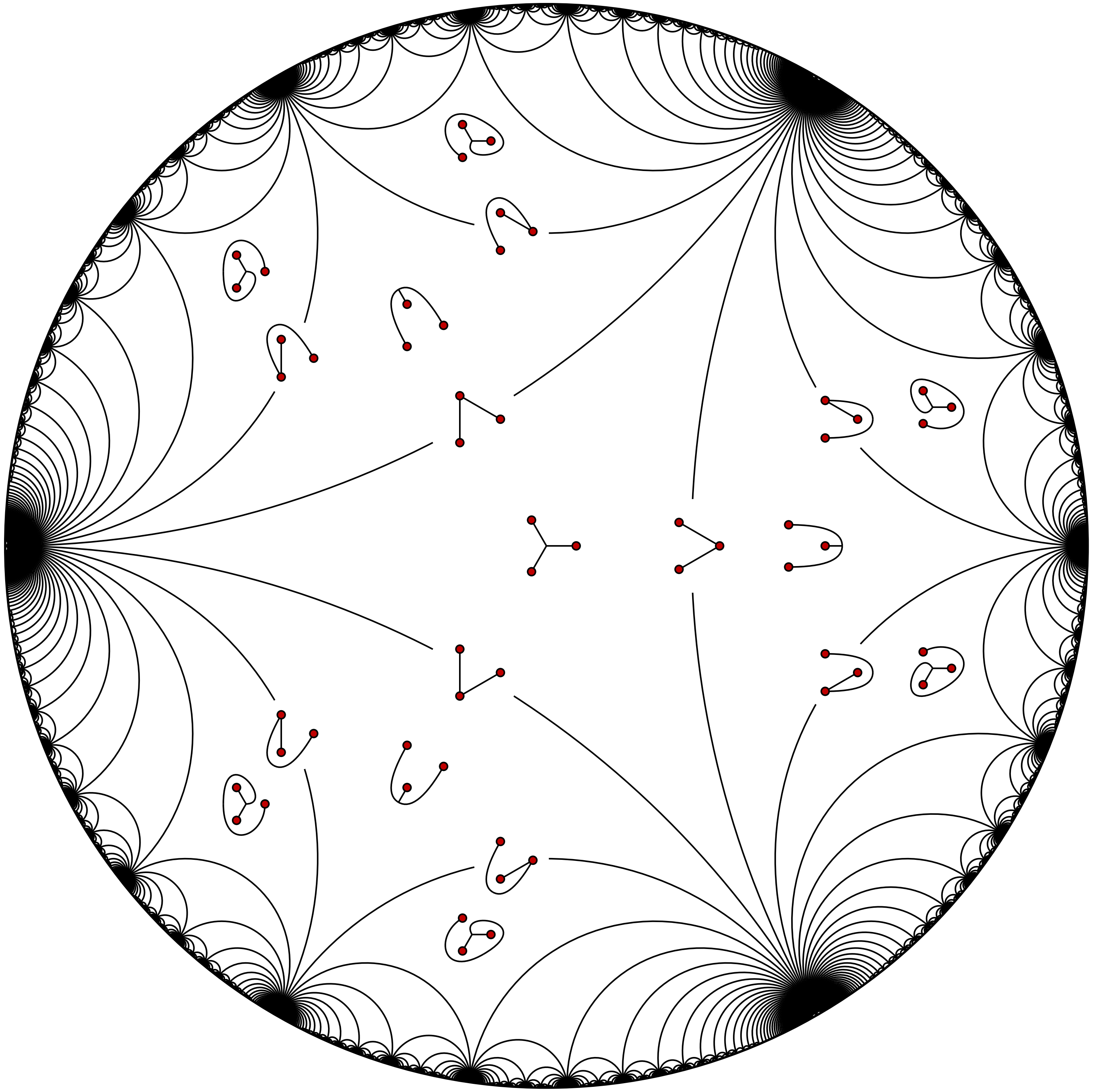
Dual to Teichmüller space.

Example

$\mathcal{T}_P$



Example



The Simplicial Map

Lifting map

Postcritically finite topological polynomial f

Image: Glendale City Trees

Lifting map

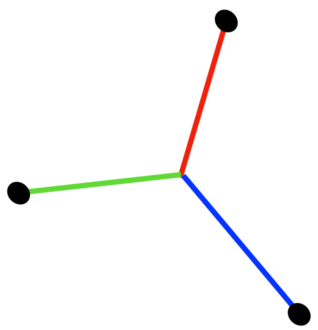
Postcritically finite topological polynomial f

Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$

Lifting map

Postcritically finite topological polynomial f

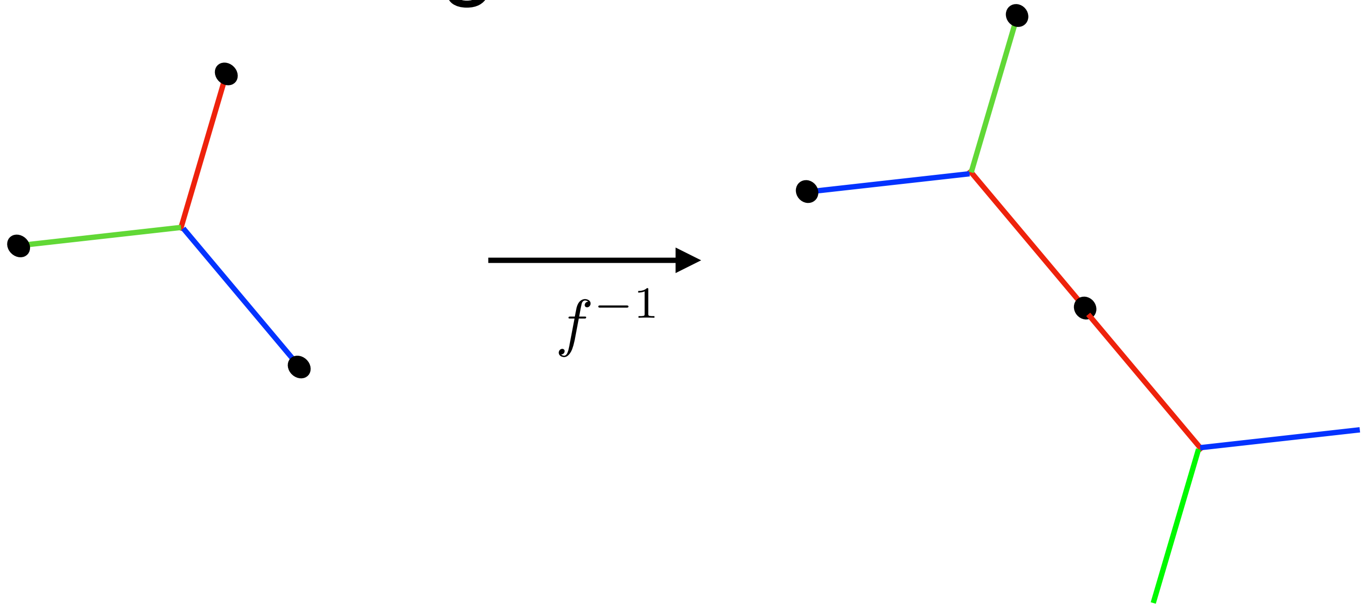
Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$



Lifting map

Postcritically finite topological polynomial f

Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$



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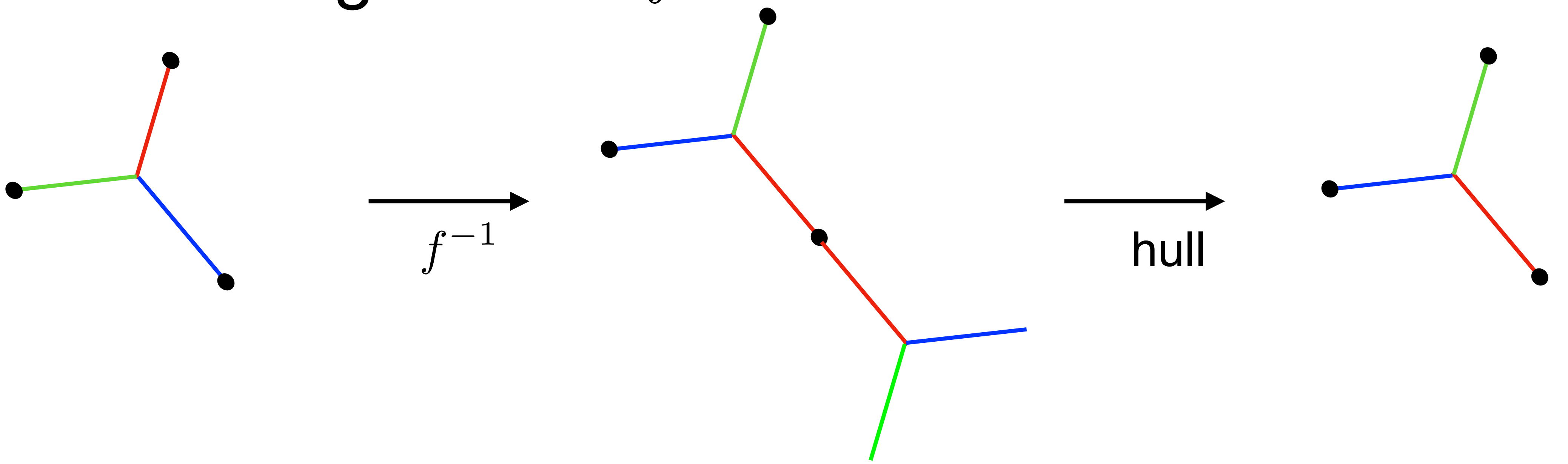
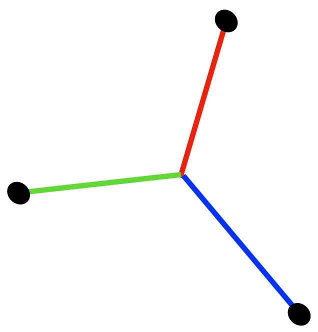


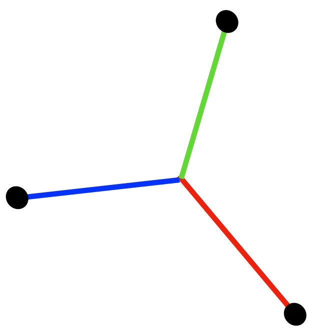
Lifting map

Postcritically finite topological polynomial f

Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$



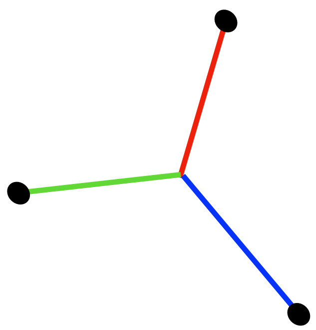
$$\xrightarrow{f^*}$$



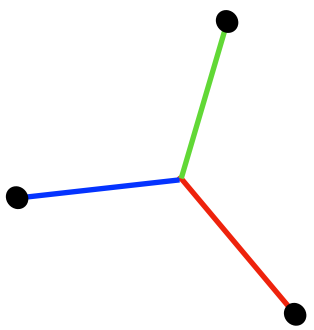
Lifting map

Postcritically finite topological polynomial f

Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$



$$\xrightarrow{f^*}$$

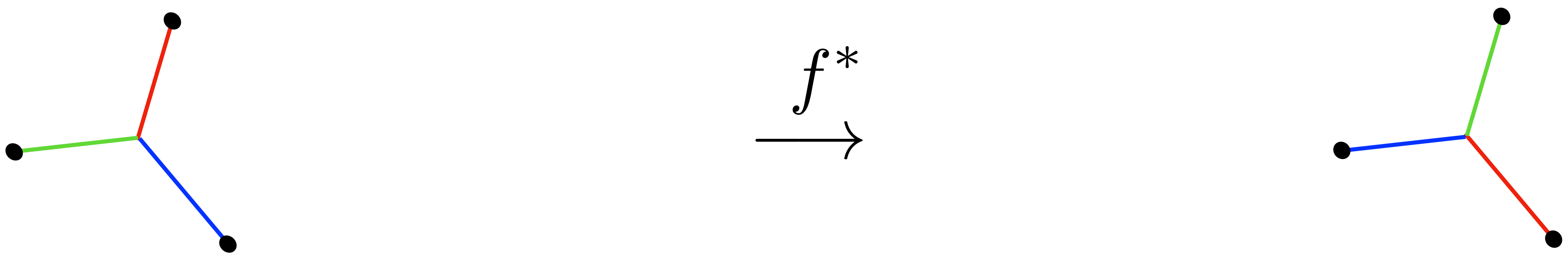


Known: there is a fixed point (Hubbard tree)

Lifting map

Postcritically finite topological polynomial f

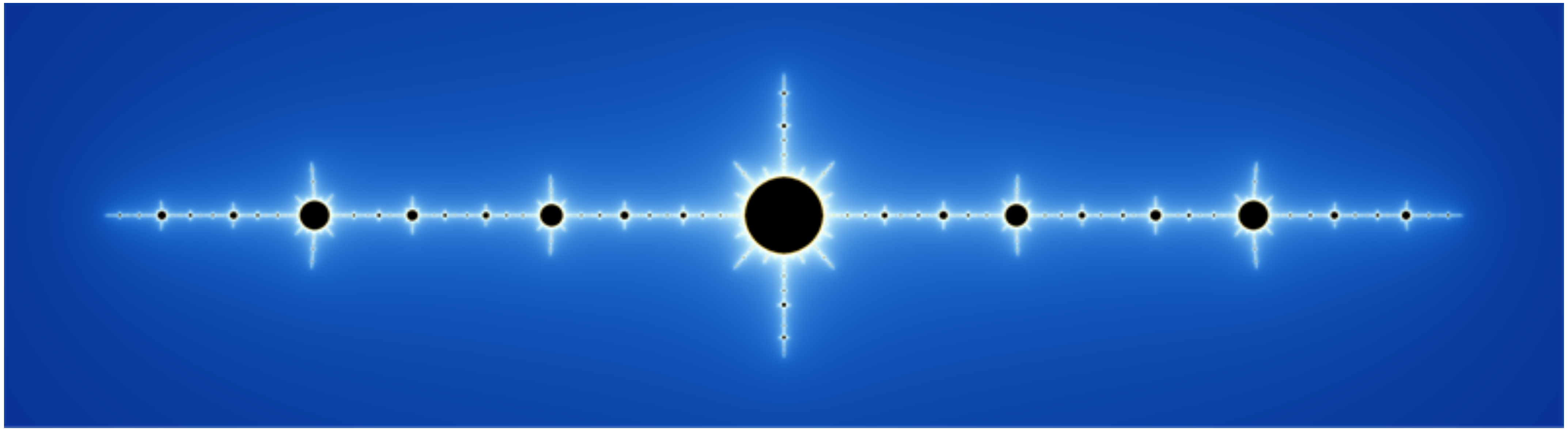
Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$



Known: there is a fixed point (Hubbard tree)

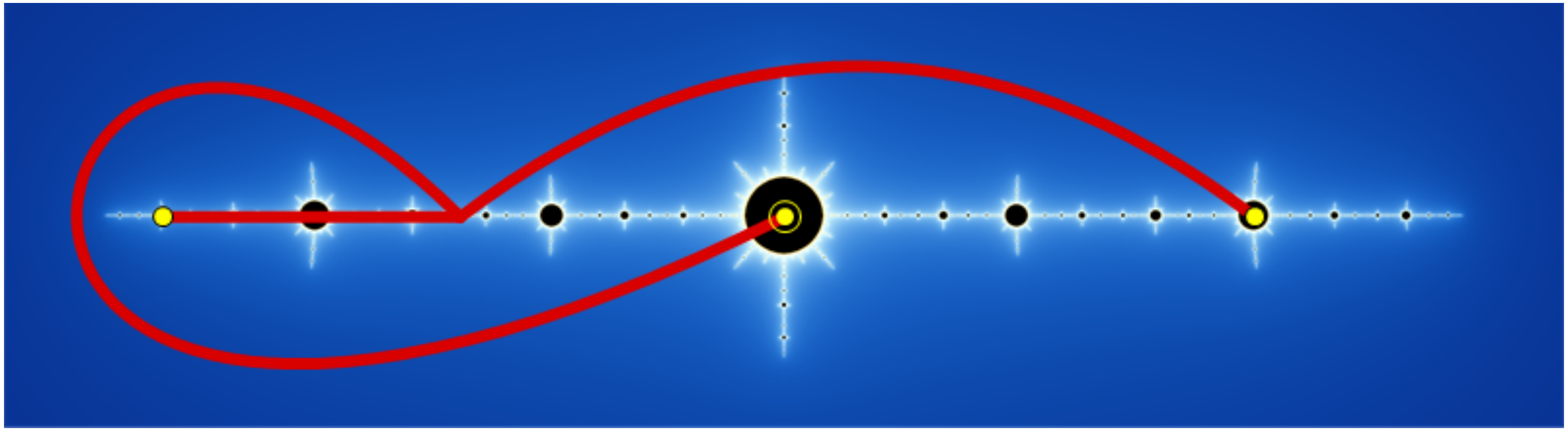
Strategy: Lift until you find it

Airplane example



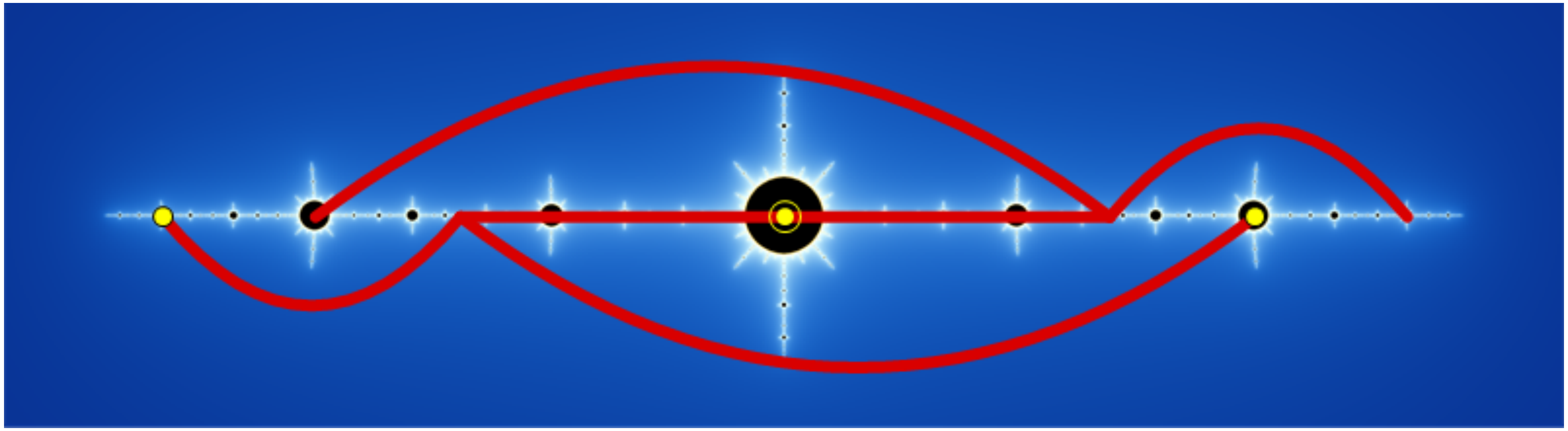
Airplane example

Choose a tree T



Airplane example

Take the preimage.



Airplane example

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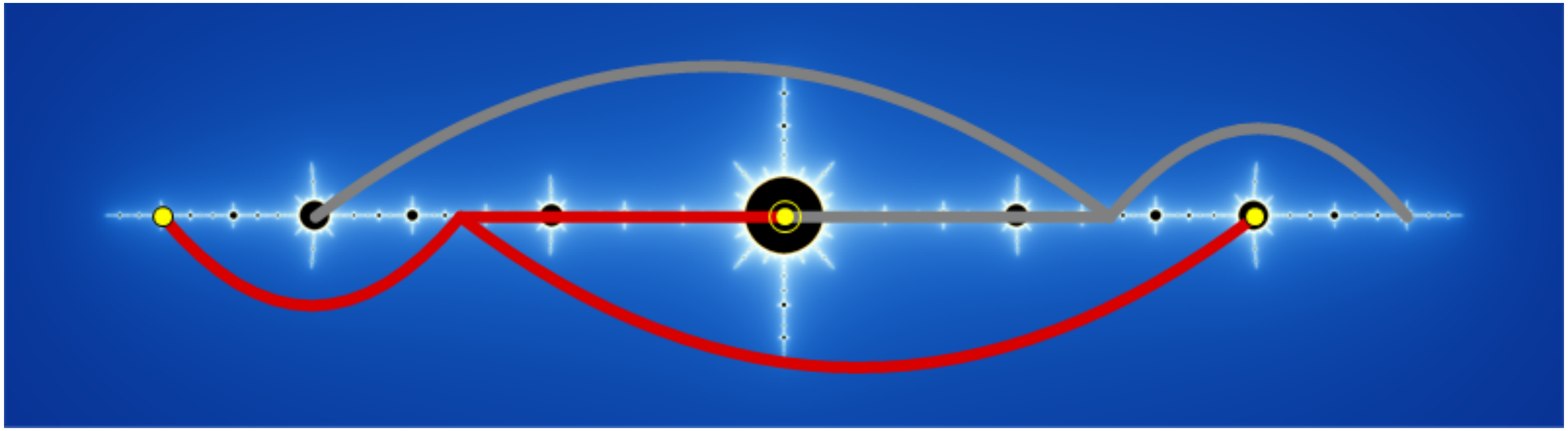
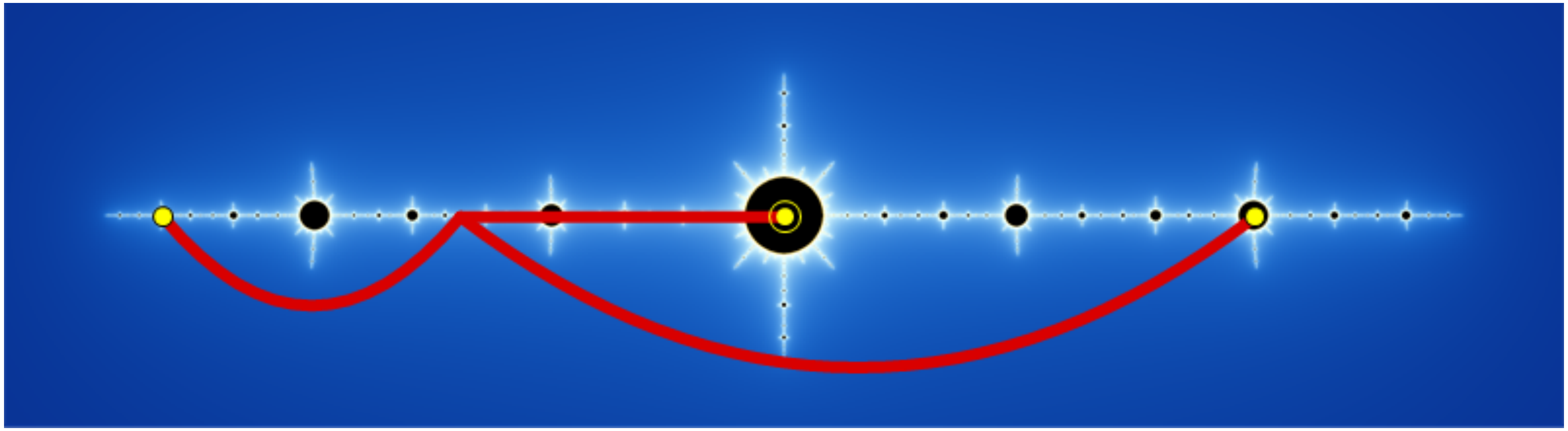


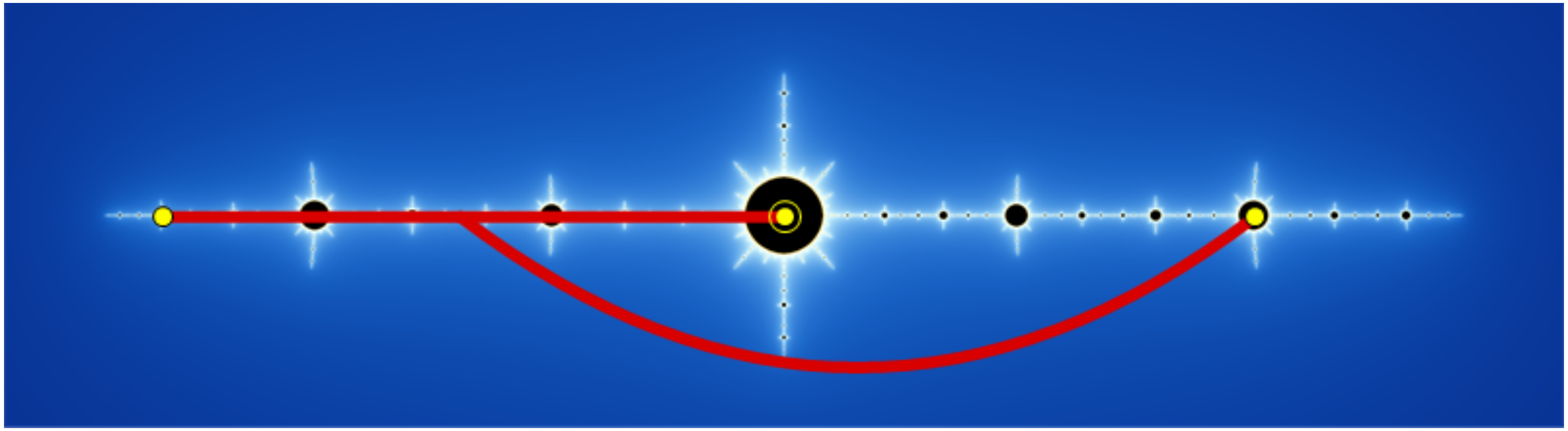
Airplane example

Take the hull.



Airplane example

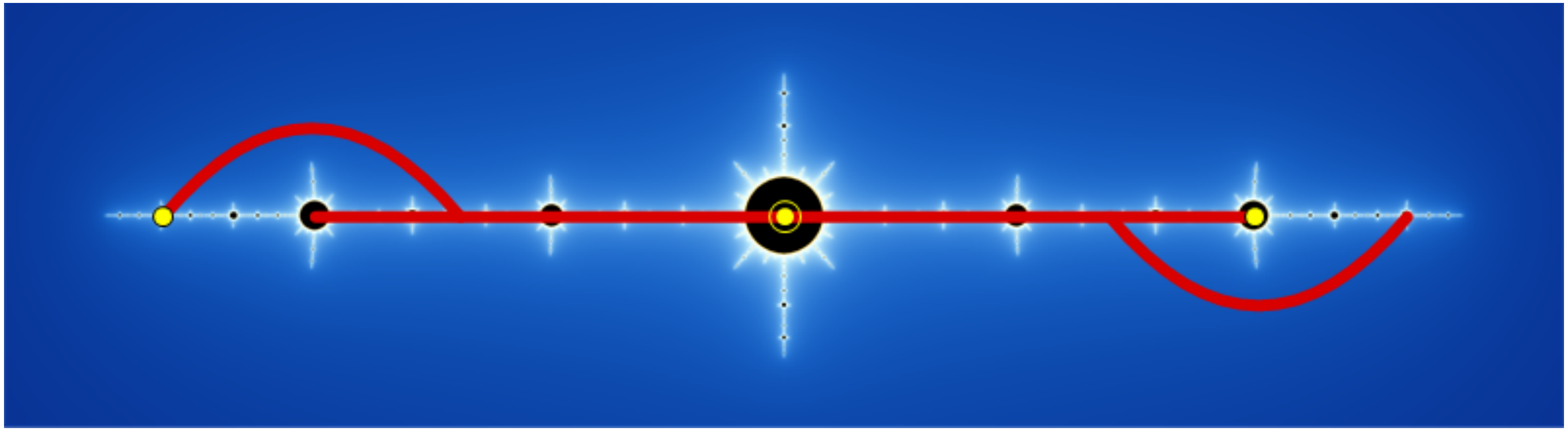
Simplify.



Not the same, repeat.

Airplane example

Take the preimage.



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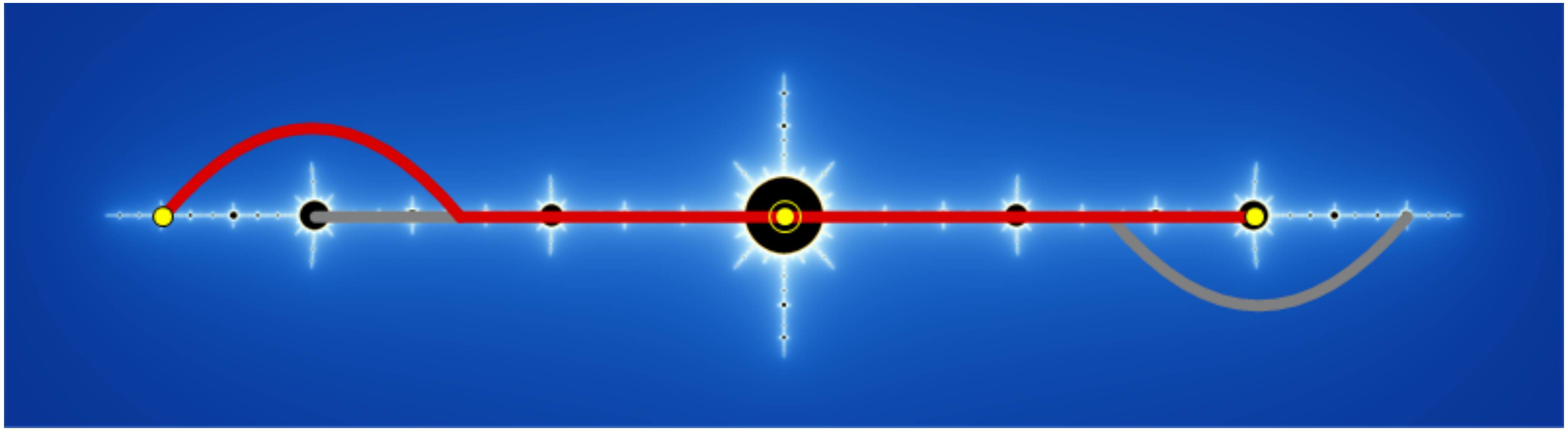
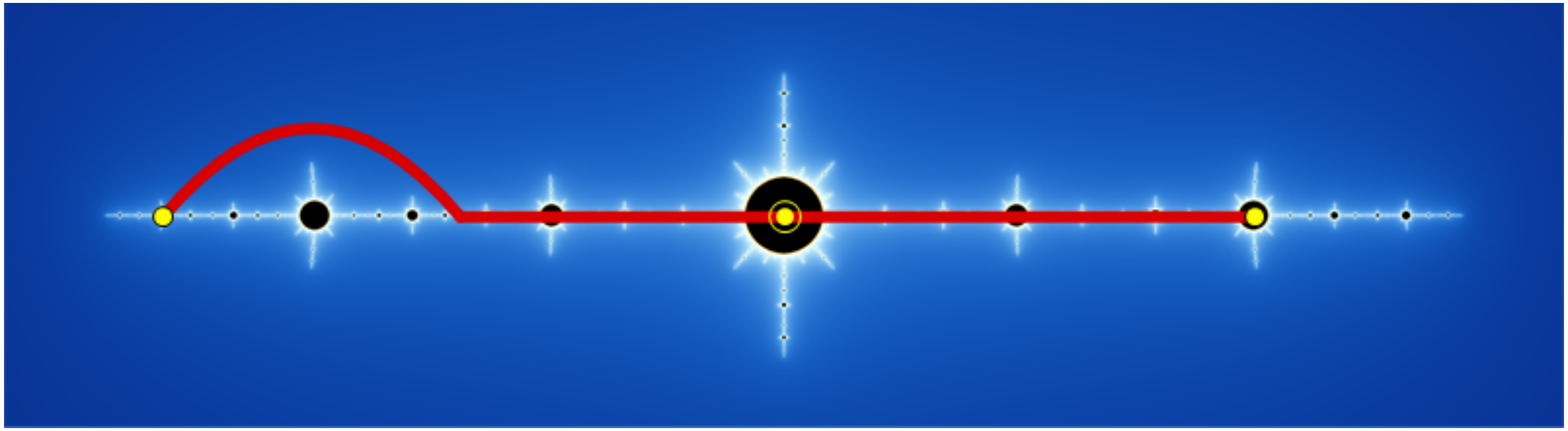


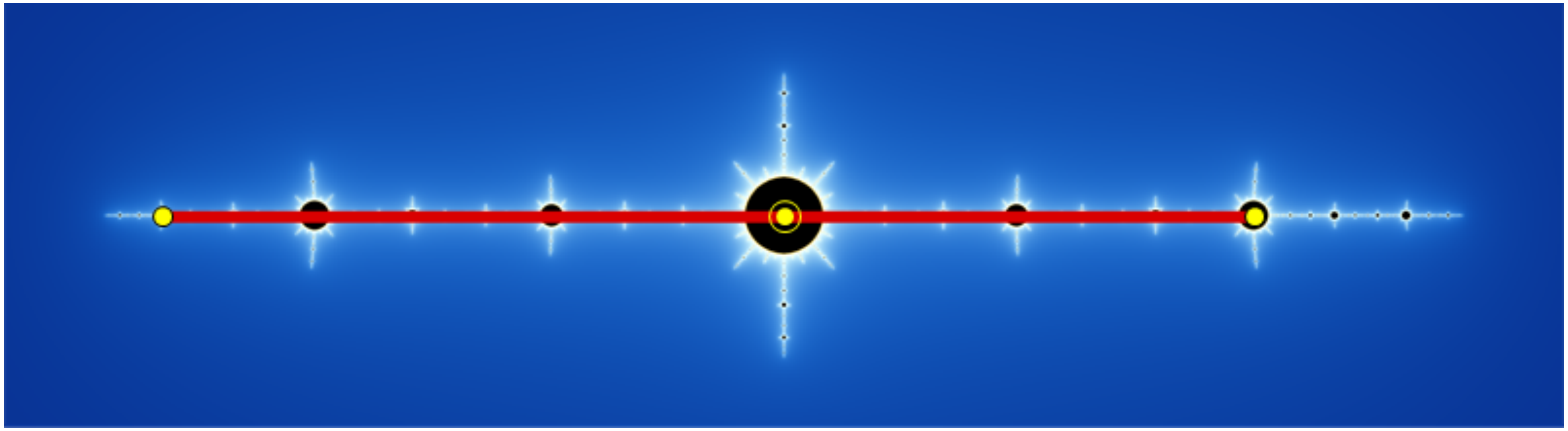
Airplane example

Take the hull.



Airplane example

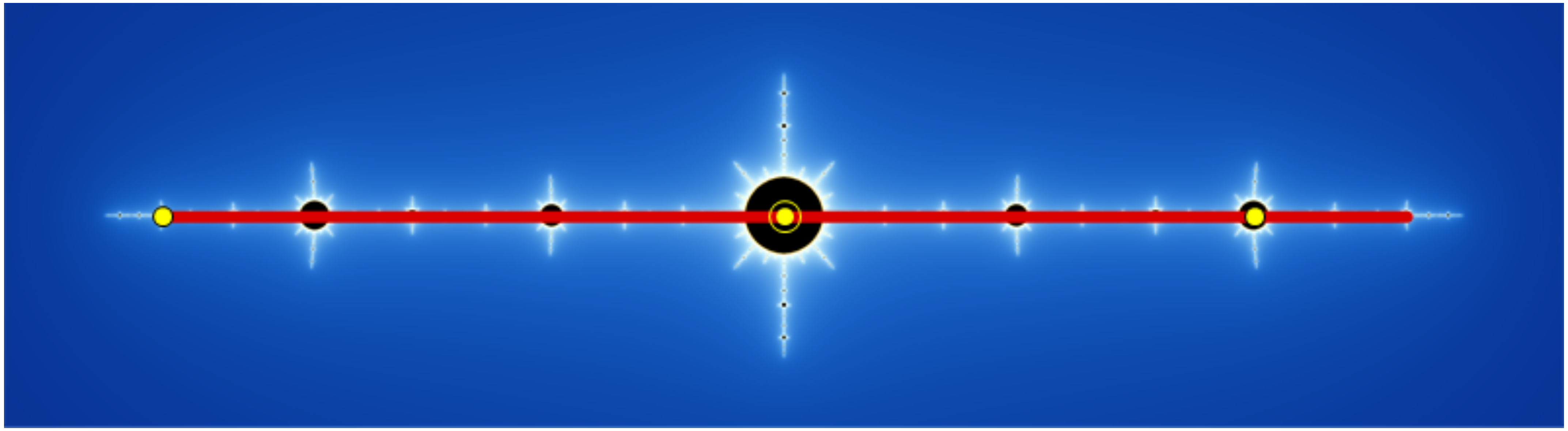
Simplify.



Not the same, repeat.

Airplane example

Take the preimage.



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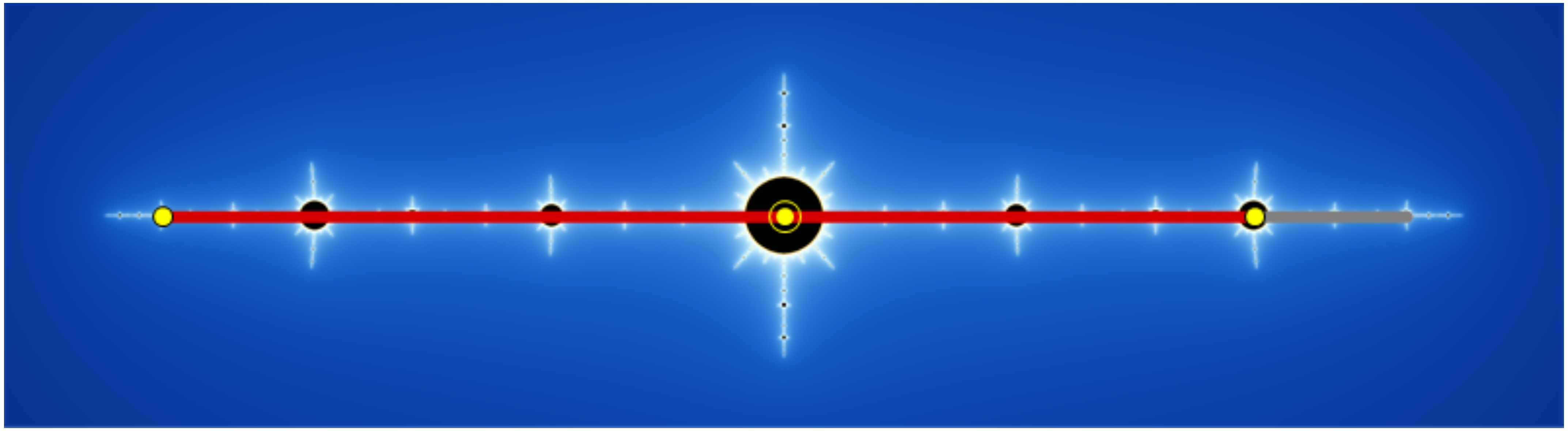
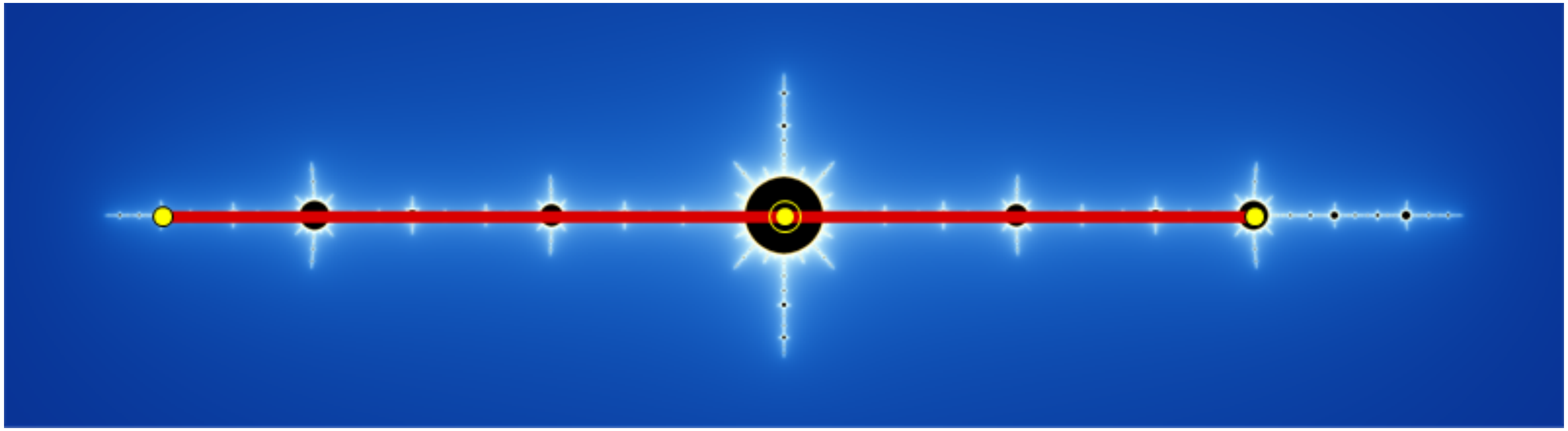


Airplane example

Take the hull.

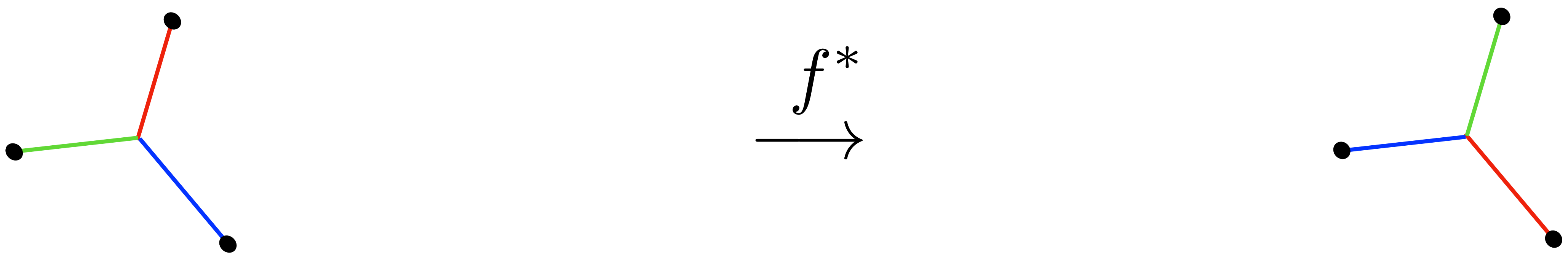


It is the same! You found an invariant tree!

Lifting map

Postcritically finite topological polynomial f

Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$



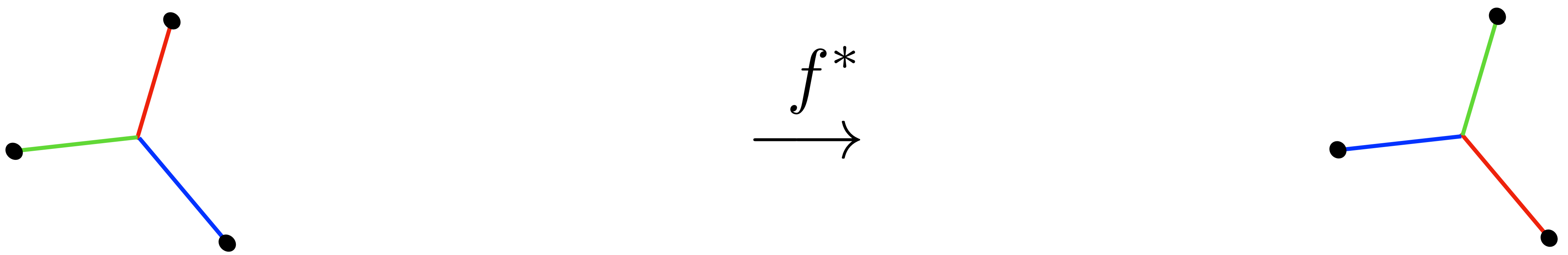
Known: there is a fixed point (Hubbard tree)

Hope: global attracting fixed point

Lifting map

Postcritically finite topological polynomial f

Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$



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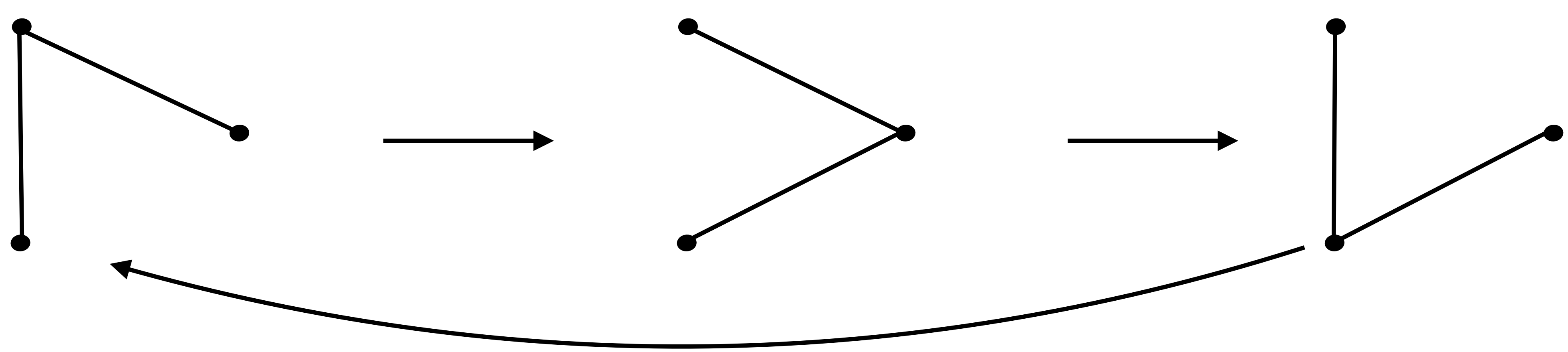
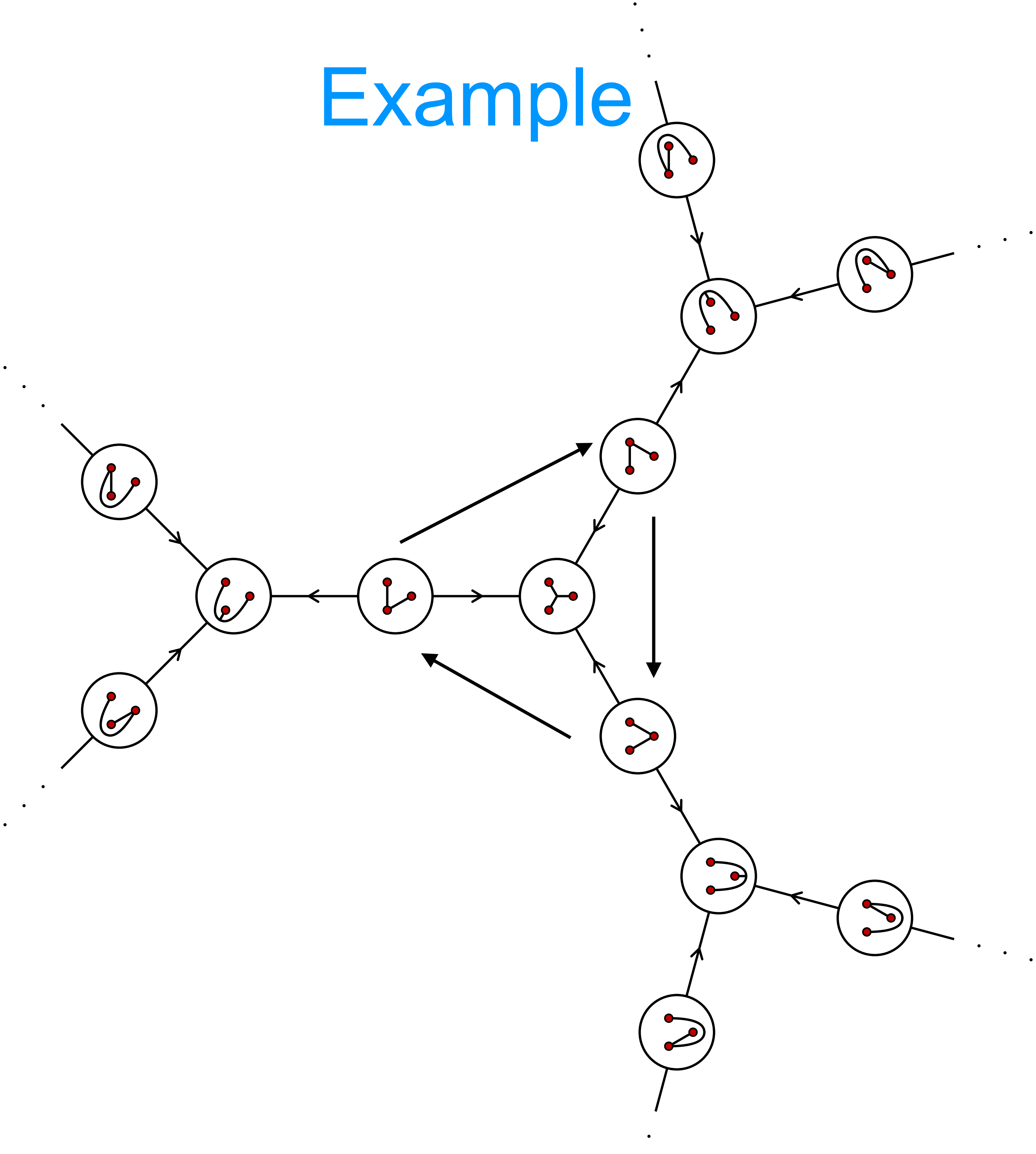


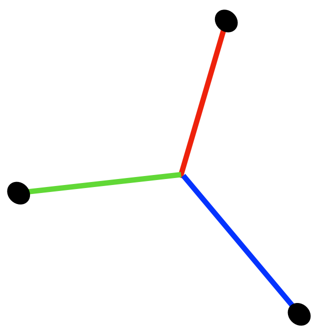
Example



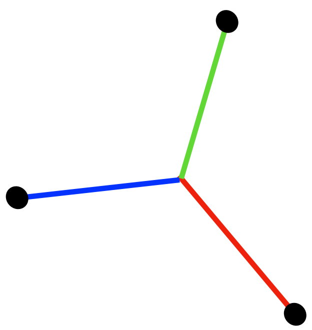
Lifting map

Postcritically finite topological polynomial f

Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$



$$\xrightarrow{f^*}$$



Known: there is a fixed point (Hubbard tree)

~~Hope: global attracting fixed point~~

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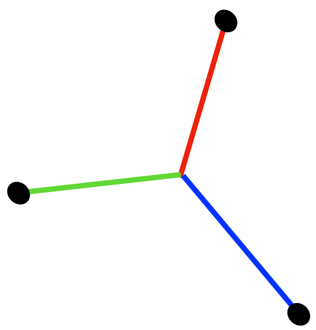
Postcritically finite topological polynomial f

Lifting: $f^* : \mathcal{T}_P \to \mathcal{T}_P$



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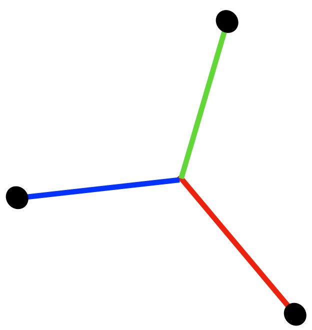

Known: there is a fixed point (Hubbard tree)

~~Hope: global attracting fixed point~~

New hope: Finite nucleus (global attracting subcomplex)

Our results

<u>Theorem</u> (Belk–Lanier–Margalit–W)
f unobstructed topological polynomial $\Rightarrow$ f^* finite nucleus

Contains the Hubbard tree

Our results

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Contains the Hubbard tree

Our results

Theorem (Belk–Lanier–Margalit–W)
f unobstructed topological polynomial $\Rightarrow$ f^* finite nucleus

Contains the Hubbard tree

Is contained in a 2-nbhd of Hubbard tree

Our results

Theorem (Belk–Lanier–Margalit–W)
f unobstructed topological polynomial $\Rightarrow$ f^* finite nucleus

Contains the Hubbard tree

Is contained in a 2-nbhd of Hubbard tree

$\rightsquigarrow$ finite check to find the Hubbard tree

Our results

Theorem (Belk–Lanier–Margalit–W)
f unobstructed topological polynomial $\Rightarrow$ f^* finite nucleus

Contains the Hubbard tree

Is contained in a 2-nbhd of Hubbard tree

$\rightsquigarrow$ finite check to find the Hubbard tree

= polynomial

Summary

Branched covers $S^2 \to S^2$ = higher degree braids

Thurston's theorem for branched covers

=

Nielsen–Thurston for mapping classes

Belk–Lanier–Margalit–W: Algorithm for polynomials

Obstructions

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Proof: f top. polynomial $\rightsquigarrow$ $f_* : \mathrm{Teich}(\mathbb{C}, P) \to \mathrm{Teich}(\mathbb{C}, P)$
pullback

Canonical obstructions

Pilgrim: An obstructed topological polynomial has a *canonical obstruction*

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Curves $\to$ 0 under lifting

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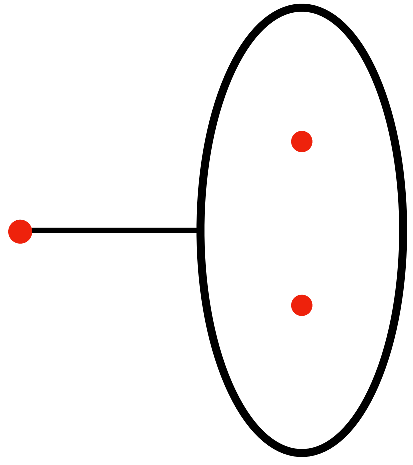
Selinger: Exterior of canonical obstruction is a polynomial

Bubble trees



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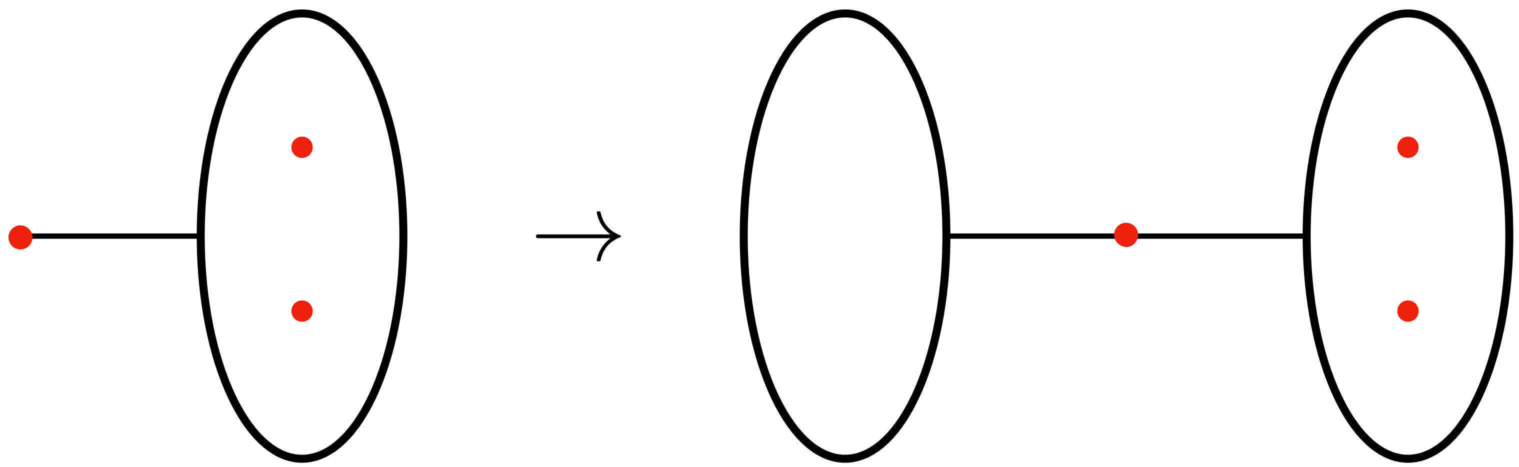

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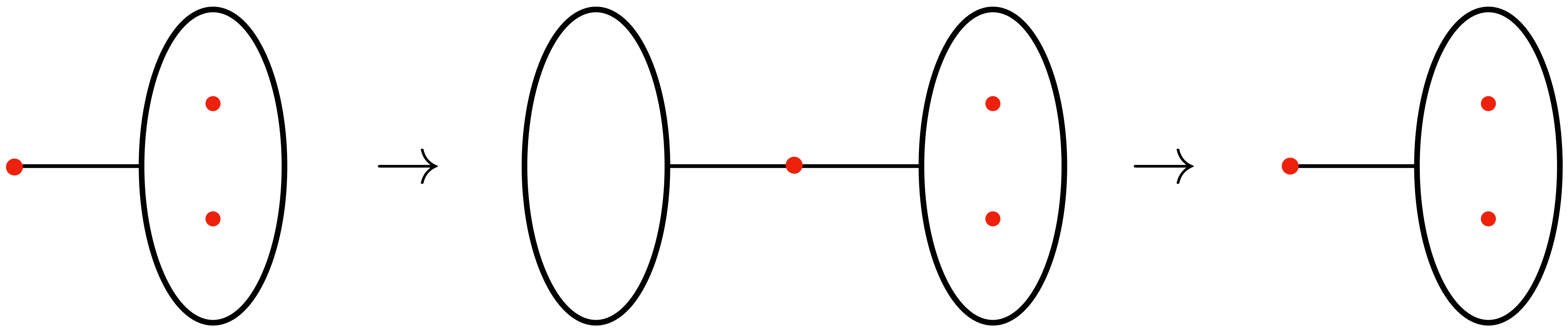

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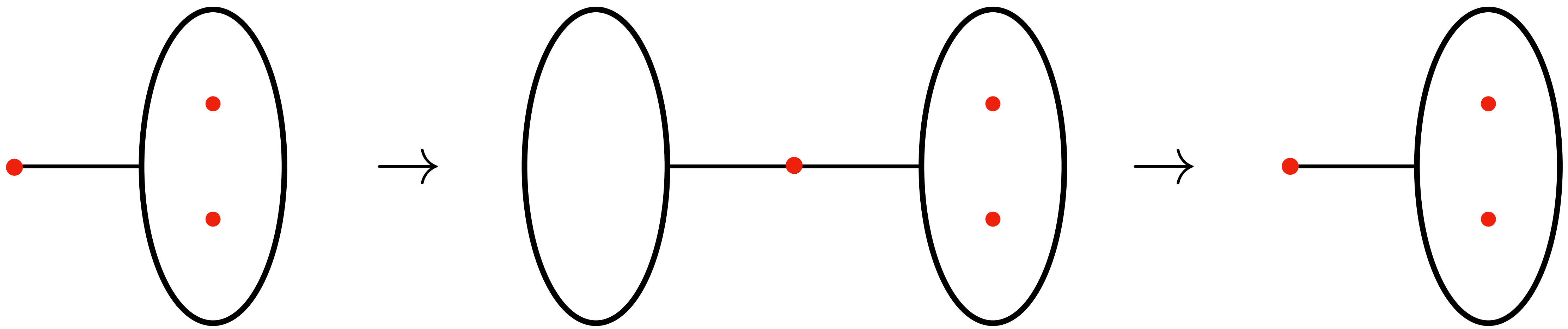
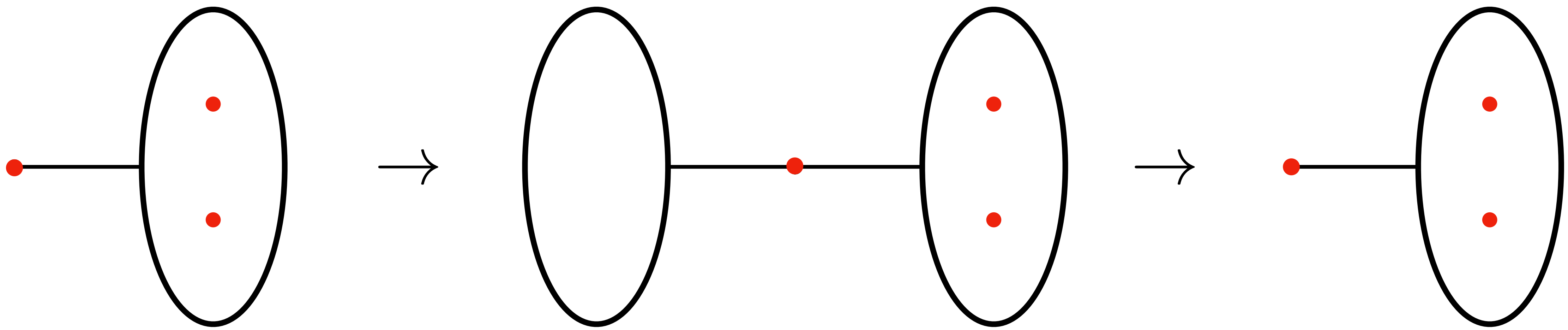


In the boundary $\mathcal{T}_P$

Bubble trees



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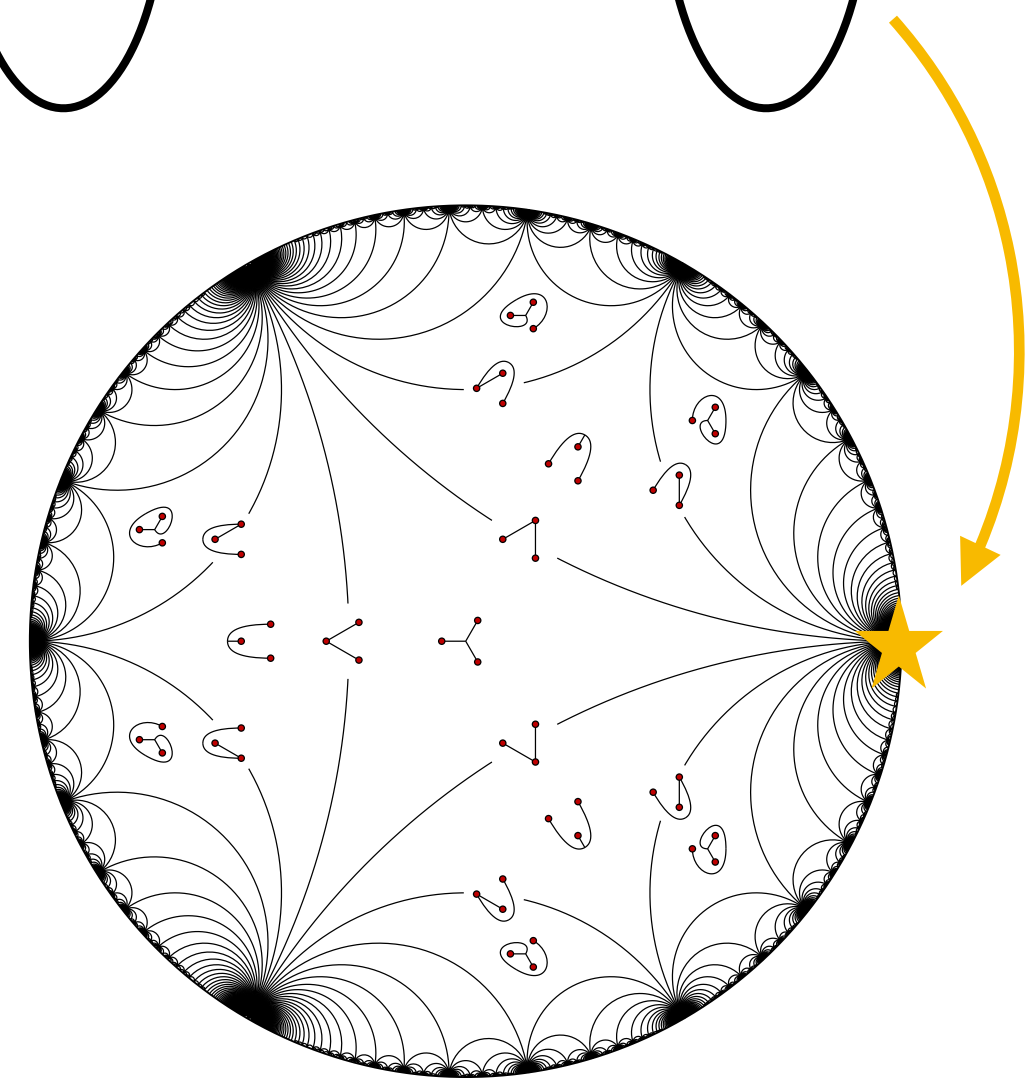

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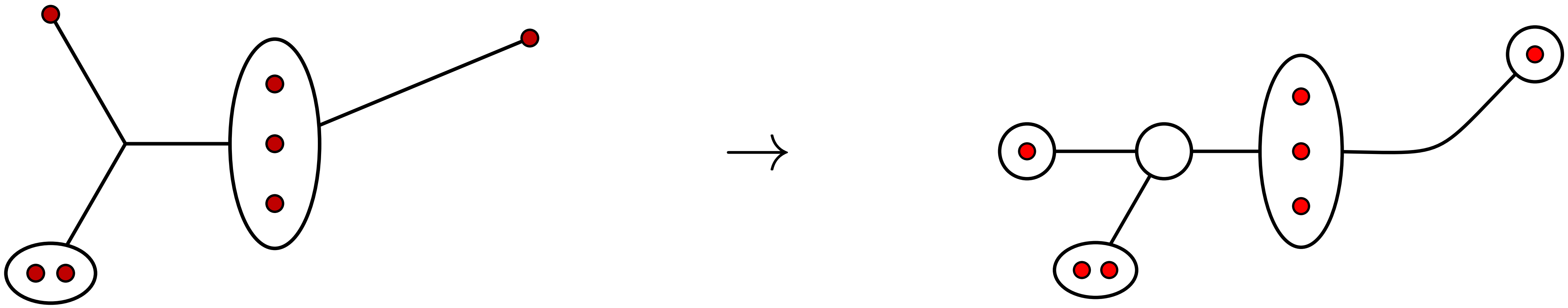

Proposition (Belk–Lanier–Margalit–W)

Every (obstructed) topological polynomial has a Hubbard bubble tree.

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Boundary of $\mathcal{T}_P$

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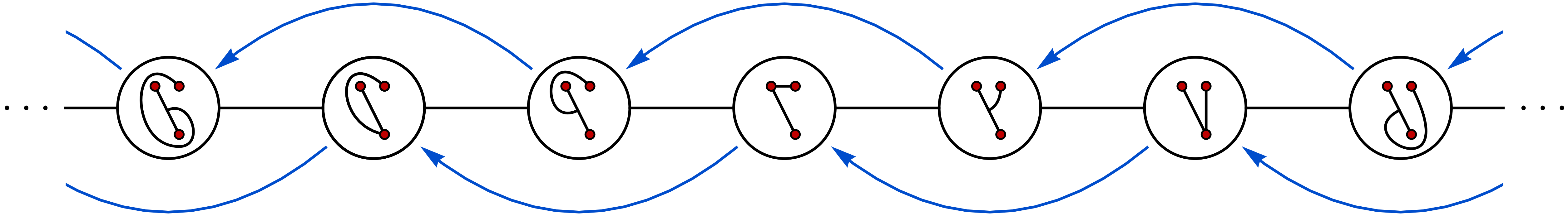
Infinite set

Infinite nucleus



Algorithm

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3. If you don’t find a Hubbard tree or canonical obstruction, return to 1.