Surface Braids and Galois Cohomology

joint w/ B. Farb & M. Kisin

Outline:

0) Motivation

1) Local Structure of Complex Varieties

2) Surface Braids & Braid Monodromy

3) Galois Cohom. Classes

Project comes out of an attempt to solve H13.

Roughly, this asks how hard to solve polynomials.

Can phrase it as being about

"branched" covers
$$\begin{tikzcd} \tilde{X} \arrow[d] \\ X \end{tikzcd} \quad \text{var.} / \mathbb{C}$$

Two key issues:

1) Problem is birational:

any invariant
$$\begin{tikzcd} \tilde{X} \arrow[d] & \supset & \tilde{X}|_U \arrow[d] \\ X & \supset & U \end{tikzcd} \quad \rightsquigarrow$$

2) Need to allow "accessory" totally ramified Galois covers branched

$$\begin{tikzcd} \tilde{X}|_E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{X} \arrow[d] \\ E \arrow[r, "\text{“accessory”}"'] & X \end{tikzcd}$$

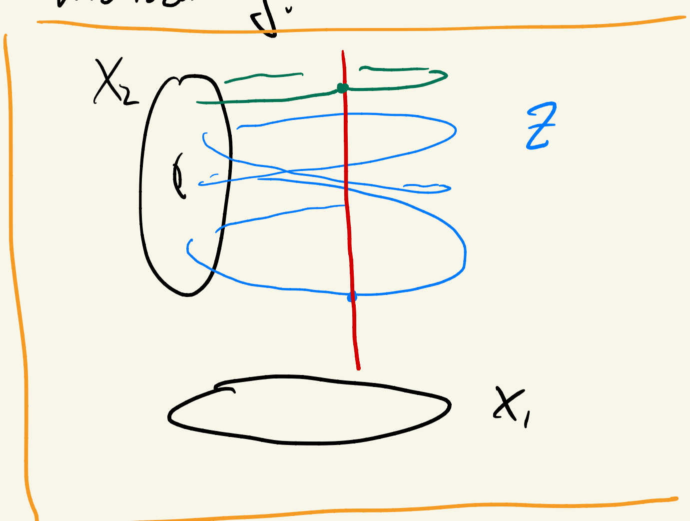
Q: What can we say about the local structure of varieties & branched covers?

Simplest non-trivial case: X_i curves

$X = X_1 \times X_2$. Arises in the varieties we care about for 1+3.

<u>Lemma</u>: Let X_1, X_2 be complex curves.

Any sufficiently small $U \subset X_1 \times X_2$ admits the structure of a curve by curve fibration w/ braid monodromy.



pf: Let $Z = X_1 \times X_2 - U$.

By shrinking X_1 & X_2, can ensure Z is a disjoint union of smooth curves Z_i s.t. $Z_i \to X_1^0$ is a covering space. (of deg n)

$\Rightarrow$ $U \to X_1$ is a bundle w/ fiber $X_{2,x} - X_2 \cap Z_{i,x}$.

i.e. we have map of fibrations

$$\begin{tikzcd}
X_2 - \{p_i\} \arrow[r] \arrow[d, hook] & \mathcal{U} \arrow[r] \arrow[d] & X_1 \arrow[d, equal] \\
X_2 \arrow[r] & X_1 \times X_2 \arrow[r] & X_1
\end{tikzcd}$$

$\Longrightarrow$ $\mathcal{U}$ is classified by a map

s.t.

$$\begin{tikzcd}
\pi_1(X_1) \arrow[r] \arrow[dr, "0"'] & \mathrm{Mod}(X_2 - \{p_i\}) \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Mod}(X_2)
\end{tikzcd}$$

(S top'l surface

Have $\mathrm{Mod}(S, z) := \pi_0 \mathrm{Diff}^+(S, z)$

$B_n(S) := \pi_1(\mathrm{UConf}_n(S))$.)

Birman exact sequence:

$$1 \to B_n(X_2) \to \mathrm{Mod}(X_2 - \{p_i\}) \to \mathrm{Mod}(X_2) \to 1$$

$\Longrightarrow$ $\mathcal{U}$ classified by

$$\pi_1(X_1) \to B_n(X_2).$$

$\square$

Fact (Artin-Griffiths):

Every complex variety X is Zariski locally an iterated curve fibration:

$$\exists\ U \subset X \quad \text{s.t.} \quad \begin{array}{ccc} C_d & \to & U \\ & & \downarrow \\ & & \vdots \\ C_2 & \to & U_2 \\ & & \downarrow \\ & & C_1 = U_1 \end{array} \qquad d = \dim X$$

w/ C_i complex curves & $U_i \to U_{i-1}$ topologically a surface bundle w/ fiber C_i.

Fact (Generalized Riemann Existence Thm):

$$G_{\mathbb{C}(X)} := \mathrm{Gal}\left(\overline{\mathbb{C}(X)}/\mathbb{C}(X)\right) \cong \varprojlim_{U \subset X} \widehat{\pi_1(U)}.$$

$\therefore$ $G_{\mathbb{C}(X)}$ is a pro-free-by-free group w/ monodromy in MCG!

1)' Local structure of Branched covers:

$$E \xrightarrow{f} X \quad \text{branched.}$$

$$\rightsquigarrow$$

max^{l} unramified cover

$$\begin{tikzcd} E \arrow[r, "\pi"] \arrow[rr, bend right, "f"'] & E^u \arrow[r, "p"] & X \end{tikzcd}$$

totally ramified w/ $\pi_1(E^u) = f_* \pi_1(E) \leq \pi_1(X)$.

<u>Def</u>: A branched cover $E \xrightarrow{f} B$ is <u>totally</u> <u>ramified</u> if $f(\pi_1(E)) = \pi_1(B)$.

In context of H13, we have tools to handle unramified covers, & E^u can again be treated as a product of curves.

Prop: $X_1 \times X_2$ — prod. of curves

$E \to X_1 \times X_2$ totally ramified.

branched G-cover

For small enough $U \subset X_1 \times X_2$ Zariski open, the fibering

$$\begin{tikzcd} X_2 - \{q_i\} \arrow[r] \arrow[d, hook] & U \arrow[r] \arrow[d] & X_1 - \{p_i\} \arrow[d] \\ X_2 \arrow[r, hook] & X_1 \times X_2 \arrow[r, two heads] & X_1 \end{tikzcd}$$

induces a fibering of $E|_U$ as

$$\begin{tikzcd} \tilde{U}_2 \arrow[r] \arrow[d, "G_2"'] & E|_U \arrow[r] \arrow[d] & \tilde{U}_1 \arrow[d, "G_1"] \\ X_2 - \{q_i\} \arrow[r] & U \arrow[r] & X_1 - \{p_i\} \end{tikzcd}$$

w/ $\tilde{U}_i \to X_i$ connected, totally ramified.

branched covers.

What do these look like?

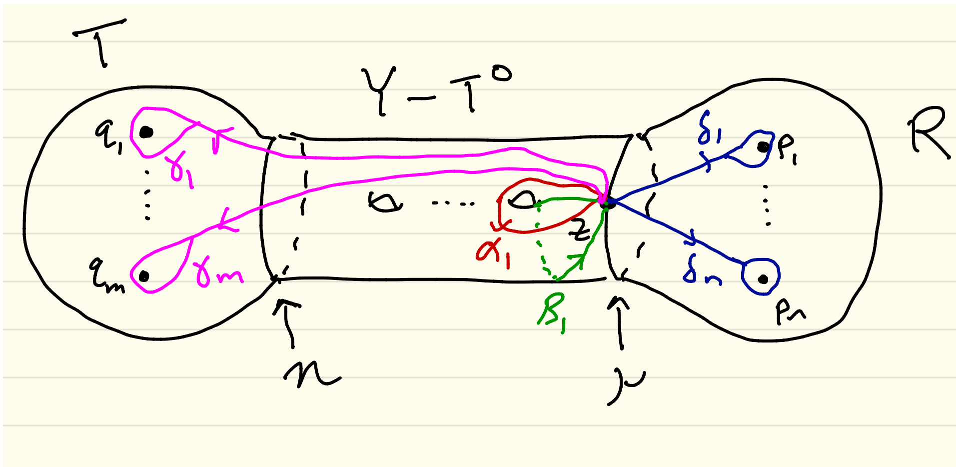


Figure 6: The surface S'. The compact subsurface $Y \subset S'$ is the surface bounded by μ on the right. A generating set for $\pi_1(S', z)$ is indicated, along with the two separating simple closed curves η and μ.

$\widetilde{\mathcal{U}}_1$

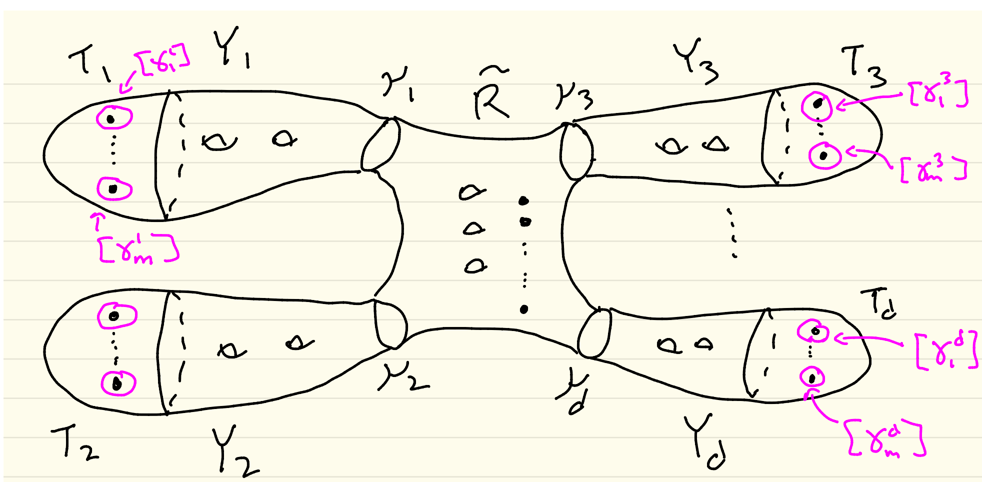


Figure 7: The surface $\widetilde{S}'$. The finite group G acts on $\widetilde{S}'$ by deck tramsformations. This action leaves $\widetilde{R}$ invariant and acts simply transitively on $\{\mu_i\}$, as well as $\{Y_i\}$ and $\{T_i\}$. Representatives for the homology classes $[\gamma_j^i]$ are indicated.

Monodromy of $\tilde{U}_2 \to E \to \tilde{U}_1$?

Let $\tilde{S} \xrightarrow{\pi} S$ be a Galois G-cover of surfaces.

Let

mapping classes which lift

$$\Lambda_\pi = \left\{ \tau \in \mathrm{Mod}(S') \;\middle|\; \exists \begin{tikzcd} \tilde{S}' \arrow[r, "\tilde{\tau}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & \tilde{S}' \arrow[d, "\pi"] \\ S \arrow[r, "\tau"'] & S \end{tikzcd} \right\}$$

$$\Rightarrow \quad \begin{tikzcd} \tilde{\Lambda}_\pi \arrow[d] & \subset & \mathrm{Mod}(\tilde{S}) \arrow[d] \\ \Lambda_\pi & \subset & \mathrm{Mod}(S) \end{tikzcd}$$

<u>Lemma</u>: 1) $\tilde{\Lambda}_\pi \cong \Lambda_\pi \times G$

2) If
$$\begin{tikzcd} \tilde{S}' \arrow[r] \arrow[d] & \tilde{F} \arrow[d] & B \arrow[l, hook'] \arrow[d, equal] \\ S' \arrow[r] & F & B \arrow[l, hook'] \end{tikzcd}$$

Then $\rho : \pi_1(B) \to \mathrm{Mod}(\tilde{S}')$
$$\begin{tikzcd} \pi_1(B) \arrow[r] \arrow[dr, "\exists"'] & \mathrm{Mod}(\tilde{S}') \\ & \Lambda_\pi \times \{1\} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\therefore \quad E \longrightarrow X_1 \times X_2$ tot. ramified

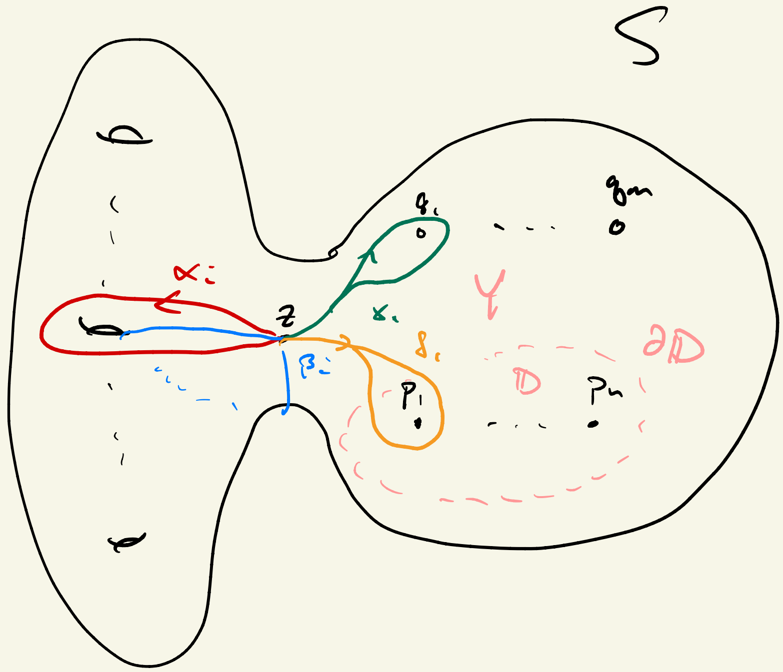
$\Rightarrow \quad \hat{U}_2 \longrightarrow E \longrightarrow \tilde{U}_1$ has braid monodromy

$$\rho : \pi_1(\tilde{U}_1) \longrightarrow \Lambda_\pi \subset B_n(X_2)$$

2) <u>Surface Braids</u> & <u>Braid Monodromy</u>

S — genus g surface w/ m punctures $q_1, \dots, q_m$

$S' = S - \{p_1, \dots, p_n\}$



$$\pi_1(S', z) = \left\langle \{\alpha_i, \beta_i\}_{i=1}^{g}, \{\gamma_i\}_{i=1}^{m}, \{\delta_i\}_{i=1}^{n} \;\middle|\; \prod_{i=1}^{g} [\alpha_i, \beta_i] = \prod_{i=1}^{m} \gamma_i \prod_{i=1}^{n} \delta_i \right\rangle$$

Birman Exact Sequence

$$1 \to B_n(S) \longrightarrow \mathrm{Mod}(S', z) \longrightarrow \mathrm{Mod}(S, z) \longrightarrow 1$$

Want to understand action $B_n(S)$ on $\pi_1(S', z)$:

(via action on generators).

$C = \{ \{\alpha_i, \beta_i\}_{i=1}^{g}, \{\gamma_j\}_{j=1}^{m-1} \}$. — gen. set for $\pi_1(S, z)$.

Pick $z \overset{\tau}{\frown} p_1$ in S. For $w \in C$, let $w' = w^{\tau}$.

So C' is a generating set for $\pi_1(S, p_1)$.

Two types of elts in $B_n(S)$:

"Local braids":

$$S = Y \cup_{\partial D} D \quad , \quad S' = Y \cup_{\partial D} D - \{p_1, \dots, p_n\}$$

gives $B_n \hookrightarrow B_n(S)$

$$\sigma_i \longmapsto \text{"local braid" } \sigma_i$$

"

"Point pushes" Push_δ for δ a curve based at a point $y \in S$.

$$\text{Push}_\delta := T_{S' \times \{0\}} \circ T^{-1}_{S' \times \{1\}}$$ for $S' \times [0,1] \to S$ a tubular neighborhood of δ.

Thm: (Bellingeri)

$B_n(S)$ is generated by local braids $\sigma_1, \dots, \sigma_{n-1}$

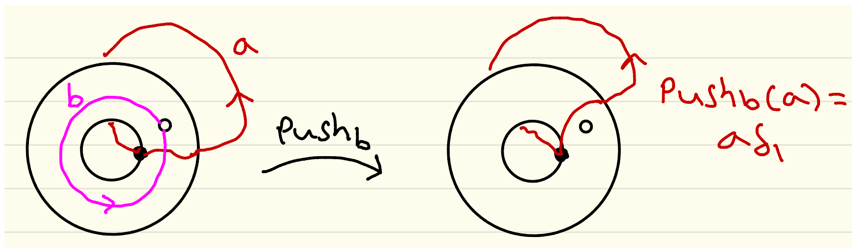
and point pushes $\{ \mathrm{Push}_{\omega} \mid \omega \in C' \}$.

(Farb, Kisin, W) (thanks to B. Tshishiku)

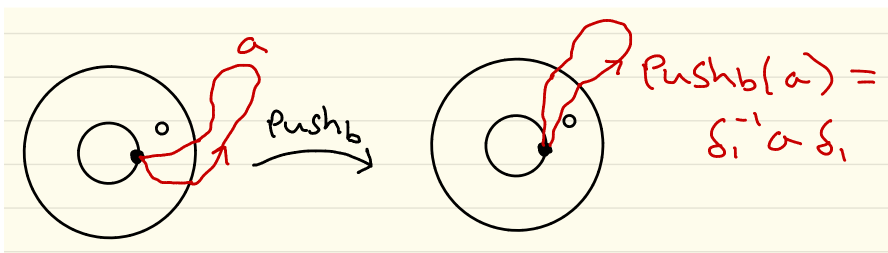
Theorem 4.12 (Action of generators of $B_n(S)$ of generators of $\pi_1(S', z)$). *With the terminology as above, and for simplicity of notation setting $\epsilon := \delta_1$, the action of a generator of $B_n(S)$ on a generator of $\pi_1(S', z)$ is trivial except for the following:*

1. $\mathrm{Push}_{\alpha'_j} : \beta_j \mapsto \beta_j\epsilon$; $\epsilon \mapsto \alpha_j\epsilon\alpha_j^{-1}$; $\delta_k \mapsto \epsilon^{-1}\delta_k\epsilon$ *for each* $k \geq 2$*; and and* $\gamma_\ell \mapsto \epsilon\gamma_\ell\epsilon^{-1}$ *for all* $1 \leq \ell \leq m$.
2. $\mathrm{Push}_{\beta'_j} : \alpha_j \mapsto \alpha_j\epsilon$; $\epsilon \mapsto \beta_j\epsilon\beta_j^{-1}$; $\delta_k \mapsto \epsilon^{-1}\delta_k\epsilon$ *for each* $k \geq 2$*; and* $\gamma_\ell \mapsto \epsilon\gamma_\ell\epsilon^{-1}$ *for all* $1 \leq \ell \leq m$.
3. $\mathrm{Push}_{\gamma'_j} : \gamma_j \mapsto \epsilon^{-1}\gamma_j\epsilon$.
4. σ_i *for* $1 \leq i \leq n-1 : \delta_i \mapsto \delta_{i+1}$ *and* $\delta_{i+1} \mapsto \delta_{i+1}\delta_i\delta_{i+1}^{-1}$.

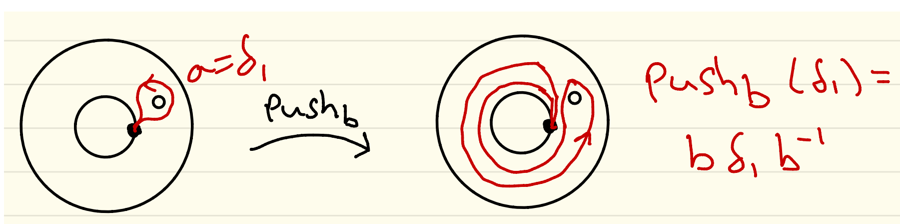
pf: Possible intersections



(a) Type I intersection and the effect of the resulting point push.



(b) Type II intersection and the effect of the resulting point push.



(c) Type III intersection and the effect of the resulting point push.

Figure 3: Three types of local intersection of an element $a \in \mathcal{C}$ and an element $b \in \mathcal{C}'$. In each diagram, the annular neighborhood of b is pictured in black, with the intersection of a curve $a \in \mathcal{C}$ with this neighborhood pictured in red. The solid point on the boundary of the annulus is the point $z \in a$, while the hollow point is the point $p_1 \in b$ being pushed by Push_b. These three types of intersections are called Type I, Type II and Type III, respectively. Also indicated in each diagram is the action of Push_b on a, written as an element of $\pi_1(S', z)$.

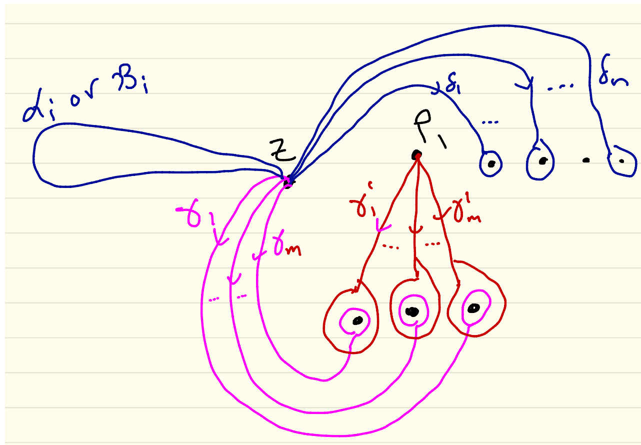


Figure 4: How the loops γ'_j intersect the loops $a \in \mathcal{C}$. Since $|\gamma_j \cap \gamma'_k|$ equals 2 when $j = k$ and equals 0 when $j \neq k$, when $j = k$ the intersection is of Type II; otherwise it is of Type 0, as are the intersections of each γ'_j with each element of $\mathcal{C} \setminus \{\gamma_j\}$.

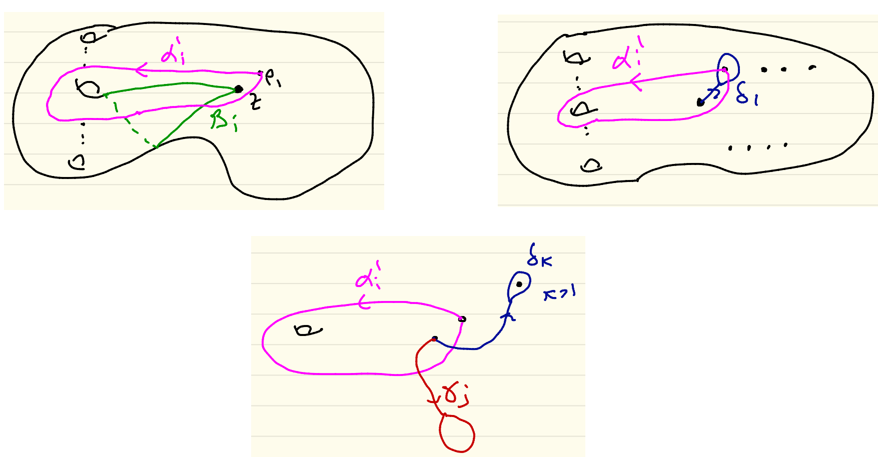


Figure 5: How the loop α'_j intersect the loops $a \in \mathcal{C}$.

Table 1: The intersection types of each element $a \in \mathcal{C}$ with each element $b \in \mathcal{C}'$. .

	α'_i	β'_i	γ'_i
α_j	Type 0	$\begin{cases} \text{Type I} & i = j \\ \text{Type 0} & i \neq j \end{cases}$	Type 0
β_j	$\begin{cases} \text{Type I} & i = j \\ \text{Type 0} & i \neq j \end{cases}$	Type 0	Type 0
γ_j	Type II	Type II	Type II
$\delta_j, j \geq 2$	Type II	Type II	Type 0
δ_1	Type III	Type III	Type 0

3) <u>Galois Cohom</u>

<u>Thm</u> (Farb-Kisin-W)

1) Let X_1, X_2 be a pair of complex curves.

Let $\widehat{\mathbb{Z}[X_i]}$ be the profinite free group generated by points of X_i.

Then $\exists$ split injection (w/ Hodge Theory can inject slightly more)

(suppressing Tate twists, all weight 2)

$$\widehat{\mathbb{Z}_\ell[X_1 \times X_2]} = \widehat{\mathbb{Z}_\ell[X_1]} \otimes \widehat{\mathbb{Z}_\ell[X_2]} \rightleftarrows H^2_{et}(G_{X_1 \times X_2}; \mathbb{Z}_\ell)$$

2) $E \to X_1 \times X_2$ totally ramified.

$\overline{X}_1 - X_1 = \{q_1, \ldots, q_m\}$

$\overline{X}_2 - X_2 = \{p_1, \ldots, p_n\}$

Then $\exists$ split lift

$$\begin{tikzcd}
 & & H^2(G_E; \mathbb{Z}_\ell) \arrow[d] \arrow[dll, bend right, "\exists"'] \\
\widehat{\mathbb{Z}_\ell[\{p_i\}]} \otimes \widehat{\mathbb{Z}_\ell[\{q_i\}]} \arrow[urr] \arrow[rr, hook] & & H^2(G_{X_1 \times X_2}; \mathbb{Z}_\ell)
\end{tikzcd}$$

pf <u>sketch</u>:

For 1) $\forall \{p_1, \ldots, p_n\} \subset X_1$, $\{q_1, \ldots, q_m\} \subset X_2$, $\exists$

compatible injections

$$\mathbb{Z}_\ell[\{p_i\}] \otimes \mathbb{Z}_\ell[\{q_j\}] \longrightarrow H^2_{et}(U; \mathbb{Z}_\ell(-2))$$

for any sufficiently small $U \subset X_1 - \{p_i\} \times X_2 - \{q_j\}$.

By Lemma above, $\exists$ fibering w/ Braid Monodromy

$$\begin{array}{ccc} U_2 \to U & \subset (X_1 - \{p_i\}) \times (X_2 - \{q_i\}) \\ \downarrow & \downarrow \\ U_1 \subset & X_1 - \{p_i\} \end{array}$$

Serre SS for these fiberings

gives

$$\begin{array}{ccc} H^2_{et}(U; \mathbb{Z}_\ell(-2)) & \cong & H^1_{et}(U_1; H^1_{et}(U_2)) \\ \downarrow & & \downarrow \\ H^2_{et}((X_1 - \{p_i\}) \times (X_2 - \{q_i\})) & \cong & H^1_{et}(X_1 - \{p_i\}) \otimes H^1_{et}(X_2 - \{q_i\}) \end{array}$$

Need :

<u>Lemma</u>: $V \xrightarrow{\pi} W$ map of $\mathbb{Z}[\Gamma]$-modules w/ W free as $\mathbb{Z}$-module

Then π splits Γ-equivariantly over $\pi(V^{\Gamma})$.

pf. W free ab $\Rightarrow$ $\pi(V^{\Gamma})$ free ab

$\Rightarrow$ $V^{\Gamma} \twoheadrightarrow \pi(V^{\Gamma})$ splits as $\mathbb{Z}$-modules

& thus $\mathbb{Z}[\Gamma]$-modules.

$$\Rightarrow \quad \begin{tikzcd} V^{\Gamma} \arrow[r, two heads] \arrow[d, hook] & \pi(V^{\Gamma}) \arrow[l, bend right] \arrow[d, hook] \\ V \arrow[r, "\pi"'] & W \end{tikzcd} \quad \text{in } \mathbb{Z}[\Gamma]\text{-mod} \qquad \square$$

$U_1 \simeq K(\Gamma, 1)$ $\quad\quad$ $\Gamma = \pi_1(U_1)$.

Let $V = H^1(U_2)$, $\quad W = H^1(X_2 - \{q_i\})$.

w/ $V \xrightarrow{\pi} W$ $\quad$ Γ-equivariant & $\Gamma \curvearrowright W$ trivially.

Braid Monodromy Thm $\Longrightarrow$ $V^\Gamma \supset \mathbb{Z}_\ell[\{q_i\}]$

Lemma $\Rightarrow$ $H^1(U_1; H^1(U_2)) \longrightarrow H^1(U_1; W) \cong H^1(U_1) \otimes W$

splits over $\pi(\mathbb{Z}_\ell[\{q_i\}]) \cong \mathbb{Z}_\ell[\{q_i\}]$

$\Rightarrow$ $H^1(U_1; H^1(U_2)) \supset H^1(U_1; \mathbb{Z}_\ell) \otimes \mathbb{Z}_\ell[\{q_i\}]$

$\supset \mathbb{Z}_\ell[\{p_i\}] \otimes \mathbb{Z}_\ell[\{q_i\}]$

$\square$

<u>Application</u>: • Characteristic classes to try prove analogue of H13 for $\mathbb{Z}_\ell$-local systems.
(w/ const. coeffs)

• Bloch–Kato $\implies$ torsion char. classes in Galois cohom **can't** prove H13 for finite covers.