Rewrite Systems in 3-free Artin groups

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Artin Groups

Let $S = \{s_1, \dots, s_n\}$ be a finite set. For each pair of of elements in this set s_i and s_j choose an $m_{i,j} \in \{2, 3, \dots \infty\}$.

Definition

The *Artin group*, A_S is the group with presentation

$$A_S = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \dots}_{m_{ij} \text{ terms}} = \underbrace{s_j s_i s_j \dots}_{m_{ij} \text{ terms}} \ \text{for all } i \neq j \rangle \, .$$

If $m_{i,j} = \infty$, then there is no group relation between s_i and s_j.

Examples

Example

$$A_S = \langle a, b, c | ab = ba, bcb = cbc, cacac = acaca \rangle$$

Examples include:

- Braid groups
- Free groups
- Free abelian groups and other right-angled Artin groups
- Free products and direct products of other Artin groups

The Word Problem

In a group generated by a set S, a *word* is a finite sequence of letters in $S \cup S^{-1}$.

Two word are considered equivalent if they represent the same group element.

A *solution to the word problem* is a finite time algorithm for determining if a given word is equivalent to the identity.

The Word Problem

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Open Problem

In general, it is unknown whether or not Artin groups have solvable word problem.

Cases where the word problem has been solved

1. FC-type Artin groups, including braid groups, spherical-type Artin groups and right-angled Artin groups (Artin 1947, Garside 1969, Deligne 1972, Brieskorn-Saito 1972, Birman-Ko-Lee 1998, Altobelli 1998, Altobelli-Charney 2000)
2. 2-dimensional Artin groups (VanWyck 1994)
3. Sufficiently large Artin groups (Appel-Schupp 1983, Peifer 1996, Holt-Rees 2012, Holt-Rees 2013)
4. Euclidean Artin groups (Digne 2006, Digne 2012, McCammond-Sulway 2017)

Theorem (Blasco-García, Cumplido, MW)

Let A_S be an Artin group where $m_{i,j} \neq 3$ for all i, j. This is called a 3-free Artin group.

There is a finite-time algorithm that solves the word problem for 3-free Artin groups.

In dihedral Artin groups

A dihedral Artin group is an Artin group with 2 generators.

$$A_S = \langle s, t \;\mid\; \underbrace{sts\ldots}_{m \text{ terms}} = \underbrace{tst\ldots}_{m \text{ terms}} \rangle \,.$$

Critical words (Mairesse, Matheus 2006)

In a dihedral Artin group there a words, called critical words, such that there a multiple possible geodesic words representing this group element. These words have $p + n = m$ where p is the length of longest alternation of the form $stst\dots$ and n is the length of the longest alternation of the form $s^{-1}t^{-1}s^{-1}\dots$

Example

(m=5)

- $ststs$
- $(ststs)st^2s^3$
- $(sts)st^2s^3t^{-4}(t^{-1}s^{-1})$

τ moves

There is an involution τ which can be applied to critical words.

Example

(m=5)

- $ststs \overset{\tau}{\mapsto} tstst$
- $(ststs)st^2s^3 \overset{\tau}{\mapsto} ts^2t^3(tstst)$
- $(sts)st^2s^3t^{-4}(t^{-1}s^{-1}) \overset{\tau}{\mapsto} (t^{-1}s^{-1})ts^2t^3s^{-4}(tst)$

How does this work?

(m=5) case: Let $\Delta = ststs$. Repeatedly conjugate by Δ

$$(ststs)st^2s^3 \to (ststs)st^2s^3\Delta^{-1}\Delta \to (ststs)\Delta^{-1}ts^2t^3\Delta \to ts^2t^3(tstst)$$

Key Properties of τ

The map τ satisfies the following properties for any critical word w in a dihedral Artin groups A_S (Brien 2012, Holt & Rees 2012)

1. $\tau(w)$ is also critical, $\tau(w) =_G w$.
2. If $l[w] \in \{s, s^{-1}\}$, then $l[\tau(w)] \in \{t, t^{-1}\}$.

Rightward Reducing Sequence(RRS) of τ moves in large type

Holt and Rees (2012) show that you can solve the word problem for large type Artin groups ($m_{i,j} \geq 3$ for all i, j). Their algorithm involves repeated application of τ moves in a rightword sequence.

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Example (RRS in large type for $m_{a,b} = m_{b,c} = m_{c,d} = 5$)

$$(ababa)(cbcb)(dcdc)(d^{-1})$$
$$\downarrow$$
$$(babab)(cbcb)(dcdc)(d^{-1})$$
$$(baba)(bcbcb)(dcdc)(d^{-1})$$
$$\downarrow$$
$$(baba)(cbcbc)(dcdc)(d^{-1})$$
$$(baba)(cbcb)(cdcdc)(d^{-1})$$
$$\downarrow$$
$$(baba)(cbcb)(dcdcd)(d^{-1})$$

Theorem

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Proof strategy:

1. Show that if w does not admit an RRS, $t \in S$ and

$$wt \overset{RRS}{\rightsquigarrow} w'$$

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1. Show that if w does not admit an RRS, $t \in S$ and
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then w' does not admit an RRS
2. Define a recursive function ϕ such that $\phi(ws) = \phi(\phi(w)s)$ and $\phi(wt) = w'$.
3. Show that length of $\phi(w)$ is equal the length of $\phi(u)$ if w and u are equivalent words in the group.

How does it work in the 3 free case?

If $m_{s,t} \geq 3$ we define *pseudo-2-generated(P2G)* word in s, t, to be a word w such that $f[w], l[w] \in \{s, s^{-1}, t, t^{-1}\}$ and where all the letters in w not in $\{s, s^{-1}, t, t^{-1}\}$ can be pushed via commutations with individual letters either to the left or to the right.

Example ($m_{x,s} = m_{y,s} = m_{y,t} = m_{z,t} = 2$)

Then $sxytysyzt$ is equivalent to $xy^3(stst)z$.

P2G Critical words

We call the remaining 2-generated word in the middle $\hat{w}$ and the letters which cannot be pushed to the rleft but can be pushed to right form β_w.

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We call the remaining 2-generated word in the middle $\hat{w}$ and the letters which cannot be pushed to the rleft but can be pushed to right form β_w.

We say that a word is *P2G critical* if it is a P2G word such that $\hat{w}$ is a critical word. We extend the definition of τ to P2G words by applying τ to $\hat{w}$.

Example ($m_{s,t} = 4$, $m_{x,s} = m_{y,s} = m_{y,t} = m_{z,t} = 2$)

$\tau(sxytysyzt) = xy^3\tau(stst)z = xy^3(tsts)z$

Return to Rightward Reducing Sequences

In 3-free Artin groups, we can now define an RRS

Data:

- word $w = \text{---} w_1 w_2 \dots w_{k+1} g^{-1} \text{---}$
- w_1 and $u_{i+1} = l[\tau(\hat{u}_i)]\beta_i w_{i+1}$ P2G critical for $i < k$.
- All letters in w_{k+1} commute with g and $l[\tau(u_k)] = g$

$$\text{---} w_1 w_2 \dots w_{k+1} g^{-1} \text{---}$$
$$\downarrow$$
$$\text{---} \tau(w_1) w_2 \dots w_{k+1} g^{-1} \text{---}$$
$$\text{-----} l[\tau(\hat{u}_1)]\beta_1 w_2 \dots w_{k+1} g^{-1} \text{---}$$
$$\downarrow$$
$$\text{-----} l[\tau(u_k)] w_{k+1} g^{-1} \text{---}$$
$$\text{-----} w_{k+1} g g^{-1} \text{---}$$

RRS example

Example

(RRS) in 3-free Artin group. $m_{a,b} = m_{b,c} = 5$

$m_{a,c} = m_{a,x} = m_{a,y} = m_{b,y} = m_{z,c} = 2$

$$(axbaybca)(bcb)(z^3a)(c^{-1})$$
$$\downarrow$$
$$(xy(babab)c)(bcb)(z^3a)(c^{-1})$$
$$xybaba(bcbcb)(z^3a)(c^{-1})$$
$$\downarrow$$
$$xybaba(cbcbc)(z^3a)(c^{-1})$$
$$xybabacbcb(cz^3a)(c^{-1})$$
$$\downarrow$$
$$xybabacbcb(z^3ac)(c^{-1})$$

Results

Let A_S be a 3-free Artin group

Theorem (Blasco-García, Cumplido, MW)

Let w be a word that does not admit an RRS and let t be a letter. Then either wt is already geodesic or, there is an RRS that can be applied to wt that will result in a geodesic word.

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Let w be a word that does not admit an RRS and let t be a letter. Then either wt is already geodesic or, there is an RRS that can be applied to wt that will result in a geodesic word.

Theorem (Blasco-García, Cumplido, MW)

The set of geodesic words in A_S is exactly those words that do not admit an RRS

Conclusion

Let A_S be a 3-free Artin group. There there exists an algorithm which solves the word problem, without increasing the length of the word at any step.

About the 3-free condition

If $m_{s,t} \geq 4$, it becomes much easier to "trap" letters and make it impossible to have P2G critical words.

Example

Consider *stxst* with $m_{s,t} = 4$. In order for this to be P2G word, x must commute with both s and t.

If $m_{s,t} = 3$ then *stxs* can be P2G critical even if x does not commute with t.

Thank you!