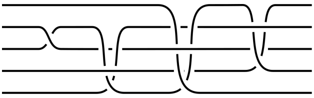
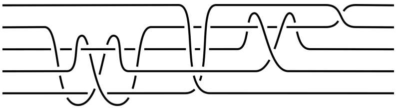
Braid factorizations and exotic complex curves





Kyle Hayden

March 25, 2022

algebraic symplectic topological homological

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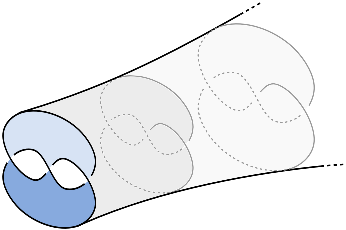
holomorphic smooth homotopical

Goal: Use braided objects to study the phase transition between rigidity and flexibility (especially in dimension 4)

Abhyankar-Moh-Suzuki (1975): Any algebraic embedding $\mathbb{C} \hookrightarrow \mathbb{C}^2$ is equivalent to $\mathbb{C} \times 0$ under an algebraic automorphism of $\mathbb{C}^2$.

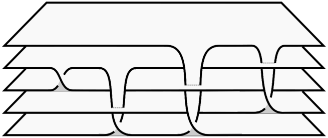
Neumann (1989): Any smooth algebraic curve $V \subset \mathbb{C}^2$ with connected link at infinity L_∞ is isotopic to a standard Seifert surface for L_∞ extended cylindrically to infinity.

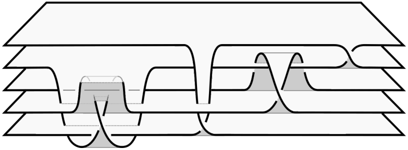


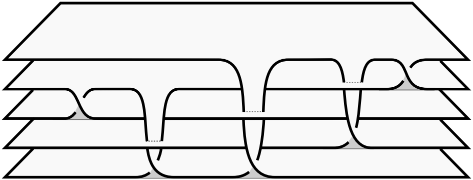
Freedman (1985): $\exists$ smooth $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ that are isotopic to $\mathbb{R}^2 \times 0$ via homeomorphisms but not diffeomorphisms of $\mathbb{R}^4$.

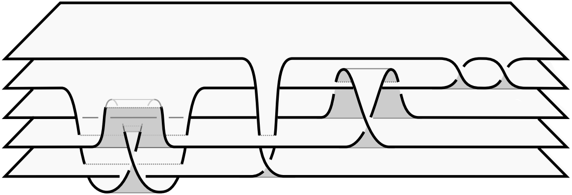
Theorem (H, 2021)

There are smooth complex curves in $\mathbb{C}^2$ that are isotopic via ambient homeomorphisms but not diffeomorphisms.

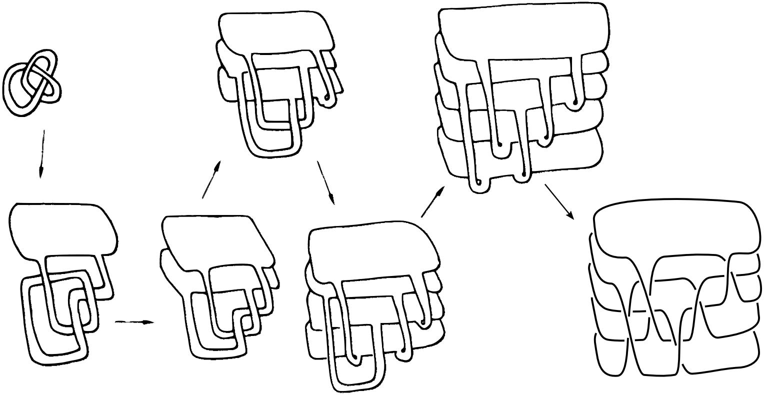








Rudolph '84: Surfaces in B^4 are often* isotopic to *braided surfaces.*



*often = no local maxima

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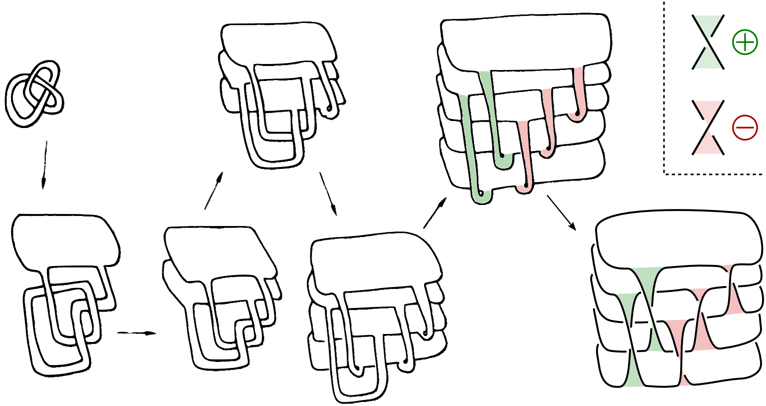
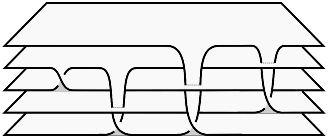
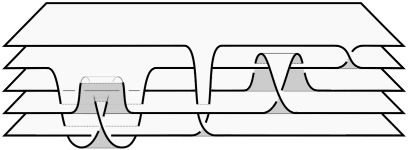


*often = no local maxima

Informal 3D definition

braided surface = a finite collection of parallel disks joined by half-twisted bands (with "ribbon" intersections)





Formal 4D definition

A smooth surface Σ in $B^4 \approx D^2 \times D^2$ is *braided* if the first-coordinate projection $D^2 \times D^2 \to D^2$ restricts to a branched covering $\Sigma \to D^2$.

Theorem (Rudolph, 1983)

Every positively braided surface in B^4 is isotopic to $V \cap B^4$ for a smooth complex curve $V \subset \mathbb{C}^2$.

Theorem (Boileau-Orevkov, 2001)

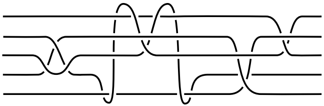
Every properly embedded complex curve in $B^4 \subset \mathbb{C}^2$ is isotopic to a positively braided surface.

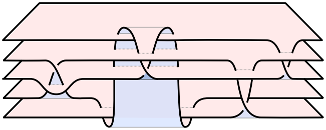
Note: They prove it for symplectic surfaces with transverse boundary.

$$\left\{\begin{matrix}\text{Bounded complex} \\ \text{curves in } B^4 \subset \mathbb{C}^2\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{pos. braided} \\ \text{surfaces}\end{matrix}\right\} \longleftrightarrow \left\{\beta = \prod_k w_k \sigma_{i_k}^{+1} w_k^{-1}\right\}$$

quasipositive braid = product of conjugates of pos. generators $\sigma_i \in B_n$, i.e. $\beta = \prod_{k=1}^{\ell} w_k \sigma_{i_k} w_k^{-1}$

Example:

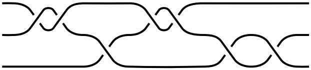


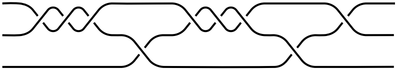


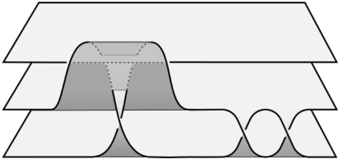
$$\beta = (\sigma_2\sigma_3\sigma_2^{-1})(\sigma_1^{-2}\sigma_2\sigma_3\sigma_4^2\sigma_3^{-1}\sigma_2\sigma_3\sigma_4^{-2}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2)(\sigma_3^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_3)(\sigma_4^{-1}\sigma_3\sigma_4)$$
$$= (w_1\boldsymbol{\sigma_3}w_1^{-1})(\text{———}\ w_2\ \text{———}\ \boldsymbol{\sigma_2}\ \text{———}\ w_2^{-1}\ \text{———})(-\ w_3 - \boldsymbol{\sigma_1} - w_3^{-1} -)(w_4\boldsymbol{\sigma_3}w_4^{-1})$$

Inequivalent factorizations can yield inequivalent surfaces.

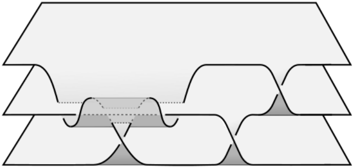
Example (Auroux): $(\sigma_2^{-2}\sigma_1\sigma_2^2)\sigma_1^2 = (\sigma_1^{-3}\sigma_1\sigma_3)\sigma_1\sigma_2 \in B_3$



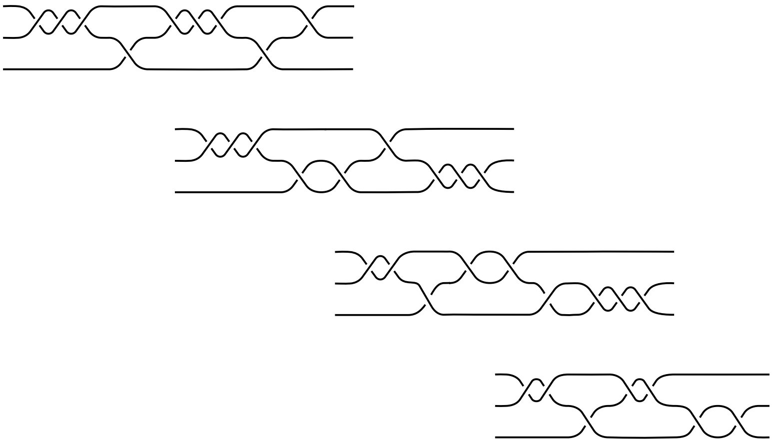




$\neq$



$(\sigma_1^{-3}\sigma_1\sigma_3)\sigma_1\sigma_2 =$



$= (\sigma_2^{-2}\sigma_1\sigma_2^2)\sigma_1^2$

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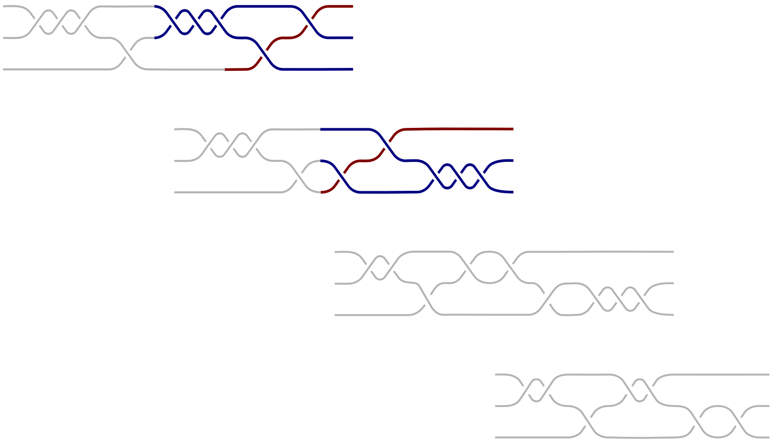

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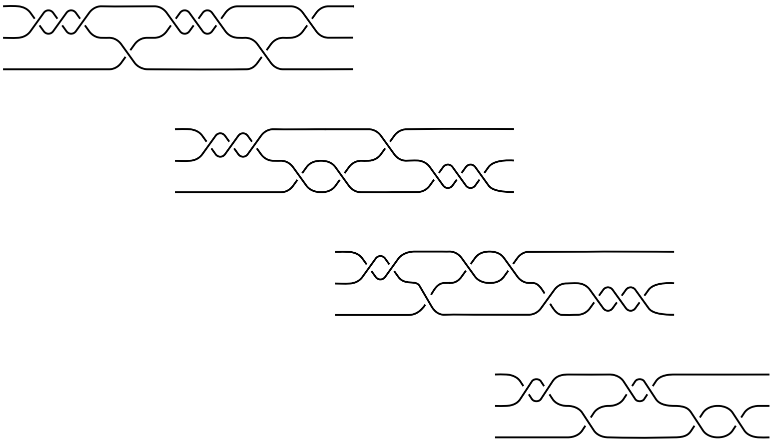

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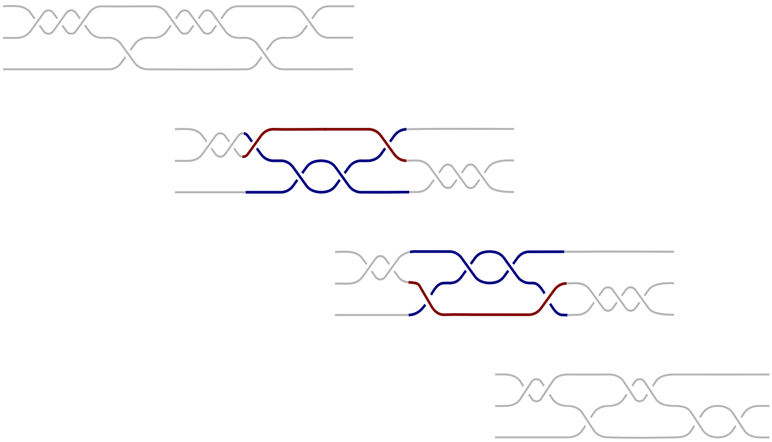

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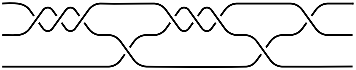

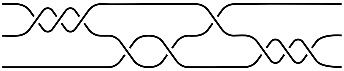

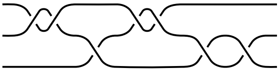

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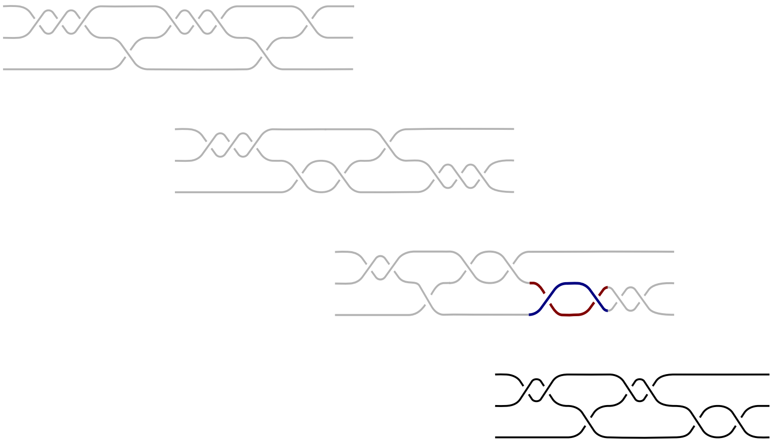

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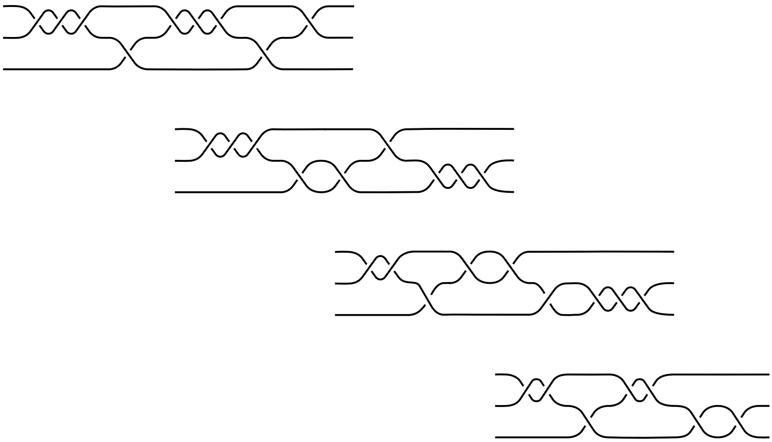

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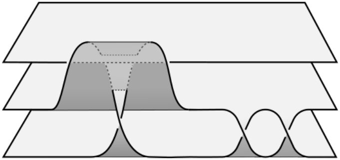

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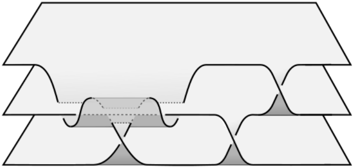
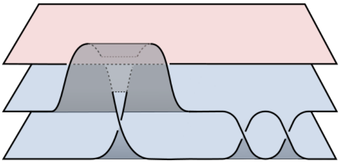


Distinguished by π_0.



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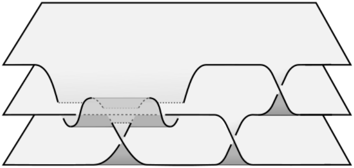

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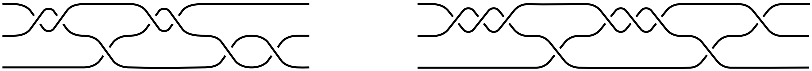
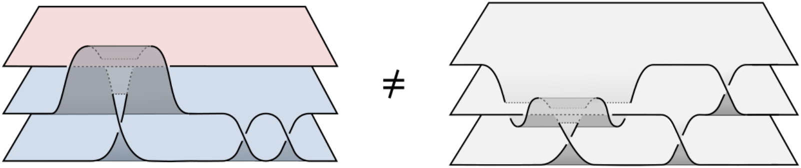


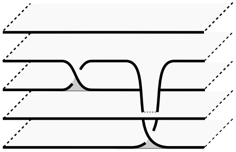


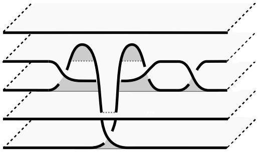
But there *are* operations that preserve the isotopy type! E.g., apply braid relations (anti)symmetrically to w_k and w_k^{-1} in $\prod_{k=1}^{\ell} w_k \sigma_{i_k} w_k^{-1}$.

Two main "nontrivial" moves:

- *global conjugation:* $\alpha_1 \cdots \alpha_\ell \;\longleftrightarrow\; (w\alpha_1 w^{-1}) \cdots (w\alpha_\ell w^{-1})$
- *Hurwitz moves:*

$$\alpha_1 \cdots \alpha_k \alpha_{k+1} \cdots \alpha_\ell \;\longleftrightarrow\; \alpha_1 \cdots (\alpha_k \alpha_{k+1} \alpha_k^{-1}) \cdot \alpha_k \cdots \alpha_\ell$$





Related settings:

- The Symplectic Isotopy Problem (S.I.P.) = *Every smooth symplectic surface in $\mathbb{C}P^2$ is isotopic to a complex curve.*

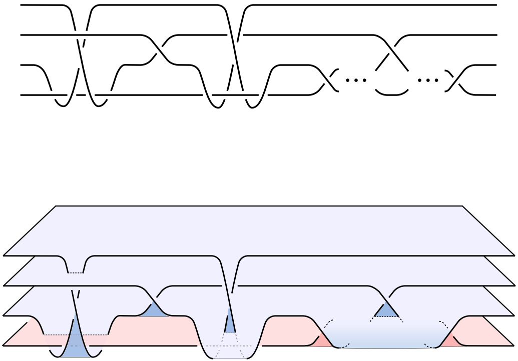
  Kharlamov-Kulikov: S.I.P. $\iff$ The full twist $\Delta_d^2 \in B_d$ has a unique quasipositive factorization up to Hurwitz equivalence.

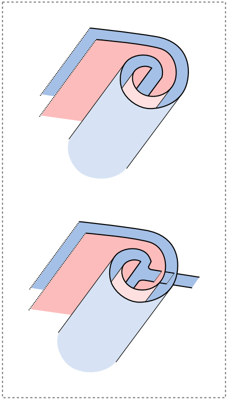
  braid index d = degree; algebraic length $\ell = d^2 - d$

- **Baykur-Etnyre-H-Hedden-Kawamuro-Van Horn-Morris:** Much of Rudolph + Boileau-Orevkov's correspondence extends to arbitrary Stein surfaces

Example (Baykur–Van Horn-Morris): ∞'ly many factorizations!

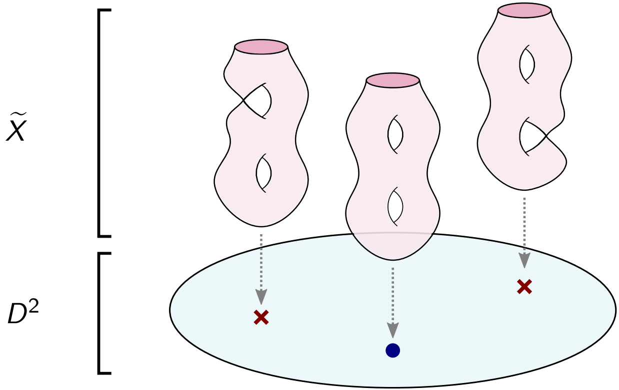
$$\beta_n = (\sigma_1^{-2}\sigma_3^{-1}\sigma_2\sigma_3\sigma_1^2)\sigma_2(\sigma_1^2\sigma_3\sigma_2\sigma_3\sigma_1^{-1})(\sigma_1^n\sigma_2\sigma_1^{-n}) \in B_4$$





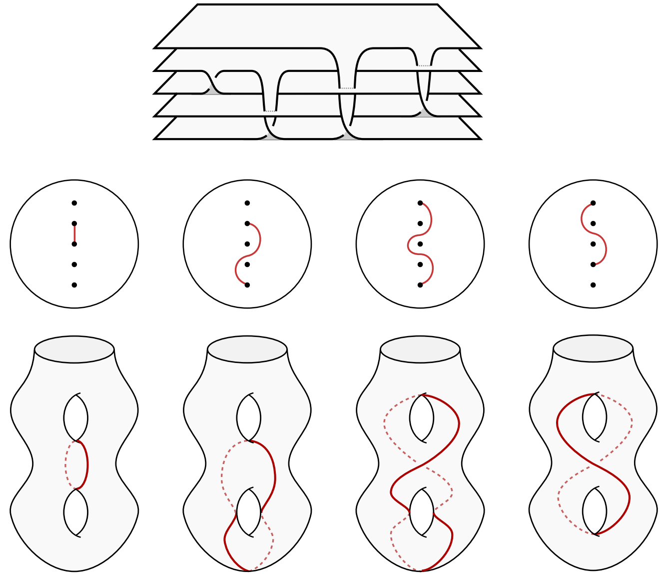
$\widetilde{X}$ = branched double cover of B^4 over a pos. braided surface

$\implies \widetilde{X}$ has the structure of *Lefschetz fibration*, $f : \widetilde{X} \to D^2$.



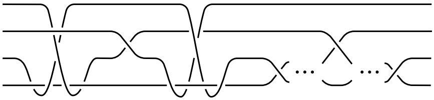
Braid index fixes regular fiber type. Bands determine singular fibers.

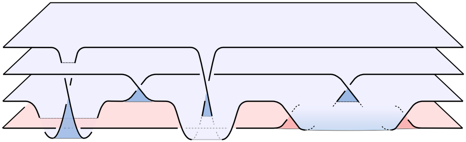
Example:

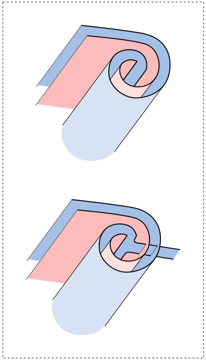


Example (Baykur–Van Horn-Morris): ∞'ly many factorizations!

$$\beta_n = (\sigma_1^{-2}\sigma_3^{-1}\sigma_2\sigma_3\sigma_1^2)\sigma_2(\sigma_1^2\sigma_3\sigma_2\sigma_3\sigma_1^{-1})(\sigma_1^n\sigma_2\sigma_1^{-n}) \in B_4$$







These surfaces Σ_n are distinguished by $H_1(\widetilde{X}_n) \cong \mathbb{Z}/n\mathbb{Z}$.

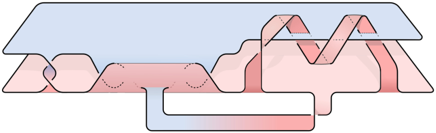
Most examples seem to arise by studying relations in the braid group or among products of Dehn twists commuting with hyperelliptic involution.

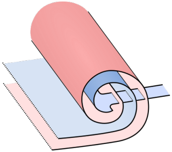
These types of relations are often useful for building distinct 4-manifolds via Lefschetz fibrations.

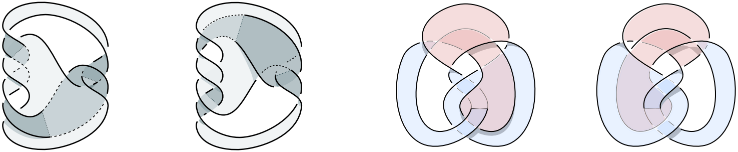
Turn this around:

1. Start with interesting 4-manifolds.
2. Try to realize them as branched covers of surfaces in B^4.
3. Try to show the surfaces are isotopic to pos. braided surfaces.
4. Do something with them!

There's no good reason for this to work... but it does.

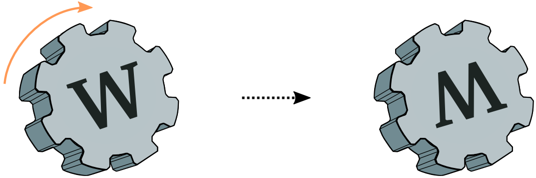




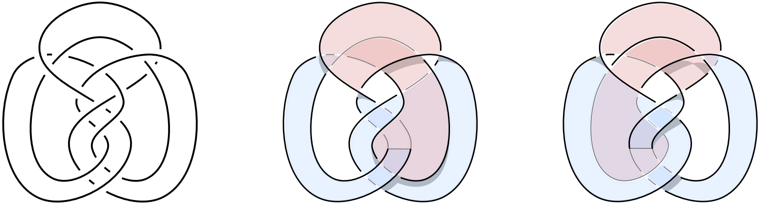


Side question: Why do many well-known 4-manifolds arise as double branched covers of interesting complex curves in B^4 in interesting ways?

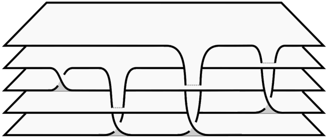
Step 1. The "positron cork" W is a contractible 4-manifold with involution τ of ∂W that extends to a homeo (but not diffeo) of W.



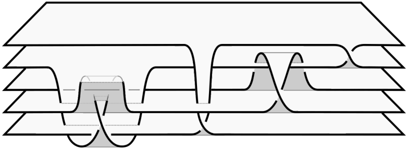
Step 2. W is the branched cover of B^4 over these disks:



Step 3. The disks are isotopic to positively braided surfaces.



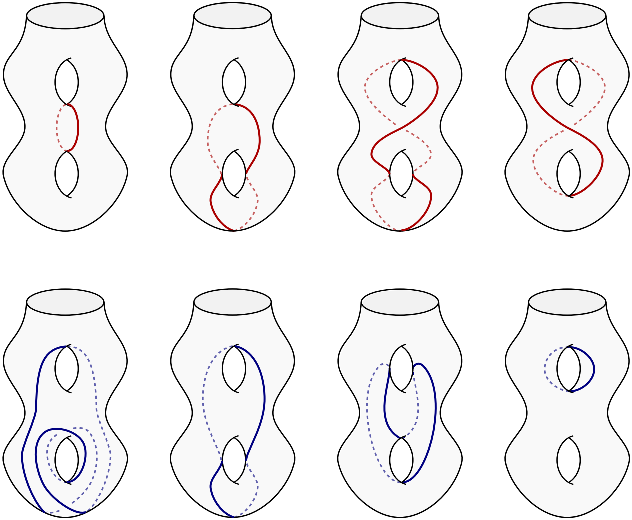
D



D'

$$\beta = \sigma_3(\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^{-1})(\sigma_4\sigma_3^{-1}\sigma_2\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1})(\sigma_4^{-1}\sigma_3\sigma_2\sigma_3^{-1}\sigma_4)$$

$$\beta' = (\sigma_3\sigma_2^{-1}\sigma_1^{-2}\sigma_3^{-2}\sigma_2\sigma_3^2\sigma_1^2\sigma_2\sigma_3^{-1})(\sigma_4\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1})(\sigma_3^2\sigma_2\sigma_3^{-2})\sigma_4$$



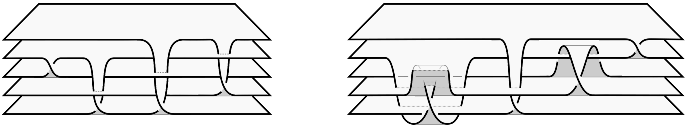
(This monodromy substitution realizes the positron cork twist.)

Theorem (H, 2021)

There are smooth complex curves in $\mathbb{C}^2$ that are isotopic via ambient homeomorphisms but not diffeomorphisms.

Proof sketch.

- Construct isotopy $D \to D'$ through homeomorphisms of B^4

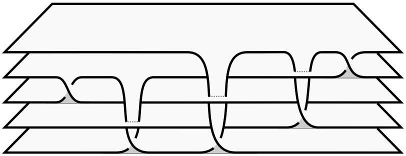


D D'

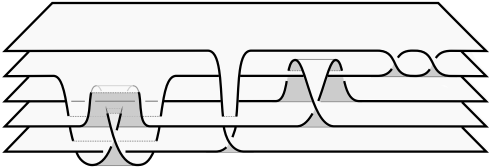
Use Conway-Powell (cf Freedman) and $\pi_1 \cong \mathbb{Z}$ for $B^4 \backslash D$ and $B^4 \backslash D'$.

- Enlarge D, D' into annuli A, A' by adding extra band at end.

A

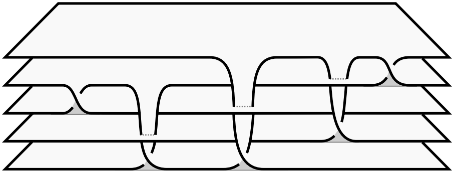


A'

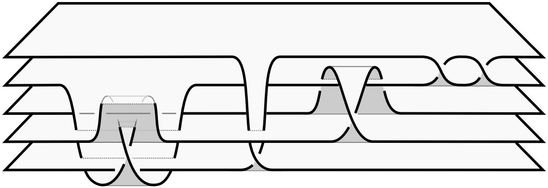


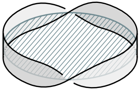
- Claim: Branched cover of A doesn't contain any smoothly embedded S^2 of square -2, yet branched cover of A' does.

A

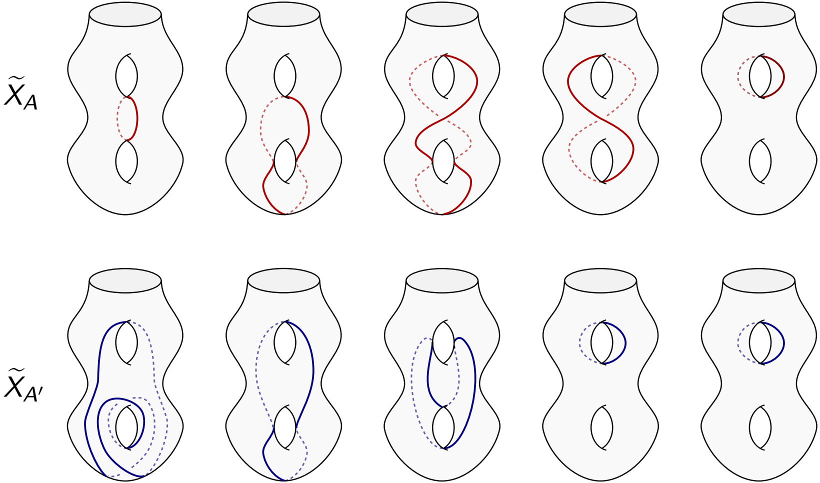


A'





Can rule out (-2)-spheres for $\widetilde{X}_A$ using adjunction inequality (Seiberg Witten theory). Can explicitly see the (-2)-sphere in $\widetilde{X}_{A'}$.



$\implies \widetilde{X}_A$ and $\widetilde{X}_{A'}$ are not diffeo

$\implies (B^4, A)$ and (B^4, A') are not diffeo. $\square$*

*These are properly embedded curves in the compact unit B^4. Use Fatou-Bieberbach domains to make properly embedded in $\mathbb{C}^2$.

Thank you!