Case-Based Decision Theory and Financial Markets

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Chapter 1. Introduction

The case-based decision theory has been recently proposed by Gilboa and Schmeidler (1995) as an alternative theory for decision-making under uncertainty. Differently from the expected utility theory, the case-based decision theory models decisions in situations of structural ignorance, in which neither states of the world, nor their probabilities can be naturally derived from the description of the problem. It is assumed that a decision-maker can only learn from experience, by evaluating an act based on its past performance in similar circumstances. An aspiration level is used as a bench-mark in the evaluation process. It distinguishes results considered satisfactory (those exceeding the aspiration level), which make the alternative more attractive, from the unsatisfactory ones, which influence negatively the evaluation of the alternative.

Similarity considerations play an important role in case-based reasoning. The evaluation of an alternative depends not only on its own performance, but also on the utility realizations achieved from *similar* alternatives in *similar* circumstances.

Although the case-based decision theory has been applied in several economic contexts[1], it has not been used to model decision-making in financial markets up to now. Still, a model of financial markets, in which expected utility maximization is replaced by case-based reasoning is of interest for several reasons. First, it allows to gain a deeper understanding of the case-based decision theory itself. In such a model the operationalization of theoretical concepts such as the aspiration level, the past experience of the decision-maker, as well as his similarity perceptions becomes necessary. It is therefore possible to examine the influence of these concepts on individual behavior. The results achieved further allow to interpret these concepts in an economically meaningful way.

Second, the application of the case-based decision theory to financial markets contributes to the literature on behavioral finance, by describing the dynamics of portfolio holdings and asset prices in a market with case-based investors. The analysis of the behavior implied by case-based reasoning allows for comparisons to the predictions of the standard financial theory, as well as to the results obtained from alternative decision theories[2].

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1 This literature is reviewed in section 2 of the introduction.

2 The implications of an abstract decision theory can only be understood by applying it to a specific economic

Introducing case-based decisions into a model of financial markets further allows to compare the qualitative results on price dynamics and portfolio holdings to empirical findings. Hence, the case-based decision theory might allow to explain empirical observations which are inconsistent with standard asset pricing theories.

Last, but not least, in an asset market the performance of case-based reasoning can be compared to the performance of alternative decision rules, especially to those of expected utility maximization. This comparison is useful, especially for the analysis of competitive environments in which the influence of a given decision rule on market prices is determined by its past performance. If the case-based decision-makers consistently lose money compared to rational individuals, then their influence on the market processes will vanish in the limit. If, on the other hand, individuals using case-based reasoning are not driven out of the market, then the influence of their behavior on prices and returns cannot be neglected.

It follows that the application of the case-based decision theory to models of asset markets can lead to interesting results both in the field of decision theory and of financial markets. However, before turning to the construction of formal models, the place of the case-based reasoning in the field of decision theory has to be discussed and its applicability to financial markets examined.

The decision theory has developed rapidly during the last sixty years. The expected utility theory, axiomatized by von Neumann and Morgenstern (1947) and Savage (1954), was proposed both as a normative and a descriptive theory. Experimental findings, however, attack its empirical validity, whereas alternative models criticize its normative foundations. The case-based decision theory proposed by Gilboa and Schmeidler (1995) also emerges from this discourse, by criticizing fundamental concepts of the expected utility theory and proposing a new framework for decision-making under uncertainty. In order to understand its place among the numerous decision theories proposed in the literature, the development of this field will be sketched in section 1 of this introduction, before presenting the framework of Gilboa and Schmeidler (1995, 1997 (a)) in section 2 and reviewing the literature on case-based decisions.

Since it is found that the expected utility theory combined with rational expectations cannot ex-

problem. A decision theory is evaluated according to its ability to explain empirical findings and to produce qualitatively new predictions, not obtained before. Financial markets represent a perfect "testing" field for decision theories, first, because the implications of expected utility maximizations are very well studied in this context, and second, because of the vast quantity of empirical data available.

plain a range of phenomena observed in the market data, alternative theories for decision-making under uncertainty have been applied to the analysis of financial markets in the literature. In section 3, I shortly review two major approaches: the noise trader approach, which assumes expected utility maximization with biased beliefs, and the assumption of alternative preferences, such as ambiguity-aversion or loss aversion, which criticize the conceptual foundations of the expected utility maximization. Since the case-based decision theory questions the mere possibility of reasoning about states of nature and forming beliefs about their probabilities, it represents a new framework for modelling decisions in financial markets, which substantially differs from the two approaches discussed.

The applicability of the case-based decision theory to financial markets is further discussed. In section 4, I argue that the structural ignorance of the decision-maker assumed in the case-based decision theory might well describe the complex structure of financial markets.

Since the case-based decision theory was only recently developed, not much is known about its behavioral implications. I, therefore, discuss how case-based decisions can incorporate some of the psychological biases observed in experimental and real financial markets. I further compare the type of learning modelled by the expected utility theory to those implied by the case-based decision theory.

Section 5 presents an overview of the thesis.

1.1 Developments in the Theory of Decision-Making under Uncertainty

The expected utility theory as proposed by von Neumann and Morgenstern (1947) and Savage (1954) is the most prominent theory for decision under uncertainty. According to this theory uncertainty is represented as a set of states of the world. It is either assumed that the probability of the occurrence of these states is *objectively* given (as in the case of betting on an outcome of throwing a fair coin) or that the decision-maker is able to ascribe *subjective* probabilities to each of the states (as in the case of betting on an outcome of a horse race). The decision under uncertainty is then regarded as a choice among a set of acts, each of which represents a probability distribution over state-contingent outcomes. The expected utility theory proposes a representation of the preferences over such acts: the utility of an act is computed as the weighted sum of the utility of the state-contingent outcomes, the weights being the state probabilities. Von Neumann and Morgenstern (1947), Anscombe and Aumann (1963) and Savage (1954) each propose a system of axioms imposed on preferences which are equivalent to an expected utility representation.

It seemed at the time that these results have put an end to a long discussion in which the existence of objective probabilities has been opposed to the thesis that all probabilities are subjective. Keynes (1921, p. 4), who holds the objectivistic view, writes ”[...] in the sense important to logic probability is not subjective. It is not that's to say, subject to human caprice. A proposition is not probable because we think it so. When once the facts are given which determine our knowledge, what is probable or improbable in these circumstances has been fixed objectively, and is independent of our opinion”. Keynes suggests that the reasoning about probabilities should obey the laws of logic, and, hence, be rational in the following sense: given individual's knowledge[3] about some propositions h and the knowledge of propositions ascertaining some probability relation between h and p, the individual will rationally entertain a probabilistic belief in p, Keynes (1921 p. 16). Nevertheless, the notion of rationality and hence also of probability is relative to the knowledge h a person has, Keynes (1921, p. 32).

A person who behaves boundedly rational might however fail to use his knowledge to establish these logical connections. He will, therefore, either have no or a wrong notion of the probability he is trying to estimate. Moreover, although Keynes derives the probabilistic beliefs from logic conclusions, he argues that these beliefs do not always exist and are not always measurable and comparable to one another, Keynes (1921, p. 33), a statement which seems to oppose his objectivistic view.

De Finetti (1937) and Ramsey (1926), on the other hand, represent the subjectivistic view. Ramsey (1926, p. 67-68) criticizes the relativity view of Keynes. He himself proposes a behavioristic view of probability beliefs: ”the degree of belief is a casual property of it, which we can express vaguely as the extent to which we are prepared to act on it”, Ramsey (1926, p. 71). Hence, the method to measure beliefs consists in asking the person how he would act for different payoffs. This idea is later used by Savage (1954) to elicit subjective probabilities.

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3 Keynes (1921, p. 12) differentiates between direct (obtained by experience) and indirect (obtained by argument) knowledge. He only permits the derivation of the probability of p, given the knowledge of h (which may be direct or indirect) by indirect knowledge, hence by logical argument.

De Finetti (1937) supports the thesis that *all* probabilistic beliefs are subjective. Even in the case of games of chance, such as roulette and throwing a fair coin, does he criticize the view that the readily agreed upon equal possibility of all outcomes should be seen as a demonstration of the existence of objective probability distributions. De Finetti (1937, p. 112) argues that this ”objectivity” might have its reason in a common psychological perception of symmetry, which has nothing to do with objective considerations. ”[...] any event whatever can only happen or not happen, and neither in one case nor in the other can one decide what would be the degree of doubt with which it would be ”reasonable” or ”right” to expect the event before knowing whether it has occurred or not”, De Finetti (1937, p. 113).

The work of von Neumann and Morgenstern (1947), Anscombe and Aumann (1963) and Savage (1954) shows that both views, the objectivistic and the subjectivistic, lead to the same representation of preferences. The expected utility theory further establishes a connection between rationality and probabilistic reasoning: every rational person should have a system of preferences, which lead him to make decisions as if he had some subjective probabilities in mind. Moreover, it allows to establish a link between the knowledge of a person and her probability perceptions. Starting with a prior and updating it in a Bayesian way, the decision-maker can learn the ”true” objective probability distribution in the limit. Hence, the expected utility theory provides a theoretical support for the ideas expressed by Keynes.

The expected utility theory has turned out to be a very useful and easily applicable instrument for modelling decision-making under uncertainty. In the mean time the applications to financial markets, insurance, contingent contracts, bargaining, to mention only a few, are so numerous that it would be impossible even to try to list them here. This shows that the expected utility theory has become a valuable and indispensable part of the economic methodology.

However, no sooner was this theory developed that the first criticisms began to emerge. Ellsberg (1961) constructed a thought experiment which questioned the intuition of $P2$, the axiom, called ”the sure-thing principle” in the theory of Savage. His experiment showed that in making decisions people do not necessarily behave as if they ascribed subjective probabilities to states of the nature. The idea that people evaluate situations in which they *have* information about probability distributions differently from situations in which such information is (partially) *missing* made the economists aware of a phenomenon called ”ambiguity-aversion” and gave rise to a new branch of the literature on decision-making. By relaxing[4] the sure-thing principle and the independence axiom the axiomatizations of preferences proposed by Schmeidler (1989), Gilboa and Schmeidler (1989), Gilboa (1987), Wakker (1989) and Sarin and Wakker (1992) are able to describe decision-making in situations, in which the decision-maker considers multiple priors, (i.e. he considers more than one probability distribution as possible) or in which the perception of probabilities is described by a capacity[5]. In the first case, the decision-maker uses different probability distributions to evaluate acts, which are not comonotonic.. Nevertheless, his beliefs can be described by a set of probability distributions. In the second case, in which the beliefs are described by a capacity, a probability distribution consistent with the capacity might not even exist. Hence, the question of the existence of probabilities posed by Keynes (1921) is raised again.

The non-expected utility theories allow to capture the idea that people might value information about state probabilities and underestimate utility realizations in states the probabilities of which are not known. Ambiguity-aversion does not only allow to explain the typical behavior observed in the Ellsberg paradox. It can also account for paradoxes observed in financial markets, which are based on the assumption that decision-makers assign additive probabilities to the possible states of nature.

Objective probabilities also do not stand the test of experiments. Allais (1953) constructs the first experiment violating the ”independence axiom” of von Neumann and Morgenstern (1947). It is found that the perception of probabilities is a non-linear one, people underweight large probabilities and overweight small ones, Gonzalez and Wu (1999). The rank-dependent theories (called so, because the evaluation of a payoff and the assignment of probabilities depend on the rank of the payoff under the act considered), such as Quiggin's (1982) rank-dependent utility, Yaari's (1987) dual theory of choice under risk, Tversky and Kahneman's (1992) cumu-

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[4] For instance, by assuming that the independence axiom or the sure-thing principle only hold for comonotonic acts (acts which rank states according to their consequences in the same way), as in Schmeidler (1989) and Gilboa (1987) or only for mixtures with constant acts, as in Gilboa and Schmeidler (1989). Wakker (1989) uses a version of his, Wakker (1984), ”cardinal coordinate independence” axiom, also restricted to comonotonic acts. Sarin and Wakker (1992) propose an axiom, which is similar to stochastic dominance in the context of additive probabilities, see Camerer and Weber (1992, p. 351).

[5] A capacity is a function, which assigns to each event a number between 0 and 1. It satisfies the monotonicity property, i.e. the number assigned to a union of two events should weakly exceed the number assigned to each of these events. In contrast to probability, a capacity need not be additive.

lative prospect theory[6] allow for representations of such weighting functions. It turns out that it is impossible to distinguish empirically, whether the utility function is non-linear and the perception of probabilities is non-biased or whether the probabilities are biased and the individual is risk-neutral.

An interesting feature of the model of Tversky and Kahneman (1992), not present in the other decision theories, is the introduction of a reference point, representing the status quo. Payoffs are treated as gains and losses relative to this reference point and evaluated differently according to this distinction. Losses are weighted stronger than gains and the individual exhibits risk-aversion for gains and is risk-loving for losses. In this sense, the prospect theory is similar to the case-based decision theory, in which payoffs are also evaluated with respect to an aspiration level. Nevertheless, the development of the case-based decision theory represents a completely new approach to decision-making under uncertainty.

1.2 The Case-Based Decision Theory

Although the discussion about multiple priors and rank-dependent utility raises the old question of whether probabilities are well-defined and exist, the main concepts of the theory of Savage remain unchanged. The probability distribution is replaced by the more general concept of a capacity or by a probability weighting function, but the description of the world by means of states of nature and the representation of acts as vectors of state-contingent outcomes persists.

In 1995 Gilboa and Schmeidler propose a new theory for decision-making under uncertainty which criticizes these concepts per se — the case-based decision theory. They argue that the theory of Savage can describe well only situations, in which the states of nature and their probabilities are well known or can easily be inferred[7]. In contrast, decision problems in which either the states of the world are not clearly stated, or they are so numerous and complex that probabilities could hardly be attributed to them in a sensible way, are not captured by the expected utility theory.

6 An axiomatization of the cumulative prospect theory is provided by Wakker and Tversky (1993).

7 An alternative theory for decision making under uncertainty, which criticizes the same issues is formulated by Easley and Rustichini (1999). The idea and the framework are very similar to those of Gilboa and Schmeidler, but similarity considerations are not introduced and it is assumed that the decision maker can observe the realizations of all available acts, independently of his actual choice.

Gilboa and Schmeidler (1995) handle this problem by proposing a new framework: the description of the decision situation is called a *problem*. To "solve" the problem the decision-maker has to choose an *act* out of a given set of *available acts*, known to him. The decision-maker does not have any knowledge of states of the world, state-contingent outcomes and their distributions. Nor does he try to infer them from the statement of the problem he faces. Instead, he uses his *memory*, in which information about *past cases* is collected. A case is defined as a triple consisting of a problem encountered, an act chosen to solve this problem and a utility realization consequently experienced. The utility realizations are evaluated according to an *aspiration level*. A utility realization which exceeds the aspiration level is considered satisfactory and increases the evaluation of the act which has lead to it and vice versa. The sum of the net utility realizations (utility realization less the aspiration level) observed when choosing a certain act determines its *cumulative utility*. It is postulated that the decision-maker chooses the act with maximal cumulative utility.

This simple way to make decisions might however turn out to be impractical, should an act have never been chosen before, or should the problem a decision-maker faces differ from those encountered in the past. Therefore, Gilboa and Schmeidler (1995, 1997 (a)) introduce the notion of *similarity*, which plays an important role in the case-based decision-making. Problems, acts and utility realizations can be perceived as similar. The evaluation of an act is then affected by its similarity to the acts chosen in the past and by the similarity of past problems to the current one. For instance, buying an asset at a price of 10 might be considered similar to buying the same asset at a price of 9,50. If an investor has achieved a satisfactory (unsatisfactory) return by buying this asset at a price of 10 in the past, then this result will positively (negatively) affect his evaluation of the act to buy the asset at a price of 9,50[8].

Differently from the similarity concept defined by Rubinstein (1988), which is a binomial relation (two objects are either similar or dissimilar), in the case-based decision theory the similarity is a trinomial relation (object α is more similar to α' than to α''). Therefore, it can be represented by a similarity function. This similarity function serves to determine the weights with

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8 At the moment I am vague about the distinction between acts and problems. In this example it might be that the act is defined as "buy asset X at a price of 10" or that the act is specified simply as "buy asset X" and the price enters the description of the decision problem encountered. Indeed, in the case-based decision theory this does not create a problem, since the similarity function is specified for problem-act pairs. I will be more explicit about this issue in chapter 2. See also Gilboa and Schmeidler (2001 (a), p. 51).

which *each* of the past net utility realizations enters the evaluation of an act[9].

To illustrate the main concepts of the case-based decision theory suppose that the set of available acts is given by $\mathfrak{A}$ with α denoting a representative act. T past experience is captured by the memory, which consists of cases encountered in the past. A case is a triple of a problem encountered, ρ_τ, an act actually chosen to solve this problem, α_τ and a utility realization actually observed as a consequence[10], u_τ. Hence, the memory at time t has the form:

$$M_t = \left(\left(\rho_\tau; \alpha_\tau; u_\tau\right)\right)_{\tau=1\ldots t-1}. \tag{1.1}$$

The proposed representation of preferences is of the form: for each α and $\alpha' \in \mathfrak{A}$

$$\begin{aligned} \alpha &\succeq \alpha', \text{ iff} \\ U_t(\alpha) &\geq U_t(\alpha'), \text{ with} \end{aligned}$$

$$U_t(\alpha) = \sum_{\tau=1}^{t-1} s\left((\rho;\alpha);(\rho_\tau;\alpha_\tau)\right)\left[u_\tau - \bar{u}_t\right], \tag{1.2}$$

where $s\left((\rho;\alpha);(\rho_\tau;\alpha_\tau)\right)$ denotes the similarity between the problem-act pair encountered at time τ and the problem-act pair to be evaluated and $\bar{u}_t$ stays for the aspiration level of the decision-maker at time t[11]. $U_t(\alpha)$ is called the cumulative utility of act α at time t. Note that satisfactory results enter the cumulative utility of an act with a positive sign, whereas unsatisfactory realizations have a negative impact. Moreover, since in general the similarity function depends in a non-degenerate way on the act considered, different aspiration levels lead to different evaluations of the acts available, even if the memory of two decision-makers is identical.

1.2.1 Axiomatic Representation

Gilboa and Schmeidler (1997 (a)) show that the representation (1.2) with $\bar{u}_t = 0$ is equivalent to axioms imposed on the preferences of a decision-maker over acts, given a set of observed cases, i.e. for a given memory. Assume that the utility function, associated with the payoffs of the acts, is given[12], so that u_τ indeed denotes a utility realization experienced at time τ. Suppose

9 Of course, it might be that the similarity between certain acts and/ or problems is considered to be 0. The respective utility realizations will then have no effect on the evaluation of the acts in the current situation.

10 The case-based decision theory also allows teh consideration of hypothetical cases. I neglect this issue at the moment and discuss this aspect in chapter 3, section 7 and in chapter 5, section 5. Note that the time index can also be interpreted just as the number of cases present in the memory. Hence, hypothetical reasoning is also captured by this representation.

11 The indexation by t allows for updating of the aspiration level depending on the past problems encountered and realizations observed.

12 Gilboa, Schmeidler and Wakker (2002) propose an axiomatization which derives the utility function $u(\cdot)$, the

that the memory is of length $(t-1)$ and is given by (1.1). The set of acts is given by $\mathfrak{A}$ and the problem at hand is ρ. The similarity function of the decision-maker over the set of acts $\mathfrak{A}$ can be elicited by changing the utility realizations of the different problem - act pairs in the memory and by asking the decision-maker about his preferences over the acts for different utility vectors $u \in \mathbb{R}^{t-1}$. Denote this preference relation by $\succsim_u$ for a given u. Following axioms are required, see Gilboa and Schmeidler (1997 (a), p. 51-52):

$\mathcal{A}1$: *Preorder:* $\succsim_u$ is complete and transitive;

$\mathcal{A}2$: *Continuity:* $\{\succsim_u\}_{u\in\mathbb{R}^{t-1}}$ is continuous in u, i.e. for each sequence $(u_k)_{k=1}^{\infty} \to u$, such that

$$\alpha \succ_{u_k} \alpha' \text{ for all } k$$
$$\alpha \succ_u \alpha'$$

holds.

$\mathcal{A}3$: *Additivity:* for all u and $v \in \mathbb{R}^{t-1}$ and for all α, $\alpha' \in \mathfrak{A}$,

$$\alpha \succ_u \alpha' \text{ and } \alpha \succsim_v \alpha'$$

imply $\alpha \succ_{u+v} \alpha'$.

$\mathcal{A}4$: *Neutrality:* for $u = (0...0)$

$$\alpha \sim \alpha' \text{ holds for all } \alpha,\ \alpha' \in \mathfrak{A}.$$

$\mathcal{A}5$: *Diversity:* for all distinct acts α, α', α'', $\alpha''' \in \mathfrak{A}$ there exists a vector $u \in \mathbb{R}^{t-1}$, so that:

$$\alpha \succ_u \alpha' \succ_u \alpha'' \succ_u \alpha'''.$$

The five axioms imply the existence of $s\left((\rho;\alpha);(\rho_\tau;\alpha_\tau)\right)$, such that

$$\begin{aligned} \alpha &\succeq \alpha', \text{ iff} \\ \sum_{\tau=1}^{t-1} s\left((\rho;\alpha);(\rho;\alpha_\tau)\right) u_\tau &\geq \sum_{\tau=1}^{t-1} s\left((\rho;\alpha');(\rho;\alpha_\tau)\right) u_\tau. \end{aligned}$$

Moreover, if there are more than four acts available, the similarity function is unique in the following sense: if $s\left((\rho;\alpha);(\rho_\tau;\alpha_\tau)\right)$ represents the preference relation $\succeq$, then so does each s', such that:

$$s'\left((\rho;\alpha);(\rho_\tau;\alpha_\tau)\right) = \beta s\left((\rho;\alpha);(\rho_\tau;\alpha_\tau)\right) + w_\tau$$

with $\beta > 0$ and $(w_\tau)_{\tau=1}^{t-1} \in \mathbb{R}^{t-1}$, Gilboa and Schmeidler (1997 (a), p.52-53).

The first axiom, which defines the preference order, is standard in decision theory and without it

cumulative utility representation and the similarity function directly from preferences.

a functional representation is virtually impossible. The second axiom requires that preferences do not change discontinuously when small changes in the utility realizations are observed. It insures the continuity of the functional representation. The additivity-axiom is central for the additive representation of the preferences by the cumulative utility as given in (1.2). It requires that if two vectors of utility realizations u and v both support the choice of an act α, then so should their sum do. In a similar manner, in which the additivity in expected utility theory is violated by behavior typical for the Ellsberg paradox, $\mathcal{A}3$ would be violated by a decision-maker whose similarity function is not independent of the utility realizations observed, see Gilboa and Schmeidler (1993, p. 15) for an example.

The neutrality axiom defines the aspiration level as $\bar{u} = 0$. Achieving 0 as a utility realization from all acts chosen does not allow to differentiate among them, since 0 is a "neutral" result — it makes the decision-maker "neither sad, nor happy", see Gilboa and Schmeidler (1997 (a), p.52). Note that since $\mathcal{A}4$ defines the aspiration level to be 0, the utility realizations u and v in $\mathcal{A}3$ are understood as "net" utility realizations, i.e. utility minus the aspiration level. Indeed, the representation proposed would not be valid, should in $\mathcal{A}3$ u and v represent utility realizations and the aspiration level be different from 0. However, it is easily seen that each utility function $u' = u + b$ in combination with an aspiration level $\bar{u}' = b$ would satisfy the axioms for a constant $b \in \mathbb{R}$, as long as the utility realizations u are replaced by the "net" utility realizations $u' - \bar{u}'$ everywhere in the axioms. Hence, differently from the von Neumann Morgenstern representation, the utility function in the case-based decision theory can only be subjected to linear affine transformations *together* with the aspiration level, so as to leave the behavior of the decision-maker unchanged.

The diversity axiom $\mathcal{A}5$ is only a technical one, it is not implied by the representation (1.2). It precludes the case in which one act is always ranked between other two, independently of the utility realizations observed. Gilboa and Schmeidler (1997 (a), p. 52) give an example of the acts "sell 100 shares", "sell 200 shares" and "sell 300 shares" and argue that for this particular acts $\mathcal{A}5$ might seem controversial. Still, it is not needed for settings in which less than 4 acts are considered, since then it is trivially satisfied. Moreover, Gilboa and Schmeidler (1997 (a), p. 60-61) demonstrate that indeed, without this axiom a representation of preferences by (1.2) is not guaranteed.

Note that the definition of similarity is "context"-dependent[13], i.e. the perception of similarity is determined relative to the memory M_t. Gilboa and Schmeidler (1997 (a), p. 54) indicate conditions for a memory-independent similarity relation. However, since the similarity should be independent of the vector of utility realizations observed, the description of acts and problems present in the memory is essential for the elicitation of similarity perceptions. For instance, the act "buy 10 shares of Telecom" might be represented in different contexts as "buy an asset", "buy a risky asset", "buy shares of a telecommunication branch", etc. and these representations will induce possibly different notions of similarity on the set of problem-act pairs.

One would like to associate the definition of similarity with behavior remaining constant under certain circumstances. To simplify matters, consider a setting, in which all problems are considered identical and similarity among acts only has to be determined. Suppose that $s(\alpha;\alpha') = 1$, hence, α and α' are completely similar. However, this does not mean that the decision-maker will be indifferent between α and α' in all contexts. Indeed, this will be the case only for contexts, containing cases in which either α or α', but no other act has been chosen, i.e. for memories:

$$M_t = \left((\rho; \alpha; u_\tau)_{\tau \in S \subset \{1 \ldots t-1\}} ; (\rho; \alpha'; u_\tau)_{\tau \in \{1 \ldots t-1\} \setminus S} \right).$$

Since $s(\alpha;\alpha') = 1$, the decision-maker will indeed be indifferent between α and α' for all vectors $u \in \mathbb{R}^{t-1}$. If, on the other hand, the memory also contains cases in which a different act α'' has been chosen and if $s(\alpha;\alpha'') \neq s(\alpha';\alpha'')$, the indifference between α and α' will no longer hold[14], since their cumulative utilities will be influenced differently by the realizations of α''.

I have presented only one possible axiomatization of the case-based decision theory, which, I think, provides useful insights into the assumptions needed to derive the preference representation and summarizes the cases which will be used in this thesis. Apart from the axiomatization presented above several axiomatizations have been proposed in the literature. The first of them goes back to Gilboa and Schmeidler (1995), who only consider similarity among problems, but not among acts. Hence, the similarity function takes the form $s(\rho;\rho')$. On the other hand, similarity on the set of utility realizations can also be included, so as to embed the notion that from

[13] Gilboa and Schmeidler (1997 (a)) call the set $((\rho_\tau;\alpha_\tau))_{\tau=1}^{t-1}$ a context.

[14] In a similar manner, in expected utility theory, we connect equal likelihood of states s and s' to equal willingness to engage in a lotterie which delivers x (x'), if s (s') occurs and y (y'), if not s (not s') occurs, as long as $x \succ y$ implies $x' \succ y'$. Still, we do not expect this willingness to remain identical for both states, if the signs of the inequalities are different. I thank Uzi Seagal for pointing out this analogy to me.

similar acts chosen in similar situations *similar* (and not *identical*) results are expected, see Gilboa and Schmeidler (2001 (a), p.52). The axiomatization, which generalizes these three approaches and is based on similarity among cases is derived in Gilboa and Schmeidler (2001 (a), pp. 62-90). In all these works the utility function $u(\cdot)$ is assumed to be given. Gilboa, Schmeidler and Wakker (2002) present an axiomatic representation in which the utility function is also derived from the preferences.

The axiomatic foundation of the case-based decision theory serves several purposes. First, it avoids postulating the existence of such constructs as similarity or aspiration level. Instead, they are derived from observed behavior and therefore gain ”cognitive significance”, see Gilboa and Schmeidler (1993, p. 9). Furthermore, in an experimental setting, results about the typical form of the similarity function and typical rules for adapting the aspiration level can be elicited by simply observing the choices made. Artifactual results arising from different interpretations of these concepts by subjects and experimenters can be avoided in this way.

Second, the formulation of the axioms in terms of observable characteristics, such as binary choices, allow to test the theory in an experimental setting. By presenting the subject a set of past cases and by eliciting his preferences for multiple vectors of utility realizations, one can examine whether the exhibited behavior violates the axioms imposed. Since the axiomatization precludes some types of behavior, the case-base decision theory can be subject to falsification in the sense of Popper (1966).

Third, the axioms allow to judge the reasonability of a theory, since they present the assumptions on the preferences (behavior) of a decision-maker that are necessary for a given functional representation to hold. Apart from their descriptive power (do people behave according to the prescriptions of the theory?) their normative power (would people *wish* to behave according to the axioms of the theory, if they knew and understood them?[15]) can be analyzed.

1.2.2 Applications of the Case-Based Decision Theory

Since the case-based decision theory is a relatively new concept, there are still few applications of it to economic problems. The case-based decision theory has been applied in the consumer

[15] Gilboa and Schmeidler (2001, pp. 9-12) view a normative theory as a theory describing ”second-order reality”, i.e. describing decisions with respect to preferences a decision maker would wish to have.

theory, Gilboa and Schmeidler (1997 (b), 2001 (b)), Gilboa and Pazgal (2001), theory of voting, Aragones (1997), production theory, Jahnke, Chwolka and Simons (2001), social learning, Blonski (1999), cooperation in games, Pazgal (1997), herding behavior, Krause (2003) and choices among lotteries, Gayer (2003).

Gilboa and Schmeidler (1997 (b), 2001 (b)) and Gilboa and Pazgal (2001) analyze a repeated consumer problem, consisting in choosing one of the available goods (or consumption bundles). Gilboa and Schmeidler (1997 (b)) and Gilboa and Pazgal (2001) assume a constant aspiration level and a memory, consisting of all past cases encountered. Gilboa and Schmeidler (1997 (b)) consider the case of deterministic utility realizations connected with the choice of a consumption good. They discuss the influence of similarity between acts on the decisions made. They demonstrate that positive similarity can be associated with complementarity among goods, whereas negative similarity is connected with substitutability between goods. Gilboa and Pazgal (2001) are interested in the influence of the position of the aspiration level on choices among goods with random utility realizations. They show that high aspiration levels lead to ”brand switching” behavior, an effect observed in empirical market data.

In contrast to these works, the setting of Gilboa and Schmeidler (2001 (b)) allows for adaptation of the aspiration level towards the consumer surplus experienced. Assuming constant utility realizations of each commodity chosen, the impact of a price increase on consumer’s behavior is analyzed. Since a price increase lowers the experienced utility realization, it initially leads to switching behavior. Since, however, the aspiration level is subsequently adapted downwards, the decision-maker becomes eventually satisfied with one of the acts and his choice remains constant afterwards. It is further shown that the reaction of a consumer may be different, depending on whether a price increase occurs suddenly or in a gradual manner.

Aragones (1997) studies a model of voting, in which voters assign negative net utilities to the policies adopted by the parties. He models an interaction between parties adopting a specific policy and voters choosing a party depending on their policy. He shows that, with two parties, the dynamic process of interaction forces the parties to adopt distinct policies, which they do not alter over time. The voters are endogenously divided into three categories, depending on their aspiration level. One category always opts for the first party, another one always chooses the second party, whereas a third category exhibits switching behavior.

Pazgal (1997) examines the behavior of case-based decision-makers, who engage in a game, but are not even informed about the strategic nature of the situation. He shows that the usage of the ”ambitious-realistic” rule for adaptation of aspiration levels, proposed by Gilboa and Schmeidler (1996), allows the players to learn to coordinate on a Pareto-optimal equilibrium in a coordination game.

Blonski (1999) introduces case-based decisions into a model of social learning. Similarity is used to describe the structure of the society. He examines the effects of social structures on the optimality of choices. The results show that different similarity functions imply different stable states and may positively or negatively influence the learning process in the society.

Jahnke, Chwolka and Simons (2001) analyze the adaptation process of a firm, who faces a kinked demand function, but does not know its exact form. The firm can choose the price of the product, as well as the capacity in each period of time. The capacity chosen determines, which part of the demand function is relevant for the decision of the consumers. The decision process of the firm is modelled using the case-based decision theory. It is assumed that the information of the firm (or the problem it faces) at each time consists of its optimization problem, the decision made in the last period and the parameters estimated from previous choice. The possibility to engage in hypothetical reasoning, as well as the (de facto[16]) deterministic utility realizations of the available acts allow the firm to learn and choose the optimal combination of price and capacity despite its short memory.

Krause (2003) models social learning in financial markets. In the tradition of herding behavior models, he assumes that the investors sequentially decide on one of the two available alternatives with independent random utility realizations. No market is modelled. The memory of an investor consists of cases experienced by investors who have made their decisions prior to him. A problem is defined by the identity of the investor who made a particular choice and by the time passed since the decision was made. These two dimensions determine the similarity between a problem present in the memory and the problem at hand, more recent problems being attached higher similarity. Krause (2003) conducts some simulations of the dynamic of choices in such an economy. He identifies herding behavior. However, since he assumes that both alternatives are equally good, it is not possible to judge the optimality of choice arising from case-based

[16] The demand is distributed according to a Poisson process, but Jahnke, Chwolka and Simons (2001) assume that the period between two subsequent decisions is sufficiently long, so that the Law of Large Numbers is applicable.

decisions in his model.

Gayer (2003) analyzes the impact of similarity considerations in a lottery choice problem. She argues that the notion of a probability distribution can be sharpened only with experience. So, people might know how to interpret a probability of $\frac{1}{3}$, but not be able to work with probabilities such as $0,34526$. Hence, instead of computing the expected utility of a lottery using the real probability, they use a notion of similarity between the known probability $\frac{1}{3}$ and $0,34526$ to evaluate a lottery, in which the latter probability is assigned to an outcome. She shows that if the similarity function evolves with time, making the distinction of probabilities more and more precise, the decision-maker learns to work with the real probability distributions. However, she conducts some simulations which demonstrate that overweighting of small and underweighting of large probabilities can occur in the model. Hence, this particular form of a probability weighting function can be explained by assuming such kind of case-based learning about probability distributions.

Although few, these applications demonstrate the potential of the case-based decision theory to describe human behavior in complex economic environments, in which the states of nature and the respective probability distributions are not naturally defined and in which, therefore, learning from experience plays an essential role. Nevertheless, up to now the case-based decision theory has not been applied to model decision-making in financial markets[17].

1.3 Decision Theory and Financial Markets

The expected utility theory combined with the assumption of correct or rational expectations is still most commonly used to describe behavior in financial markets. A range of observed phenomena, however, seem to be inconsistent with the theoretical results and to question the predictive power of this approach.

The standard theory of asset pricing is based on the condition of no-arbitrage. Two securities, which bear the same payoffs in each state of nature should have the same price, Eichberger and Harper (1997, p. 118). However, violations of this condition are observed in real, as well as in experimental markets. Rosenthal and Young (1990) find that shares issued by the same

[17] The model of Krause (2003) is more in the spirit of the literature on social learning and herding behavior, in which an act can have two realizations, which are exogenously given. Markets and prices are not modeled.

company, but traded in different markets have significantly different prices over long periods. Similar findings are reported by Lamont and Thaler (2002): they present a case, in which holders of a share of company A are expected to receive x shares of company B, but the share of A costs less than x times a share of B.

Whereas in real markets arbitrage might be hampered by transaction costs and short-sale constraints, in experimental settings it is possible to imitate perfect markets. Rietz (1998) and Oliven and Rietz (1995), however, demonstrate that violations of arbitrage freedom occur in experiments as well. The efforts taken by experimenters to indicate profit opportunities to the subjects involved and to teach them what trading strategies to use to eliminate arbitrage did not help to remove the violations of the no-arbitrage condition. A professional arbitrageur had to be inserted into the market to insure that the law of one price holds.

The capital asset pricing model (CAPM), Sharp (1964), Lintner (1965), predicts that, given assets with normally distributed payoffs, the investors in the economy should choose efficient portfolios, i.e. portfolios with minimal variance for a given mean. Moreover, given that there is a riskless asset in the market, each investor should invest one part of his endowment into the market portfolio and the rest into the riskless asset, implying the two-fund-separation theorem. Both implications are violated in the experiments of Kroll, Levy and Rappoport (1988). Moreover, low rates of diversification are often observed in real markets, Kang and Stulz (1997), Coval and Moskowitz (1999), Tesar and Werner (1995). Instead of holding the market portfolio, investors seem to exhibit preferences for national or even local assets, the so-called home bias.

Asset market models imply that expected returns of the assets in the market can differ only to account for the risk-aversion of the investors. Hence, a risky asset will in general exhibit higher expected returns than a riskless one, if investors are risk-averse. Mehra and Prescott (1985) find that the mean returns of securities exceed significantly those of bonds over the whole period, for which returns have been registered. They estimate the risk-aversion, which would be consistent with such behavior on the side of investors and find values which seem to be extremely high. This paradox has entered the literature under the name of ”equity premium puzzle”.

The efficient market hypothesis, formulated by Fama (1970), states that price movements should be unpredictable. Hence, trade on information available in the market should not be profitable, since such information is already taken into account by the market participants. Especially, past prices should not entail any information about future price movements. The empirical evidence, however, demonstrates that information is often priced incorrectly. DeBondt and Thaler (1985), Chopra, Lakonishok and Ritter (1992) find overreacting to information, whereas Bernard (1992), Bernard and Thomas (1989,1990), Loughran and Ritter (1995), Ikenberry, Lakonishock and Vermaellen (1995) and Womack (1996) find that investors react too slowly to news. Since prices adapt with time, so as to reflect the information correctly, price movements become predictable, violating the implications of the efficient market hypothesis.

The efficient market hypothesis further suggests that ”buy and hold” is the best strategy to follow. Since prices follow a random walk, winnings and losses are due merely to incoming new information, hence to chance and not to the proficiency of a trader. Therefore, no active trading can make an investor better-off. Nevertheless, Odean (1999) finds that investors trade too often. Not only are these trades not justified by new information, but traders engaging in them lose money even if transaction costs are not taken into account. Similar results are obtained by Barber and Odean (2001 (a), 2001 (b)).

The financial theory predicts that prices should reflect information about fundamentals. Hence, prices should only change when new information becomes available. It is however found that observed asset price volatility cannot be explained by changes in dividend payments (which would reflect changes in fundamentals) or by new information, Roll (1984), Shiller (1981, 1999).

One of the approaches taken to explain the phenomena observed in financial markets consists in assuming that some of the investors in the market have biased beliefs, while keeping the assumption of expected utility maximization. For instance, DeLong, Shleifer, Summers and Waldmann (1990 (a)), as well as Shleifer and Vishny (1997) show how noise traders, who misperceive the variance of a risky asset, can generate arbitrage possibilities in a market where the rational arbitrageurs are fully invested. The presence of positive feedback traders, who mistakenly believe that asset price movements are positively correlated, can lead to price bubbles, as DeLong, Shleifer, Summers and Waldmann (1990 (b)) demonstrate. The representativeness bias, (i.e. the usage of short time-series as if they were representative of the population) is used by Barberis, Shleifer and Vishny (1998) to explain under- and overreaction in financial markets. The assumption about the existence of chartists (positive feedback traders) and fundamentalists (who ignore prices and trade only on signals about future returns) in the market generates positive autocorrelation in the short-run and negative autocorrelation in the long-run, see Cutler, Poterba and Summers (1990). The excessive rate of trades is explained by the overconfidence of traders who overestimate the precision of the signals they receive, as in Odean (1998), or, in a dynamic set-up, by the self-attribution bias, as in Daniel, Hirshleifer and Subrahmanyam (1998) and Gervais and Odean (2001).

Another class of models assumes that investors have alternative preferences, different from expected utility maximization. For instance the presence of ambiguity-averse investors in the market can lead to results, which are inconsistent with the results obtained under expected utility maximizations and can explain some of the anomalies observed empirically. One of the first observations made by Dow and Werlang (1992) is that the optimal portfolio chosen by a Choquet expected utility maximizer is not sensitive to price changes at points of complete insurance for a non-degenerate interval of prices. This observation contradicts the result of Arrow (1965), who shows that such intervals consist only of single points for an expected utility maximizer. Moreover, Mukerji and Tallon (2003) demonstrate that such ”portfolio inertia” *characterizes* the perception of ambiguity.

Epstein and Wang (1994) construct an infinite horizon equilibrium model with a representative agent in which beliefs are represented by a multiple prior. They show that the result of Dow and Werlang (1992) remains valid, in that price is undetermined in an equilibrium. They conclude that high price volatility can result out of this indeterminacy. Nevertheless, as Epstein and Wang (1994, p. 310) themselves note, these results do not necessarily imply price indeterminacy in a model with heterogenous agents. Even in a representative agent model, price indeterminacy requires the presence of non-diversifiable risk[18]. In a model with heterogenous agents with identical multiple priors Chateauneuf, Dana and Tallon (2000) demonstrate that equilibrium allocations are comonotonic (hence, satisfy the sure-thing principle) and the equilibrium price supporting the allocation is unique. Similar to the representative agent setting, aggregate uncertainty is required to induce price indeterminacy in an equilibrium, as shown by Dana (2000).

Whereas willingness to diversify is equivalent to a concave von Neumann Morgenstern utility function in the context of expected utility maximization, Chateauneuf and Tallon (2002) show

[18] Figures 6 and 7 in Rigotti and Shannon (2001, p. 41-42) provide an illustration of this fact.

that with Choquet expected utility different notions of preference for diversification are supported by different characteristics of the decision problem — concave utility index is in general neither necessary, nor sufficient for such preferences to emerge. An important role plays the specific perception of ambiguity, i.e. the specific capacity of the decision-maker. Therefore, non-additive priors and ambiguity-aversion can also explain phenomena, such as underdiversification, see Uppal and Wang (2003) and the "home bias", i.e. the fact that investors hold undiversified portfolios, in which foreign securities are underrepresented, see Epstein and Miao (2003). The results are based on the assumption that some of the assets (e.g. the foreign ones) are ambiguous. Ambiguity-averse investors, hence, choose to hold a lower share of ambiguous assets in their portfolio compared to a portfolio of an ambiguity-neutral investor. Excessive returns, as compared to the prediction of the CAPM, can also be explained by the fact that individuals are ambiguity-averse and require an "uncertainty" premium, see Kogan and Wang (2002). In a similar way, Chateauneuf, Eichberger and Grant (2002) show that pessimistic investors, who overestimate the probability of low returns of a risky asset, prefer a riskless investments and can, therefore, provide an explanation for the equity premium puzzle.

The cumulative prospect theory also offers explanations of financial market paradoxes, see Camerer (1998). Thaler and Johnson (1990) provide evidence that people are more willing to choose risky acts, after they have gained money, than after having made losses. Benartzi and Thaler (1995) model an individual portfolio choice problem and show that loss aversion can make an investor reluctant to invest in risky assets. Hence, they suggest that the prospect theory might help to explain the equity premium puzzle. Barberis, Huang and Santos (2001) construct a market model, in which investors are interested not only in maximizing expected utility, but also in their financial wealth in each period of time. The utility derived from financial wealth in a given period is computed using concepts from the prospect theory, losses being weighted higher than gains. Both gains and losses are measured relative to a reference level (e.g. a past price). The model explains the equity premium puzzle, high volatility of stock prices and predictability of returns.

The higher sensitivity for losses than for gains in the prospect theory can account for investors holding too long stocks losing value and selling quickly stocks that have gained in value recently, see Shefrin and Statman (1985). This prediction is consistent with empirical studies: Odean

(1998 (b)) finds that investors indeed sell winners quickly, but hold losers too long. Experiments conducted by Weber and Camerer (1998) show similar effects.

Like the expected utility theory, the prospect theory and the Choquet expected utility the case-based decision theory is derived from a set of axioms, which translate the set of preferences consistent with the theory into observable patterns of behavior and allow to test the theory in an experimental setting. The new approach of Gilboa and Schmeidler however deviates significantly from another theories for decision-making under uncertainty discussed up to this point, since it replaces the framework of the expected utility theory by a new one. The description of financial markets in terms of states of nature and state-contingent outcomes has persisted in the literature for more than thirty years. The non-expected utility theories, like Choquet expected utility and the prospect theory question the existence (or the uniqueness) of probability distributions over the outcomes, but do not criticize the main concepts per se. It is therefore necessary to ask the question whether the new framework introduced by the case-based decision theory is appropriate in the context of financial markets. Furthermore, I will analyze whether the kind of behavior described by the case-based decision theory can capture some of the psychologic phenomena observed in financial markets. This will be the topic of the next section.

1.4 Case-Based Reasoning in Models of Financial Markets

Gilboa and Schmeidler (1995) find that the theory they propose should be regarded more as a complement to the expected utility theory, than as an alternative to it. They suggest that the case-based decision-making may be appropriate in situations in which decision-makers have little information about the problem and are unable to form beliefs about the possible outcomes and their probabilities. It seems that the decision problems in financial markets hardly meet this criterion. Financial economists regard it as natural to think in terms of states of nature (leading to contingent payoffs) and state probabilities. Since the works of Arrow (1970, p. 98) it has been assumed that the expected utility framework naturally fits the description of an asset in terms of a probability distribution over state-contingent outcomes.

A thorough consideration of this framework, however, shows that it is in no way natural to formulate the states of nature in a financial market. Indeed, besides the problem of deciding, which payoffs of a security should be considered possible, the question of correlation among the payoffs of different assets arises. Hence, it is not a solution of the problem to identify the states of the world with the payoffs an asset renders[19].

Moreover, in a market environment payoffs are determined not only by the dividends paid by an asset, but also by the capital gains and therefore, by the equilibrium prices, which themselves depend on the expectations of the market participants. The well known beauty contest used by Keynes (1936) to describe the expectation formation in asset markets illustrates this point. As Arthur (1995, p. 23) notes ”[w]here forming expectations means predicting an aggregate outcome that is formed in part from others' expectations, expectation formation can become self-referential. The problem of logically forming expectations then becomes ill-defined, and rational deduction finds itself with no bottom ground to stand upon”. Moreover, in economies with heterogenous investors the computational problems connected with determining the equilibrium with rational expectations are still not solved by the economic science.

Heterogeneity of beliefs is a natural characteristic of real markets. Furthermore, beliefs tend to change with time and these changes are not necessarily consistent with Bayesian updating. The rules used to make decisions given some beliefs often violate the expected utility maximization. But then it is questionable whether states of the world, which would have to include changing beliefs as well, can be naturally formulated and asset payoffs assigned to them. In such an environment it might be much more natural for a decision-maker to evaluate portfolios of assets according to their past performance. As a consequence of his critique of rational expectations Arthur (1995) proposes an inductive approach to modelling decisions in financial markets. Similarly to the approaches of Sargent (1993) and of Evans and Honkapohja (2001), the idea of Arthur is based on adaptive expectations, which presupposes that investors know the model of the economy, except for its parameters, which they can estimate as statisticians from market data.

The case-based decision theory also provides a model of inductive learning. However, differently from the approaches cited above, it does not presuppose knowledge of the structure of the economy and of possible models which can explain the data. Instead, a situation of ”structural ignorance” is modelled and the decision-maker uses inductive reasoning based on his experience, his aspiration level and his perception of similarity to evaluate the available alternatives.

[19] See Bossert, Pattanaik and Xu (2000, p.296) for a discussion of the problems connected with the construction of states.

The case-based decision theory relies on the concepts of memory, aspiration level and similarity. Although these concepts seem very natural, they need not reflect the way in which people perceive a decision problem. Some situations might be well described in the standard framework of states and state-contingent payoffs and it might be quite counterintuitive to try to represent the information available as cases. If an investor is experienced and well acquainted with the market situation, if he expects to be able to make correct forecasts about the future, based on the data he possesses, then he will possibly find the Savage framework to be the appropriate decision rule to follow. Such an investor might be confused, if asked to formulate his experience in terms of a memory containing past cases and similarity among problems. In contrast, if the situation is new, or considered unpredictable, if the investor does not have enough knowledge about the structure of the market or about the behavior of other market participants, he might prefer to base his behavior on his own or on other's experience. In short, if an investor already knows the structure of payoffs and can compute the optimal act, an aspiration level becomes useless to him. On the other hand, if he is not aware of possible payoffs and their correlation, then an aspiration level used to distinguish ”good” from ”bad” results, a perception of similarity between assets, as well as memory reflecting knowledge of past cases may compensate for this lack of more general information and be useful concepts to base one's decisions upon.

The difference between the expected utility maximization and the case-based decision theory consists not only in the information and structural knowledge they presuppose, but also in the processing and usage of this information. The question therefore arises, whether it is possible to indicate case-based reasoning in financial markets. Up to now, experiments designed to test the case-based decision theory have not been constructed. Nevertheless, it is possible to analyze whether the case-based decision theory allows to capture some of the psychologic biases commonly observed. The next paragraphs will therefore deal with the question of how the kind of behavior, described by the case-based decision theory is connected to the behavior observed in financial markets and how the main elements of the model can be interpreted in this context[20].

1.4.1 Anchoring

Case-based decision-makers learn from information contained in past cases. Hence, a case-

[20] I consider only those phenomena, which seem to be captured by the case-based decision-theory. For complete reviews on psychological phenomena in financial markets see Rabin (1998), Shiller (1999) and Hirshleifer (2001).

based investor uses past prices and dividends to draw conclusions about the future performance of a portfolio in the future. This behavior is regarded as irrational from the point of view of the efficient market hypotheses, which states, see Fama (1970), that asset prices should move as a random walk and therefore past prices could not be used to predict future returns[21]. However, empirical evidence shows that usage of information which seems to be irrelevant for the decision at hand is often observed.

When asked to make quantitative predictions, subjects in experiments use available pieces of information as a bench-mark. Moreover, they tend to do so even if the information available is of little or even no relevance to the problem stated, see Tversky and Kahneman (1974). This phenomenon is called anchoring.

Shiller (1999) suggests that the same phenomenon might be observed in financial markets, where investors, lacking a better rule for predicting returns or future prices, use the current values as a bench-mark for their assessments. ”Values in speculative markets, like the stock market, are inherently ambiguous. Who would know what the value of Dow Jones Industrial Average should be? [...] There is no agreement upon economic theory that would answer these questions. In the absence of any better information, past prices (or asking prices or similar objects or other simple comparisons) are likely to be important determinants of prices today”, Shiller (1999, p. 1315) writes. But this is exactly the way in which a case-based investor uses information of similar past cases to make forecasts about future prices and returns.

Effects of anchoring are indeed found in financial markets. Shiller, Kon Ya and Tsutsui (1996) analyze the price crash of the Nikkei between 1989 and 1992. When asked to assess whether the stock prices in Japan were too high, American and Japanese investors gave significantly different answers on average. Shiller (1999, p. 1316) suggests that this effect may be due to anchoring: whereas American investors used the American stock prices (which were relatively low at the time) as a bench-mark and concluded that Nikkei was overvalued, Japanese investors were used to high price-earning ratios and claimed that the stock prices reflected correctly the fundamental values. Further evidence of anchoring in financial markets is provided by Shiller

[21] Should the expected direction of the next price movement be different from 0, there would be risk-neutral investors who would use this information to buy (for an upward movement) or to sell the asset (for a downward movement). Hence, the price would rise or fall and the excess returns present in the market would disappear. See Samuelson (1965) and Mandelbrot (1966) for analytical proofs of the efficient market hypothesis.

(1984).

Shiller (1990) further finds that the models used by investors to predict future prices are ”extremely simple”, e.g. ”notion that large price drops should be followed shortly by a reversal” and the usage of ”more established theories” is ”rare”, (Shiller 1990, p. 57). The case-based decision theory allows to incorporate such learning from past price patterns in a formal model.

1.4.2 Reference Levels

Case-based decisions require a specification of an aspiration level, which has two functions: it determines the ex-ante expectations of the investors about the performance of the available acts and it defines which utility realizations are considered satisfactory. Whereas the usage of an aspiration level in the framework of expected utility maximization would not influence the behavior of a decision-maker[22], case-based decisions are sensitive to changes of the aspiration level, since they alter the evaluation of results already experienced. Although not modelled in the standard decision theory, reference and aspiration levels seem to influence decision-making in experimental and real markets.

Northcraft and Neale (1987) find that anchoring might distort the evaluation of a result already achieved. Rabin (1998, p. 13) states: ”Instead of utility at time t depending solely of present consumption [...], it may also depend on a ”reference level”[...] determined by factors like past consumption or expectation of future consumption”.

Empirical studies show that such kind of behavior is not atypical for financial markets. DeGeorge, Patel and Zeckhauser (1999) find that managers adjust earnings to meet threshold levels such as zero, past levels and level forecast by analysts, suggesting that investors are sensitive not only to the level of actual earnings, but also to deviations from a certain reference level. Richardson, Teoh and Wysocki (1999) note that a similar phenomenon exists in the analysts' forecasts, which are pessimistic in the short run, so that prices are likely to exceed the forecast level. Lewellen, Lease and Schlarbaum (1977, p. 308) report that 42% of the investors they interview choose major public indices as a criteria to evaluate portfolio performance, whereas ”another 45% report[...] that they had internalized instead a personal standard of return as an

[22] The introduction of an aspiration level is equivalent to a linear-affine transformation of the von Neumann Morgenstern utility function, which does not influence the preferences of an expected utility maximizer.

amalgam of experience, evidence and concepts of ”fair” yields”.

The empirical findings show that a reference or aspiration level can influence decisions in financial markets in two ways: first, it may change evaluation of an outcome already experienced and second, it provides a bench-mark for forming expectations. Both roles of aspirations are captured by the case-based decision theory.

Both passages cited above suggest that the aspiration level depends on past evidence observed. The case-based decision theory allows for such updating of the aspiration level. Although, the aspirations have to be chosen arbitrary at first (since no information about possible utility realization is available), the case-based investors will in general update it in a way that reflects their past experience. However, since there is little evidence on how reference levels are determined, see Rabin (1998, p. 15) and since there are few works trying to describe economic behavior, when reference levels are updated over the time[23], we must be careful when interpreting results, achieved by assuming some kind of updating of the aspiration level.

1.4.3 Representativeness Heuristic

The case-based decision theory further implies that in the initial periods the decision-maker relies on a relatively small series of data he possesses and acts as if it were representative of the population from which it is drawn. Thus at earlier periods a case-based decision-maker behaves as if he were chasing a trend in the price movements. This phenomenon is called representativeness heuristic: Tversky and Kahneman (1974) find that people try to categorize events and infer probabilities from this categorization. A typical error resulting from this kind of behavior is to assume that small samples accurately represent the population, Tversky and Kahneman (1971). People tend to see patterns in short random sequences and rely on these patterns when making decisions. ”Because we underestimate the frequency of a mediocre financial analyst making lucky guesses three times in a row, we exaggerate the likelihood that an analyst is good, if she is right three times in a row”, Rabin (1998, p. 25). Although a case-based decision-maker does not think in terms of probabilities, with little experience his behavior can initially exhibit the representativeness heuristic.

[23] See for instance Ryder and Heal (1973), who assume a first-order autocorrelation process for the reference level and Duesenberry (1949), who assumes, that the aspiration level at a given period is set to be the highest utility level achieved up to this period.

Kroll, Levy and Rappoport (1988) report an experiment consisting in choosing an optimal portfolio of three risky assets with known means, variances and covariances, whose returns were not serially correlated. The participants in the experiment were students, who had taken a one-year course in statistics. However, the students seemed not to rely on the statistical information presented to them. Instead, they used the time series available, trying to infer some patterns and trends from past returns.

Such kind of behavior is similar to the one suggested by the theory of Gilboa and Schmeidler (1995). Since a case-based decision-maker has no model of the structure of the economy, knowledge of parameters is of little use for him. Hence, it is possible that his predictions about future returns and, therefore, his behavior depend on information which is not representative for the process of asset returns.

Shiller (1990, p. 59) finds this kind of behavior in the housing markets of three major cities in the USA, in which housing prices have recently risen significantly: "It is peculiar, then that there is so little apparent interest in quantitative data about fundamentals. There is instead a feeling in most cities that housing prices cannot decline". But the representativeness heuristic seems to be common in financial markets, as well. Shiller (1990, p. 63) reports that 47% of the individual investors and 28% of the institutional investors he interviewed find that a 15% jump in the price of a share recommended by a broker is a "strong" or "positive" evidence for the broker's ability to choose profitable investments. Further evidence is provided by DeBondt (1993), who finds that small investors' sentiments follow price movements. Bange (2000) finds that small investors tend to use feedback strategies, i.e. buy shares which have performed well in the near past. Ippolito (1992) and Sirri and Tuffano (1998) present evidence that investors extrapolate the performance of mutual funds. By assuming that the decision-maker bases his decisions on small (possibly not representative) samples in the initial periods, the case-based decision theory is able to capture such effects.

1.4.4 Belief Perseverance and Confirmatory Bias

The similarity function of a decision-maker states which cases are considered to be relevant for the problem at hand and determines the weights of the past utility realizations for the evaluation of each act available. If some of these weights are 0, the decision-maker might neglect available information, which objectively seen could improve his decision, see chapter 3. If some of these weights are even negative, then positive past evidence may be interpreted as negative. This can lead to phenomena such as belief perseverance, i.e. refusal to process information, which may oppose one's views, and confirmatory bias, which describes the fact that people tend to interpret additional evidence as supporting their beliefs.

The phenomena of belief perseverance and confirmatory bias have been observed in the experiments of Lord, Ross and Lepper (1979), who provide evidence that identical information given to people who already differ in their opinion leads to even more severe polarization. Similar results are documented by Plous (1991) and Darley and Gross (1983). The more complex the problem and the more ambiguous the evidence that subjects receive, the more pronounced are the effects, Lord, Ross and Lepper (1979, p. 2099).

Applied to case-based decisions, belief perseverance means that an investor, who has already come to the conclusion that a portfolio α is a "good" choice, may just ignore information about assets yielding higher returns than his portfolio, for instance by assigning these cases a similarity weight of 0. Alternatively, subject to the confirmatory bias, he may apply negative similarity weights to these cases and therefore interpret this information as justifying his actual choice. Odean (1999, p. 1295) finds that investors "do not [...] routinely look up the performance of a security they sold several months ago and compare it to the performance of a security they bought in its stead". Hence, a decision-maker may persist in following an unprofitable strategy, despite the fact that contrary evidence is easily available to him.

1.4.5 Learning

The next point to be made concerns learning. Economists usually argue that biases, found in experiments are eliminated, once subjects become more experienced in solving the particular kind of problems, or once their mistakes are explained to them. Hence, (at least) in the long run economic agents should be expected to behave rationally. For a case-based decision-maker this need not be the case, as subsequent analysis will show. Is it plausible to assume that learning could lead to suboptimal decisions?

Experiments show that learning does not necessarily lead to application of the principles learned. Kahneman and Tversky (1982), as well as the experimental results of Kroll, Levy and Rappoport

(1988) find that statistical knowledge does not help subjects to perform better in experiments. Wilson and LaFleur (1995) demonstrate that even a conscious reasoning process per se does not necessarily lead to better decisions.

”Learning can even sometimes tend to exacerbate errors”, notes Rabin (1998, p. 32), for instance, by making the decision-makers overconfident. ”Additional information can lead to an illusion of knowledge and foster overconfidence, which leads to biased judgements”, write Barber and Odean (2001 (b), p. 6). This is confirmed by several studies, see for instance, Hoge (1970), Slovic (1973), Peterson and Pitz (1988). Barber and Odean (2001 (a), 2001 (b)) find that certain classes of investors, to whom large data bases are available, tend to become overconfident and to lower their profits significantly.

Hence, especially in complex systems, such as financial markets, which, according to Odean (1999, p. 1295), are ”a noisy place to learn”, it cannot be expected that decision-makers behave fully rationally even if they have been making similar decisions for a long time. The case-based decision theory allows to examine the influence of parameters, such as aspiration level, similarity function and memory, on learning and indicate conditions under which optimal learning is possible in the limit. On the other hand, it also allows for patterns which are inconsistent with rational learning and might therefore better describe empirical evidence from financial markets.

1.4.6 Learning through Induction

A different issue I would like to address concerns the *type* of learning modelled by the case-based decision theory. It has already been mentioned that a case-based decision-maker learns from his experience about possible utility realizations and their frequencies. Instead of starting with a predefined set of models of the environment and trying to learn which of these models is correct, the case-based decision-maker reasons inductively using a subjective similarity relation. This kind of implicit induction dating back to Hume (1748) is based on his claim that ”from similar causes we expect similar effects” and represents a starting point for the case-based decision theory, Gilboa and Schmeidler (2001 (a), p. 184). Differently from the so called explicit induction introduced by Wittgenstein (1922, 6.363), which consists in ”accepting as true the *simplest* law that can be reconciled with our experience”, it does not define rules or laws. Although rules can be implicitly present in the case-based reasoning, for instance as a summary of multiple cases or as an indicator of similarity between cases, see Gilboa and Schmeidler (2001 (a), p. 107-108), rule-based reasoning is considered less flexible than the case-based decision theory, since it does not give any recipe for behavior in case a rule is contradicted by experience[24][25].

It is well known that induction does not give a satisfactory answer to the question of how knowledge is acquired from the point of view of the philosophy of science, see Musgrave (1993). Nevertheless, for the purpose of economic theory it creates a new possibility to model knowledge acquisition. Instead of learning being purely deductive, as in the expected utility theory, where the set of consistent models is constrained more and more with incoming information, but in which no new models outside the initial set can be learned, an inductive method is proposed, which does not rely on prior structural knowledge.

Poincare, see Keynes (1921, p. 285), distinguishes two notions of ignorance. Probability itself represents ignorance, but this ignorance concerns only the *outcome* of a stochastic process. Poincare compares this kind of ignorance to the ignorance about the *laws* governing processes, which are considered deterministic per se. It seems that whereas the standard decision theories take the nomologic knowledge of laws for granted and allow learning only about facts or probabilities by means of logic conclusions, (ontologic knowledge), the case-based decision theory models exactly the *acquisition* of nomologic knowledge[26]. In fact, the expected utility theory treats these two types of knowledge identically. The same framework is used to model learning about a realization of a random variable and learning a probability distribution or learning a model of an economy. The conceptual problem that the expected utility theory faces in this respect is that new structural meta-knowledge is required each time a new meta-law has to be learned.

I am far away from concluding that the case-based decision theory is superior to the standard decision theories and especially to expected utility maximization. In order to draw a similar conclusion, one should ask the question whether the case-based decision theory itself allows for differentiating between nomological and ontological knowledge. The inductive learning

[24] In this sense rule-based reasoning is similar to Bayesian updating, which also does not specify the behavior in case an unforeseen event (with a prior probability of 0) occurs.

[25] Gilboa, Lieberman and Schmeidler (2004) provide a comparison of the performance of case-based and rule-based reasoning based on the housing market in Tel Aviv. They demontsrate that case-based reasoning outperforms rule-based reasoning for relatively small and for relatively large data sets. In contrast, rule-based reasoning is superior for data sets of intermediate size.

[26] This classification of knowledge is to be found in Keynes (1921, p. 288).

modeled by the case-based decision theory is constrained by the fact that similarity perceptions are used to make decisions, but similarity relations only become known with experience[27]. But then it follows that the case-based decision theory does not conceptually differentiate between learning about facts and learning about laws, either. Hence, the case-based decision theory allows for an alternative way of learning, which cannot replace but only complement the methods of the expected utility theory. Applying this theory in a specific economic model can, therefore, provide new insights into the functioning of economic systems and lead to a better understanding of the case-based decision theory itself.

1.5 Overview of the Thesis

It has been shown that the case-based decision theory can incorporate some important psychological biases, which human behavior exhibits. Furthermore, the case-based decision theory allows to model decisions which are not based on a definition of states of nature and on the knowledge of state-contingent payoffs. Hence, it offers an alternative framework for modelling decisions in complex systems, such as financial markets. This thesis applies the case-based decision theory to financial markets, the first aim being to *describe* the behavior of case-based investors. This is done in the context of an individual portfolio choice problem, as well as in a market environment with endogenous prices. The thesis studies the question of how the choices of case-based investors differ from those of expected utility maximizers with correct beliefs. Further, the dynamic of asset prices and portfolio choices is analyzed.

The second aim of this thesis is an *explanative* one. The qualitative implications of case-based decision-making in financial markets are compared to the empirically observed phenomena, in order to study whether the presence of case-based investors could explain some of the empirical findings.

Afterwards, the somewhat *normative* issue of performance of case-based decisions relative to expected utility maximization is discussed. The analysis addresses the question of whether case-based investors can survive and influence market prices in the presence of expected utility maximizers.

In order to answer these questions, the thesis is organized as follows. Chapter 2 addresses some

[27] Gilboa and Schmeidler (1993) call this effect second-order induction.

conceptual issues connected with the application of the case-based decision theory to financial markets. It first discusses the description and operationalization of the exogenous characteristics of the decision situation, such as the decision problem, the set of available acts and the memory of a decision-maker. It is found that case-based decisions are sensitive to the particular description of problems and acts, since it might influence the perception of similarity. I discuss whether the case of endogenous memory (i.e. memory that contains only cases actually experienced by the investor) as opposed to exogenous memory (i.e. cases in memory are independent of the decision made by the investor) is more interesting and appropriate in the context of financial markets. I further address the evolution of memory in the context of an economy with financial markets populated by case-based decision-makers.

Further, the concepts derived by means of axioms are considered. The literature on aspiration levels and their adaptation in economic environment is sparse and few adaptation rules have been formulated and studied. I, therefore, decide to concentrate on constant aspiration levels throughout most of the thesis and to analyze the influence of aspiration levels by means of comparative statics. Adaptation of aspirations is used only in the context of an individual portfolio choice problem in order to test the robustness of the results achieved and to explore whether the results of Gilboa and Schmeidler (1996) can be transferred to a decision problem with similarity considerations.

The similarity function is the last issue discussed in this chapter. I suggest to use the Euclidean distance between portfolios or alternatively between price-portfolio pairs to measure similarity. In a portfolio choice problem with exogenous prices I suggest to consider only similarity between acts (portfolios), whereas problems are considered identical. In a market environment, similarity between problem-act / price-portfolio pairs is introduced. The similarity function is assumed to decrease in the Euclidean distance, take on values between 0 and 1 and assign a similarity of 1 to identical objects. No other assumptions on the similarity function seem to be justified either by the case-based decision theory, or by the analysis of the decision situation.

Chapter 3 concentrates on the analysis of a portfolio choice problem with case-based decisions. In a first instance the results derived by Gilboa and Schmeidler (1996) and Gilboa and Pazgal (2001) are presented and interpreted in the context of a portfolio choice problem. Gilboa and Schmeidler (1996) find an adaptation rule which guarantees that a case-based decision-maker who acts in a stationary environment learns to choose the optimal (from the point of view of expected utility maximization) portfolio in the limit. Gilboa and Pazgal (2001) analyze the influence of a constant aspiration level on repeated choice in a stationary environment. They show that high aspiration levels lead to constant switching among available acts. In chapter 3, these results are reinterpreted in the context of portfolio choice. Investors who update their aspiration level in the ”ambitious-realistic” way proposed by Gilboa and Schmeidler (1996) eventually learn to make correct decisions. Investors with high constant aspiration levels switch constantly among the available portfolios, never learning to choose the optimal one. Hence, they trade too much and, by failing to hold the optimal portfolio all the time, they loose money in general. Case-based investors with high aspiration levels, therefore, exhibit behavior similar to those of ”overconfident” investors in the sense of Odean (1998). However, differently from the overconfident investors, who overestimate the meaning of incoming information, the behavior of case-based decision-makers is predetermined by the negative evaluation of the past returns achieved..

Investors with low constant aspiration level in general exhibit a constant but suboptimal behavior. Since they find the returns of their portfolio satisfactory, they do not have an incentive to try another one. Low aspiration levels can therefore explain such effects as holding underdiversified portfolios and especially the home-bias, as well as unused arbitrage possibilities.

Similar results are achieved by assuming that the investor sets his aspiration level equal to the maximal or to the minimal utility realization achieved, or to a linear combination of both. If the weight put on the maximal utility realization is sufficiently high, the investor behaves as if he has a relatively high aspiration level in the limit and vice versa. Hence, the results are robust for this kind of aspiration level updating.

Further analysis shows how the results change, if the investors are able to observe the realizations of all portfolios at each time and not only of the portfolio actually chosen. If the investor is allowed to make observations for an infinitely long time, he learns (given the i.i.d. structure of returns) to choose the best portfolio in the limit. However, if the investor is only allowed to observe all realizations for a finite number of periods, this helps him to act optimally only if his aspiration level is relatively low. If his aspiration level is chosen sufficiently high, learning for a finite period of time does not improve his performance.

This analysis is carried through with a very special form of similarity function — it is assumed that all problems are identical and all acts are completely different from each other. Hence, the utility realization, given that a portfolio is chosen does not contain any information used to evaluate the performance of other portfolios. This assumption is relaxed in section 8 of chapter 3 in order to analyze the influence of similarity considerations on portfolio choice. It is again assumed that problems are identical, but the similarity between portfolios / acts is modelled by a function which is decreasing in the Euclidean distance between portfolios situated on a simplex. It is assumed that the similarity function is concave in the Euclidean distance. First, the case of constant aspiration level is considered. It is shown that a case-based investor chooses a diversified portfolio in each period only if his aspiration level is sufficiently low. For high aspiration levels the investor switches to one of the undiversified portfolios in finite time and never diversifies again afterwards.

Next, the issue of learning is analyzed. For the "ambitious-realistic" rule proposed by Gilboa and Schmeidler (1996), it is shown that the decision-maker learns to choose the best *non-diversified* portfolio, but his choice is not globally optimal in the limit. Crucial for this result is the assumption of a concave similarity function. Allowing for convexity regions in the similarity function makes the investor experiment among more portfolios and improves his limit choice. Nevertheless, the learning is not optimal but in the limit case, in which the similarity function collapses to the identity indicator function, as in the model of Gilboa and Schmeidler (1996).

The results of this chapter show that although investors making case-based decisions can learn to make optimal choices if their aspiration level is updated in a special way, they in general act suboptimally. Moreover, their behavior can help explain some of the phenomena observed in real and experimental asset markets.

The next step is, therefore, to model a market with case-based decision-makers, the first aim being to describe how prices and portfolio holdings evolve in such a market. In a second step, it will be interesting to know whether the presence of case-based decision-makers in the market can explain empirical phenomena, which are inconsistent with expected utility maximization.

The analysis of a market presupposes a definition of a competitive market equilibrium in an economy populated by case-based decision-makers. In chapter 4, an asset market equilibrium in the context of an overlapping generations (OLG) economy populated by case-based decision-makers is introduced. It is shown that the equilibrium exists under mild conditions imposed on the utility function and the initial endowments of the old investors in period 1. The uniqueness of the equilibrium is, however, not guaranteed. Moreover, it is demonstrated that the demand of the case-based decision-makers can be very insensitive to price changes when prices are low. This leads to the existence of degenerate equilibria, in which the prices of some of the assets are 0. In equilibrium, these assets are not demanded. It is possible to state conditions which exclude the existence of such equilibria, or at least insure that a non-degenerate equilibrium exists, but these conditions are not always economically meaningful, since they require the existence of investors with very low aspiration levels. Furthermore, the existence of such degenerate equilibria might help model and explain phenomena such as arising and bursting of bubbles or the persistence of arbitrage possibilities in the market.

Chapter 5 studies the asset price dynamics in an OLG model with case-based decision-makers. At first, it is assumed that the decision-makers can only choose between two undiversified portfolios — one completely riskless, the other consisting only of risky assets. The memory of an investor contains only cases actually experienced by his predecessors with the same aspiration level. It turns out that the memory and the highest aspiration level in the economy are the major factors which determine the dynamic of prices and portfolio holdings in equilibrium. With one-period memory the economy remains in a stationary state with constant asset prices and constant portfolio choices over the time, as long as the aspiration levels are relatively low. The portfolio holdings are not necessarily optimal at the equilibrium prices and the price of the risky asset in general differs from the price under rational expectations. Higher aspiration levels lead to two-state-cycles, which can be stochastic (for intermediate aspiration levels) or deterministic (for high aspiration levels). The price dynamic exhibits small bubbles: a one-period price rise, which does not depend on the value of the dividend realization, is followed by a (one-period) price fall, caused by a low dividend realization. Predictability of asset returns obtains in the model, since the price rise does not depend on the dividend realizations. The case-based decision-makers with high aspiration levels, who constantly switch among the available portfolios, cause excess price volatility. Moreover, arbitrage possibilities can persist in a market with case-based investors. It is found that high aspiration levels have the same effects on prices as overconfidence: they increase the trading volume and the price volatility. Furthermore, the traders with high aspiration levels incur losses, since they tend to buy at high prices and sell at low prices.

The analysis is then carried out for the case, in which the memory of the investors encompasses all past cases in the economy from period 0 on. The price dynamic and the portfolio holdings in equilibrium are determined. The results for different aspiration levels are similar to those derived with one period memory: whereas low aspiration levels lead to satisficing behavior and constant prices, high aspirations imply stochastic cycles. If, however, the highest aspiration level in the economy is chosen appropriately, the investors with this aspiration level learn to choose the optimal portfolio as the time evolves. Hence, although the economy exhibits two-state cycles, with time the cycles vanish and the investors with high aspirations behave optimally at the equilibrium asset prices. Nevertheless, it is not guaranteed that the asset price coincides with the price under rational expectations. Indeed, the investors with lower aspiration levels in general fail to learn to behave optimally. Hence, in the stationary state of the economy the expected returns of the risky asset might be too high, compared with the returns of the riskless bond and still a positive share of the investors would be holding bonds in each period of time. This result, which is independent of the curvature of the utility function of the case-based decision-makers, might help to explain the equity premium puzzle. The analysis of an asset market with case-based investors further allows to generalize the theorem of Gilboa and Pazgal (2001), formulated for an individual choice problem, to a market environment..

In a next step, the assumption that the memory of the investors can contain only cases actually experienced by their predecessors with the same aspiration level is relaxed. I construct an example, in which investors can choose between the two undiversified portfolios and the market portfolio. The market portfolio dominates weakly the two other available portfolios. Allowing the investors to observe the utility realization of the market portfolio in each period of time, in addition to the cases actually experienced, leads to the choice of the market portfolio in the limit, but only if the aspiration level is chosen sufficiently low. For relatively high aspiration levels, the non-diversified portfolios are almost surely chosen with strictly positive frequencies. Hence, hypothetical reasoning need not lead to optimal choices in general, especially, if no dominance relationship is present among the acts available.

The assumption that the investors can only choose between the two undiversified portfolios seems to be too severe. Therefore, section 6 of chapter 5 considers the case in which diversification is allowed and a similarity function is used to evaluate portfolios not chosen before. In contrast to chapter 3, the similarity function is now assumed to depend not only on the portfolio chosen, but also on the problem encountered, where a problem is associated with the price of the risky asset in the economy[28]. The results do not change qualitatively, as long as the similarity function is concave[29]. However, with similarity considerations a bubble on the risky asset can *endogenously emerge and persist for a number of periods*. The price of the asset rises in periods, in which the dividend realization is low, which can explain phenomena such as the internet bubble, during which no dividend payments occurred. The bubble bursts with probability 1 in finite time and never reappears.

Having described the price dynamic in an economy populated solely by case-based decision-makers, a somewhat normative approach is adopted next in order to discuss the issue of survival of case-based decision-makers in the market. This issue is crucial for the understanding and interpretation of the descriptive results obtained. Indeed, the influence of case-based investors on prices can be persistent over time only if they are able to survive in a financial market in the presence of expected utility maximizers (or even of expected utility maximizers with rational expectations). Only in this case would it be meaningful to explain the emergence of empirically observed phenomena by the presence of case-based investors. Should, on the other hand, the case-based decision-makers vanish with time, then they could only have temporary influence on the market. In chapter 6, therefore, I model an economy populated by case-based decision-makers and expected utility maximizers. Whereas the share of wealth invested into the risky asset by the case-based decision-makers is increasing in its price, this share is falling in the price of the asset for the expected utility maximizers. A replicator dynamic is introduced, which selects for the type of investors performing better than the average of the society. The state in which only expected utility maximizers are present in the market is stationary for all values of the parameters. For relatively low aspiration levels, there are also stationary states in which the case-based decision-makers and the expected utility maximizers coexist in the market (in arbitrary proportions), hold identical portfolios and achieve identical returns in each period of time. However, these stationary states are empirically indistinguishable from an economy with

[28] This assumption seems appropriate, since all other parameters describing the portfolio choice problem of an investor remain constant over time.

[29] This assumption is only needed in the case of long memory.

expected utility maximizers with rational expectations.

I, therefore, study the dynamic of the system for higher aspiration levels and analyze the stability of the stationary state, in which only expected utility maximizers are present in the market. Under the assumption of a linear utility function it is possible to identify cases in which the proportion of the expected utility maximizers decreases in expectation in some surrounding of the stationary state. Hence, the stationary state is not stable and the case-based decision-makers survive almost surely in a positive proportion. This is due to the fact that near the stationary state the expected returns of both types of investors are equal, whereas the portfolio of the case-based decision-makers is less risky. Since the replicator dynamic is concave, it selects for the strategy of the case-based decision-makers.

If the aspiration level of the case-based decision-makers is relatively high, then they can cause excessive volatility, bubbles and predictability of returns in a market in which the share of the expected utility maximizers is not too large. For relatively low aspiration levels, the case-based decision-makers can drive the expected utility maximizers out of the market for a finite number of periods. In these periods a risky asset with positive fundamental value is traded at a price of 0. Hence, arbitrage possibilities are present in the market, which can be used by the expected utility maximizers, even though their wealth share is 0. In the first period in which the risky asset pays a high dividend the share of the expected utility maximizers recovers to a strictly positive level.

The assumption of a linear utility function is next relaxed. It is found that for any coefficient of relative risk-aversion between $[0;1)$, there are values of the parameters, for which the case-based decision-makers survive almost surely in the market. Only expected utility maximizers with a logarithmic utility function are able to drive the case-based decision-makers to extinction for all values of the parameters.

Chapter 7 summarizes the results of the thesis and outlines directions for future research.

Chapter 2. Conceptual Issues

The case-based decision theory has been mainly discussed in the literature as an axiomatic theory for decision-making under structural ignorance, see Gilboa and Schmeidler (2001 (a)) . It has been applied to economic problems in several contexts, but not to financial markets. The application of the case-based decision theory, especially in a market environment and in the context of financial markets poses some conceptual issues, which have to be discussed before constructing a formal model.

The axioms which support this theory and were presented in the introduction show that the elicitation of preferences and similarity perceptions from the choices of the decision-maker are dependent on several elements, which are assumed to be given in the derivation of the theory. Such elements are the set of available acts and the problems which the decision-maker faces and their exact description, as well as the composition of the memory. In a model, in which the case-based decision theory is applied to an economic problem, the specification of these elements has to be done in advance and will play an important role for the results derived.

A second issue concerns the characteristics of a case-based decision-maker, which can be elicited by the means of the axioms, i.e. his aspiration level and his similarity function. Up to my knowledge, there are few empirical results on aspiration level adaptation and similarity perceptions in economic environment, see Lant (1992) for a study on aspiration adaptations and Zizzo (2002), Buschena and Zilberman (1995, 1999) and Rubinstein (2003) for experiments on similarity perceptions[30]. None of these works discusses the problems in the context of asset markets. Since empirical evidence is missing, the question of how aspiration levels are chosen, what rules for adapting the aspiration level are modeled and how similarity considerations can be implemented into a model of a financial market will have to be discussed.

Up to now the case-based decision theory has only been applied to individual choice problems. Embedding case-based decisions into a market environment requires establishing a connection between prices and decisions, which will enable defining a market equilibrium. Such a definition will especially concern the specification of the memory of an investor and its possible

[30] There is, of course, a vast experimental work done by psychologists on similarity perceptions. I will not make an attempt to review this literature here, but see Tversky (1977) for references.

dependence on prices and individual choices in the economy.

2.1 Problems and Acts

The case-based decision theory describes a decision situation in terms of two components — the problem faced and the set of available acts. The memory of a case-based decision-maker, which determines his preferences, is also formulated in terms of problems and acts. Therefore, the description of these two components will play a major role for the decision to be made.

The problem is identified with the description of the decision situation. However, this description does not consist of states of nature, as in the expected utility theory. Nor does it include the possible realizations of an act, as is usually assumed. Hence, in contrast to the set of states of nature in the expected utility theory, a problem cannot be a complete description of the situation, ”leaving no relevant aspect undescribed”, Savage (1954, p. 9). The problem can be possibly seen as a formulation of the aims of the decision-maker, combined with the description of relevant characteristics of the situation, about which the decision-maker is informed. Consider for instance the problem faced by an investor in a financial market. It could be described as: ”invest a certain amount of wealth today, so as to enable future consumption”. Note that maximization of expected payoffs or utility cannot be embedded as an aim in this problem, since the investor wouldn’t have the knowledge to solve such a problem. True, he will be searching for an ”optimal act”, but this evaluation will not take place based on the prospects of the act, about which he has no information, but on the basis of cases present in his memory, i.e. on past realizations.

This problem formulation stated above is quite vague and does not allow to distinguish between situations in which an investment has to be made. It can be used if all problems encountered are regarded as identical, as for instance in Gilboa and Schmeidler (1996, 1997 (b), 2001 (b)) and Pazgal (1997). However, it clearly does not capture important characteristics of the decision situation, which could help the investor make a decision. A problem could be, for instance, characterized by the time at which the decision is made, the wealth to be invested, the prices for consumption goods in the economy, as well as the asset prices, the investor engaged in solving the problem[31], in short, by any feature which is independent of the actions taken by the decision-

[31] This becomes relevant, if the decision maker has to rely on the experience of other investors.

maker[32]. So, for instance, Blonski (1999, p. 63) and Krause (2003) define a decision problem by the period in which the decision is made and the agent involved in the decision-making process. Jahnke, Chwolka and Simons (2001) represent a problem by the aim of the firm (its optimization problem) combined with the past information available about choices and inferred parameter values.

In this thesis, a problem will in general be characterized by the (rather vague) aim of the investor to obtain future consumption by making investments today. The period in which the decision is made, the personality of the investor (captured by his aspiration level), as well as asset prices will be used to differentiate among problems in some of the models. The dependence on the time period might account for the fact that investors have limited memory and can, therefore, only remember a certain number of recent cases. Alternatively, it might capture the idea that investors regard the environment as changing and rely only on relatively recent cases to make decisions. The personality of an investor involved in a certain case might determine the relevance of his experience for the choice of another investor or the credibility ascribed to this evidence. Asset prices represent an important part of the description of the market situation. Buying an asset in a booming market might be quite different from buying the same asset when the prices decrease. Therefore, it seems to me that prices should also enter the description of a problem in the context of a financial market model.

Once included into the description of a problem, the features listed above will influence the similarity perceptions of an investor. Since I assume that the similarity with respect to investors' personality and with respect to the period of time at which the problem is encountered is either 0 or 1 it is more convenient to capture these factors by the contents of the memory of an investors. Hence, the memory contains only cases in which problems assigned a similarity of 1 to the current problem are encountered. In contrast, asset prices enter explicitly the definition of a problem in the context of an asset market model and, therefore, appear as an argument of the similarity function.

The description of acts is not less problematic than the description of the problem. Nevertheless, the question of how acts can be ascribed to a given decision situation, as well as which acts should be considered relevant, is a question, which poses itself in the applications of any decision theory.

[32] Like the states in the expected utility theory, the problem is seen as exogenously given and independent of the act chosen by the decision maker.

Therefore, it is solved in the same way as in the applications of the expected utility theory to financial markets. I identify acts with portfolios to be chosen. Restrictions, such as short sales or diversification constraints, are introduced in some of the models and the influence of such restrictions is discussed.

Notice, however, the difference between the case-based decision theory and the expected utility representation of acts. In the latter, a portfolio can always be identified by a vector of state-contingent outcomes. In the case-based decision theory, on the contrary, a portfolio is only characterized by its composition, by its perceived similarity to other portfolios and by its past returns contained in the memory. Hence, differently from the expected utility theory, the case-based decision theory is much more sensitive to effects of labelling. Since the similarity function is elicited in a given context (with formulated past problems and acts chosen), which does not comprise the utility achieved, the decisions might be very sensitive in regard to changing the ”nicknames” of the acts. The expected utility theory does not allow for such frame-dependency: all acts have to be evaluated based only on their state-contingent outcomes, labels are irrelevant for the decision. Similarly, the model of Easley and Rustichini (1999), which is close in spirit to the case-based decision theory, explicitly prevents labelling effects by introducing the requirement that each act is evaluated solely by its past utility realizations. On the one hand, since framing of decisions is observed in experiments, see Tversky and Kahneman (1981), it might be seen as an advantage of the case-based decision theory to be able to accept framing in absence of more structured information about the acts, than their ”nicknames”. On the other hand, however, the dependence on labels makes a formal model much less rigorous and the results obtained become unequivocal if multiple act descriptions appear natural in the context of a model discussed.

2.2 Memory

The memory of the decision-maker represents the information he uses when solving the decision problem at hand. The memory contains past problems, choices made and utility realizations obtained. In deriving the preference relation from the set of axioms the memory is assumed to be exogenously given. In real situations, however, the memory will not be exogenous, but will be determined by the previous choices of the decision-makers. In this case, I will speak of endogenous memory. The analysis of decisions with exogenous memory is relatively straightforward in the case-based decision theory and is described by maximization of the cumulative utility. In an individual choice problem, learning the random realizations of *all* available acts at *each* period of time, leads to optimal choices in the limit, as long as the environment is stationary, see proposition 3.8, chapter 3. Exogenous memory is used by Gayer (2003), who assumes that all possible lotteries are present in the memory in such a way that the observed frequencies of their realizations reflect the actual probability distributions. However, most of the work concentrates on decision-making with endogenous memory. So, for instance, Gilboa and Schmeidler (1996, 1997 (b), 2001 (b)), Pazgal (1997), Gilboa and Pazgal (2001) all use the cases actually encountered by a decision-maker to construct his memory. On might argue that this assumption requires a decision-maker to start with absolutely no information in period 1. An alternative interpretation could be, however, that the experience possessed by the decision-maker does not allow him to differentiate between the available acts (i.e. he is indifferent among them). In both cases, this assumption seems to be close in spirit to the Laplace principle of insufficient reason. Nevertheless, the assumption of endogenous memory has the advantage that no ad hoc assumptions about the constitution of the memory are required and experience can be modelled as a natural product of repeated decision-making. I will, therefore, assume through most of the text that only choices actually made and utility realizations actually obtained are present in the memory.

This assumption prevents investors from using hypothetical cases of the type "if I had invested in asset α' (instead of α) I would have received a utility realization of u' (instead of u)". Such hypothetical reasoning is used by Jahnke, Chwolka and Simons (2001) to model the adjustment process of a firm which faces incomplete information about the demand function and has to choose optimal capacity and price. In their model, such hypothetical reasoning is possible only for certain capacity - price constellations and is justified by the fact that the decision of the consumers can be predicted, if only one of the quantities (price or capacity) is changed, whereas the other one is kept constant.

Although the case-based decision theory allows for such hypothetical reasoning, see Gilboa and Schmeidler (2001 (a), pp. 93-95), excluding it can be justified by limited memory and limited capacity required to process the whole information about prices and dividends in the market. Moreover, an investor might feel that his actual experience is more important for his decision than hypothetical cases and ignore them, by assigning a similarity of 0 between such hypothetical cases and the problem at hand.

Of course, the case of completely endogenous memory is an extreme case, which can be used rather as a bench-mark, than as a sensible description of reality. Therefore, where appropriate, I will allow for partial hypothetical reasoning and discuss the differences in results. It turns out that partial hypothetical reasoning helps the investor to make better decisions only in the case, in which the aspiration level is relatively low. In general, the behavior of decision-makers with high aspiration levels does not change qualitatively by introducing hypothetical reasoning.

The fact that the behavior of an individual depends on his memory and his memory on his behavior, makes his decisions path- and history- dependent. Moreover, since the number of cases present in the memory grows as more decisions are made, experience and hence information accumulates over the time. Starting with little or no information initially, the individual has to choose at random in the initial periods. Therefore, results on limit behavior seem to be more appropriate, than analysis of few periods only.

In an individual portfolio choice, the memory naturally contains cases encountered by this particular investor. In a market environment matters are different. Since I consider an overlapping generations model, in which each investor makes a single investment decision, it is not possible to assume that an investor learns from his own past decisions. Therefore, I introduce learning from past generations. This approach is similar to the one adopted by Krause (2003), who allows the decision-makers to learn not only from their own past decisions, but also from those made by other market participants. He then introduces a similarity function on the set of investors to account for different evaluation of personal and indirect experience.

In my model each investor is identified by his aspiration level. Therefore, I assume that each investor's memory contains only cases encountered by his predecessors with the same aspiration level. Of course, this assumption is rather arbitrary. However, it may be argued that an investor might perceive similarity among problem/act pairs based on the aspiration level of the investor from whom he receives the information about the particular case. Assuming a similarity function, which assigns a similarity of 1 to two cases, in which investors with identical aspiration levels are involved and 0 otherwise, leads to identical results. A more general approach is proposed by Blonski (1999), who is interested in learning by the means of social communication.

He models the social structure of the society using a similarity function on the set of individuals, which assumes only values 0 and 1 and which describes the set of those members of the society, whose experience an individual takes into account, when making his decision. Since, however, the problems of social learning lie outside the scope of this work, I neglect this kind of dependence in this thesis.

Independently of the length of the memory m, i.e. of the number of cases contained in it, I assume that the investors learn the *most recent* m cases. This assumption implies that investors assign more weight to recent experience than to cases lying far away in the past. However, I do not assume a gradual discounting of the past, as Krause (2003) and Gilboa and Schmeidler (2001 (b)) do. Instead, it is assumed that the capacity of the memory is restricted to m and all cases remembered are considered equally relevant to the problem at hand[33].

Remembering the last case in an overlapping generations model leads to the following phenomenon: since the young consumers learn the utility realization derived by their direct predecessor, and since this utility realization is increasing in the price of the portfolio held, the young investors are more willing to choose the same portfolio as their predecessors, the higher the price of this portfolio is. Hence, their demand for an asset is in general non-monotonic in its price, increasing at low prices and decreasing at high prices. Learning from the direct predecessor thus makes preferences for assets price-dependent and can lead to 0-asset prices in equilibrium. It seems that this paradox can be resolved by excluding the last case from the memory of the young investors. However, this assumption only leads to a globally decreasing demand for assets, but does not prevent equilibrium prices from falling to 0.

In the individual portfolio choice problem I concentrate only on the case of long memory, i.e. the investor remembers all past cases, in order to compare my findings to the results obtained in the literature, where long memory is commonly used, see Gilboa and Schmeidler (1996, 1997 (b), 2001 (b)), Gilboa and Pazgal (2001), Pazgal (1997). In the case of an asset market I distinguish between the two extreme cases of long memory and one-period memory. The assumption of one-period memory is made by Blonski (1999), as well as by Jahnke, Chwolka and Simons (2001), who assume that the memory contains only (hypothetical) cases from the last period. Again, these two extremes are not to be seen as a good description of real investors, but they provide

[33] In other words, the similarity of the last m cases to the problem at hand is 1, whereas the similarity of cases lying more than $t-m$ periods back in the past to the investment problem at time t is 0 .

an intuition of how learning, investors' behavior and asset prices are influenced by the length of memory[34]. In the evolutionary model discussed later I assume that the memory of the case-based investors is short. On the one hand, this assumption makes the computations tractable. On the other hand, the fact that case-based decision-makers survive even if their memory contains one case only, means that they would also be able to survive, if the length of their memory grows and hence the available information increases.

After discussing the concepts of problems, acts and memory, which are assumed as exogenous in the axiomatic derivation of the case-based decision theory, I now turn to the two characteristics of the decision-maker, aspiration level and similarity function which are derived from the axioms.

2.3 Aspiration Level

The aspiration level is one of the main concepts of the case-based decision theory. Its existence is embedded in $\mathcal{A}4$ Neutrality and its value can be derived from observed choices. In fact, the aspiration level is the utility realization $\bar{u}$, which, if observed in each of the cases present in the memory, would make the decision-maker indifferent among all acts.

2.3.1 The Concept of Aspiration Level in Psychology

The concept of "aspiration level" goes back to Dembo (1931) and is used in social psychology to describe a personal standard applied to evaluate achievements or situations. The aspiration level determines the envisaged proficiency of an individual at a given task and is influenced by past experience and comparison to others, see Festinger (1942) and Frey, Daunenheimer, Parge and Haisch (1993, p. 82). The aspiration level has two main functions: it sets a "level of future performance in a familiar task which an individual [...] explicitly undertakes to reach", Frank (1935, p. 119), hence, it defines the motives of behavior, Gebert and Rosenstiel (1992, p. 51) and is used for selfevaluation, Heckhausen (1989, p. 172).

The evolution of the aspiration level over time is mainly due to observed sequences of successes and failures while performing a task. Here, success and failure refer to situations, in which

[34] In his talk at the RUD conference in Evanston (IL) in June 2004, Itzhak Gilboa expressed the idea that relatively long, as well as relatively short memory provide the most natural environments for the usage of case-based reasoning. Whereas with short memory, rules cannot yet be formulated, with long memory, rules are not needed anymore.

the actual performance exceeds, respectively falls below, the aspiration level, Lewin, Dembo, Festinger and Sears (1944, p. 334). The difference between the aspiration level and the actual performance, called attainment discrepancy, explains best the adaptation of aspiration levels observed in the data. So, Jucknat (1937) and Festinger (1942) find that aspirations are adapted upwards, when the attainment discrepancy is positive and vice versa.

This, so called ”typical” reaction is also found by McClelland (1958) and Atkinson and Litwin (1960). Moreover, they demonstrate that persons motivated by the wish to succeed choose aspiration levels which are relatively high, but realistic. On the other hand, persons, whose motivation is to avoid failure are found to choose either extremely high or extremely low aspirations. Heckhausen (1963) and Moulton (1965) show that this group can be subdivided into subjects with high general motivation, who usually choose very high aspiration levels and subjects with low general motivation, who tend to set their goals relatively low.

Hence, the experimental evidence shows that realistic aspiration levels are often observed, but extremely high or extremely low values are typical for subjects trying to avoid failure.

The tendency to adapt the aspiration level and the speed of the adaptation are related to its initial level. Sears (1940) and Irwin and Mintzer (1942) find that subjects with low aspiration levels adapt their aspirations more strongly in the direction of the discrepancy, than do subjects with relatively high aspiration levels. These findings can be interpreted in terms of realism of the goals set, where realism corresponds to aspirations being attainable and closely related to actual performance, see Lewin, Dembo, Festinger and Sears (1944, p. 345). Preston and Bayton (1941) further show that even if subjects are pressed to distinguish between their hopes and their actual goals[35], the actual goals tend to be biased upwards, suggesting that even objective statements exhibit some irrealism.

The aspiration level is further influenced by factors, such as socio-economic background, similarity between decision situations and group standards. A more favorable socio-economic situation tends to reduce the aspiration level, see Gould (1941). The more similar two decision situations are, the more relevant is the performance and hence the aspiration level adopted for the earlier experienced one for the setting of goals in the second situation, Jucknat (1937). The

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[35] This distinction is made necessary by the fact, that the question about the aspiration level might be interpreted in different ways, for instance, as a minimum to overrich, as an expectation of the actual achievement or as the performance hoped for, see Lewin, Dembo, Festinger and Sears (1944, p. 344).

influence of performance of others on the aspiration level is exactly opposite to the influence of one's own performance: if an individual significantly outperforms a group, he will in general tend to decrease his aspiration level, see Frey, Daunenheimer, Parge and Haisch (1993, p. 82) and vice versa.

Two main differences have to be mentioned in the understanding of the concept of aspiration level in the social psychology and in economics, represented by the case-based decision theory. First, in social psychology it is understood that the task is *familiar* and therefore the individual can sensibly decide what aspirations he might associate with it. In experiments, subjects are usually presented the possible outcomes and even the estimated probability to obtain a certain level of proficiency. In case-based decision theory on the opposite, it is assumed that the decision-maker has no or little structural knowledge about the situation. Hence, the choice of an aspiration level cannot be based on the description of the task alone.

The second difference refers to the elicitation of aspiration levels in experiments. Whereas in social psychology the notion of aspiration level in experimental settings is defined as the goal of the subject with respect to a known task, as *reported to the experimenter*, see Heckhausen (1989, p. 172), the theory of Gilboa and Schmeidler (1997 (a)) relies on revealed preferences in order to elicit cognitive constructs.

2.3.2 Aspiration Levels in the Economic Literature

The concept of aspiration level in economics is first proposed by Simon (1957) and March and Simon (1958) with the introduction of the term satisficing behavior. The main idea behind this concept is that a decision-maker who acts in a complex system and whose cognitive abilities are constrained might search for an alternative whose performance is good enough (in the sense that it exceeds a predefined level) rather than optimal.

Concerning the adaptation of aspirations, March (1994, p. 22) writes: ”The history is important because aspiration levels – the dividing line between good enough and not good enough – are not stable. In particular, individuals adapt their aspirations [...] to their experience.” March further suggests that aspirations adapt more rapidly upwards, which gives the individuals stimuli to search for better alternatives. Not satisfaction, but the dissatisfaction should be seen as the driving force of the economy, see March and Simon (1958, p. 71).

Empirical investigations on adaptation rules in economic environment are rare. Lant (1992) examines the adaptation of aspirations in organizations. She finds that adaptation into the direction of the discrepancy between the aspirations and the realized payoff can better explain the evidence than rational and adaptive expectations do. Her aim is, however, to test different concepts of expectations formation rather than different rules for adaptation of the aspiration level against each other. The findings of Lant are verified by Mezias, Chen and Murphy (2002) in the context of financial services organizations. They also find that aspirations are adapted towards a combination of the past aspiration level and the performance achieved. They further record a dependence of aspiration levels on the performance of other companies. Shapira (2001) examines the effects of aspirations on the risk taking behavior of government bond traders. He finds that bond traders are more willing to trade when they haven't achieved their aspirations and prefer to keep their positions unchanged, once the aspiration level has been reached. Here, the aspirations are identified with the targets set. Hence, the results provide insights into a behavior of an individual with a constant aspiration level.

Rainwatter (1994) and Easterlin (2003) examine the change of aspirations with respect to income and living standards. They find that increased income and growing consumption lead to an increase of aspirations. Moreover, the magnitudes of growth in aspirations corresponds to the magnitude of consumption growth. Hence, instead of becoming satisfied with their achievements, households become more demanding over time, at least with respect to income and consumption[36].

The economic literature has not presented many models incorporating adaptation of aspiration levels. It is usually assumed that aspirations are adapted in the direction of the current consumption level. Most of the models combine rational expectations and expected utility maximization with aspiration adaptation. In these models, consumers take into account that increasing consumption first and decreasing it thereafter will decrease their utility, because of increased aspiration levels. These models propose or implicitly assume certain models for adaptation of aspiration levels, e.g. Ryder and Heal (1973) suggest to use the formula

$$\bar{u}_t = \beta u_{t-1} + (1 - \beta)\, \bar{u}_{t-1},$$

where $\beta \in (0; 1)$ denotes the speed of adaptation in the direction of the attainment discrepancy[37].

36 Easterlin (2003) finds that in the family domain, aspirations do not increase over time.

37 Gilboa and Schmeidler (2001 (b)) use an adaptation rule, which is similar to the one proposed by Ryder and

Duesenberry (1949), on the other hand, implicitly assumes that the rule is given by:

$$\bar{u}_t = \max_{\tau < t} u_\tau,$$

i.e. the aspirations are set at the highest utility realization observed. There are also empirical findings that people prefer not to get accustomed to a level of consumption they know they cannot maintain, see Rabin (1998, p. 15, footnote 8 and the literature cited there). However, this literature presupposes that individuals are conscious of the process of adaptation of their aspiration level and that they are informed about the future consumption stream. In contrast, the case-based decision theory makes none of these assumptions.

Sauermann and Selten (1962) and Selten (1996) propose an aspiration adaptation theory. The source of bounded rationality in their model is the presence of multiple goals, which are incomparable and cannot be represented by a functional subject to optimization. It is therefore suggested that an aspiration level is used with respect to each goal. Starting with a vector of such aspiration levels, the decision-maker searches for alternatives which perform better than the preset aspiration level with respect to at least one of the goals. The aspiration level is then adapted upwards with respect to this particular goal. The (mental) search continues, until an increase of any of the aspiration levels is impossible. This process leads to a choice of an undominated alternative, but in general, different initial aspiration levels lead to distinct results. Differently from the case-based decision theory, the theory of Sauermann and Selten presupposes the knowledge of the payoffs of each alternative with respect to each of the goals.

Aspiration levels are used in most similar manner to the one proposed by the case-based decision theory in the literature on reinforcement learning in games. Three approaches can be distinguished in this literature, see Bendor, Mookherjee and Ray (2001): constant aspiration levels, which are, however, consistent with the long-run average payoffs of the players[38], as in Bendor, Mookherjee and Ray (1992, 1995); adaptation of aspiration levels in the direction of attainment discrepancy, as suggested by Ryder and Heal (1973), but with rare permutations, see Karandikar, Mookherjee, Ray and Vega-Redondo (1998); aspirations based on observations of other market participants, as in Dixon (2000), where firms engaging in Cournot competition adapt their aspirations towards the average profits in the market and in Palomino and Vega-Redondo (1999),

Heal (1973). They assume however, that the adaptation towards the last encountered utility realization takes place, only if this realization exceeds a certain level.

38 To insure consistency of actual play and aspiration levels, a new equilibrium concept has to be defined and its existence shown. This is done in Bendor, Mookherjee and Ray (1992, 1995).

where the players participating in a Prisoners' Dilemma game adapt their aspiration levels towards the average payoff of the population. Most of the results indicate convergence towards a cooperative outcome of the game, even if this outcome is not a Nash equilibrium.

The sparse literature on aspiration levels in economics shows that the concept is relatively new and that more research is needed in order to understand its implications on economic behavior. Some of the problems, might be connected with the different interpretations of the concept used and with the necessity of elicitation of the aspiration level using self-reports and the difficulties arising from it. The case-based decision theory avoids these problems by deriving the aspiration level directly from the axioms and thus relating a cognitive concept directly to exhibited behavior.

2.3.3 The Operationalization of the Concept of Aspiration Level in the Applications of the Case-Based Decision Theory

The aspiration level in the case-based decision theory can be seen as a bench-mark which helps the decision-maker separate "good" from "bad" performance. Since he only observes the utility realizations of some of the acts for a given problem, it is impossible for him to know what the best solution of the problem would have been, either in an ex-ante, or in an ex-post sense[39]. It is, therefore, the aspiration level that indicates that an observed utility realization is satisfactory (if it exceeds the aspiration level) or unsatisfactory (if it lies below it). Satisfactory utility realizations enter the evaluation of an act positively and vice versa.

The aspiration level has also another function — if an act cannot be evaluated by its own past performance, or by the performance of similar acts in similar problems, the decision-maker assumes that its cumulative utility is 0, hence this act is evaluated as if it would result in utility exactly equal to the aspiration level. Hence, apart from discriminating between good and bad outcomes, the aspiration level determines the prior expectations of the decision-maker about acts he has no information about. Indeed, with an empty memory, the cumulative utility of all acts is 0 and therefore all acts are assigned utility realizations equal to the aspiration level.

Of course, the aspiration level needn't remain constant over the time, but can be updated, so as to take into account the experience of the decision-maker. However, neither the value of the

[39] Observing the realizations of all acts would enable a verification of an ex-post, but not of an ex-ante optimality.

aspiration level, nor the updating rule are determined by the theory and have therefore to be specified in the context of the model[40].

The works of Gilboa and Schmeidler (1997 (b)), Gilboa and Pazgal (2001), Blonski (1999) and Krause (2003) use a constant aspiration level to derive their results. Whereas the case of a constant aspiration level seems quite irrealistic on empirical grounds, it has its theoretical advantages, since no ad hoc assumption about adaptation rules are needed. Moreover, the possibility to do comparative statics varying the aspiration level can provide us with useful insights into how the aspiration level influences the results. Therefore, constant aspiration level will be assumed in most parts of this thesis.

In the context of individual portfolio choice, this assumption will be relaxed. This seems appropriate for two reasons: first, in a partial model the analysis of updating rules is simplified by the fact that prices and income are assumed exogenous and fixed over the time. Hence, apart from the case of constant aspiration level, the formula proposed by Duesenberry (1949), as well as a version of it, in which the aspiration level at t is set equal to a linear combination of the best and the worst results achieved till time t is examined. Second, the work of Gilboa and Schmeidler (1996), later applied to cooperation games by Pazgal (1997), has identified an adaptation rule, which applied in a stationary environment leads to an optimal choice from the point of view of expected utility maximization. Hence, this case can be used as a bench-mark to evaluate alternative adaptation rules. Furthermore, the validity of this rule will be tested in a model with similarity considerations, a case not analyzed in the literature.

Whereas I keep the aspiration level of the investors constant in an asset market model with case-based investors, I allow for heterogenous aspiration levels across the economy. It is, therefore, not only possible to examine the influence of an overall increase in aspiration levels throughout the economy, but also to compare the behavior and performance of investors with different aspiration levels under identical conditions.

In the last part on the evolutionary fitness of case-based decisions I assume that all case-based decision-makers have identical and constant aspiration levels over the time, an assumption made to simplify calculations and to make results tractable. Again, comparative statics allows to

[40] The axiomatization of the case-based decision theory, proposed by Gilboa and Schmeidler (1997, p.53) allows for adaptation of the aspiration level towards a linear combination of the utility realizations observed.

analyze the influence of an increase or decrease of the aspiration level on the dynamics of the economy.

2.4 Similarity

The theory of case-based decisions allows to incorporate similarity considerations into a formal model. However, similarity is a relatively new concept in economics, hence, it is not a priori clear how similarity should be operationalized in different contexts. Therefore, the first question to ask is what does ”similar” mean in the context of financial markets.

The case-based decision theory works with similarity functions, which allow a numerical representation of perceived similarity. Up to now, there has been little research regarding the form and the characteristics of similarity functions. This is the next issue to be discussed in this section.

2.4.1 Similarity — A Philosophical View

Similarity, without doubt, affects our judgements and therefore influences our decisions. Nevertheless, although ”there is nothing more basic to thought and language than our sense of similarity”, Quine (1969, p. 6), it seems to be very difficult to give a precise definition of similarity, which could be used in a formal model. Philosophical thought has been dwelling on this question for a long time and still, the problem does not seem to have a straightforward solution.

The problem is that similarity can be neither defined by means of sets, nor by means of logical structures. Indeed, suppose that one could divide the objects into sets, such that all objects, which are similar to each other are in one such set. One easily sees that as long as objects are compared with respect to all possible criteria, each object should either be placed into a separate set, or all objects will be considered similar in some sense and will build a unique set, thus leading the notion of similarity ad absurdum.

A representation of similarity by means of a logical structure is also problematic. Suppose you want to evaluate alternative α. If you know that alternative α' has performed well and you consider α and α' to be similar, you will probably think that α will also perform well. Now suppose that you also know that alternative α'' has performed badly and you consider α'' to be dissimilar to α. Will this make your believe that α will perform well stronger? Probably yes, if you feel that this confirms the (perceived) law that alternatives that perform poorly are not similar to α, therefore that alternatives which are similar to α cannot perform poorly. It is however paradoxical that an alternative, which is considered to be dissimilar to α should be used to predict the performance of α[41].

The main problem is, of course, that since mathematical concepts obey the laws of logic, one necessarily embeds this paradox into each mathematical model of similarity. And indeed, the theory of Gilboa and Schmeidler (1997 (a)) also bears this paradox in itself.

Although a philosophical definition of similarity is problematic, it still does not mean that similarity is a concept that couldn't be implemented in an economic model. In fact, as long as we are interested only in individual perception of similarity, it is enough to elicit the similarity function of an individual (by asking him questions of the type "is α similar to α'?" or "is α more similar to α' than to α''?"), in order to model his behavior. However this behavioristic modelling does not give any clues up to which similarity perceptions are sensible or what should be understood under similarity in economics.

2.4.2 Similarity in Economics

Similarity does not belong to the standard concepts of economic theory. Rubinstein (1988) suggests a theory of decision-making based on similarities between lotteries. The concept of similarity he uses is a binomial one. Two objects are either similar or not similar. Similarity is defined on intervals of real numbers and two numbers are considered similar, if and only if the difference between these two numbers is less than a given number s, and dissimilar else. This definition of similarity can be used to explain some of the experimentally observed behavior of choices between lotteries, Buschena and Zilberman (1995, 1999), as well as hyperbolic discounting, Rubinstein (2003). Since however the similarity relation is not transitive, it leads to preferences with intransitive indifference relations[42]. Of course, this intransitivity of preferences is due to the assumption of binomial similarity relation and can be avoided by introducing a "more similar than" relation.

[41] This argument is known as the Hempel's puzzle. Its original version is that since "each black raven tends to confirm the law, that all ravens are black, so each green leave, being a non-black non-raven, should tend to confirm the law that all non-black things are non-ravens, that is, again, that al ravens are black", Quine (1969, p. 5).

[42] Kajii (1996) provides a characterization of preferences consistent with the similarity concept of Rubinstein.

Such a relation is proposed by Tversky (1977), who derives a similarity function from a set of axioms. Similarity is seen as possession of common attributes, still symmetry is not implied by the axioms and is generally violated empirically. The approach of Nehring and Puppe (2003) is similar to that of Tversky (1977). Their similarity concept is a trinomial relation, interpreted as "α is more similar to α'' than is α'". It satisfies reflexivity, symmetry and transitivity and expresses the idea that α and α'' have more common attributes than α' and α''. Moreover, this trinomial relation is representable by a similarity function. However, the similarity relation is not complete, i.e. it is not defined for all triples of acts and therefore similarity judgements are not always possible.

An alternative theory is the case-based decision theory proposed by Gilboa and Schmeidler (1997 (a)). They assume a similarity function defined on problem – act pairs, which is also based on a "more similar than" relation. They suggest that the cumulative utility of an act depends not only on the utility realizations observed when the same act was chosen in the past, but also on utility realizations of similar acts chosen in similar decision problems. An axiomatic representation of this functional form is provided. Nevertheless, the axiomatization does not give any clue about the characteristics a similarity function should have and their impact on the decisions made[43].

Few models have used this concept of similarity up to now. Gilboa and Schmeidler (1997 (b)) show how similarity between goods can be interpreted in terms of complementarity and substitutability in a consumer choice problem. Blonski (1999) models social learning and represents social structures by similarity considerations. He shows that different similarity functions, associated with "star"-communication structures, Δ-neighborhood structures and complete information, imply different stable states of the dynamical learning process. Gayer (2003) shows how similarity considerations may affect the perception of lotteries and lead to overestimation of low probabilities and underestimation of high probabilities. She proposes to measure the similarity between lotteries in terms of distance. In contrast to the models presented in this thesis, she assumes an endogenous similarity function, which becomes finer, as the memory grows. She shows that if the similarity function depends on the distance, its concavity insures that the

[43] Billot, Gilboa and Schmeidler (2004) provide an axiomatization of a similarity function which is an exponential function of the Euclidean distance. However, they do not provide any reasons for the choice of this particular form of the similarity function. Nor are there applications of this similarity function in the litearture providing insights into how its form influences decisions.

decision-maker will be able to learn the correct distribution.

This research shows that the results obtained are very sensitive to the form of the similarity function assumed. It will turn out that the form of the similarity function plays a major role for the behavior of case-based investors in financial markets, as well.

2.4.3 Similarity in Financial Markets

Suppose that you ask an investor, who has lost a reasonable amount of money on dot.com assets, whether he would be ready to invest in a dot.com company now. You would probably receive a firm no, even, if the company you are proposing is a sound one. Not necessarily because the unlucky investor has looked up the performance of the company and has analyzed the pro and contras, but because his experience has taught him that dot.com assets can lead to significant losses and he obviously finds the investment you are proposing to be quite similar to the bad choice he has made last time. Similarity can, therefore, play major role in asset choice decisions.

The most natural concept of similarity (from the point of view of standard financial economics) is the concept of covariance. However, since the case-based decision theory presupposes no knowledge about state-contingent outcomes and their distribution, it does not seem appropriate to use the concept of covariance to model similarity in case-based decision-making[44].

Since the information available to a case-based decision-maker consists of a problem and a set of acts, it is reasonable to use the description of acts to elicit a notion of similarity. In this sense similarity may refer to the fact that assets of firms in the same industry are regarded as similar, as contrasted to firms from different industries. For instance, an investor may consider shares of BMW and Renault to be more similar to each other, than the shares of Renault and Telecom. Similarity perceptions might also include the nationality of an asset. Thus, an investor might find that Telecom is more similar to BMW than to Renault. Other characteristics, such as maturity or being derivatives of the same underlying asset can also influence similarity perceptions in financial markets.

The above discussed characteristics allow only for a comparison of individual assets, but not

[44] There seems to exist a certain connection between the notions of similarity and correlation, as observed by Matsui (2000). Still, since his states of nature do not necessarilly correspond to the states of nature as considered in the standard financial economics, the interpretation of his result is ambiguous.

of portfolios. However, it is easy to imagine, how similarity might refer to the comparison of the structure of two portfolios. For instance, an investor may consider a portfolio, consisting of 20% risky assets and 80% bonds to be more similar to a riskless portfolio, consisting of 100% bonds, than a portfolio, consisting only of risky assets.

To formalize these ideas, note that each portfolio, consisting of at most K assets can be represented as a point in a K-dimensional simplex Δ^{K-1}. Now, similarity between portfolios can be modelled as a continuous decreasing function of the Euclidean distance between the points in this simplex:

$$\begin{aligned} s(\alpha;\alpha') &= f(\|\alpha-\alpha'\|) \text{ with} \\ f'(\|\alpha-\alpha'\|) &< 0. \end{aligned}$$

In order to include similarity considerations between assets (such as maturity, or belonging to the same industry), it is enough to modify the simplex, allowing the distance between its vertices to vary with the degree of similarity. Figure 1 illustrates this.

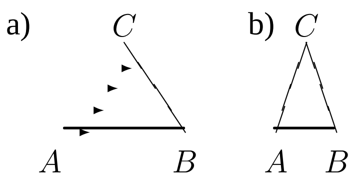
Such a similarity function applies, of course, only to situations which are considered to be identical and in which only similarity between acts matter.

Still, the market situation might also influence the evaluation of different acts. Buying an asset in a market boom might be quite different from buying the same asset, when prices fall. The characteristics of a given decision situation are captured in the notion of a problem. In a financial market, asset prices seem to bear the most important information about the decision situation and will, therefore, influence similarity perceptions.

In a model of case-based decision-making in financial markets, these two aspects of similarity — similarity between problems and between acts — are captured in a single similarity function:

$$s((\rho;\alpha);(\rho';\alpha')),$$

which is to be interpreted as the degree of similarity of choosing act α in problem ρ to choosing act α' in problem ρ'. It has already been shown that the acts α can be situated on a metric space, depending on how similar they are perceived to one another. Since it seems to me that the major characteristic of a portfolio choice problem is represented by the prices in the economy, I propose to identify the problem with a price vector $(p_1...p_K)$ and to represent a problem - act pair in $\Delta^{K-1} \times \Delta^{K-1}$. Again, taking the Euclidean distance as measure of similarity, the similarity



a) A, B and C are mutually dissimilar.
b) A and B are similar, A and C, as well as B and C are mutually dissimilar.

Figure 1

function can be written as:

$$s\left(\left(p;\alpha\right);\left(p';\alpha'\right)\right)=f\left(\left\|\left(\alpha;p\right)-\left(\alpha';p'\right)\right\|\right),$$

where $f\left(\cdot\right)$ is continuous and decreasing.

The axiomatization of Gilboa and Schmeidler (1997 (a)) implies that the similarity function is unique up to an affine-linear transformation, and therefore the similarity function can be normalized, so that it takes on only values between 0 and 1 where a value of 1 means that two objects are ”identical” or ”completely similar”:

$$s\left(\left(p;\alpha\right);\left(p;\alpha\right)\right)=1,$$

whereas 0 can be interpreted as ”having nothing in common”, or being ”completely different”, depending on the context.

Of course, the notion of similarity, as well as the specific similarity function used by an investor will influence his behavior in a financial market, since they will determine his evaluation of acts. Moreover, the similarity function may (as well as the aspiration level) evolve with the time, reflecting the fact that the decision-maker learns about (dis)similarities among alternatives, which he was not aware of, see Gilboa and Schmeidler (2001 (a), Chapter 19) and Gayer (2003). In this thesis, I will assume that the similarity function remains constant over the time and will explore the influence of the form of the similarity function on investors' behavior.

2.5 Conclusion

This chapter has discussed the conceptual issues connected with the application of the case-based decision theory to financial markets. Most of the concepts are relatively new in economics and therefore the literature discussing them is sparse. The freedom connected with the possibility to vary the parameters and the specification of the model has both its advantages and disadvantages. On the one hand, the possibility to use arbitrary methods for aspiration adaptations and to introduce memory, containing arbitrary cases, allows for a very rich model. On the other hand, it bears the risk of losing any explanatory power, if the specifications used have no economically meaningful interpretations. In the models presented in the next chapters of this thesis I will therefore try to control for the assumptions made about aspiration levels, contents of the memory and similarity perceptions by varying them, so as to test the robustness of results. This will also allow to examine the influence of these concepts on the behavior of economic subjects, as well as to give them an interpretation in the context of the model used.

Chapter 3. Portfolio Choice and Case-Based Reasoning

Having analyzed the issues of operationalization of the concepts of the case-based decision theory in the context of financial markets, it is now possible to construct a specific model in which the meaning and the influence of the memory, the aspiration level and the similarity function of an investor on his decisions can be studied. In this chapter, I apply the case-based decision theory to a standard portfolio-choice problem. I assume that the process of returns is i.i.d. and model the behavior of the investor in the limit. Standard portfolio choice theory predicts that the investor would choose the portfolio that maximizes his expected utility, provided he knows the correct distribution of returns. Should the decision-maker have imprecise knowledge of the distribution, expected utility theory assumes that he will use Bayesian learning and will be able to learn the correct distribution with time. In contrast, a case-based decision-maker neither has the correct probability distribution in mind, nor does he like a Bayesian have a prior, which he updates, as more information becomes revealed. Instead, he learns from past observations, accumulating information about what is possible, as opposed to using the incoming information to exclude some possibilities (as a Bayesian would). The value of the aspiration level, as well as the way it is updated determine his behavior. I concentrate on limit behavior for two reasons: first, the initial decisions of the case-based decision-makers depend crucially on the random realizations of the return process and second, I am interested in studying the possibility of learning to choose the optimal portfolio over time.

I first use the results derived by Gilboa and Schmeidler (1996) and Gilboa and Pazgal (2001) for general decision problems by applying them to a portfolio-choice problem. Gilboa and Schmeidler (1996) construct a method for updating the aspiration level that leads to optimal behavior in the limit, as long as the utility realizations are i.i.d. In Gilboa and Pazgal (2001), the case of a constant aspiration level is considered in a model of consumer choice. An aspiration level, exceeding the highest mean utility realization of an act makes the decision-maker switch infinitely often among the available acts. For low values of the aspiration level, the choice of the decision-maker remains constant over time, but is in general suboptimal, since being satisfied with the utility realizations obtained, he does not have an incentive to try another, possibly superior, acts.

In a next step, I analyze alternative rules for updating the aspiration level. The max (min) -rule sets the aspiration level in each period equal to the maximal (minimal) achieved utility realization up to this period. A linear combination of these two rules attaches a weight β to the minimal realization and a weight $(1-\beta)$ to the maximal one. It can be shown that if β is relatively low, then the decision-maker behaves as if he had a relatively high aspiration level in the limit, i.e. he switches constantly among the portfolios available to him. If β is relatively high, he behaves in a satisficing manner. However, there is an interval of values of β for which the results are path-dependent.

The derived results can then be interpreted in light of the empirical evidence from financial markets. Satisficing behavior can help explain unused arbitrage possibilities, low diversification and the home-bias, whereas updating the aspiration level upwards can be associated with overconfidence and lead to frequent, but unprofitable trades.

Up to this point, I assume that the investor's information consists only of the utility realizations of the portfolios he has actually chosen. In the stringent analysis, I relax this assumption in two ways. First, I consider the case in which all portfolio realizations are observed and included in the evaluation in each period of time. In this case, the aspiration level becomes irrelevant for the behavior of an investor, he learns the correct distribution of returns and chooses the optimal portfolio in the limit. Restricting the ability of the investor to learn all returns for only a finite number of periods improves his choice (compared to the situation of endogenous memory) if his aspiration level is relatively low, but does not lead to optimal decisions if his aspiration level is relatively high.

Another possibility to relax the assumption that the investor learns only about the utility realizations of the acts actually chosen consists in introducing a similarity function among acts. I define similarity between two portfolios as a decreasing function in the Euclidean distance between these two portfolios (assuming that portfolios are situated on a simplex). Once a portfolio is chosen and its utility realization observed, the investor assigns the same utility realization to every other available portfolio, but weighted by the similarity between the portfolio actually chosen and the portfolio evaluated.

I examine again the model of repeated portfolio-choice, assuming first that the similarity function is concave in the distance between portfolios. It turns out that when the decision-maker starts with a diversified portfolio, he diversifies only for a finite number of periods, if his aspiration level is relatively high. Afterwards, he chooses one of the undiversified acts in each period of time. Moreover, if his aspiration level exceeds the mean returns of both undiversified portfolios, then he switches infinitely often between them in the limit. Hence, similarity considerations in combination with high aspiration level lead to a combination of the two effects — underdiversification and frequent trading.

Next, I analyze the issue of learning in this setup. I apply the "ambitious-realistic" adaptation rule, introduced by Gilboa and Schmeidler (1996), to the portfolio-choice problem with similarity. With a concave similarity function, this rule allows the decision-maker to learn to choose the optimal non-diversified portfolio. Convexities in the similarity function improve learning, by increasing the number of portfolios from which the optimal one is selected. Nevertheless, global optimality can be achieved only in the limit, in which no similarity between distinct portfolios is present and the number of available acts is finite, as in the model of Gilboa and Schmeidler (1996).

The rest of the chapter is organized as follows. I start by presenting the general setting of Gilboa and Schmeidler (1996) in section 1. In section 2, the model of Gilboa and Schmeidler is applied to a portfolio-choice problem. Section 3 describes the sufficient conditions, under which a case-based decision-maker behaves like an expected utility maximizer, as stated by Gilboa and Schmeidler (1996). Section 4 presents the implications of a constant aspiration level on the behavior of a decision-maker, derived by Gilboa and Pazgal (2001), while section 5 examines max-min adaptation rules. In section 6, possible interpretations of the results in the context of the portfolio-selection problem are proposed. In section 7, the model is adapted in order to allow the decision-maker to acquire additional information. Section 8 analyzes the portfolio-choice problem with similarity considerations, assuming a concave similarity function and a constant aspiration level. The adaptation rule of Gilboa and Schmeidler (1996) is applied to the portfolio-choice problem with similarity. Section 9 concludes. The proofs of all results are stated in the appendix.

3.1 Case-Based Decision-Making

Consider a decision-maker who faces an identical problem ρ in each time period $t = 1, 2, ...$

Let $\mathfrak{A}$ be a finite set of possible acts, among which the decision-maker can choose in each period of time. The utility resulting from the choice of an act $\alpha \in \mathfrak{A}$ is a random variable $\mathfrak{U}_\alpha$ with a distribution function $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$, which can be interpreted as a conditional distribution of utility, given that act α is chosen. It is assumed that these distributions have finite expectations μ_α, finite variance σ_α and bounded supports Λ_α. One can think about the acts in terms of the theory of Savage (1954): the decision-maker chooses an act and thus, indirectly, a probability distribution over results.

The main difference between the model of Savage (1954) and the one of Gilboa and Schmeidler (1996) concerns the information available to the decision-maker. Whereas in the setting of Savage the (subjective) probability distributions $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$ are known to the decision-maker, in the model of Gilboa and Schmeidler (1996) the individual is not even aware of the possible realizations he can expect from a given act. To be precise, the decision-maker's information at the beginning of period $t = 1$ consists only of the problem ρ to be solved and the set of possible acts $\mathfrak{A}$. After an act α_t has been chosen at time t, the decision-maker learns the utility realization u_t of this act. Thus, at the beginning of period $(t + 1)$ he has information, consisting of triples of a problem to be solved, act chosen and a utility realization achieved by choosing the act for each of the previous periods:

$$M_{t+1} = \{(\rho_1; \alpha_1; u_1) \ldots (\rho_t; \alpha_t; u_t)\} .$$

M_{t+1} represents the memory of the decision-maker at time $(t + 1)$. Since, however, all problems are identical ($\rho_\tau = \rho$ for each τ), only the tuples $(\alpha_\tau; u_\tau)$ are used to make decisions.

Now suppose that the decision-maker has an aspiration level, which is used as a bench-mark (in a way to be specified below) to assess the utility realizations of the different acts. This aspiration level need not remain constant over time, but may be updated in some way depending on the history observed. Call the aspiration level of the decision-maker at time t — $\bar{u}_t$. Then the behavior of a decision-maker can be characterized by an infinite vector of triples $((\bar{u}_t; \alpha_t; u_t))_{t=1,2,\ldots}$, defining the aspiration level, the act chosen and the utility achieved from this act for each period of time t. Call the set of such vectors S_0 and let ω be a typical element. Then S_0 can be written as:

$$S_0 = \left\{\omega = ((\bar{u}_t; \alpha_t; u_t))_{t=1,2\ldots}\right\} \subset (\mathbb{R} \times \mathfrak{A} \times \Lambda)^{\mathbb{N}}$$

where $\bar{u}_t$ is a real number, α_t is one of the possible acts from $\mathfrak{A}$, the realizations u_t are drawn from the set $\Lambda = \cup_{\alpha \in \mathfrak{A}} \Lambda_\alpha$ of all possible realizations, and $\mathbb{N}$ is the set of natural numbers. Thus, S_0 represents the set of all possible decision-paths an analyst may expect to observe.

Of course, if a decision rule is specified, then the set of possible paths will reduce to paths consistent with this particular rule. For instance, the behavior of an expected utility maximizer, who knows, that the act $\alpha^* \in \mathfrak{A}$ yields the maximal expected utility, can be represented by:

$$S_{EU} = \left\{ \omega = \left((\bar{u}_t; \alpha_t; u_t) \right)_{t=1,2\ldots} = \left((\bar{u}_t; \alpha_t = \alpha^*; u_t) \right)_{t=1,2\ldots} \right\} = \left(\mathbb{R} \times \{\alpha^*\} \times \Lambda_{\alpha^*} \right)^{\mathbb{N}}$$

The set of chosen acts is thus reduced to $\{\alpha^*\}$ and the set of possible realizations to Λ_{α^*}.

Alternatively, if the decision-maker has no information about the distributions $(\Pi_\alpha)_{\alpha \in \mathfrak{A}}$, he can only condition his behavior (i.e. the choice of an act and of an aspiration level) on the information he already possesses. The case-based decision theory proposed by Gilboa and Schmeidler (1995) states that the decision-maker chooses in each period that act which has the maximal cumulative utility up to the current period. I will write $\bar{u}_t$, α_t, u_t[45] for the projections of S_0 on $\mathbb{R}$, $\mathfrak{A}$ and Λ (for a specific ω), respectively, and will denote by $U_t(\alpha)$ the cumulative utility (to be specified below) of an act α at the beginning of period t. Then the set of possible decision-paths for a decision-maker, acting in accordance with the case-based decision theory, reduces to:

$$S_1 = \left\{ \omega \in S_0 \mid \alpha_t \in \arg\max_{\alpha \in \mathfrak{A}} U_t(\alpha) \;\; \forall t \geq 1 \right\}.$$

Given that all problems are identical, the concept of cumulative utility of an act is specified as follows:

$$U_t(\alpha) = \sum_{\tau=1}^{t-1} s(\alpha; \alpha_\tau)(u_\tau - \bar{u}_t),$$

where $s(\alpha; \alpha_\tau)$ denotes the perceived similarity between the acts α and α_τ. In sections 3–7, I will assume that no similarity considerations influence the decision. Two acts are only considered similar, if they are identical:

$$s(\alpha; \alpha_\tau) = \left\{ \begin{array}{l} 1, \text{ if } \alpha = \alpha_\tau \\ 0, \text{ if } \alpha \neq \alpha_\tau \end{array} \right\}.$$

In this case, it is convenient to formulate the cumulative utility of an act is the following way: define by $C_t(\alpha)$ the set of all periods up to period t, in which act α has been chosen, i.e.:

$$C_t(\alpha) = \{\tau < t \mid \alpha_\tau = \alpha\}$$

[45] All the variables introduced in the model depend of course on the chosen path ω. I suppress this dependence in the notation for convenience. All probabilities are defined with respect to the distribution of ω.

Denote by

$$U_t(\alpha) = \sum_{\tau \in C_t(\alpha)} (u_\tau - \bar{u}_t)$$

the cumulative utility of an act α at time t, given that the aspiration level at time t is $\bar{u}_t$. The interpretation of the cumulative utility is simple: the decision-maker sums up the "net" utility realizations of an act (utility achieved at some period, less the aspiration level at time t) over all the periods at which the act has been chosen up to the present. Then he compares the cumulative utilities over the set $\mathfrak{A}$ and chooses the act with the maximal cumulative utility.

In the next section, I present a possible application of this model in financial markets.

3.2 The Portfolio-Choice Problem

Consider an investor facing the problem ρ of investing a single unit of wealth[46] in each period $t = 1, 2....$ He can choose among portfolios consisting of two assets[47] denoted by a and b, with random returns δ_a distributed on an interval $\left[\underline{\delta}_a; \bar{\delta}_a\right]$ and δ_b distributed on an interval $\left[\underline{\delta}_b; \bar{\delta}_b\right]$, respectively.

Assume that the distributions of returns are identical and independent in each period t, although δ_a and δ_b might be correlated in a single period. Suppose that the assets are not perfectly divisible and that short-sales are not allowed, or allowed only up to a certain limit only so that only a set, consisting of a finite number of portfolios $\mathfrak{A} = \{\alpha_1...\alpha_n\}$ is available, with α_i denoting the share of asset a in the portfolio[48]. Thus, if the decision-maker chooses a portfolio α and the realizations of the random returns are δ_a and δ_b, he will achieve a return of

$$\alpha\delta_a + (1 - \alpha)\delta_b$$

Denote the utility function of the investor by $u(\cdot)$. Then the distributions of δ_a and δ_b determine the distribution functions $(\Pi_\alpha)_{\alpha \in \mathfrak{A}}$ of the utility realizations of all possible acts α. If u is well-

46 We can assume, that the decision-maker has an exogenous income and that he uses the returns of his investment for consumption. However, it is also possible to think of a decision-maker, desiring to choose the alternative with the highest return per amount of wealth invested.

47 The arguments stated do not depend on the number of assets. The introduction of more assets would only change the type of the variable α defined below from scalar to vector, but would leave all the results unchanged.

48 The set of portfolios should also entail the budget restriction of the portfolio-choice problem. Since the portfolios are defined by the share of wealth the decision maker invests into asset a, assuming that the rest of the budget is invested in b, the budget restriction is satisfied for each of the available portfolios.

defined, bounded and continuous on

$$\left[\alpha\underline{\delta}_a + (1-\alpha)\,\underline{\delta}_b; \alpha\bar{\delta}_a + (1-\alpha)\,\bar{\delta}_b\right]$$

for all $\alpha \in \mathfrak{A}$, then the distribution Π_α of $u(\cdot)$ for a given α is characterized by a finite mean μ_α and finite variance σ_α, i.e. they have the same properties as in the model of Gilboa and Schmeidler (1996). Moreover, differently from the setting of Gilboa and Schmeidler (1996) the distributions $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$ have bounded and closed supports Λ_α. As in the classic theory of portfolio-choice, Markowitz (1952, 1959), one would like to determine the portfolio chosen by the decision-maker.

It is clear that in the initial periods the behavior of the decision-maker is guided by chance, as he has no or little information about possible payoffs and their distributions. However, as time evolves, his behavior is determined by his memory which, on its turn, depends on the choices made. It is reasonable to ask whether the behavior of the decision-maker will exhibit some constant patterns in the limit. Hence, the limit frequencies with which each of the acts is chosen as the number of periods becomes large are of particular interest.

Consider an individual facing the portfolio-choice problem formulated above, who acts according to the case-based decision theory, so that $\omega \in S_1$. For a given path $\omega \in S_1$ denote by

$$|C_t(\alpha)| = |\{\tau < t \mid \alpha_\tau = \alpha\}|$$

the number of times the act α has been chosen up to period t. Dividing by the number of periods, we obtain the relative frequency of an act α up to period t:

$$\frac{|C_t(\alpha)|}{t}$$

The behavior of the decision-maker in the long run can be characterized by the limit of the relative frequencies for each $\alpha \in \mathfrak{A}$ as $t \to \infty$. If such a limit exists, it will be denoted by

$$\pi(\alpha) = \lim_{t\to\infty} \frac{|C_t(\alpha)|}{t}$$

and referred to as the relative frequency of α on ω. Of course, the existence and the values of $\pi(\alpha)$ will depend on the way the decision-maker changes his aspiration level $\bar{u}_t$ with time. The effect of using particular updating rules is analyzed in sections 3, 4 and 5.

3.3 Case-based Decision Theory and Expected-Utility Maximization

In this and the subsequent two sections, the limit behavior of a decision-maker acting in accordance with case-based decision theory will be explored for different rules of updating the aspiration level. The behavior of an expected utility maximizer, who, in the same set-up, would choose the act with the maximal expected utility at each period of time, can be used as a benchmark. I first state two theorems by Gilboa and Schmeidler (1996), which identify conditions under which the case-based decision-making approaches expected utility maximization in the limit. Next, I analyze the behavior of a decision-maker who does not satisfy these conditions.

It is intuitively clear (see also proposition 3.4) that a relatively low aspiration level can lead to a path, on which a sole suboptimal act is chosen forever. Therefore, a special adaptation rule is necessary to guarantee that a case-based decision-maker behaves like an expected utility maximizer. It consists in making the aspiration level convergent towards $\max_{\alpha \in \mathfrak{A}} \mu_\alpha$.

The first result of Gilboa and Schmeidler (1996) states that starting from a sufficiently high aspiration level and gradually updating it in the direction of the highest average utility achieved, leads with a probability close to one to a path on which the case-base decision-maker imitates the behavior of an expected utility maximizer in the limit. Formally, let

$$X_t(\alpha) = \frac{\sum_{\tau \in C_t(\alpha)} u_\tau}{|C_t(\alpha)|}$$

denote the average utility of an act α up to period t, if this act has been chosen at least once (i.e. $|C_t(\alpha)| \neq 0$) and write

$$X_t = \max_{\alpha \in \mathfrak{A}} \{X_t(\alpha) \mid |C_t(\alpha)| > 0\}$$

for the maximal achieved average utility among all the acts chosen at least once. The idea of Gilboa and Schmeidler (1996) consists in assuming that, starting from an initial aspiration level $\bar{u}_1$, the aspiration level is updated in the following manner:

$$\begin{aligned} &\bar{u}_1 = \bar{u}_1 \\ &\bar{u}_t = \beta \bar{u}_{t-1} + (1-\beta) X_t \text{ for } t \geq 2, \end{aligned} \tag{3.3}$$

i.e., given the initial aspiration level, the level for the next period is determined as a linear combination of the aspiration level of the previous period and the highest average utility achieved, for a chosen parameter $\beta \in (0;1)$. In this case, the set of possible decision paths reduces to:

$$\Phi = \Phi(\beta; \bar{u}_1) = \left\{ \omega \in S_1 \mid \begin{array}{l} \bar{u}_1 = \bar{u}_1 \text{ and} \\ \bar{u}_t = \beta \bar{u}_{t-1} + (1-\beta) X_t \text{ for } t \geq 2 \end{array} \right\},$$

so that only paths on which both the decision is made according to the case-based decision theory and the aspiration level is updated according to (3.3) are considered.

Gilboa and Schmeidler (1996, p.11) define a probability distribution on Φ in the following way: define a σ-algebra on S_0, i.e. the algebra generated by the Borel σ-algebra on each copy of $\mathbb{R}$ and of Λ and by the algebra $2^{\mathfrak{A}}$ on each copy of $\mathfrak{A}$. Let $\Sigma = \Sigma(\beta; \bar{u}_1)$ be the induced σ-algebra on Φ. A probability measure P on Σ is called consistent with $(\Pi_\alpha)_{\alpha \in \mathfrak{A}}$ if for every $t \geq 1$ and $\alpha \in \mathfrak{A}$, the conditional distribution of u_t, given that $\alpha_t = \alpha$ is Π_α and $u_t(\alpha)$ is independent of $((\bar{u}_\tau; \alpha_\tau; u_\tau))_{\tau < t}$

Theorem 3.1 *Gilboa and Schmeidler (1996, p. 11). Let there be given a set* $\mathfrak{A} = \{\alpha_1 ... \alpha_n\}$*, probability distributions* $(\Pi_\alpha)_{\alpha \in \mathfrak{A}}$ *and a parameter* $\beta \in (0;1)$ *as above. Then, for each* $\varepsilon > 0$*, there exists a* $\bar{u}_0 \in \mathbb{R}$*, such that for each* $\bar{u}_1 \geq \bar{u}_0$ *and each probability measure* P *on* $(\Phi(\beta; \bar{u}_1); \Sigma(\beta; \bar{u}_1))$*, consistent with* $(\Pi_\alpha)_{\alpha \in \mathfrak{A}}$ *:*

$$P \left\{ \omega \in \Phi \mid \exists \pi \left(\arg\max_{\alpha \in \mathfrak{A}} \mu_\alpha \right) = 1 \right\} \geq 1 - \varepsilon$$

This result can be made even stronger[49]. Suppose that the decision-maker still follows the rule given by (3.3), but in some periods he becomes ambitious and sets his aspiration level to a number exceeding the highest average by a constant $h > 0$. Then, independently of the initial aspiration level, he will behave in the limit like an expected utility maximizer with probability 1. Formally, let $N \subset \mathbb{N}$ be the set of periods, in which the decision-maker is ambitious. Then the rule he applies to update his aspiration level can be stated as follows:

$$\begin{array}{l} \bar{u}_1 = \bar{u}_1 \\ \bar{u}_t = \beta \bar{u}_{t-1} + (1-\beta) X_t \text{ for } t \geq 2,\ t \notin N \\ \bar{u}_t = X_t + h \text{ for } t \geq 2,\ t \in N \end{array} \tag{3.4}$$

In this case, the set of possible decision paths becomes:

$$\Phi_1 = \Phi_1(\beta; \bar{u}_1; N; h) = \left\{ \omega \in S_1 \mid \begin{array}{l} \bar{u}_1 = \bar{u}_1 \text{ and} \\ \bar{u}_t = \beta \bar{u}_{t-1} + (1-\beta) X_t \text{ for } t \geq 2,\ t \notin N \\ \bar{u}_t = X_t + h \text{ for } t \geq 2,\ t \in N \end{array} \right\}$$

Theorem 3.2 *Gilboa and Schmeidler (1996, p.12). Let there be given* $\mathfrak{A} = \{\alpha_1 ... \alpha_n\}$*,* $(\Pi_\alpha)_{\alpha \in \mathfrak{A}}$*,* $\beta \in (0;1)$ *as above,* $N \subset \mathbb{N}$*,* $h > 0$ *and* $\bar{u}_1 \in \mathbb{R}$*. If* N *is infinite but sparse, then for each prob-*

[49] The necessary conditions, which insure that expected utility maximizing obtains in the limit, are: (i) the convergence of the aspiration level to the maximal mean; (ii) a relatively slow convergence, Gilboa and Schmeidler (1996, p. 14).

ability measure P on $(\Phi_1(\beta; \bar{u}_{1;} N; h); \Sigma(\beta; \bar{u}_1; N; h))$, consistent with $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$

$$P\left\{\omega \in \Phi_1 \mid \exists\pi\left(\arg\max_{\alpha\in\mathfrak{A}} \mu_\alpha\right) = 1\right\} = 1$$

Hence, Gilboa and Schmeidler (1996) state two possible updating rules, for which the behavior of a case-based decision-maker with high probability conforms to the expected utility hypothesis in the limit. An investor, faced with the portfolio-choice problem from above, will behave with high probability (or even almost surely) as if he were maximizing expected utility with frequency 1 in the limit, as long as he is following (3.3) or (3.4). The relatively high initial aspiration level or the adaptation of the initial aspiration level upwards insure that all acts will be tried a sufficient number of times, so that the utility realizations observed by the decision-maker are representative of the underlying stochastic process. Hence, by starting with no information at all, the investor will, in the end, have the same (or almost the same) information as an informed expected utility maximizer, but in a somewhat different form — instead of a distribution, he will have a set of raw data, with almost the same distribution as the one perceived by the expected utility maximizer. Note that the case-based decision-maker need know nothing about objective (or subjective) distributions, nor have any knowledge in statistics. He does not concern himself with the existence of objective probabilities or the fact, whether or not they vary over the time. Neither does he pay any attention to possible autocorrelation in his set of data. However, given that the probability distributions $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$ have the properties given above, he will act as if he were informed about them.

The two updating rules (3.3) and (3.4) are interpreted by Gilboa and Schmeidler (1996, p. 2-3) as both ”realistic” and ”ambitious”. In this context, ”realism” means that the decision-maker adapts his aspiration level towards the average of the already observed utility realizations. Thus, he increases or decreases his aspiration level depending on whether the observed realizations exceed or fall short of his expectations. The term ”ambitiousness” refers either to the fact that the initial aspiration level is set high, or to the idea that in some periods the aspiration level is increased by a constant. This ambitiousness insures that the decision-maker will not choose a suboptimal act forever, without trying a better one. The combination of ”realism” and ”ambitiousness” guarantees that the aspiration level eventually converges to $\max_{\alpha\in\mathfrak{A}} \mu_\alpha$. Therefore, since only the act with maximal mean utility is perceived satisfactory in the limit, the decision-maker chooses it with frequency 1.

Now, I turn to explore the limit behavior of a decision-maker, who applies different rules for updating his aspiration level.

3.4 The Case of Constant Aspiration Level

In this section, the case of a constant aspiration level is analyzed. The results concerning the case of a relatively high aspiration level, stated in propositions 3.1 and 3.2, are special cases of the more general theorem, proved in Gilboa and Pazgal (2001, p. 125). Since, however, the argument of the proof seems to be important for understanding and interpreting other results as well, I state these as separate propositions and deliver a proof which differs from the one in Gilboa and Pazgal (2001).

Consider a decision-maker acting in accordance with the case-based decision theory, whose aspiration level remains constant over time. In this case, the space of possible decision paths reduces from S_1 to:

$$\Phi_2 = \Phi_2\left(\bar{u}_1\right) = \left\{\omega \in S_1 \mid \quad \bar{u}_t = \bar{u}_1 \text{ for all } t \in \mathbb{N} \ \right\}$$

for some $\bar{u}_1 \in \mathbb{R}$.

Define a probability distribution P on Φ_2 to be consistent with $\left(\Pi_\alpha\right)_{\alpha\in\mathfrak{A}}$ as a distribution, which induces the same probability on u_t as Π_α given that $\alpha_t = \alpha$ is chosen, in such a way that u_t is independent of the memory up to time t. The next propositions state that it the relative position of $\bar{u}_1$ to the maximal mean utility $\max_{\alpha\in\mathfrak{A}} \mu_\alpha$ determines the relative frequencies of choices, implied by this distribution.

In the first proposition, $\bar{u}_1$ is set sufficiently high, so that all possible returns achieved when choosing an act are negative. This is possible, because of the assumption that the supports of the returns are bounded. Denote by

$$\Phi_2' = \Phi_2'\left(\bar{u}_1\right) = \left\{\omega \in S_1 \mid \quad \bar{u}_t = \bar{u}_1 > u \text{ for all } u \in \Lambda \text{ and all } t \in \mathbb{N} \ \right\}$$

the set of possible decision paths.

Proposition 3.1 *Let the constant aspiration level $\bar{u}_1$ of a decision-maker exceed all possible utility realizations of the available acts, i.e. $u < \bar{u}_1$ for all $u \in \Lambda$. Then, each probability distribution P on Φ_2' consistent with $\left(\Pi_\alpha\right)_{\alpha\in\mathfrak{A}}$ satisfies:*

$$P\left\{\omega \in \Phi_2' \mid \forall \alpha, \tilde{\alpha} \in \mathfrak{A}\ \exists \pi\left(\alpha\right) \text{ and } \pi\left(\tilde{\alpha}\right), \text{s.t. } \frac{\pi\left(\alpha\right)}{\pi\left(\tilde{\alpha}\right)} = \frac{\bar{u}_1 - \mu_{\tilde{\alpha}}}{\bar{u}_1 - \mu_\alpha}\right\} = 1.$$

Next, it is shown that the same result also holds if the mean utilities of the acts do not exceed the aspiration level. Let Φ_2'' be the corresponding set of possible decision-paths:

$$\Phi_2'' = \Phi_2''(\bar{u}_1) = \left\{\omega \in S_1 \mid \quad \bar{u}_t = \bar{u}_1 > \mu_\alpha \text{ for all } \alpha \in \mathfrak{A} \text{ and all } t \in \mathbb{N} \ \right\}.$$

Proposition 3.2 *Let the constant aspiration level $\bar{u}_1$ of the decision-maker exceed the mean utilities of all the acts available, i.e. $\mu_\alpha < \bar{u}_1$ for all $\alpha \in \mathfrak{A}$. Then each probability distribution P on Φ_2'', consistent with $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$ satisfies:*

$$P\left\{\omega \in \Phi_2'' \mid \forall\alpha, \tilde{\alpha} \in \mathfrak{A}\ \exists\pi(\alpha) \ \textit{and}\ \pi(\tilde{\alpha}), \textit{s.t.}\ \frac{\pi(\alpha)}{\pi(\tilde{\alpha})} = \frac{\bar{u}_1 - \mu_{\tilde{\alpha}}}{\bar{u}_1 - \mu_\alpha}\right\} = 1.$$

A decision-maker with a constant, but relatively high aspiration level will not be satisfied with any of the acts available to him. Hence, (as long as the act of doing nothing is not considered), in the limit all available acts are chosen with positive frequency. Although the decision-maker eventually acquires enough observations from each act, so as to be able to infer their distributions, he uses this information in a different manner than an expected utility maximizer. Instead of choosing the act with the highest mean utility, he chooses all acts with a frequency inversely proportional to their "net"-mean utility, i.e. the difference between the mean utility and the aspiration level. In the portfolio choice problem, this means that the investor most frequently chooses the portfolio which yields the highest mean utility, but, at times, he also chooses inferior portfolios (or even portfolios dominated in the sense of zero-order stochastic dominance).

Now suppose that the aspiration level becomes even lower, so that at least one act has a mean utility exceeding the aspiration level. Again, define the corresponding set of decision paths as:

$$\Phi_2''' = \Phi_2'''(\bar{u}_1) = \left\{\omega \in S_1 \mid \quad \bar{u}_t = \bar{u}_1 < \mu_\alpha \text{ for some } \alpha \in \mathfrak{A} \text{ and all } t \in \mathbb{N} \ \right\}.$$

Then the following result applies:

Proposition 3.3 *Let the constant aspiration level $\bar{u}_1$ lie below the mean utility μ_α of at least one of the acts α. Write $\tilde{A}$ for the set of acts, for which this condition is satisfied. Suppose as well that $\mu_\alpha - \bar{u}_1 \neq 0$ holds for each $\alpha \in \mathfrak{A}$. Then, on almost all paths ω, one of the acts from $\tilde{A}$ is chosen with frequency 1 in the limit :*

$$P\left\{\omega \in \Phi_2''' \mid \ \exists\, \alpha(\omega) \in \tilde{A} \ \textit{such that}\ \pi(\alpha(\omega)) \ = 1 \ \textit{on}\ \omega\right\} = 1.$$

For completeness, the obvious result for the case when all realizations of all possible acts exceed the aspiration level is also stated:

$$\Phi_2^{IV} = \Phi_2^{IV}(\bar{u}_1) = \left\{\omega \in S_1 \mid \quad \bar{u}_t = \bar{u}_1 < u \text{ for all } u \in \Lambda \text{ and all } t \in \mathbb{N} \ \right\}$$

denotes the corresponding set of decision paths.

Proposition 3.4 *Let the constant aspiration level* $\bar{u}_1$ *lie below all possible realizations of the available acts, i.e.* $\bar{u}_1 < u$ *for all* $u \in \Lambda$*. Then, on almost all paths* ω*, one of the acts* $\alpha \in \mathfrak{A}$ *is chosen with frequency* 1 *in the limit:*

$$P\left\{\omega \in \Phi_2^{IV} \mid \exists\, \alpha(\omega) \in \mathfrak{A} \text{ such that } \pi(\alpha(\omega)) = 1 \text{ on } \omega\right\} = 1.$$

Given that the initial aspiration level is sufficiently low, on almost every path ω one of the acts will be chosen with frequency 1 in the limit. However, it is not necessarily the act with the highest mean utility. Moreover, on two different paths of return realizations, the limit choices of the decision-maker may differ. Indeed, the acts with mean utility below the aspiration level are chosen with frequency 0, but this does not guarantee optimality in the long run, as there is a positive probability that the act with the highest mean utility might never be chosen. In the context of portfolio-selection this means that the investor may be constantly choosing an inefficient portfolio, or even a dominated portfolio, thus leaving arbitrage possibilities unused.

3.5 Max-Min Updating Rules

Suppose now that instead of keeping his aspiration level constant over time, the investor updates it in accordance with the utility realizations he observes. If the investor is optimistic, he may, for instance, believe that in the future he will continue to achieve the best result he has observed up to the present. One could alternatively interpret this kind of adaptation of the aspiration level as a self-attribution bias, the tendency of people to ascribe their failures to bad luck and their successes to their personal abilities[50]. An investor acting in this way will then think that the high returns he has observed are due to his ability to pick the right assets at the right time, whereas the low returns are due to chance. High payoffs, therefore, increase the expectations of such an investor (as captured by his aspiration level) and lower his evaluation of future returns. The rule capturing this phenomenon is given by:

$$\bar{u}_t = \left\{ \begin{array}{l} \bar{u}_1 < \max_{u \in \Lambda} u, \text{ for } t = 1 \\ \max\left\{\bar{u}_1; \max_{\tau < t} u_\tau\right\}, \text{ for } t > 1, \end{array} \right\}, \tag{3.5}$$

i.e. the decision-maker sets his aspiration level equal to the maximal utility achieved up to period t[51]. Write

$$\Phi_3 = \left\{\omega \in S_1 \mid \bar{u}_t = \left\{ \begin{array}{l} \bar{u}_1 < \max_{u \in \Lambda} u, \text{ for } t = 1 \\ \max\left\{\bar{u}_1; \max_{\tau < t} u_\tau\right\}, \text{ for } t > 1, \end{array} \right\}\right\}$$

[50] See Miller and Ross (1975) and Langer and Roth (1975) for a description and analysis of the self-attribution bias.

[51] If $\bar{u}_1$ exceeds the upper boundary of Λ, no updating of the aspiration level takes place and, consequently, the result of proposition 3.1 holds.

for the set of possible decision paths consistent with this updating rule. Then (again defining a probability measure P on Φ_3, consistent with $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$) the following result applies:

Proposition 3.5 *Consider a decision-maker who updates his aspiration level according to (3.5). Assume that $\sigma_\alpha > 0$ for all $\alpha \in \mathfrak{A}$. Then, each probability distribution P on Φ_3 consistent with $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$ satisfies:*

$$P\left\{\omega \in \Phi_3 \mid \forall\alpha, \tilde{\alpha} \in \mathfrak{A}\ \exists\pi(\alpha) \textit{ and } \pi(\tilde{\alpha}), \textit{s.t. } \frac{\pi(\alpha)}{\pi(\tilde{\alpha})} = \frac{\max_{u\in\Lambda} u - \mu_{\tilde{\alpha}}}{\max_{u\in\Lambda} u - \mu_\alpha}\right\} = 1.$$

Alternatively, one can consider a pessimistic investor, who sets his aspiration level at the lowest possible level of utility achieved until the current period. Unlike the self-confident investors, this investor believes that his own mistakes are the reason for low profits, whereas the high returns are only due to chance[52]. His adaptation rule is captured by:

$$\bar{u}_t = \left\{\begin{array}{l} \bar{u}_1, \text{ for } t = 1 \\ \min\{\bar{u}_1; \min_{\tau<t} u_\tau\}, \text{ for } t > 1, \end{array}\right\}. \tag{3.6}$$

No matter how high the initial aspiration level of such an investor might be, he will always end up choosing one act with frequency one in the limit. Let

$$\Phi_4 = \left\{\omega \in S_1 \mid \bar{u}_t = \left\{\begin{array}{l} \bar{u}_1 > \min_{u\in\Lambda} u, \text{ for } t = 1 \\ \min\{\bar{u}_1; \min_{\tau<t} u_\tau\}, \text{ for } t > 1, \end{array}\right\}\right\}$$

denote the set of consistent decision paths[53].

Proposition 3.6 *Consider a decision-maker, who updates his aspiration level according to (3.6). Then, each probability distribution P on Φ_4, consistent with $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$ satisfies:*

$$P\{\omega \in \Phi_4 \mid \exists\, \alpha(\omega) \in \mathfrak{A}, \textit{ such that } \pi(\alpha(\omega)) = 1 \textit{ on } \omega\} = 1$$

The updating rules (3.5) and (3.6) represent extreme cases of very optimistic and very pessimistic decision-makers. It is also of interest to consider an intermediate case, in which the investor takes into account the minimum, as well as the maximum achieved. His pessimism is captured by the weight β he assigns to the minimal realization achieved and his optimism by the weight $(1-\beta)$ of the best realization. The cases $\beta = 0$ and $\beta = 1$ correspond then to the updating rules (3.5) and (3.6), respectively. Then for sufficiently low values of β, the decision-maker chooses all acts with positive frequency in the limit, whereas for β near 1, satisficing behavior emerges.

[52] Sociologists have found that men and women explain their misfortunes differently. Whereas men tend to explain their mistakes by bad luck and their successes by their own abilities, women seem to do the contrary. This is probably also the reason why most of the overconfident investors are men, as Barber and Odean (2001 (b)) find.

[53] If $\bar{u}_1$ lies below the lower boundary of Λ, then no updating of the aspiration level takes place and, consequently, the result of proposition 3.4 holds.

Proposition 3.7 *Let $\sigma_\alpha > 0$ for all $\alpha \in \mathfrak{A}$. Consider the adaptation rule*

$$\begin{array}{l} \bar{u}_1 = \bar{u}_1 \in \left[\min\limits_{u \in \Lambda} u; \max\limits_{u \in \Lambda} u\right] \\ \bar{u}_t = \beta \min\left\{\bar{u}_1; \min\limits_{\tau<t} u_\tau\right\} + (1-\beta)\max\left\{\bar{u}_1; \max\limits_{\tau<t} u_\tau\right\} \text{ for } \forall t \geq 2 \end{array}. \tag{3.7}$$

and denote by Φ_5 the set of consistent decision paths. Then for all values of $\beta \in [0;1]$ such that:

$$\beta < \frac{\max\limits_{u \in \Lambda_\alpha} u - \mu_\alpha}{\max\limits_{u \in \Lambda_\alpha} u - \min\limits_{\alpha \in \mathfrak{A}} \min\limits_{u \in \Lambda_\alpha} u} \text{ for each } \alpha \in \mathfrak{A}, \tag{3.8}$$

each probability distribution P on Φ_5 consistent with $(\Pi_\alpha)_{\alpha \in \mathfrak{A}}$ satisfies:

$$P\left\{\omega \in \Phi_5 \mid \frac{\pi(\alpha')}{\pi(\alpha'')} = \frac{\beta \min\limits_{\alpha \in \mathfrak{A}} \min\limits_{u \in \Lambda_\alpha} u + (1-\beta)\max\limits_{\alpha \in \mathfrak{A}} \max\limits_{u \in \Lambda_\alpha} u - \mu_{\alpha'}}{\beta \min\limits_{\alpha \in \mathfrak{A}} \min\limits_{u \in \Lambda_\alpha} u + (1-\beta)\max\limits_{\alpha \in \mathfrak{A}} \max\limits_{u \in \Lambda_\alpha} u - \mu_{\alpha''}}\right\} = 1.$$

Thus, if the decision-maker is relatively optimistic, his aspiration level becomes sufficiently high in the limit and he switches infinitely often among all possible acts, with a frequency inversely proportional to the net mean utility of the act in the limit. It is however not true that for all values of β higher than those in (3.8), one act will be chosen with frequency 1 on almost each path ω. In fact, this will be the case only if

$$\beta > \frac{\max\limits_{\alpha \in \mathfrak{A}} \max\limits_{u \in \Lambda_\alpha} u - \mu_\alpha}{\max\limits_{\alpha \in \mathfrak{A}} \max\limits_{u \in \Lambda_\alpha} u - \min\limits_{\alpha \in \mathfrak{A}} \min\limits_{u \in \Lambda_\alpha} u} \text{ for some } \alpha \in \mathfrak{A}. \tag{3.9}$$

For values of β between (3.8) and (3.9), however, the frequencies with which the acts are chosen are path-dependent. For instance, even if β does not satisfy (3.9), it is possible that the act chosen in the first period yields a low utility realization, and, as a result, the aspiration level is updated downwards. The next act chosen may then turn out to be satisfactory (relative to the low aspiration level) and, thus, will be chosen forever. If, however, the first act chosen has a relatively high realization, then the aspiration level is updated upwards, possibly rendering all of the acts unsatisfactory, and causing permanent switching among them.

To summarize, the results of this and the preceding section show that high (or increasing over time) aspiration levels lead to permanent switching among the acts, whereas low (or decreasing over time) aspiration levels induce satisficing behavior in the limit.

3.6 Interpretation of the Results

In this section, the results derived above are compared to the empirical evidence from financial markets. It will be shown that some of the observed phenomena are consistent with the theoretical results obtained.

3.6.1 Choosing an Efficient Portfolio

The CAPM, suggested by Sharpe (1964) and Lintner (1965) generates strong predictions not only with respect to portfolio holdings but also with respect to asset prices in equilibrium. The early empirical studies in this area have been sharply criticized by Roll (1977), who argued that only the mean-variance-efficiency of the chosen portfolios, and not their optimality could be tested. Kroll, Levy and Rappoport (1988), thus conduct an experiment which is designed to test whether subjects choose mean-variance-efficient portfolios if mean, variance and covariances of the available risky assets are known. The experiment is designed in a way which excludes any serial correlation between the returns of the assets. This makes the comparison between the experimental and the theoretical results particularly simple. Kroll, Levy and Rappoport (1988, p. 509) refute the following three hypothesis:

(a) Changes in the covariance structure affect the choice of a portfolio;

(b) Allowing subjects to borrow and lend at the same rate leads to a movement in the direction of a common, optimal risky portfolio;

(c) The chosen portfolios become more efficient, as subjects gain more experience and obtain feedback.

Hence, the behavior observed is inconsistent with the predictions of the portfolio-choice theory, as formulated by Markowitz (1952, 1959) and Tobin (1958) and, therefore, ultimately with the CAPM on which it is based. The results observed in the experiment are however consistent with the behavior of case-based decision-makers with constant low aspiration level.

First, the proofs of propositions 3.3 and 3.4 show that the covariance between the assets does not play any role for the limit behavior of the decision-maker. This is due to the fact that the result of the portfolio chosen in a certain period is evaluated independently of the realizations of the portfolios which were not chosen in this period.

Note also that for different subjects with the same aspiration level, the choice of portfolio in the limit need not be the same, even if borrowing and lending at the same rate is allowed. As has already been mentioned, the limit behavior in the case of low aspiration level depends strongly on the first periods, in which the behavior of the investors is guided by chance. Thus, even identical subjects may end up with different (suboptimal) portfolios of risky assets in the limit.

As stated above, a case-based investor with a relatively low aspiration level will not choose an optimal portfolio in general, even if he is allowed to repeat his choice infinitely often. This result is due to the fact that the feedback obtained is interpreted as positive, relative to the aspiration level and is consistent with the falsification of hypothesis (c).

3.6.2 Arbitrage Restrictions

A rather ”strange” feature of the case-based decision-makers with relatively low aspiration level is that they may choose a portfolio which is strictly dominated by another portfolio in the set of the possible acts with frequency 1 in the limit. Thus, a market in which case-based investors operate would not be safe from arbitrage. Empirical data show, see, for instance, Figlewski (1989) that unpredictable volatility, indivisibility of assets, transaction costs and imperfect financial markets can render arbitrage strategies too costly. However, neither of these elements drives the results in the model above. Nevertheless, some experimental studies demonstrate that arbitrage opportunities may occur even in perfect financial markets.

Rietz (1998) constructs an experimental market with two states of the world and no aggregate uncertainty. He finds, see Rietz (1998, p. 2), that ”[a]arbitrage opportunities were prevalent and extremely robust. Exactly why traders did not exploit the opportunities remains a mystery. They simply did not take advantage of profitable opportunities even when they had been shown how, when they had considerable experience, when they could also engage in direct portfolio trading and when they could sell short.” Even though the set-up is quite simple and the subjects are literally taught how to exploit arbitrage possibilities, a professional arbitrageur is needed to draw the prices to their arbitrage-free values.

In another empirical study by Oliven and Rietz (1995), the non-arbitrage conditions on prices are tested in the context of a futures-market. Again, the result is that investors fail to use even the simplest possibilities for arbitrage. The highest number of violations — 37,7%, Oliven and Rietz (1995, p. 2) — is observed in the behavior of the so-called price-takers (investors who accept an outstanding offer or bid), whereas the price-makers (those who set offers or bids) fail to use an arbitrage possibility in only 5,39% of the time.

Whereas the observed behavior is inconsistent with expected utility maximization, case-based investors with relatively low aspiration levels might indeed fail to use arbitrage possibilities. As long as they consider the profits achieved in the past satisfactory, they do not consider it necessary to experiment with different portfolios and engage in arbitrage activities.

An interesting result is that in the experiment of Oliven and Rietz (1995) the number of violations of the non-arbitrage condition increases with the income class reported. This might be explained by the findings of Gould (1941), who examines the dependence of aspiration level on socio-economic factors. His results show that aspiration levels decrease with more favorable social and economic position. Hence, high-income investors might have less incentives to experiment and might therefore be more viable to violations of non-arbitrage conditions.

The reason for not using arbitrage possibilities in this case is due to the fact that the decision-maker is not informed about the existence of better acts. In section 7 it will be shown that supplying the decision-maker with additional information can lead to optimal behavior.

3.6.3 Diversification and Familiarity Effects

Even after asset markets have been largely globalized, investors seem not to understand and use the advantages of diversification. Tesar and Werner (1995) analyze aggregate trade with assets in several countries (USA, Canada, Great Britain) and show that the percentage of domestic assets held is higher than the optimal one. Coval and Moskowitz (1999) find a similar bias in the USA, where investors seem to prefer local firms to firms operating in other states and regions. Several possible explanations have been proposed in the literature, see Coval and Moskowitz (1999) and Lewis (1999). Transaction costs and cross-border frictions are the first to be considered, as in Black (1974) and Stulz (1981). However, Tesar and Werner (1995) show that taking them into account does not remove the effect completely. Lewis (1999) claims that insuring against home-risks (e.g. inflation-risk) cannot explain the home-bias, either. There is some evidence that the informational structure of the economy might be connected with the preferences to invest in home or foreign companies, see Grinblatt and Keloharju (2001), Kang and Stulz (1997) and Coval and Moskowitz (1999). They argue that since investors are better informed about local than foreign companies, it is rational for them to invest more in local companies[54]. It is, however,

[54] For models on assymetric information in this context see Brennan and Cao (1997) and Coval (1999).

unclear why information per se should lead to a buy-and-hold decision. Analyzing Japanese companies, Kang and Stulz (1997) find that those preferred by foreign investors are not only large (and thus better known and communicating more information), but also well-performing and low-risk.

Suppose then that before choosing a portfolio for the first time, the investor receives information about past returns of some of the assets available. Assuming that he is better informed about domestic and large foreign companies, he is more likely to receive information about the returns on their assets. As long as he considers the information to be equivalent to real experience (see section 7) and provided that his aspiration level is relatively low, he will prefer buying the portfolio consisting of the shares of those companies he has been informed about. For low values of the aspiration level (or, alternatively, relatively high returns), the cumulative utility of the portfolio chosen remains positive over the time. The investor will, therefore, continue to choose this particular asset forever, without considering others, especially if he has no (or only vague) information about their utility realizations.

Another explanation stated in the literature, e.g. Huberman (1998), is that investors exhibit some kind of preferences for local assets. It is well known that people prefer acts which are familiar to them. However, provided that the investor believes that one of the acts yields a higher profit, it is not quite clear why he should choose the more familiar and the more unprofitable one (ignoring patriotic feelings and wishes to invest in the domestic industry to support it). Thus, familiarity effects can be taken into account in those cases in which the acts are equally evaluated from the point of view of expected returns. Suppose, therefore, that a case-based decision-maker still evaluates the acts according to their cumulative utility in each period. If, however, two acts have the same cumulative utility, he chooses the one containing more familiar assets. Suppose as well that the decision-maker has no initial information about the returns of the assets. In this case, the cumulative utility of all the acts is 0 in period $t = 1$. The first choices will be therefore based on familiarity preferences. If, furthermore, the aspiration level of the investor is relatively low, this kind of behavior may lead him to choose an underdiversified portfolio containing a high share of (familiar) domestic assets.

To summarize, the assumptions of case-based decisions combined with a relatively low constant aspiration level can help explain, why investors fail to choose an efficient portfolio in the long run. These assumptions also imply that unexploited arbitrage possibilities might be present in the market and can describe situations in which investors neglect foreign assets and fail to diversify their portfolios optimally.

3.6.4 Excessive Trading

Up to this point, the implications of the case-based decision theory in the case of low aspiration level have been analyzed. Now consider the case of high aspiration level. Propositions 3.1 and 3.2 show that investors with a relatively high aspiration level switch among the acts in the long run, choosing all of them with positive probability. They violate optimality by trading more often than it would be rational, given correct beliefs. Odean (1999) finds that the frequency of trades is extremely high in financial markets. He shows that excessive trading is not eliminated even after controlling for liquidity demands, tax-loss selling and portfolio-rebalancing. It leads to lower returns for the investors engaging in it, which do not disappear, even if transaction costs are ignored. Odean calls such investors overconfident.

Case-based investors with high aspiration level also choose to trade more frequently than rational, thus lowering their returns. Whereas "overconfident" investors are usually modelled as decision-makers, who believe that the signals they receive about future returns are more precise, than they actually are, as in Hirshleifer, Subrahmanyam and Titman (1994), Hong and Stein (1999), Daniel, Hirshleifer and Subrahmanyam (1998, 2001), Odean (1998 (a)), the behavior of case-based decision-makers is only influenced by information about past returns. Their suboptimal behavior is not due to the fact that they lack information. In fact, choosing each act for an infinite number of times, they receive enough information in order to learn the distribution and the mean utility of each portfolio. The evaluation of the experienced payoffs as dissatisficing relative to the unrealistically high expectations captured by the aspiration level causes the excessive trading in this model.

As proposition 3.3 shows, the "optimal" constant aspiration level in the model should fulfill:

$$\max_{\alpha \in \mathfrak{A}} \mu_\alpha > \bar{u}_1 > \max_{\alpha \in \mathfrak{A} \setminus \{\arg\max \mu_\alpha\}} \mu_\alpha$$

Once it becomes larger than $\max_{\alpha \in \mathfrak{A}} \mu_\alpha$, the investor engages into excessive trading and lowers their utility from trade compared to the optimum. Note as well that the higher the aspiration level of the decision-maker, the lower his average utility in the long run becomes. As one can easily verify from propositions 3.1 and 3.2, it follows that, if $\bar{u}_1 \to \infty$, then the limit frequencies of all the acts are equal, since for all α and $\tilde{\alpha} \in \mathfrak{A}$:

$$\lim_{\bar{u}_1 \to \infty} \frac{\pi(\alpha)}{\pi(\tilde{\alpha})} = \lim_{\bar{u}_1 \to \infty} \frac{\mu_{\tilde{\alpha}} - \bar{u}_1}{\mu_\alpha - \bar{u}_1} = 1$$

holds. Hence, the acts with high mean utility will be chosen more frequently for smaller values of $\bar{u}_1$ than for higher aspiration levels, implying that increasing the aspiration level leads to lower mean utility.

Barber and Odean (2001 (b)) show that investors who have previously experienced superior performance tend to switch to on-line-trading and trade more actively, thus decreasing their profits. One of the possible reasons suggested by Barber and Odean (2001 (b), p. 6) is the self-attribution bias, the tendency of people to ascribe their failures to luck and their successes to their personal abilities. Propositions 3.5 and 3.7 show that the empirical findings of Barber and Odean (2001 (b)) can be reproduced by the behavior of case-based decision-makers with a linear utility function, who update their aspiration levels towards the highest utilities they have achieved in the past. The aspiration level becomes higher as the time passes and in the limit the investors trade too much, achieving lower mean profits, than would be optimal[55].

Shapira (2001) analyzes the observed behavior of government bond traders. He finds that bond traders who have achieved the goals set to them tend to trade less in order to take less risk. Traders who fail to fulfill the targets become more aggressive and trade more taking greater risks. Shapira explains his findings by the prospect theory of Kahneman and Tversky (1979). Propositions 3.5 and 3.7 provide an alternative explanation: if the trader perceives the acts he has tried as unsatisfactory with respect to his aspiration level (alias set goals), he will switch between the acts, until he finds a satisfactory one, or infinitely often, if no act can satisfy him. Having achieved his targets, he will then settle on the successful act and stop trading.

3.7 Collecting Additional Information

Up to this point, it has been assumed that the decision-maker receives no other information apart from the utility realizations of the acts he chooses. This assumption seems quite restrictive

[55] In this model prices and returns are exogenously given and therefore needn't coincide with their equilibrium values. Nevertheless, it can be shown, (see chapter 5) that traders with relatively high aspiration levels trade too much and achieve lower profits in a market environment with endogenous prices and returns as well. I thank Michael Waldman for this comment.

in the context of financial markets, where abundant information about past returns of assets is available. In this section, the case in which the decision-maker acquires additional information about the acts from the set $\mathfrak{A}$ is considered. Two assumptions are made: first, the information collected regards only past utility realizations; second, the decision-maker treats the information he receives as absolutely reliable and uses it when calculating the cumulative utility of an act as if he had chosen this act in some previous period and achieved the utility realization he has been informed about[56].

Consider a decision-maker who receives information about the realizations of all possible acts in each period of time $t = 1, 2, ...$ Some additional notation is necessary in order to discriminate between the knowledge of the decision-maker and his actual behavior. Let $k_t(\alpha)$ denote the information of the decision-maker about the realization of the act α in period $t \in \mathbb{N}$ and denote by

$$k_t = (k_t(\alpha_1) ... k_t(\alpha_n))$$

the information of the decision-maker about the realizations in period t (note that the realization of the act actually chosen in period t is also contained in the information about period t). Then the set of possible paths can be defined as:

$$S_0^I = \left\{ \omega = ((\bar{u}_t; \alpha_t; u_t; k_t))_{t=1,2...} \right\} \subset \left(\mathbb{R} \times \mathfrak{A} \times \Lambda \times (\times_{\alpha\in\mathfrak{A}}\Lambda_\alpha)\right)^{\mathbb{N}}$$

In this context, the cumulative utility of an act α in period t is defined as:

$$U_t^I(\alpha) = \sum_{\tau=1}^{t-1} [k_\tau(\alpha) - \bar{u}_t]$$

The decision-maker chooses in each period the act with maximal cumulative utility so that the set of possible paths reduces to:

$$S_1^I = \left\{ \omega \in S_0^I \mid \alpha_t \in \arg\max_{\alpha\in\mathfrak{A}} U_t^I(\alpha) \ \forall t \geq 1 \right\}.$$

Define a σ-algebra on S_0^I, i.e. the algebra generated by the Borel σ-algebra on each copy of $\mathbb{R}$ of Λ and of $(\times_{\alpha\in\mathfrak{A}}\Lambda_\alpha)$ and by the algebra $2^{\mathfrak{A}}$ on each copy of $\mathfrak{A}$. Let Σ be the induced σ-algebra on S_1^I. Let the probability distribution P on S_1^I be consistent with $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$. Again, the behavior of the decision-maker can be characterized by the frequencies with which the acts are chosen

56 The case-based decision theory of Gilboa and Schmeidler (1997 (a), 2001 (a)) allows for a more general treatment of the assessment of information. It is, for instance, possible to value personal experience more than the indirect, acquired by the information available, or vice versa. This would require the introduction of a similarity function between information and experience. The analysis of sections 3 – 5 has been carried through with the implicit assumption that the similarity between direct and indirect experience is 0. In the current section, it is assumed that this similarity is 1.

when the number of periods becomes infinitely large.

Proposition 3.8 *Consider a decision-maker who in each period of time receives the information about the utility realizations of all acts $\alpha \in \mathfrak{A}$ in this period. Then, independently of his aspiration level $(\bar{u}_t)_{t=1,2,\ldots}$, on almost all paths of return realizations he will choose the act with the highest mean utility with frequency 1:*

$$P\left\{\omega \in S_1^I \mid \exists\pi\left(\arg\max_{\alpha\in\mathfrak{A}} \mu_\alpha\right) = 1\right\} = 1.$$

The result of the proposition states that a case-based decision-maker who possesses and is able to process the whole past information about the problem he is facing behaves as an expected utility maximizer in the limit. Moreover, this result depends neither on the value of the aspiration level, nor on the way it is updated.

Note that although the case-based decision-maker eventually learns to choose the optimal portfolio, his learning differs from the Bayesian one. It is possible to elicit the "beliefs" of the case-based decision-maker in the following way: since the cumulative utility of an act α is given by

$$U_t^I(\alpha) = \sum_{\tau=1}^{t-1} [k_\tau(\alpha) - \bar{u}_t],$$

the decision-maker obviously acts in a frequentist way, ascribing to each act those possible realizations which he has actually observed and weighting them by the frequency with which they have occurred.. The difference of this approach from the Bayesian learning is two-fold. First, instead of starting with a prior on the set of possible outcomes and updating it to eliminate outcomes which never occur, the case-based decision-maker starts with the "prior" that each act yields a utility realization equal to his aspiration level $\bar{u}_1$. After the first period, the decision-maker observes the utility realizations of all portfolios and "updates" his beliefs in a non-Bayesian manner, since now he *adds* to his "prior beliefs" the possibility that an act α yields the utility realization $k_1(\alpha)$. He assigns this realization a probability of 1 in the evaluation of the act. Note that in this period he assigns a probability of 0 to any realization different from $k_1(\alpha)$, which cannot happen to a Bayesian who starts with a correct prior and, therefore, never assigns a 0 probability to a set with a positive objective probability.

The assumption of proposition 3.8 suggests that the decision-maker is able to process and acquire unlimited amount of information without incurring costs. While it is not quite realistic to imagine that a decision-maker collects the whole information available in every single period of time, collecting information for few periods may be a very realistic assumption, if one considers an investor who has not faced the problem before. Before choosing a portfolio for the first time, he may try to gather information about the realizations single portfolios had in the near past. However, once he becomes more familiar with the problem, he might begin to rely on his experience and stop collecting further information.

In the next two propositions, the case of an investor who receives information about all realizations only for a finite number of periods $\mathcal{K}$ is analyzed. After period $\mathcal{K}$, the decision-maker receives only information about the utility realizations of the acts he actually chooses. Assume, as above that information and direct experience are treated equally when evaluating the acts. Using the notation from above define the new set of possible decision paths as:

$$\begin{aligned} S_0^K &= \left\{\omega = \left(((\bar{u}_t;\alpha_t;u_t;k_t))_{t=1,\ldots\mathcal{K}};((\bar{u}_t;\alpha_t;u_t))_{t=\mathcal{K}+1,\mathcal{K}+2\ldots}\right)\right\} \subset \\ &\subset \left(\mathbb{R}\times\mathfrak{A}\times\Lambda\times\left(\times_{\alpha\in\mathfrak{A}}\Lambda_\alpha\right)\right)^{\mathcal{K}}\times\left(\mathbb{R}\times\mathfrak{A}\times\Lambda\right)^{\mathbb{N}/\{1\ldots\mathcal{K}\}} \end{aligned}$$

In this case the cumulative utility is defined as:

$$U_t^K(\alpha) = \left\{ \begin{array}{ll} \sum_{\tau=1}^{t-1}\left[k_\tau(\alpha)-\bar{u}_t\right] & \text{for } t \le \mathcal{K} \\ \sum_{\tau=1}^{\mathcal{K}}\left[k_\tau(\alpha)-\bar{u}_t\right] + \sum_{\tau\in C_t^{\mathcal{K}}(\alpha)} u_\tau & \text{for } t > \mathcal{K} \end{array} \right\}$$

with $C_t^{\mathcal{K}}(\alpha)$ denoting the set of periods between period $\mathcal{K}$ and period t, in which the act α has been chosen:

$$C_t^{\mathcal{K}}(\alpha) = \{\mathcal{K} < \tau < t \mid \alpha_\tau = \alpha\}.$$

Consider first the case of constant aspiration level, i.e. $\bar{u}_t = \bar{u}_1$ for all $t = 1, 2, \ldots$. The set of possible decision paths then reduces to:

$$S_1^K(\mathcal{K};\bar{u}_1) = \left\{\omega \in S_0^K \mid \begin{array}{l} \alpha_t \in \arg\max_{\alpha\in\mathfrak{A}} U_t^K(\alpha)\ \forall t \ge 1 \\ \bar{u}_t = \bar{u}_1\ \forall t \ge 1 \end{array} \right\}.$$

Define a probability measure P on S_1^K, consistent with $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$. The first result shows that if a decision-maker with a relatively low aspiration level receives information for a sufficiently large number of periods $\mathcal{K}$, then he will be rational with probability arbitrarily close to 1.

Proposition 3.9 *Consider a decision-maker with a constant aspiration level $\bar{u}_1$ such that the "net" mean utility of all acts in some subset of $\mathfrak{A}$ — $\tilde{A}$ is positive, i.e.:*

$$\mu_\alpha - \bar{u}_1 > 0, \text{ if } \alpha \in \tilde{A}.$$

Let the decision-maker receive information about the utility realizations of all acts for T periods. Then, for each $\varepsilon > 0$, there exists an integer $\mathcal{K}$, such that

$$P\left\{\omega \in S_1^K(T;\bar{u}_1) \mid \exists\pi\left(\arg\max_{\alpha\in\mathfrak{A}}\mu_\alpha\right) = 1\right\} \ge 1-\varepsilon$$

for all $T \ge \mathcal{K}$.

Arbitrage possibilities and underdiversification are eliminated with a high probability in a market in which investors with relatively low aspiration level are allowed to learn. This is, however, not true when investors with high aspiration levels are considered.

Proposition 3.10 *Consider a decision-maker with a constant aspiration level $\bar{u}_1$, such that the "net" mean utility of all acts $\alpha \in \mathfrak{A}$ is negative, i.e.:*

$$\mu_\alpha - \bar{u}_1 < 0, \text{ for all } \alpha \in \mathfrak{A}.$$

Let the decision-maker receive information about all the realizations of the acts for $\mathcal{K}$ periods. Then for all finite $\mathcal{K}$

$$P\left\{\omega \in S_1^K\left(\mathcal{K};\bar{u}_1\right) \mid \forall\alpha,\tilde{\alpha} \in \mathfrak{A}\, \exists\pi\left(\alpha\right) \text{ and } \pi\left(\tilde{\alpha}\right), \text{s.t. } \frac{\pi\left(\alpha\right)}{\pi\left(\tilde{\alpha}\right)} = \frac{\bar{u}_1 - \mu_{\tilde{\alpha}}}{\bar{u}_1 - \mu_\alpha}\right\} = 1$$

holds.

The result of the proposition is not due to the fact that the investor has not enough information about the possible realizations and their distributions. Indeed, if $\mathcal{K}$ is sufficiently large, then at period $\mathcal{K}$ the decision-maker chooses the optimal act with probability is close to 1. However, since even the best act seems to him unsatisfactory, after period $\mathcal{K}$, the investor will switch among all available acts as in the case in which he cannot gather additional information. Thus, we should not expect that investors with high aspiration levels learn to behave optimally, unless they are able to proceed the whole past information available in the market.

In an analogous way it is possible to describe the behavior of a decision-maker who updates his aspiration level to the maximal utility achieved in the past.

Proposition 3.11 *Consider a decision-maker with an aspiration level updated according to*[57]

$$\bar{u}_t = \left\{\begin{array}{l} \bar{u}_1 < \max_{u\in\Lambda} u, \text{ for } t=1 \\ \max\left\{\max\limits_{\alpha\in\mathfrak{A}} k_{t-1}\left(\alpha\right);\bar{u}_{t-1}\right\}, \text{ for } 1 < t \leq \mathcal{K}+1 \\ \max\left\{u_{t-1};\bar{u}_{t-1}\right\}, \text{ for } t > \mathcal{K}+1 \end{array}\right\}. \tag{3.10}$$

Denote the set of decision-paths consistent with this updating-rule by $S_2^K\left(\mathcal{K}\right)$. Then each probability distribution P on $S_2^K\left(\mathcal{K}\right)$, which is consistent with $\left(\Pi_\alpha\right)_{\alpha\in\mathfrak{A}}$, satisfies:

$$P\left\{\omega \in S_2^K\left(\mathcal{K}\right) \mid \forall\alpha,\tilde{\alpha} \in \mathfrak{A}\, \exists\pi\left(\alpha\right) \text{ and } \pi\left(\tilde{\alpha}\right), \text{s.t. } \frac{\pi\left(\alpha\right)}{\pi\left(\tilde{\alpha}\right)} = \frac{\max_{u\in\Lambda} u - \mu_{\tilde{\alpha}}}{\max_{u\in\Lambda} u - \mu_\alpha}\right\} = 1.$$

If we think of investors using the updating rule (3.10) as being subjected to the self-attribution bias, we again reach the conclusion that investors who increase their expectations to unrealistically high levels switch more often between the acts than it would be optimal and lose money in

[57] The initial value of the aspiration level $\bar{u}_1$ does not influence the behavior of the investor in the limit, as long as it is below the upper boundary of Λ. If it exceeds this upper boundary, then no updating of the aspiration level takes place and the result is as in proposition 3.10.

general. Hence, even in the case in which information about long return sequences is available, the behavior of these investors is similar to the behavior of "overconfident" investors described by Odean (1999). Barber and Odean (2001 (b), p. 7) relate the overconfidence of online investors to the fact that "[o]nline investors have access to vast quantities of investment data". The abundant information makes the traders believe that they have better skills than the average investor and trade more than it would be rational. The last proposition shows that if a trader suffers from the self-attribution bias, (in the sense that he ascribes his success to his own ability, and losses to chance), additional information will not help him to learn to behave rationally.

3.8 Portfolio Choice with Similarity Considerations

Up to now, I have assumed that the utility yielded by a portfolio does not influence the evaluation of the portfolios not chosen in the given period. It is, however, possible that the utility achieved by choosing one portfolio affects the evaluation of another one because of similarity perceptions.

In chapter 2, the issue of introducing similarity perceptions into a model of financial markets has been discussed. I now present a model of individual portfolio choice in which similarity considerations are integrated. By using this model, the impact of the form of the similarity function on the individual decision-making can be examined. The results are then compared to those obtained without similarity considerations and to the results of the standard theory, which assumes a fully informed expected utility maximizer.

Differently from the analysis above, the introduction of a similarity function allows to consider an infinite number of available acts. Suppose, therefore, that $\mathfrak{A} = [0;1]$. Hence, diversification is allowed, but short-sales are forbidden.. Assume that the similarity function satisfies[58]

$$\begin{aligned} s(\alpha;\alpha) &= 1 \\ s(\alpha;\alpha') &= s(\alpha';\alpha) \\ s(0;1) &\in [0;1) \end{aligned}$$

for all α, $\alpha' \in [0;1]$. Let s further depend only on the distance between α and α' and assume that it is strictly decreasing and concave in $\|\alpha - \alpha'\|$. Figure 2 provides an example of such a

[58] Gilboa and Schmeidler (2001 (a), p.144) comment that the second characteristic of the similarity function required in the model, symmetry, is not natural in some applications. Moreover, they define a different notion of symmetry in the context of their model. However, since in this model the *distance* between the portfolios determines the similarity, the similarity function is assumed to by symmetric.

similarity function.

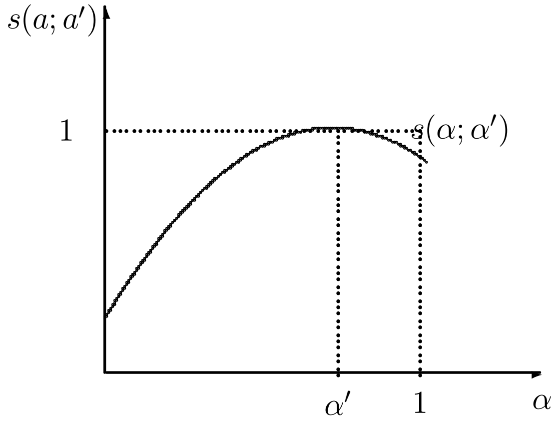


Figure 2

Recall that the memory of the investor at time t is represented by the set of cases encountered until period t:

$$M_t = ((\rho_\tau; \alpha_\tau; u_\tau))_{\tau=1,2...t-1}\,.$$

M_t is assumed to be endogenously determined at each time t, i.e. M_t contains only cases with portfolios actually chosen and utility realizations actually observed by the investor.

3.8.1 The Case of Constant Aspiration Level

Assume first that the aspiration level $\bar{u}$ of the investor is constant over the time. Given a problem ρ and a memory of length t, the decision rule of a case-based decision-maker consists in choosing the act with maximal cumulative utility $U_t\,(\alpha)$ with:

$$U_t\,(\alpha) = \sum_{\tau=1}^{t-1} s\,(\alpha; \alpha_\tau)\,[u_\tau - \bar{u}]\,.$$

Let $\alpha_1 = \bar{\alpha}$ denote the act chosen in the first period and assume that $\bar{\alpha}$ is a strictly diversified portfolio, i.e. $\bar{\alpha} \in (0;1)$. Φ_6 describes the set of possible paths:

$$\Phi_6 = \left\{ \omega \in S_1 \mid \begin{array}{l} \bar{u}_t = \bar{u} \text{ for all } t = 1, 2, ... \\ \alpha_1 = \bar{\alpha} \end{array} \right\}.$$

Let P denote a probability measure consistent with $(\Pi_\alpha)_{\alpha \in [0;1]}$ as described in section 3. It is easily seen that:

Lemma 3.1 *If $\bar{u} < \mu_{\bar{\alpha}}$, then the expected time, for which the investor will hold $\bar{\alpha}$ is infinite. If $\bar{u} > \mu_{\bar{\alpha}}$, then the investor will almost surely switch in finite time to a different act α with $\alpha = 0$ for $\bar{\alpha} > \frac{1}{2}$ and $\alpha = 1$ for $\bar{\alpha} < \frac{1}{2}$.*

The next proposition shows how an investor will behave if his aspiration level is higher than the mean utility of the initially chosen portfolio.

Proposition 3.12 *Let the similarity function $s(\alpha; \alpha')$ of an investor be concave in $\|\alpha - \alpha'\|$ with $s(1;0) \in [0;1)$ and the memory of the investor be endogenous and contain all past cases up to the current period. If $\bar{u} > \mu_{\bar{\alpha}}$ and*

- $\bar{u} > \max\{\mu_0; \mu_1\}$, *then*
  $$P\left\{ \omega \in \Phi_6 \mid \exists \pi(\alpha) : [0;1] \to [0;1] \textit{ and } \frac{\pi(0)}{\pi(1)} = \frac{\mu_1 - \bar{u}}{\mu_0 - \bar{u}}, \pi(\alpha) = 0 \textit{for } \alpha \notin \{0;1\} \right\} = 1;$$
- $\mu_0 > \bar{u} > \mu_1$, *then*
  $$P\{\omega \in \Phi_6 \mid \exists \pi(\alpha) : [0;1] \to [0;1] \textit{ and } \pi(0) = 1, \pi(\alpha) = 0 \textit{for } \alpha \neq 0\} = 1;$$
- $\mu_1 > \bar{u} > \mu_0$, *then*
  $$P\{\omega \in \Phi_6 \mid \exists \pi(\alpha) : [0;1] \to [0;1] \textit{ and } \pi(1) = 1, \pi(\alpha) = 0 \textit{for } \alpha \neq 1\} = 1;$$
- $\mu_1 > \bar{u}$ *and* $\mu_0 > \bar{u}$, *then*
  $$P\left\{ \omega \in \Phi_6 \mid \exists \pi(\alpha) : [0;1] \to [0;1] \textit{ and } \begin{array}{ccc} \pi(1) = 1, & \pi(\alpha) = 0 & \text{for } \alpha \neq 1 \\ & \textit{or} & \\ \pi(0) = 1, & \pi(\alpha) = 0 & \text{for } \alpha \neq 0 \end{array} \right\} = 1.$$

An investor whose aspiration level exceeds the mean utility of the initially chosen portfolio will only diversify for a finite number of periods. Afterwards he will either choose one of the undiversified portfolios forever or switch between them, depending on whether he finds their mean utility satisficing or not. Note that the result that an investor will diversify only for a finite number of periods, provided that his aspiration level is relatively high, does not depend on the assumption that the investor can remember all past cases. The statement remains true, even if the investor can remember the last m cases, where m is finite. It is essential, however, that the investor remembers only cases that really occurred, i.e. that his memory is endogenous.

With two assets only, similarity between the two undiversified portfolios can be normalized to 0. Hence, only similarity perceptions with respect to the portfolio structure pay a role for the limit behavior. This feature is due to the indeterminacy of similarity function with respect to affine-linear transformations. To discuss the role of similarity between assets, the introduction of a third asset is necessary. Hence, suppose that apart from assets a and b, a third asset $\hat{a}$ is present in the economy. Its dividend payments are denoted by $\delta_{\hat{a}}$ and are distributed on an interval $\left[\underline{\delta}_{\hat{a}}; \bar{\delta}_{\hat{a}}\right]$. A portfolio consisting of the three assets a, b and $\hat{a}$ is now described by two variables $(\alpha; \hat{\alpha}) \in \mathfrak{A} = [0; 1]^2$, denoting the proportion of a and of $\hat{a}$ in the portfolio. Thus, choosing a portfolio $(\alpha; \hat{\alpha})$ results in a utility realization of

$$u\left(\alpha\delta_a + \hat{\alpha}\delta_{\hat{a}} + (1 - \alpha - \hat{\alpha})\,\delta_b\right)$$

If u is well-defined, bounded and continuous on

$$\left[\alpha\underline{\delta}_a + \hat{\alpha}\underline{\delta}_{\hat{a}} + (1 - \alpha - \hat{\alpha})\,\underline{\delta}_b; \alpha\bar{\delta}_a + \hat{\alpha}\bar{\delta}_{\hat{a}} + (1 - \alpha - \hat{\alpha})\,\bar{\delta}_b\right]$$

for all $(\alpha; \hat{\alpha}) \in \mathfrak{A}$, then the distribution Π_α of $u(\cdot)$ for a given α is characterized by a finite mean, $\mu_{\alpha;\hat{\alpha}}$, finite variance and bounded supports.

One can imagine the portfolios consisting of a, b and $\hat{a}$ situated on a triangle. The similarity between portfolios is then represented by the distance between the corresponding points. Because of the uniqueness of the similarity function with respect to linear affine transformations, it is possible to normalize the distance between a and $\hat{a}$ so that $s\left((1; 0); (0; 1)\right) = 0$. I assume as above that s depends only on the Euclidean distance between two points:

$$s\left((\alpha; \hat{\alpha}); (\alpha'; \hat{\alpha}')\right) = f\left(\|(\alpha; \hat{\alpha}) - (\alpha'; \hat{\alpha}')\|\right),$$

where f is strictly decreasing and s is concave[59].

[59] Since $s = f\left(\left\|(\alpha; \hat{\alpha}) - \left(\alpha'; \hat{\alpha}'\right)\right\|\right)$, for s to be concave, it is necessary that the decreasing function f is not too convex. To see this, denote the Euclidean distance functional by e and note that

$$s'' = f''\left(e'\right)^2 + e''f'.$$

Since $e'' > 0$ and $f' < 0$,

$$-\frac{e''f'}{\left(e'\right)^2} > 0$$

and $s'' < 0$ holds, as long as

$$f'' < -\frac{e''f'}{\left(e'\right)^2}.$$

.

Suppose that the initially chosen portfolio is strictly diversified with $(\alpha_1; \hat{\alpha}_1) \in (0;1)^2$ and note that in analogy to lemma 3.1 the strict monotonicity of the similarity function in the Euclidean distance implies that if $\mu_{\alpha_1;\hat{\alpha}_1} < \bar{u}$ (where $\bar{u}$ denotes the constant aspiration level), then $(\alpha_1; \hat{\alpha}_1)$ is abandoned almost surely in finite time and one of the undiversified portfolios $(0;1)$, $(1;0)$ or $(0;0)$ is chosen. The concavity of the similarity function further implies that one of the undiversified portfolios will be chosen in each period of time afterwards. In order to focus on the influence of similarity between the corner portfolios on limit behavior, I will concentrate on the choices between the three corner portfolios and neglect the diversified ones. Hence, it is convenient to denote the undiversified portfolios $(1;0)$, $(0;1)$ and $(0;0)$ by a, $\hat{a}$ and b, respectively. Write s_{ab} and $s_{\hat{a}b}$ for the similarity between a and b and $\hat{a}$ and b.

Denote, as above, the set of all possible paths on which case-based decision are made by S_1. Let Φ_6' denote a subset of S_1, on which the first chosen portfolios is an undiversified one and the aspiration level is constant and exceeds the maximal mean utility of an undiversified portfolio:

$$\Phi_6' = \{\omega \in S_1 \mid \bar{u}_t = \bar{u} > \max\{\mu_a; \mu_{\hat{a}}; \mu_b\} \text{ for all } t = 1, 2, \dots \text{ and } (\alpha_1; \hat{\alpha}_1) \in \{a; \hat{a}; b\}\}.$$

Let P again denote a probability distribution on Φ_6' which is consistent with $\left(\Pi_{(\alpha;\hat{\alpha})}\right)_{(\alpha;\hat{\alpha})\in[0;1]^2}$.

Proposition 3.13 • *If*

$$s_{ab} + s_{\hat{a}b} \geq 1,$$

then on Φ_6', the frequencies with which the portfolios are chosen in the limit satisfy almost surely with respect to P:

$$\frac{\pi(a)}{\pi(\hat{a})} = \frac{\mu_a - \bar{u}}{\mu_{\hat{a}} - \bar{u}}$$
$$\pi(\alpha; \hat{\alpha}) = 0, \text{else.}$$

• *If*

$$s_{ab} + s_{\hat{a}b} < 1,$$

then on Φ_6', the frequencies with which the corner portfolios are chosen in the limit satisfy almost surely with respect to P:

$$\frac{\pi(a)}{\pi(\hat{a})} = \frac{(1 - s_{\hat{a}b})(\mu_{\hat{a}} - \bar{u})}{(1 - s_{ab})(\mu_a - \bar{u})}$$
$$\frac{\pi(b)}{\pi(\hat{a})} = \frac{(\mu_{\hat{a}} - \bar{u})}{(1 - s_{\hat{a}b})(\mu_b - \bar{u})}$$
$$\frac{\pi(b)}{\pi(a)} = \frac{(\mu_a - \bar{u})}{(1 - s_{ab})(\mu_b - \bar{u})},$$

$$\pi(\alpha;\hat{\alpha}) = 0, \text{ else.}$$

The proposition shows that depending on the relationship between s_{ab} and $s_{\hat{a}b}$, similarity between the corner portfolios can have different effects on the limit choice of the investor. As long as the sum of s_{ab} and $s_{\hat{a}b}$ is larger than 1, the negative impact exhibited by portfolios a and $\hat{a}$ on the cumulative utility of b is so large that b is never chosen after some finite period $\bar{t}$. If, however, their sum is lower than 1, b is still an optimal choice during a positive fraction of time. The frequencies with which a and $\hat{a}$ are chosen depend on how similar they are to b. Especially, the portfolio which is more similar to b is chosen less frequently, since its cumulative utility suffers more from the negative net utility realizations of b. Clearly, for $s_{\hat{a}b} = s_{ab}$, this effect disappears. Still, as long as $0 < s_{\hat{a}b} = s_{ab}$ holds, the utility realizations of b negatively affect the cumulative utilities of a and $\hat{a}$ and their frequencies are therefore lower than in the case in which no similarity between a and b and $\hat{a}$ and b is perceived.

3.8.2 Learning with Similarity

The model of Gilboa and Schmeidler (1996) poses the question whether a case-based decision-maker is able to learn to choose the optimal (expected utility maximizing) act if the same problem is repeated an infinite number of times. They find a rule for adapting the aspiration level that indeed implies optimal behavior in the limit, see section 3. The combination of realism and ambitiousness leads to optimal choice, formally, for each $\varepsilon > 0$, the probability of choosing one of the acts $\alpha \in \arg\max\{\mu_\alpha \mid \alpha \in [0;1]\}$ with frequency 1 is at least $(1-\varepsilon)$, provided that the initial aspiration level is chosen to be sufficiently high. Especially, to prove that optimal learning is possible, Gilboa and Schmeidler (1996) select the initial aspiration level $\bar{u}_1$ in such a way that it remains above

$$R = 2\max_{\alpha\in[0;1]} u\left(\bar{\delta}_a\alpha + \bar{\delta}_b(1-\alpha)\right) - \min_{\alpha\in[0;1]} u\left(\underline{\delta}_a\alpha + \underline{\delta}_b(1-\alpha)\right)$$

during the first T_0 periods. T_0 is chosen so as to assure that during the first T_0 periods all acts (out of the finite set $\mathfrak{A}$) are chosen at least C times each. At the same time, C is sufficiently large to guarantee that after C choices the average utility of each act a is close to its mean utility μ_a. The choice of the initial aspiration level, therefore, precludes the possibility that a decision maker becomes satisfied with a suboptimal act and never chooses an alternative one.

The theorem assumes a very specific similarity function, for which

$$\begin{aligned} s(\alpha;\alpha') &= 1, \text{ if } \alpha = \alpha' \\ s(\alpha;\alpha') &= 0, \text{ if } \alpha \neq \alpha'. \end{aligned}$$

In other words, two acts are only considered similar, if they are identical.

It is interesting to know whether this adaptation rule also works for more general similarity functions. Suppose as above that the similarity function is concave. Suppose as well that the first act chosen is $\alpha_1 = \bar{\alpha} \in (0;1)$. As above, Φ denotes the set of paths on which a case-based decision-maker adapts his aspiration level according to (3.3):

$$\Phi = \left\{ \omega \in S_1(\beta;\bar{u}_1) \mid \begin{array}{l} \bar{u}_1 = \bar{u}_1 \\ \bar{u}_t = \beta\bar{u}_{t-1} + (1-\beta) X_t \text{ for } t \geq 2 \end{array} \right\}. \tag{3.11}$$

Let P be a probability measure on Φ, which is consistent with $(\Pi_\alpha)_{\alpha\in[0;1]}$, as in Gilboa and Schmeidler (1996, p.11). It is clear that for sufficiently high initial aspiration level an investor following updating rule (3.3) will switch at some time to act 1 or 0. Moreover, to enable optimal learning, the initial aspiration level has to be set higher than R, which exceeds the maximal possible realization of the initially chosen portfolio $\bar{\alpha}$. This implies that the investor switches to 0 or 1 at $t = 2$. Let $u_1(\bar{\alpha})$ denote the utility realization of portfolio $\bar{\alpha}$ in period 1.

Proposition 3.14 *Suppose that s is concave and strict monotonically decreasing in the Euclidean distance between acts. Define $\tilde{\Phi}$ as*

$$\tilde{\Phi} = \{\omega \in \Phi \mid u_1(\bar{\alpha}) \leq \max\{\mu_0;\mu_1\}\}.$$

For each $\varepsilon > 0$ there exists a $\bar{u}_0$ such that for any $\bar{u}_1 > \bar{u}_0$:

$$P\left\{\omega \in \Phi \mid \exists\pi\left(\arg\max_{\alpha\in\{0;1\}} \mu_\alpha\right) = 1\right\} \geq (1-\varepsilon) P\left(\tilde{\Phi}\right)$$

$$P\left\{\omega \in \Phi \mid \begin{array}{ll} \text{for each } \alpha \in \mathfrak{A} & \exists\pi(\alpha) \text{ such that} \\ \frac{\pi_0}{\pi_1} = \frac{\mu_1 - u_1(\bar{\alpha})}{\mu_0 - u_1(\bar{\alpha})} & \text{and} \\ \pi(\alpha) = 0 & \text{for } \alpha \notin \{0;1\} \end{array}\right\} \geq (1-\varepsilon)\left[1 - P\left(\tilde{\Phi}\right)\right],$$

holds.

A similar result obtains if the second rule (3.4) proposed by Gilboa and Schmeidler (1996) is applied and the initial aspiration level is selected to exceed the maximal possible realization of the initially chosen act, or if the aspiration level is updated in an ambitious way in the first period:

$$\bar{u}_2 = X_2 + h.$$

As above, denote by

$$\Phi_1 = \Phi_1(\beta; \bar{u}_1; N; h) = \left\{ \omega \in S_1 \mid \begin{array}{c} \bar{u}_1 = \bar{u}_1 \text{ and} \\ \bar{u}_t = \beta \bar{u}_{t-1} + (1-\beta) X_t \text{ for } t \geq 2,\, t \notin N \\ \bar{u}_t = X_t + h \text{ for } t \geq 2,\, t \in N \end{array} \right\}$$

the set of decision paths and let P denote the probability measure on Φ_1 consistent with $(\Pi_\alpha)_{\alpha \in [0;1]}$.

Proposition 3.15 *Suppose that s is concave and strict monotonically decreasing in the Euclidean distance between acts. Assume that either*

$$\bar{u}_1 > u\left(\bar{\alpha}\bar{\delta}_a + (1-\bar{\alpha})\bar{\delta}_b\right)$$

or $1 \in N$. Define $\tilde{\Phi}_1$ as

$$\tilde{\Phi}_1 = \{\omega \in \Phi_1 \mid u_1(\bar{\alpha}) \leq \max\{\mu_0; \mu_1\}\}.$$

Then:

$$P\left\{\omega \in \Phi_1 \mid \exists \pi \left(\arg \max_{\alpha \in \{0;1\}} \mu_\alpha\right) = 1\right\} = P\left(\tilde{\Phi}_1\right)$$

$$P\left\{\omega \in \Phi_1 \mid \begin{array}{ll} \textit{for each } \alpha \in \mathfrak{A} & \exists \pi(\alpha) \textit{ such that} \\ \frac{\pi_0}{\pi_1} = \frac{\mu_1 - u_1(\bar{\alpha})}{\mu_0 - u_1(\bar{\alpha})} & \textit{and} \\ \pi(\alpha) = 0 & \textit{for } \alpha \notin \{0;1\} \end{array}\right\} = 1 - P\left(\tilde{\Phi}_1\right),$$

holds.

Two effects combine to prevent efficient learning. First, the concavity and the strict monotonicity of the similarity function and the initially high aspiration level imply that the diversified portfolio $\bar{\alpha}$ is abandoned in the second period. As long as the average utility of both undiversified portfolios lies below the aspiration level, the concavity of the similarity function forces the investor to overvalue the negative impact on diversified portfolios. Hence, only non-diversified portfolios are selected after $t = 2$.

Second, although $\bar{\alpha}$ is never chosen again, its initial realization influences the evolution of the aspiration level. Especially, if it exceeds the maximal mean of the undiversified portfolios, the aspiration level converges towards $u_1(\bar{\alpha})$ and both $\alpha = 0$ and $\alpha = 1$ seem unsatisfactory in the limit.

Since the proofs of propositions 3.14 and 3.15 heavily rely on the concavity of the similarity function, I now explore how results change if the similarity function is convex over some range[60]. I make the following assumptions:

Assumption 1: Suppose that $s(\alpha; \alpha')$ is concave in $\|\alpha - \alpha'\|$ for $\|\alpha - \alpha'\| \in \left(-\frac{1}{l}; \frac{1}{l}\right)$ for some $l \in \mathbb{N}$, $l > 1$ and all $\alpha' \in [0;1]$ and $s(\alpha; \alpha') = 0$ outside this interval. Moreover, assume that $s(\alpha; \alpha')$ is continuous, so that $s\left(\alpha' - \frac{1}{l}; \alpha'\right) = s\left(\alpha' + \frac{1}{l}; \alpha'\right) = 0$ for all α'.

60 Note, that a continuous similarity function that has a maximum at $s(a; a) = 1$ cannot be convex everywhere.

Assumption 2: Let $\alpha_1 = 0$ and let the investor choose the act which is next to the act he chose last, if indifferent:

$$\alpha_t = \arg \min_{\alpha \in \arg\max U_t(a)} \left\{ \left\| \arg \max_{\alpha \in [0;1]} \{U_t(\alpha)\} - \alpha_{t-1} \right\| \right\}$$

(In other words, suppose he chose $\alpha_1 = 0$ last and the cumulative utility of α_1 has fallen below 0. Since the similarity between 0 and $\alpha \geq \frac{1}{l}$ is 0, the investor is indifferent among all $\alpha \geq \frac{1}{l}$ and by the above assumption he should choose $\alpha = \frac{1}{l}$, the act next to 0.)

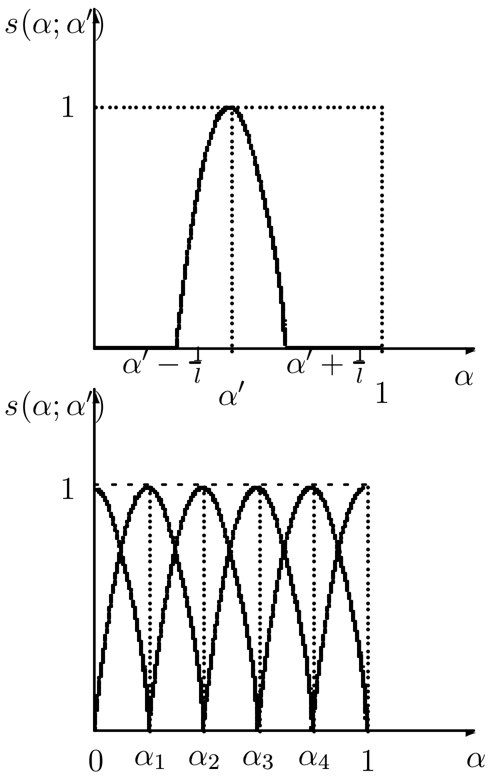


Figure 3

Figure 3 illustrates these two assumptions.

Now consider an investor who updates his aspiration level according to (3.4) and satisfies assumptions 1 and 2. Denote the corresponding set of paths by Φ_7:

$$\Phi_7 = \left\{ \omega \in \Phi_1 \mid \begin{array}{l} \alpha_1 = 0 \\ \alpha_t = \arg\min_{\alpha \in \arg\max U_t(a)} \left\{ \left\| \arg\max_{\alpha \in [0;1]} \{U_t(\alpha)\} - \alpha_{t-1} \right\| \right\} \end{array} \right\}.$$

Proposition 3.16 [61]*Suppose that Assumptions 1 and 2 hold. For all $\bar{u}_1 \in \mathbb{R}$ and all $\beta \in (0;1)$,*

[61] An analogous result, ascertaining that for a sufficiently high initial aspiration level the best of the acts $\left\{0; \frac{1}{l}; \frac{2}{l}...1\right\}$

on almost all possible paths in Φ_7 *(with respect to a probability distribution on* Φ_7 *consistent with* $(\Pi_\alpha)_{\alpha\in[0;1]}$*), an investor who updates his aspiration level according to (3.4) will choose the act*

$$\arg\max_{\alpha}\left\{\mu_\alpha \mid \alpha\in\left\{0;\frac{1}{l};\frac{2}{l}...\frac{l-1}{l};1\right\}\right\}$$

with frequency 1.

The result shows that convexities of the similarity function improve the learning process. Instead of just learning the better one of the two corner acts 0 and 1, the investor can now learn to choose the best of the $l+1$ acts (*including* 0 and 1). Therefore, he can do better, the more intervals l he distinguishes. As $l\to\infty$, the intervals on which the similarity function is positive, shrink to single points and the similarity function approximates the special degenerate case used by Gilboa and Schmeidler (1996): $s(\alpha;\alpha')=1$, if $\alpha=\alpha'$ and $s(\alpha;\alpha')=0$, else. Note, however that although the act chosen with frequency 1 converges towards the optimal one, in the limit, learning becomes impossible, since with an uncountable number of acts, choosing each act for infinitely many periods becomes impossible. However, choosing l to be sufficiently large, allows the investor's limit choice to approximate expected utility maximization with an arbitrary degree of accuracy.

3.9 Conclusion

I suggest to use the case-based decision theory of Gilboa and Schmeidler (1995, 1996) to model investors' behavior in financial markets. This could be reasonable, since the case-based decision theory generalizes the expected utility theory by von Neumann and Morgenstern (1947) and Savage (1954), while capturing some of the psychological biases, observed in real financial markets. Gilboa and Schmeidler (1996) propose sufficient conditions, under which the long run behavior predicted by the case-based decision theory coincides with the optimal behavior of an expected utility maximizer. Removing these conditions and assuming constant aspiration level can lead to a suboptimal limit behavior. A relatively low aspiration level induces a satisficing behavior, whereas an excessively high aspiration level forces the investor to trade too often. Similar results are achieved by allowing the decision-maker to update his aspiration level in an optimistic (towards the best achieved utility realization) or in a pessimistic (towards the worst achieved utility realization) way.

is chosen with frequency 1 with arbitrarily high probability, can be derived for the decision rule (3.11).

If the information about the utility realizations of all acts is available to the decision-maker in each period of time, then he is able to learn to behave optimally in a stationary environment. If, however, receiving information is restricted to a finite number of periods, optimal behavior can only be achieved for relatively low aspiration levels. This means that phenomena like failing to use arbitrage opportunities, to diversify optimally or to choose an efficient portfolio result from lack of information, rather than from suboptimal behavior. Information, however, cannot help resolve the problem of excessive trading as long as the number of information periods remains finite. Nevertheless, the case-based decision theory allows for optimal learning even in the case of constant aspiration level: there is a range of values of the aspiration level for which the case-based decision-maker behaves optimally in the long run, choosing the act with the highest mean utility.

Introducing a similarity function on the set of portfolios does not change the results qualitatively. Whereas for investors with constant low aspiration level there is a positive probability that they would hold the initially chosen (and possibly suboptimal) diversified portfolio forever, investors with relatively high aspiration levels exhibit switching behavior. Especially, with a concave similarity function a decision-maker with an aspiration level exceeding the mean utility of the initially chosen portfolio chooses one of the undiversified portfolios from some period on. If his aspiration level also exceeds the mean utilities of both undiversified portfolios, he chooses these portfolios with positive frequencies in the limit. Moreover, a concave similarity function prevents learning the optimal portfolio in the setting of Gilboa and Schmeidler (1996) for a decision-maker with a concave utility function. By allowing for convexities, the quality of learning is improved and the limit choice can become arbitrary close to the expected utility maximizing act.

This is a first attempt to apply the case-based decision theory to model human-behavior in financial markets, which leaves a large number of questions open. First, the assumption that the distributions $(\Pi_\alpha)_{\alpha\in\mathfrak{A}}$ remain constant over the time should be relaxed. This will allow to consider a market consisting of case-based decision-makers whose behavior will make the prices of the assets correlated over the time. Second, it would be interesting to look at the implications which the existence of case-based investors in the market may have on other market participants. These two questions will be discussed in the next chapters of this thesis.

Appendix

Proof of proposition 3.1:

First it is shown that if all utility realizations are below the aspiration level, all available acts must be chosen infinitely many times. Since the number of acts is finite, at least one act must be chosen for an infinite number of periods. Let this be the act α. Consider an act $\tilde{\alpha}$ which is chosen L times. As all possible realizations lead to a negative "net"-utility, the cumulative utility of α will become infinitely low in the limit. At the same time, the cumulative utility of $\tilde{\alpha}$ will not fall below

$$L\left[\min_{u\in\Lambda_{\tilde{\alpha}}} u - \bar{u}_1\right].$$

Hence, there is a time $T > L$, such that for each $t > T$

$$U_t\left(\alpha\right) < U_t\left(\tilde{\alpha}\right)$$

and still act α is chosen. This contradicts the case-based decision theory, which states that in each period the act with the highest cumulative utility has to be chosen. Therefore, since $\tilde{\alpha}$ was chosen arbitrarily, each act will be chosen infinitely often.

Next, it is stated that the quotient of the cumulative utilities of each two acts converges to 1. Indeed, let α and $\tilde{\alpha}$ be two distinct acts. Denote by $\varepsilon_t\left(\alpha;\tilde{\alpha}\right)$ the difference of the cumulative utilities of α and $\tilde{\alpha}$ at time t:

$$\varepsilon_t\left(\alpha;\tilde{\alpha}\right) = U_t\left(\alpha\right) - U_t\left(\tilde{\alpha}\right)$$

Hence:,

$$\lim_{t\to\infty}\frac{U_t\left(\alpha\right)}{U_t\left(\tilde{\alpha}\right)} = \lim_{t\to\infty}\frac{U_t\left(\tilde{\alpha}\right)+\varepsilon_t\left(\alpha;\tilde{\alpha}\right)}{U_t\left(\tilde{\alpha}\right)}$$

Consider the difference $\varepsilon_t\left(\alpha;\tilde{\alpha}\right)$. Obviously, this difference remains constant at times at which neither α, nor $\tilde{\alpha}$ is chosen. Consider, therefore, the case in which α is chosen. Note that the minimal difference between the cumulative utilities in a period in which α is chosen must exceed 0, since else the maximization of cumulative utility would be violated. Moreover, since the least possible utility realization of $\tilde{\alpha}$ is $\min_{u\in\Lambda_{\tilde{\alpha}}} u-\bar{u}_1$, it follows that the maximal possible difference between the cumulative utilities of α and $\tilde{\alpha}$ at a time at which α is chosen is

$$-\min_{u\in\Lambda_{\tilde{\alpha}}} u + \bar{u}_1.$$

But since the net-utility realizations of α are always negative, this (maximal) difference is neutralized by the choice of α in at most

$$\left\lceil \frac{(\min_{u\in\Lambda_{\tilde{\alpha}}} u - \bar{u}_1)}{(\max_{u\in\Lambda_{\tilde{\alpha}}} u - \bar{u}_1)} \right\rceil$$

periods. An analogous argument shows that $\tilde{\alpha}$ can also be chosen for only a finite number of periods in a row. It follows that the difference between the cumulative utilities of arbitrary two acts is bounded on every possible path ω (since the possible utility realizations are finite) and, therefore,

$$\lim_{t\to\infty} \frac{U_t(\alpha)}{U_t(\tilde{\alpha})} = \lim_{t\to\infty} \frac{U_t(\tilde{\alpha}) + \varepsilon_t(\alpha;\tilde{\alpha})}{U_t(\tilde{\alpha})} = 1 \tag{3.12}$$

holds on each path. Hence, from the definition of cumulative utility it follows that on each path $\omega \in \Phi_2'$:

$$\lim_{t\to\infty} \frac{U_t(\alpha)}{U_t(\tilde{\alpha})} = \lim_{t\to\infty} \frac{|C_t(\alpha)|}{|C_t(\tilde{\alpha})|} \frac{[X_t(\alpha) - \bar{u}_1]}{[X_t(\tilde{\alpha}) - \bar{u}_1]} = 1 \tag{3.13}$$

Since (3.13) holds on each path of utility realizations, for any such path it can be written as:

$$\lim_{t\to\infty} \frac{|C_t(\alpha)|}{|C_t(\tilde{\alpha})|} \lim_{t\to\infty} \frac{[X_t(\alpha) - \bar{u}_1]}{[X_t(\tilde{\alpha}) - \bar{u}_1]} = 1.$$

According to the Strong Law of Large Numbers it follows that:

$$\frac{\pi(\alpha)}{\pi(\tilde{\alpha})} = \lim_{t\to\infty} \frac{|C_t(\alpha)|}{|C_t(\tilde{\alpha})|} = \lim_{t\to\infty} \frac{[X_t(\tilde{\alpha}) - \bar{u}_1]}{[X_t(\alpha) - \bar{u}_1]} = \frac{[\mu_{\tilde{\alpha}} - \bar{u}_1]}{[\mu_\alpha - \bar{u}_1]}$$

with probability 1.■

Proof of proposition 3.2:

If the decision-maker has an infinite time-horizon, then at least one act α must be chosen infinitely often. Let α be such an act. Then, according to the Strong Law of Large Numbers, its cumulative utility satisfies:

$$\begin{aligned}
P\left\{\lim_{t\to\infty} U_t(\alpha) = -\infty \mid \lim_{t\to\infty} |C_t(\alpha)| = \infty\right\} &= \\
P\left\{\lim_{t\to\infty} |C_t(\alpha)| \left[\frac{X_t(\alpha)}{|C_t(\alpha)|} - \bar{u}_1\right] = -\infty \mid \lim_{t\to\infty} |C_t(\alpha)| = \infty\right\} &= \\
P\left\{[\mu_\alpha - \bar{u}_1] \lim_{t\to\infty} |C_t(\alpha)| = -\infty \mid \lim_{t\to\infty} |C_t(\alpha)| = \infty\right\} &= 1
\end{aligned}$$

In contrast, if an act $\tilde{\alpha}$ is chosen a finite number of periods, say L times, its cumulative utility is limited from below:

$$\lim_{t\to\infty} U_t(\tilde{\alpha}) \geq L \left[\min_{u\in\Lambda_{\tilde{\alpha}}} u - \bar{u}_1\right].$$

Hence, on almost every path $\omega \in \Phi_2''$ there exists a period $T(\omega) > L$ after which the cumulative utility of $\tilde{\alpha}$ will always exceed that of α. But this means that the decision-maker does not obey the case-based decision theory, since he does not choose the act with maximal cumulative utility. Thus, a case-based decision-maker will choose each of the acts in $\mathfrak{A}$ for an infinite number of periods.

Now, consider the difference between the cumulative utilities of arbitrary two acts α and $\tilde{\alpha}$, $\varepsilon_t(\alpha;\tilde{\alpha})$. As long as act α is chosen, this difference represents a random walk on the half line with negative expected increment. Define $\tilde{\varepsilon}_t(\alpha;\tilde{\alpha})$ as

$$\begin{aligned}\tilde{\varepsilon}_t(\alpha;\tilde{\alpha}) &= \varepsilon_t(\alpha;\tilde{\alpha}) \text{ if } \varepsilon_t(\alpha;\tilde{\alpha}) \geq 0 \\ \tilde{\varepsilon}_t(\alpha;\tilde{\alpha}) &= 0\text{, else.}\end{aligned}$$

Such a random walk has an accessible atom at 0[62]. Moreover, each set of the type $[0;c]$ is regular, see Meyn and Tweedie (1996, p. 278). This means that the state 0 is reached in finite expected time, starting from each set of the type $[0;c]$ and especially, starting from the set $[0;\bar{u}_1 - \min_{u\in\Lambda_{\tilde{\alpha}}} u]$. Denote the supremum of these expected times by $\mathcal{N}$ and observe that it is finite according to the definition of regular sets. Note that $\bar{u}_1 - \min_{u\in\Lambda_{\tilde{\alpha}}} u$ equals the maximal possible value of $\varepsilon_t(\alpha;\tilde{\alpha})$ in a period, in which the decision-maker switches from an arbitrary α' to α. Observe as well that since the probability that $\varepsilon_t(\alpha;\tilde{\alpha}) = 0$ is 0 (for atomless distributions of δ_a and δ_b), it follows that $\tilde{\varepsilon}_t(\alpha;\tilde{\alpha}) = 0$ coincides with $\varepsilon_t(\alpha;\tilde{\alpha}) < 0$. Hence, the decision-maker switches away from α when $\tilde{\varepsilon}_t(\alpha;\tilde{\alpha}) = 0$ is reached[63]. It follows that the expected time for which an arbitrary act α is held in a row is finite and uniformly bounded from above.

It remains to show that $\varepsilon_t(\alpha;\tilde{\alpha})$ is bounded on almost each path of dividend realizations. At times at which α is chosen $\varepsilon_t(\alpha;\tilde{\alpha})$ never falls below 0, since this would contradict choosing the act with highest cumulative utility in each period. Suppose, therefore that there is a sequence of periods t', t''..., such that $\varepsilon_{t'}(\alpha;\tilde{\alpha})$, $\varepsilon_{t''}(\alpha;\tilde{\alpha})$... grows to infinity. In other words, suppose that for each $\mathcal{M} > 0$ there is a k, such that $\varepsilon_{t^n}(\alpha;\tilde{\alpha}) > \mathcal{M}$ for all $n > k$. Since $U_t(\alpha)$ has negative expected increments, it follows (as shown above) that each other act and especially $\tilde{\alpha}$ is chosen infinitely many times on almost each path of dividend realizations. But each time that the act $\tilde{\alpha}$ is chosen, the difference $\varepsilon_t(\alpha;\tilde{\alpha})$ falls below 0. If $\varepsilon_{t^n}(\alpha;\tilde{\alpha}) > \mathcal{M}$, the time needed to return to the origin is at least $\frac{\mathcal{M}}{\bar{u}_1 - \min_{u\in\Lambda_\alpha} u}$, which grows to infinity, as ε_{t^n} and, hence, $\mathcal{M}$ becomes very large. However, as has been explained above, the expected time for return to the origin 0 of

62 See Meyn and Tweedie (1996, p. 105) for a definition of an accessible atom.

63 Of course, the decision-maker might switch away from α in an earlier period if for some $\alpha' \neq \tilde{\alpha}$, $\varepsilon(\alpha;\alpha') < 0$ obtains. In this case, the expected time during which α is held is also bounded from above and is obviously less than $\mathcal{N}$.

$\tilde{\varepsilon}_t(\alpha;\tilde{\alpha})$ is finite and uniformly bounded above by $\mathcal{N}$. The Law of Large Numbers then implies that for each $\kappa > 0$ on almost each path of dividend realizations there is a period $\mathcal{K}(\omega)$, such that

$$\frac{\sum_{i=1}^{n} \tau_i}{n} \leq \mathcal{N} + \kappa$$

for all $n \geq \mathcal{K}(\omega)$, where τ_i denotes the time needed for $\tilde{\varepsilon}_t(\alpha;\tilde{\alpha})$ to reach the origin, once α has been chosen. On the other hand, the assumption that $\varepsilon_{t^n}(\alpha;\tilde{\alpha}) \to \infty$ implies that the stopping times τ_i become infinitely large as the time grows — a contradiction. Hence, almost each sequence $\varepsilon_{t'}(\alpha;\tilde{\alpha})$, $\varepsilon_{t''}(\alpha;\tilde{\alpha})$... (where t', t''... denote periods at which α is chosen) is bounded from above. A symmetric argument for $\tilde{\alpha}$ shows that $\varepsilon_t(\alpha;\tilde{\alpha})$ is bounded from below. It follows that on almost each path $\omega \in \Phi_2''$

$$\lim_{t\to\infty} \frac{U_t(\alpha)}{U_t(\tilde{\alpha})} = \lim_{t\to\infty} \frac{U_t(\tilde{\alpha}) + \varepsilon_t(\alpha;\tilde{\alpha})}{U_t(\tilde{\alpha})} = 1$$

holds. The remaining part of the proof is as in proposition 3.1.■

Proof of proposition 3.3:

In the proof of proposition 3.2, it has been shown that it is impossible that only acts from the set $\mathfrak{A}\backslash\tilde{A}$ are chosen infinitely often. This means that at least one act $\alpha \in \tilde{A}$ will be chosen infinitely often. Suppose to the contrary of the statement of the proposition that there are two acts from $\tilde{A}$, α and $\tilde{\alpha}$, which are chosen with positive frequency. It is easy to show that this leads to a contradiction.

Indeed, consider the periods $z_{1\alpha}$, $z_{2\alpha}$,...$\in \mathbb{N}$ at which the decision-maker switches to act α and denote by $z_{1\tilde{\alpha}}$, $z_{2\tilde{\alpha}}$,...$\in \mathbb{N}$ the times, at which the decision-maker switches to act $\tilde{\alpha}$. Then according to the case-based decision theory it must be that:

$$U_{z_{1\alpha}}(\alpha) \geq U_{z_{1\alpha}}(\tilde{\alpha}) \geq U_{z_{1\tilde{\alpha}}}(\alpha) \geq U_{z_{2\alpha}}(\tilde{\alpha}) \geq U_{z_{2\tilde{\alpha}}}(\alpha) \geq ...$$

But these inequalities imply that $U_t(\alpha)$, which is a random walk with positive expected increment $\mu_\alpha - \bar{u}_1 > 0$, crosses each of the infinitely many boundaries $U_{z_{k\alpha}}(\tilde{\alpha})$ from above. Since, however, there is a positive probability that a random walk with positive expected increment starting from a given point, never crosses a boundary lying below this point, see Grimmet and Stirzaker (1994, p. 144), and since the stopping times are independently distributed, it follows that the probability of infinitely many switches between α and $\tilde{\alpha}$ is 0. Hence, only one of these two acts can be chosen with positive frequency in the limit.

Alternatively, suppose that an act $\tilde{\alpha}$ from the set $\mathfrak{A}\backslash\tilde{A}$ is chosen infinitely often with an act from $\tilde{A}$. Then, with probability 1, the cumulative utility of α will become infinitely high, whereas the cumulative utility of $\tilde{\alpha}$ will become infinitely low, as the number of periods grows to infinity. Hence, choosing act $\tilde{\alpha}$ infinitely often will contradict the case-based decision theory, as well.■

Proof of proposition 3.4:

Let α be the first act chosen, i.e. $\alpha_1 = \alpha$. As its cumulative utility $U_t(\alpha)$ remains positive for all $t \in \mathbb{N}$, whereas the cumulative utility of all the other acts stays at 0, α will be the only act chosen forever.■

Proof of proposition 3.5:

It will be shown that under this updating rule the aspiration level converges to $\bar{u}$.

First, note that all available acts will be chosen infinitely often. Indeed, imagine that only one act is chosen infinitely often, call it α. Then either[64]

$$P\left\{\lim_{C_t(\alpha)\to\infty}\left\{\max_{\tau\in C_t(\alpha)} u_\tau\right\} = \max_{u\in\Lambda_\alpha} u\right\} = 1$$

or

$$\bar{u}_1 > \max_{u\in\Lambda_\alpha} u > \mu_\alpha.$$

Then, on almost each path $\omega \in \Phi_3$, there is some finite $T(\omega)$, such that $\bar{u}_t > \mu_\alpha$ holds on ω for all $t > T(\omega)$. But in this case the Strong Law of Large Numbers implies that the cumulative utility of α will converge to $-\infty$ on almost each path. Hence, since α was chosen arbitrarily, it follows from proposition 3.2 that all available acts must be chosen infinitely often. Then, the aspiration level will converge to $\max_{u\in\Lambda} u$ with probability 1, implying that on almost each path $\omega \in \Phi_3$, there is a period $T'(\omega)$, such that $\bar{u}_t > \mu_\alpha$ holds on ω for all $t > T'(\omega)$ and all $\alpha \in \mathfrak{A}$. Now apply the proof of proposition 3.2 to obtain the claim stated in the proposition.■

Proof of proposition 3.6:

Suppose, contrary to the statement of the proposition, that two acts α and $\tilde{\alpha}$ are chosen with positive frequencies in the limit. Then either

$$P\left\{\lim_{t\to\infty}\{\min u_t\} = \min_{u\in\Lambda_\alpha\cup\Lambda_{\tilde{\alpha}}} u\right\} = 1$$

64 Suppose that the aspiration level has already reached some inner point of the support Δ_α, say $\bar{u}_T$. Suppose that $\Pi_\alpha(\bar{u}_T) = \epsilon$. Then the probability of the event that $\bar{u}_T$ remains the maximal utility realization experienced as $C_t(\alpha) \to \infty$ is $\lim_{C_t(\alpha)\to\infty} \epsilon^{C_t(\alpha)} = 0$.

and $\bar{u}_t \to \min\{\Lambda_\alpha; \Lambda_{\tilde{\alpha}}\}$ on almost each path $\omega \in \Phi_4$ or

$$\bar{u}_1 < \min_{u \in \Lambda_\alpha \cup \Lambda_{\tilde{\alpha}}} u < \min\{\mu_\alpha; \mu_{\tilde{\alpha}}\}.$$

In both cases, on almost each path $\omega \in \Phi_4$, there is some finite $T(\omega)$, such that $\bar{u}_t < \min\{\mu_\alpha; \mu_{\tilde{\alpha}}\}$ holds on ω for all $t > T(\omega)$. From the proof of proposition 3.3, it follows that the investor will switch only for a finite number of times between α and $\tilde{\alpha}$. Hence, in the limit, one of the acts α and $\tilde{\alpha}$ is chosen with frequency 1.■

Proof of proposition 3.7:

Suppose, to the contrary of what the proposition states that an act $\alpha \in \mathfrak{A}$ is chosen with frequency one in the limit on some set of paths $\hat{\Phi}_5 \subset \Phi_5$, assigned a positive probability by P on Φ_5. Consider two cases:

1. If α is the first act to be chosen, then the aspiration level in the limit satisfies

$$\lim_{t \to \infty} \bar{u}_t = (1-\beta) \max_{u \in \Lambda_\alpha} u + \beta \min_{u \in \Lambda_\alpha} u$$

with probability 1 on $\hat{\Phi}_5$. Since

$$\beta < \frac{\max_{u \in \Lambda_\alpha} u - \mu_\alpha}{\max_{u \in \Lambda_\alpha} u - \min\limits_{\alpha \in \mathfrak{A}} \min_{u \in \Lambda_\alpha} u} \text{ for all } \alpha \in \mathfrak{A},$$

it follows that

$$\beta < \frac{\max_{u \in \Lambda_\alpha} u - \mu_\alpha}{\max_{u \in \Lambda_\alpha} u - \min_{u \in \Lambda_\alpha} u}, \text{ or}$$
$$\mu_\alpha < (1-\beta) \max_{u \in \Lambda_\alpha} u + \beta \min_{u \in \Lambda_\alpha} u$$

Then, on almost each path $\omega \in \hat{\Phi}_5$, there is some finite $T(\omega)$, such that $\bar{u}_t > \mu_\alpha$ holds on ω for all $t > T(\omega)$. But in this case the Strong Law of Large Numbers implies that the cumulative utility of α will converge to $-\infty$, whereas the cumulative utility of the unchosen acts remains 0 almost surely on $\hat{\Phi}_5$. Hence, on $\hat{\Phi}_5$, the case-based decision rule is almost surely violated and therefore, $P\left(\hat{\Phi}_5\right) = 0$, a contradiction.

2. If some other acts have been chosen on $\omega \in \hat{\Phi}_5$ before choosing α, then it can happen that $\min_{u \in \Lambda_\alpha} u > \min\limits_{t<T} u_t$, where T is the first period in which α is chosen. Note that if $\min\limits_{t<T} u_t$ is very low, the aspiration level may remain sufficiently low so that act α really becomes satisfactory in the limit. Suppose that

$$\min_{t<T} u_t \leq \min_{u \in \Lambda_\alpha} u,$$

and that

$$\max_{t<T} u_t \leq \max_{u\in\Lambda_\alpha} u,$$

i.e. the aspiration level at T is sufficiently low and cannot become higher than $(1-\beta)\max_{u\in\Lambda_\alpha} u + \beta \min_{t<T} u_t$, as long, as only act α is chosen. However, the assumption made insures that

$$\beta < \frac{\max_{u\in\Lambda_\alpha} u - \mu_\alpha}{\max_{u\in\Lambda_\alpha} u - \min_{\alpha\in\mathfrak{A}} \min_{u\in\Lambda_\alpha} u}, \text{ or}$$

$$\mu_\alpha < (1-\beta)\max_{u\in\Lambda_\alpha} u + \beta \min_{\alpha\in\mathfrak{A}} \min_{u\in\Lambda_\alpha} u \leq (1-\beta)\max_{u\in\Lambda_\alpha} u + \beta \min_{t<T} u_t,$$

which contradicts the assumption that $P\left(\hat{\Phi}_5\right) > 0$, as shown above.

If,

$$\min_{t<T} u_t \leq \min_{u\in\Lambda_\alpha} u$$
$$\max_{t<T} u_t > \max_{u\in\Lambda_\alpha} u,$$

then the condition

$$\mu_\alpha < (1-\beta)\max_{u\in\Lambda_\alpha} u + \beta \min_{\alpha\in\mathfrak{A}} \min_{u\in\Lambda_\alpha} u \leq (1-\beta)\max_{t<T} u_t + \beta \min_{t<T} u_t$$

contradicts the assumption that $P\left(\hat{\Phi}_5\right) > 0$.

Analogous reasoning shows that for

$$\min_{t<T} u_t > \min_{u\in\Lambda_\alpha} u$$
$$\max_{t<T} u_t \leq \max_{u\in\Lambda_\alpha} u,$$
$$\mu_\alpha < (1-\beta)\max_{u\in\Lambda_\alpha} u + \beta \min_{\alpha\in\mathfrak{A}} \min_{u\in\Lambda_\alpha} u \leq (1-\beta)\max_{u\in\Lambda_\alpha} u + \beta \min_{u\in\Lambda_\alpha} u$$

and for

$$\min_{t<T} u_t > \min_{u\in\Lambda_\alpha} u$$
$$\max_{t<T} u_t > \max_{u\in\Lambda_\alpha} u,$$
$$\mu_\alpha < (1-\beta)\max_{u\in\Lambda_\alpha} u + \beta \min_{\alpha\in\mathfrak{A}} \min_{u\in\Lambda_\alpha} u$$
$$< (1-\beta)\max_{t<T} u_t + \beta \min_{u\in\Lambda_\alpha} u$$

$P\left(\hat{\Phi}_5\right) = 0$ must hold.

Since α and $\hat{\Phi}_5$ have been chosen arbitrarily, this shows that no act will be chosen with frequency one in the limit on a set of paths with positive probability. Since the cumulative utility of an act becomes infinitely negative, if the act is chosen infinitely many times and stays finite, else, it follows that all acts, including

$$\arg\max_{\alpha\in\mathfrak{A}}\left\{\max_{u\in\Lambda_\alpha} u\right\}$$

and

$$\arg\min_{\alpha\in\mathfrak{A}}\left\{\min_{u\in\Lambda_\alpha} u\right\},$$

have to be chosen infinitely often in the limit. In this case, the limit aspiration level becomes:

$$\lim_{t\to\infty}\bar{u}_t=(1-\beta)\max_{\alpha\in\mathfrak{A}}\max_{u\in\Lambda_\alpha}u+\beta\min_{\alpha\in\mathfrak{A}}\min_{u\in\Lambda_\alpha}u$$

and the result of the proposition follows by arguments, similar to the proof of proposition 3.2.

Proof of proposition 3.8:

First, note that the aspiration level plays no role, when evaluating the available acts, since for each t:

$$\begin{aligned} U_t^I(\tilde{\alpha}) &= \sum_{\tau=1}^{t}\left[k_\tau(\tilde{\alpha})-\bar{u}_t\right]=\sum_{\tau=1}^{t}k_\tau(\tilde{\alpha})-t\bar{u}_t \\ &\text{and} \\ U_t^I(\alpha) &= \sum_{\tau=1}^{t}\left[k_\tau(\alpha)-\bar{u}_t\right]=\sum_{\tau=1}^{t}k_\tau(\alpha)-t\bar{u}_t, \end{aligned}$$

thus

$$\begin{aligned} U_t^I(\tilde{\alpha}) &\geq U_t^I(\alpha) \\ &\Updownarrow \\ \sum_{\tau=1}^{t}k_\tau(\tilde{\alpha}) &\geq \sum_{\tau=1}^{t}k_\tau(\alpha) \end{aligned}$$

Let

$$\alpha^*\in\arg\max_{\alpha\in\mathfrak{A}}\mu_\alpha$$

be one of the acts with the maximal mean utility. Consider the difference

$$\sum_{\tau=1}^{t}\left[k_\tau(\alpha^*)-k_\tau(\alpha)\right].$$

For an arbitrary act α with $\mu_{\alpha^*}>\mu_\alpha$, it is a sequence of independent and identically distributed random variables with mean $\mu_{\alpha^*}-\mu_\alpha>0$. Then, according to the Strong Law of Large Numbers

$$P\left\{\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\left[k_\tau(\alpha^*)-k_\tau(\alpha)\right]=\mu_{\alpha^*}-\mu_\alpha\right\}=1,$$

and thus:

$$P\left\{\lim_{t\to\infty}\sum_{\tau=1}^{t}\left[k_\tau(\alpha^*)-k_\tau(\alpha)\right]=\left(\mu_{\alpha^*}-\mu_\alpha\right)\lim_{t\to\infty}t=\infty\right\}=1.$$

Hence, on almost each path $\omega \in S_1^I$, there exists a $T_\alpha(\omega) \in \mathbb{N}$ such that for each $t > T_\alpha(\omega)$

$$\sum_{\tau=1}^{t} [k_\tau(\alpha^*) - k_\tau(\alpha)] > 0$$

holds on this path. Denote by $T(\omega) = \max_{\alpha \in \mathfrak{A}} T_\alpha(\omega)$ so that for each $t > T(\omega)$, the above inequality is valid on ω for each $\alpha \in \mathfrak{A}$ with $\mu_{\alpha^*} - \mu_\alpha > 0$. Thus, after period $T(\omega)$ the cumulative utility of act α^* will exceed that of all acts with mean utility less than μ_{α^*}. Hence, on almost all paths, none of the acts $\alpha \notin \arg\max_{\alpha \in \mathfrak{A}} \mu_\alpha$ will be chosen after time $T(\omega)$.■

Proof of proposition 3.9:

Consider the i.i.d. random variables $u_t(\alpha_1) ... u_t(\alpha_n)$. According to the Law of Large Numbers, for each $\varepsilon > 0$ and each $\kappa > 0$ there exists a number $\mathcal{K}_\alpha$, such that

$$P\left\{\frac{\sum_{\tau=1}^{t} u_t(\alpha)}{t} \in (\mu_\alpha - \kappa; \mu_\alpha + \kappa)\right\} \geq \left(1 - \frac{\varepsilon}{n}\right) \tag{3.14}$$

holds for each $t \geq \mathcal{K}_\alpha$. Now set $\mathcal{K} = \max_{\alpha \in \mathfrak{A}} \mathcal{K}_\alpha$. Note that the utility realizations of $\alpha_1 ... \alpha_n$ are correlated, because they all depend on δ_a and δ_b. The exact form of correlation depends on the correlation between δ_a and δ_b, which is not specified. Consider therefore the "worst" possible case, in which the sets of paths of dividend realizations on which (3.14) does not hold after time $\mathcal{K}$ are distinct for all $\alpha \in \mathfrak{A}$. Hence, the probability that (3.14) does not hold for some $\alpha \in \mathfrak{A}$ is at most ε. It follows that with probability of at least $(1-\varepsilon)$ (3.14) holds for all $\alpha \in \mathfrak{A}$. Now, if κ is chosen in such a way that

$$\kappa < \frac{\mu_{\alpha^*} - \mu_{\tilde{\alpha}}}{2}$$

for each $\tilde{\alpha}$ with $\mu_{\alpha^*} - \mu_{\tilde{\alpha}} > 0$ and

$$\kappa < \frac{\mu_{\alpha^*} - \bar{u}}{2},$$

then with probability of at least $(1-\varepsilon)$

$$\begin{aligned} U_{\mathcal{K}}^{K}(\alpha^*) &= \mathcal{K}[X_{\mathcal{K}}(\alpha^*) - \bar{u}] \geq \mathcal{K}[\mu_{\alpha^*} - \kappa - \bar{u}] > \\ &> \mathcal{K}[\mu_{\tilde{\alpha}} + \kappa - \bar{u}] \geq \mathcal{K}[X_{\mathcal{K}}(\tilde{\alpha}) - \bar{u}] = U_{\mathcal{K}}^{K}(\tilde{\alpha}) \end{aligned}$$

holds for all $\tilde{\alpha}$ with $\mu_{\alpha^*} - \mu_{\tilde{\alpha}} > 0$ and moreover,

$$U_t^K(\alpha^*) > U_t^K(\tilde{\alpha}) = U_{\mathcal{K}}^K(\tilde{\alpha}),$$

at each $t > \mathcal{K}$. Hence, act α^* is chosen in each period after time $\mathcal{K}$ with probability of at least $(1-\varepsilon)$.■

Proof of proposition 3.10:

Suppose that after $\mathcal{K}$ periods of information acquisition, act α has the highest cumulative utility $U_{\mathcal{K}}(\alpha)$. It is clear that it cannot be the only act to be chosen forever after period $\mathcal{K}$, since in this case its cumulative utility would become $-\infty$ on almost all paths ω, whereas the cumulative utility of all the other acts would remain finite. Thus, all the acts must be chosen infinitely often after period $\mathcal{K}$. Since the cumulative utility of each act in period $\mathcal{K}+1$ is finite (because of the bounded supports Λ_α), it will not have any influence on the limit in (3.12). Therefore, the conclusions of propositions 3.1 and 3.2 hold.■

Proof of proposition 3.11:

The proof is easily obtained by combining the results of propositions 3.5 and 3.10.■

Proof of lemma 3.1

Suppose first that $\bar{u} < \mu_{\bar{\alpha}}$. The cumulative utility of $\bar{\alpha}$, as long as the investor holds it, is then a random walk with differences

$$\bar{\alpha}\delta_a + (1-\bar{\alpha})\,\delta_b - \bar{u}.$$

Since the expected value of the difference is $\mu_{\bar{\alpha}} - \bar{u} > 0$ and the process starts at 0, the expected time until the first period in which the process reaches 0 is ∞. But, as long as $U_t(\bar{\alpha}) > 0$, $U_t(\alpha) = s(\alpha;\bar{\alpha})\,U_t(\bar{\alpha}) \geq U_t(\bar{\alpha})$, since $s(\alpha;\bar{\alpha}) \in [0;1]$ and, therefore, $\bar{\alpha}$ is chosen.

Now suppose that $\bar{u} > \mu_{\bar{\alpha}}$. Then, the expected increments of $U(\bar{\alpha})$ are negative. Therefore, when the process starts at 0, it will cross any finite barrier below 0 in finite time. Let t be the first period, at which $U_t(\bar{\alpha}) < 0$. Then $U_t(\alpha) = s(\alpha;\bar{\alpha})\,U_t(\bar{\alpha}) < 0$. Since $s = (\alpha;\alpha')$ is strictly decreasing in the distance between the acts, $U_t(\alpha)$ has a maximum either at 0 or at 1. Moreover, $s(1;\bar{\alpha}) > s(0;\bar{\alpha})$, iff $\bar{\alpha} > \frac{1}{2}$ and since $U_t(\bar{\alpha}) < 0$, the act lest similar to $\bar{\alpha}$ is chosen. It follows that

$$\alpha_{t+1} = \left\{ \begin{array}{ll} 1, & \text{if } \bar{\alpha} < \frac{1}{2} \\ 0, & \text{if } \bar{\alpha} > \frac{1}{2}. \end{array} \right\} ■$$

Proof of proposition 3.12

It has already been shown that for $\mu_{\bar{\alpha}} - \bar{u} < 0$, the investor switches in finite time to $\alpha = 1$ or to $\alpha = 0$. Suppose, without loss of generality that $\bar{\alpha} > \frac{1}{2}$ and, therefore, $\alpha = 0$ is chosen at some time $\bar{t}$, such that $\bar{t} = \min\{t \mid U_t(\bar{\alpha}) < 0\}$. Two cases are possible: either $\mu_0 - \bar{u} < 0$ or

$\mu_0 - \bar{u} > 0$. Define $V_t(\alpha)$ as:

$$V_t(\alpha) = \sum_{\tau \in C_t(\alpha)} [u_\tau(\alpha) - \bar{u}_t].$$

Then at time $t > \bar{t}$ such that $\alpha_{\tau-1} = 0$ for all $\bar{t} - 1 \leq \tau < t$, the cumulative utility of an act α can be written as:

$$U_t(\alpha) = V_{\bar{t}}(\bar{\alpha})\, s(\alpha; \bar{\alpha}) + V_t(0)\, s(\alpha; 0).$$

As long as $V_t(0) > 0$,

$$U_t(0) = V_{\bar{t}}(\bar{\alpha})\, s(0; \bar{\alpha}) + V_t(0) > V_{\bar{t}}(\bar{\alpha})\, s(\alpha; \bar{\alpha}) + V_t(0)\, s(\alpha; 0) = U_t(\alpha)$$

holds for each $\alpha \in [0; 1]$, where the inequality stems from the fact that

$$V_{\bar{t}}(\bar{\alpha})\, s(0; \bar{\alpha}) \leq V_{\bar{t}}(\bar{\alpha})\, s(\alpha; \bar{\alpha}) \leq V_{\bar{t}}(\bar{\alpha}) < 0$$

and

$$0 \leq V_t(0)\, s(\alpha; 0) \leq V_t(0).$$

If $\mu_0 - \bar{u} > 0$ holds, then $V_t(0) > 0$ holds infinitely long in expectation. If, however, $\mu_0 - \bar{u} < 0$, then

$$V_t(0) < \frac{V_{\bar{t}}(\bar{\alpha})\,(s(1; \bar{\alpha}) - s(0; \bar{\alpha}))}{1 - s(0; 1)} < 0$$

obtains in finite time. Let now $\bar{t}'$ denote

$$\bar{t}' = \min\left\{ t \mid V_t(0) < \frac{V_{\bar{t}}(\bar{\alpha})\,(s(1; \bar{\alpha}) - s(0; \bar{\alpha}))}{1 - s(0; 1)} \right\}.$$

Note that at $\bar{t}'$ the cumulative utility of $\alpha = 1$ is:

$$U_{\bar{t}'}(1) = V_{\bar{t}}(\bar{\alpha})\, s(1; \bar{\alpha}) + V_{\bar{t}'}(0)\, s(1; 0).$$

Moreover, since now $V_{\bar{t}}(\bar{\alpha}) < 0$, $V_{\bar{t}'}(0) < 0$ and s is concave, it follows that at $\bar{t}'$ $U_{\bar{t}'}(\alpha)$ is convex for every $\alpha \in [0; 1]$. Therefore, the optimal act is either 1 or 0. Moreover:

$$U_{\bar{t}'}(1) = V_{\bar{t}}(\bar{\alpha})\, s(1; \bar{\alpha}) + V_{\bar{t}'}(0)\, s(1; 0) > V_{\bar{t}}(\bar{\alpha})\, s(0; \bar{\alpha}) + V_{\bar{t}'}(0) = U_{\bar{t}'}(0),$$

so that $\alpha_{\bar{t}'} = 1$ is chosen.

Again, if $\mu_1 - \bar{u} > 0$, then $\alpha = 1$ will be held infinitely long in expectation, whereas if $\mu_1 - \bar{u} < 0$, then the cumulative utility of $\alpha = 1$ becomes lower than the cumulative utility of $\alpha = 0$ in finite time.

To argue by induction, suppose that an act α is only abandoned in periods $\tilde{t}$ such that $V_{\tilde{t}}(\alpha) < 0$. Suppose that this condition holds up to some time $t - 1$.

Now consider a period t such that $\alpha_{t-1} = 1$. At t, the cumulative utility of an act α is given by:

$$U_t(\alpha) = V_t(\bar{\alpha}) s(0;\bar{\alpha}) + V_t(0) s(\alpha;0) + V_t(1) s(\alpha;1).$$

If $V_t(1) < 0$, then all three terms are negative and, since the similarity function is concave, $U_t(\alpha)$ becomes convex and can only have a corner maximum. Hence, $\alpha_t \in \{0;1\}$. If, on the other hand, $V_t(1) > 0$, it follows that

$$\begin{aligned}
U_t(1) &= V_t(\bar{\alpha}) s(1;\bar{\alpha}) + V_t(0) s(0;1) + V_t(1) = V_{\bar{t}''}(\bar{\alpha}) s(1;\bar{\alpha}) + V_{\bar{t}''}(0) s(0;1) + V_t(1) \\
&> V_{\bar{t}''}(\bar{\alpha}) s(1;\bar{\alpha}) + V_{\bar{t}''}(0) s(0;1) \geq V_{\bar{t}''}(\bar{\alpha}) s(1;\bar{\alpha}) + V_{\bar{t}''}(0) s(0;1) + V_{\bar{t}''}(1) \\
&= U_{\bar{t}''}(1),
\end{aligned}$$

where $\bar{t}''$ denotes the period in which the investor last switched from an arbitrary act to $\alpha = 1$. The second inequality follows from the fact that either $\alpha = 1$ is chosen for the first time at $\bar{t}''$ and, hence, $V_{\bar{t}''} = 0$, or $\alpha = 1$ has been abandoned at some time $\bar{t}''' < \bar{t}''$, which could have only happened for $V_{\bar{t}'''}(1) = V_{\bar{t}''}(1) < 0$. Cumulative utility maximization in $\bar{t}''$ implies:

$$U_{\bar{t}''}(1) \geq U_{\bar{t}''}(\alpha)$$

for each $\alpha \in [0;1]$. Hence,

$$\begin{aligned}
& U_t(1) - U_t(\alpha) \\
=\ & V_t(\bar{\alpha}) [s(1;\bar{\alpha}) - s(\alpha;\bar{\alpha})] + V_t(0) [s(1;0) - s(\alpha;0)] + V_t(1) [1 - s(\alpha;1)] \\
=\ & V_{\bar{t}''}(\bar{\alpha}) [s(1;\bar{\alpha}) - s(\alpha;\bar{\alpha})] + V_{\bar{t}''}(0) [s(1;0) - s(\alpha;0)] + V_t(1) [1 - s(\alpha;1)] \\
>\ & V_{\bar{t}''}(\bar{\alpha}) [s(1;\bar{\alpha}) - s(\alpha;\bar{\alpha})] + V_{\bar{t}''}(0) [s(1;0) - s(\alpha;0)] + V_{\bar{t}''}(1) [1 - s(\alpha;1)] \\
=\ & U_{\bar{t}''}(1) - U_{\bar{t}''}(\alpha) \geq 0.
\end{aligned}$$

Hence, $\alpha_t = 1$ obtains. It follows that act 1 can only be abandoned in a period such that $V_t(1) < 0$.

Now consider a period t such that $\alpha_{t-1} = 0$. The cumulative utility of an act α is given by:

$$U_t(\alpha) = V_t(\bar{\alpha}) s(0;\bar{\alpha}) + V_t(0) s(\alpha;0) + V_t(1) s(\alpha;1).$$

If $V_t(0) < 0$, then all three terms are negative and, since the similarity function is concave, $U_t(\alpha)$ becomes convex and can only have a corner maximum. Hence, $\alpha_t \in \{0;1\}$. If, on the other hand, $V_t(0) > 0$, it follows that

$$\begin{aligned}
U_t(0) &= V_t(\bar{\alpha}) s(0;\bar{\alpha}) + V_t(0) + V_t(1) s(0;1) = V_{\bar{t}''}(\bar{\alpha}) s(0;\bar{\alpha}) + V_t(0) + V_{\bar{t}''}(1) s(0;1) \\
&> V_{\bar{t}''}(\bar{\alpha}) s(0;\bar{\alpha}) + V_{\bar{t}''}(1) s(0;1) \geq V_{\bar{t}''}(\bar{\alpha}) s(0;\bar{\alpha}) + V_{\bar{t}''}(1) s(0;1) + V_{\bar{t}''}(0)
\end{aligned}$$

$$= \quad U_{\bar{t}''}(0),$$

where $\bar{t}''$ denotes the period in which the investor last switched from an arbitrary act to $\alpha = 0$. The second inequality follows from the fact that either $\alpha = 0$ is chosen for the first time at $\bar{t}''$ and, hence, $V_{\bar{t}''} = 0$, or $\alpha = 0$ has been abandoned at some time $\bar{t}''' < \bar{t}''$, which could have only happened for $V_{\bar{t}'''}(1) = V_{\bar{t}''}(1) < 0$. Cumulative utility maximization in $\bar{t}''$ implies:

$$U_{\bar{t}''}(0) \geq U_{\bar{t}''}(\alpha)$$

for each $\alpha \in [0;1]$. Hence,

$$\begin{aligned}
& U_t(0) - U_t(\alpha) \\
= \; & V_t(\bar{\alpha})\left[s(0;\bar{\alpha}) - s(\alpha;\bar{\alpha})\right] + V_t(0)\left[1 - s(\alpha;0)\right] + V_t(1)\left[s(1;0) - s(\alpha;1)\right] \\
= \; & V_{\bar{t}''}(\bar{\alpha})\left[s(0;\bar{\alpha}) - s(\alpha;\bar{\alpha})\right] + V_{\bar{t}''}(0)\left[1 - s(\alpha;0)\right] + V_t(1)\left[s(1;0) - s(\alpha;1)\right] \\
> \; & V_{\bar{t}''}(\bar{\alpha})\left[s(0;\bar{\alpha}) - s(\alpha;\bar{\alpha})\right] + V_{\bar{t}''}(0)\left[1 - s(\alpha;0)\right] + V_{\bar{t}''}(1)\left[s(1;0) - s(\alpha;1)\right] \\
= \; & U_{\bar{t}''}(0) - U_{\bar{t}''}(\alpha) \geq 0.
\end{aligned}$$

Therefore, act $\alpha = 0$ is abandoned only if $V_t(0) < 0$ holds. Moreover, the argument shows that only acts 0 and 1 are chosen after period $\bar{t}$ in which the initially chosen portfolio is abandoned.

Now, consider the following process:

$$\begin{aligned}
\varepsilon_{\bar{t}}(1;0) &= V_{\bar{t}}(\bar{\alpha})\frac{\left[s(1;\bar{\alpha}) - s(0;\bar{\alpha})\right]}{1-s} \\
\varepsilon_{t+1}(1;0) &= \begin{array}{ll} \varepsilon_t + u\left(\delta_{a_{t+1}}\right) - \bar{u}, & \text{if } \varepsilon_t \geq 0 \\ \varepsilon_t + u\left(\delta_{b_{t+1}}\right) - \bar{u}, & \text{if } \varepsilon_t < 0 \end{array}.
\end{aligned}$$

$(1-s)\,\varepsilon_t(1;0)$ represents the difference between the cumulative utilities of the acts $a = 1$ and $a = 0$ after period $\bar{t}$. To see this note that

$$\begin{aligned}
& U_t(1) - U_t(0) \\
= \; & \left[V_t(1) + sV_t(0) + V_{\bar{t}}(\bar{\alpha})\,s(1;\bar{\alpha})\right] - \left[sV_t(1) + V_{\bar{t}''}(0) + V_{\bar{t}}(\bar{\alpha})\,s(0;\bar{\alpha})\right] = \\
= \; & (1-s)\left[V_t(1) - V_t(0)\right] + V_{\bar{t}}(\bar{\alpha})\left[s(1;\bar{\alpha}) - s(0;\bar{\alpha})\right]
\end{aligned}$$

and

$$\varepsilon_t(1;0) = V_t(1) - V_t(0) + V_{\bar{t}}(\bar{\alpha})\frac{\left[s(1;\bar{\alpha}) - s(0;\bar{\alpha})\right]}{1-s}.$$

If at least one of the two mean utilities μ_1 and μ_2 exceeds $\bar{u}$, then the frequencies π_0 and π_1 are obtained from proposition 3.3.

Suppose now that both μ_1 and μ_2 lie below $\bar{u}$. Note that if one of the two acts, say $a = 0$ were chosen only for a finite number of times, the cumulative utility of the other act, $a = 1$, would converge almost surely to $-\infty$, since $\mu_1 - \bar{u} < 0$. Let $\bar{t}''$ denote the last period in which $a = 0$ is chosen (depending on the path of dividend realizations). Then, it would be possible to find a period $t > \bar{t}''$ such that

$$V_t(1) < V_{\bar{t}''}(0) + V_{\bar{t}}(\bar{\alpha}) \frac{[s(0;\bar{\alpha}) - s(1;\bar{\alpha})]}{1-s}$$

holds and, therefore,

$$U_t(1) = V_t(1) + sV_{\bar{t}''}(0) + V_{\bar{t}}(\bar{\alpha})\, s(1;\bar{\alpha}) < sV_t(1) + V_{\bar{t}''}(0) + V_{\bar{t}}(\bar{\alpha})\, s(0;\bar{\alpha}) = U_t(0).$$

Hence, the assumption that act 0 is chosen only for a finite number of times contradicts the case-based decision rule on almost all possible paths of dividend realizations. Analogous reasoning shows that (because of the assumption $\mu_2 - \bar{u} < 0$), act $a = 1$ cannot be chosen for a finite number of times on a set of paths with a positive probability measure.

As long as act $a = 1$ is chosen, the difference $\varepsilon_t(1;0)$ represents a random walk on the half line with negative expected increment. An argument analogous to the one used in the proof of proposition 3.2 shows that $\varepsilon_t(1;0)$ remains bounded on almost all paths in Φ_6. It follows that

$$\lim_{t\to\infty} \frac{U_t(1)}{U_t(0)} = \lim_{t\to\infty} \frac{U_t(0) + (1-s)\,\varepsilon_t(1;0)}{U_t(0)} = 1$$

with probability 1. Hence,

$$\begin{aligned}
\lim_{t\to\infty} \frac{[V_t(1) + sV_t(0) + V_{\bar{t}}(\bar{\alpha})\, s(1;\bar{\alpha})]}{[V_t(0) + sV_t(1) + V_{\bar{t}}(\bar{\alpha})\, s(0;\bar{\alpha})]} &= 1 \\
\lim_{t\to\infty} \frac{\left[|C_t(1)| \sum_{\tau\in C_t(1)} \frac{[u_\tau - \bar{u}]}{|C_t(1)|} + s\,|C_t(0)| \sum_{\tau\in C_t(0)} \frac{[u_\tau - \bar{u}]}{|C_t(0)|} + V_{\bar{t}}(\bar{\alpha})\, s(1;\bar{\alpha})\right]}{\left[s\,|C_t(1)| \sum_{\tau\in C_t(1)} \frac{[u_\tau - \bar{u}]}{|C_t(1)|} + |C_t(0)| \sum_{\tau\in C_t(0)} \frac{[u_\tau - \bar{u}]}{|C_t(0)|} + V_{\bar{t}}(\bar{\alpha})\, s(0;\bar{\alpha})\right]} &= 1.
\end{aligned}$$

Since $|C_t(1)| \to \infty$ and $|C_t(0)| \to \infty$ on almost each path, it follows according to the Law of Large Numbers that

$$\begin{aligned}
\lim_{t\to\infty} \frac{\sum_{\tau\in C_t(1)} [u_\tau - \bar{u}]}{|C_t(1)|} &= \mu_1 - \bar{u} \\
\lim_{t\to\infty} \frac{\sum_{\tau\in C_t(0)} [u_\tau - \bar{u}]}{|C_t(0)|} &= \mu_0 - \bar{u}
\end{aligned}$$

obtain almost surely in the limit. Hence,

$$\lim_{t\to\infty} \frac{[|C_t(1)|(\mu_1 - \bar{u}) + s\,|C_t(0)|(\mu_0 - \bar{u}) + V_{\bar{t}}(\bar{\alpha})\, s(1;\bar{\alpha})]}{[s\,|C_t(1)|(\mu_1 - \bar{u}) + |C_t(0)|(\mu_0 - \bar{u}) + V_{\bar{t}}(\bar{\alpha})\, s(0;\bar{\alpha})]} = 1.$$

$$\begin{aligned}
& \lim_{t\to\infty} \frac{[|C_t(1)|(\mu_1 - \bar{u}) + s\,|C_t(0)|(\mu_0 - \bar{u}) + V_{\bar{t}}(\bar{\alpha})\, s(1;\bar{\alpha})]}{[s\,|C_t(1)|(\mu_1 - \bar{u}) + |C_t(0)|(\mu_0 - \bar{u}) + V_{\bar{t}}(\bar{\alpha})\, s(0;\bar{\alpha})]} \\
= & \lim_{t\to\infty} \frac{\left[\frac{|C_t(1)|}{|C_t(0)|}(\mu_1 - \bar{u}) + s(\mu_0 - \bar{u}) + \frac{V_{\bar{t}}(\bar{\alpha})s(1;\bar{\alpha})}{|C_t(0)|}\right]}{\left[s\frac{|C_t(1)|}{|C_t(0)|}(\mu_1 - \bar{u}) + (\mu_0 - \bar{u}) + \frac{V_{\bar{t}}(\bar{\alpha})s(0;\bar{\alpha})}{|C_t(0)|}\right]} =
\end{aligned}$$

$$= \lim_{t\to\infty} \frac{\frac{|C_t(1)|}{|C_t(0)|}(\mu_1 - \bar{u}) + s(\mu_0 - \bar{u})}{s\frac{|C_t(1)|}{|C_t(0)|}(\mu_1 - \bar{u}) + (\mu_0 - \bar{u})} = 1$$

almost surely holds (since $V_{\bar{t}}(\bar{\alpha})$ is finite on almost all paths, it does not influence the limit behavior). Therefore, the limit frequencies π_1 and π_0 satisfy

$$\frac{\pi_1}{\pi_0} = \lim_{t\to\infty} \frac{|C_t(1)|}{|C_t(0)|} = \frac{\mu_0 - \bar{u}}{\mu_1 - \bar{u}}.\blacksquare$$

Proof of proposition 3.13

First, I show that a diversified portfolio is never chosen. Indeed, suppose that up to time $\bar{t}$, only undiversified portfolios have been chosen. Then the cumulative utility of any portfolio $(\alpha; \hat{\alpha})$ is given by

$$U_{\bar{t}}(\alpha; \hat{\alpha}) = V_{\bar{t}}(a)\, s((a); (\alpha; \hat{\alpha})) + V_{\bar{t}}(\hat{a})\, s((\hat{a}); (\alpha; \hat{\alpha})) + V_{\bar{t}}(b)\, s((b); (\alpha; \hat{\alpha})).$$

Note that if an act has been chosen in the past at least once and is not chosen at time $(\bar{t} - 1)$, then its $V_{\bar{t}}$ must be negative, else it would not have been abandoned for another act. Indeed, if $(\alpha'; \hat{\alpha}') \neq (\alpha_{t-1}; \hat{\alpha}_{t-1})$ is chosen at time $\tilde{t}$, then

$$U_{\tilde{t}}(\alpha; \hat{\alpha}) \leq U_{\tilde{t}}(\alpha'; \hat{\alpha}')$$

for all $(\alpha; \hat{\alpha}) \in [0; 1]^2$ must hold at this time. As long as $V_t(\alpha; \hat{\alpha}) - V_{\tilde{t}}(\alpha; \hat{\alpha})$ is positive,

$$U_t(\alpha; \hat{\alpha}) \leq U_t(\alpha'; \hat{\alpha}')$$

still holds for all $(\alpha; \hat{\alpha}) \in [0; 1]^2$. The investor can switch to a different act, only if $V_t(\alpha; \hat{\alpha}) - V_{\tilde{t}}(\alpha; \hat{\alpha}) < 0$ holds. Hence, the first switch away from $(\alpha'; \hat{\alpha}')$ occurs at $V_t(\alpha'; \hat{\alpha}') < 0$. When at time $\tilde{t}'$, $(\alpha'; \hat{\alpha}')$ is chosen again, $V_{\tilde{t}'}(\alpha'; \hat{\alpha}') < 0$ and, therefore, the next switch away from $(\alpha'; \hat{\alpha}')$ occurs at t such that

$$V_t(\alpha; \hat{\alpha}) < V_t(\alpha; \hat{\alpha}) - V_{\tilde{t}'}(\alpha; \hat{\alpha}) < 0.$$

Hence, at $\bar{t}$, $V_{\bar{t}}(\alpha'; \hat{\alpha}') \leq 0$ must hold. Therefore, at $\bar{t}$, at most one of the corner acts can have a positive $V_{\bar{t}}$, namely the one chosen at time $(\bar{t} - 1)$. If $V_{\bar{t}}(\alpha_t; \hat{\alpha}_t) > 0$, then

$$(\alpha_{\bar{t}}; \hat{\alpha}_{\bar{t}}) = \arg \max_{(\alpha;\hat{\alpha})\in[0;1]^2} U_{\bar{t}}(\alpha; \hat{\alpha}) = (\alpha_{\bar{t}-1}; \hat{\alpha}_{\bar{t}-1})$$

and, therefore, an undiversified act is chosen again. If, on the other hand, $V_{\bar{t}}(\alpha_t; \hat{\alpha}_t) < 0$, then the function $U_{\bar{t}}(\alpha; \hat{\alpha})$ is a sum of convex functions and has, therefore, a corner optimum. Hence, again an undiversified act is chosen. It follows that starting with an undiversified portfolio, the investor never diversifies and, hence,

$$\pi(\alpha; \hat{\alpha}) = 0 \text{ for all } (\alpha; \hat{\alpha}) \notin \{a; \hat{a}; b\}.$$

Suppose now that

$$s_{ab} + s_{\hat{a}b} \geq 1$$

holds and assume that the investor has chosen to hold the undiversified portfolio consisting only of asset b in the first period,

$$(\alpha_1; \hat{\alpha}_1) = b.$$

Without loss of generality, assume

$$s_{ab} < s_{\hat{a}b}.$$

Since $\mu_b < \bar{u}$,

$$U_{\bar{t}}(b) = V_{\bar{t}}(b) < 0$$

obtains almost surely in finite time. Hence, at $\bar{t}$, $(\alpha; \hat{\alpha}) = a$ is chosen, since

$$U_{\bar{t}}(a) = s_{ab} V_{\bar{t}}(b) > s_{\hat{a}b} V_{\bar{t}}(b) = U_{\bar{t}}(\hat{a}) > V_{\bar{t}}(b) = U_{\bar{t}}(b).$$

As long as a is chosen, the cumulative utility of the portfolios consisting only of b and only of $\hat{a}$ is given by:

$$\begin{aligned} U_t(b) &= V_{\bar{t}}(b) + s_{ab} V_t(a) \\ U_t(\hat{a}) &= s_{\hat{a}b} V_{\bar{t}}(b), \end{aligned}$$

since the similarity between $\hat{a}$ and a is 0. Hence,

$$U_t(\hat{a}) > U_t(b)$$

at each such t and, especially, in the first period $\bar{t}'$ such that

$$\begin{aligned} U_{\bar{t}'}(a) &< U_{\bar{t}'}(b) \text{ or} \\ U_{\bar{t}'}(a) &< U_{\bar{t}'}(\hat{a}) \end{aligned}$$

holds. Clearly, $\bar{t}'$ is almost surely finite, since

$$\begin{aligned} U_t(a) - U_t(b) &= (s_{ab} - 1)(V_{\bar{t}}(b) - V_t(a)) \\ U_t(a) - U_t(\hat{a}) &= (s_{ab} - s_{\hat{a}b}) V_{\bar{t}}(b) + V_t(a) \end{aligned}$$

and $V_t(a)$ has negative expected increments $\mu_a - \bar{u} < 0$. Hence, in period $\bar{t}'$, act $\hat{a}$ is chosen.

As long as the investor holds portfolio $\hat{a}$, the cumulative utilities of the three undiversified portfolios satisfy:

$$\begin{aligned} U_t(b) &= V_{\bar{t}}(b) + s_{ab} V_{\bar{t}'}(a) + s_{\hat{a}b} V_t(\hat{a}) \\ U_t(\hat{a}) &= s_{\hat{a}b} V_{\bar{t}}(b) + V_t(\hat{a}) \\ U_t(a) &= s_{ab} V_{\bar{t}}(b) + V_{\bar{t}'}(a) \end{aligned}$$

Note that as long as $V_t(\hat{a}) > 0$ holds, act $\hat{a}$ is chosen, since $V_{\bar{t}}(b) < 0$ and $V_{\bar{t}'}(a) < 0$ hold. Once, however, $V_t(\hat{a}) < 0$ obtains,

$$s_{ab} + s_{\hat{a}b} \geq 1$$

implies that

$$s_{ab} V_{\bar{t}'}(a) + s_{\hat{a}b} V_t(\hat{a}) < \max\{V_t(\hat{a}); V_{\bar{t}'}(a)\}.$$

To see this, assume without loss of generality[65], $V_t(\hat{a}) \geq V_{\bar{t}'}(a)$ and note that

$$\begin{aligned} & (s_{ab} + s_{\hat{a}b}) V_t(\hat{a}) + s_{ab}\left[V_{\bar{t}'}(a) - V_t(\hat{a})\right] \\ < \quad & V_t(\hat{a}) + s_{ab}\left[V_{\bar{t}'}(a) - V_t(\hat{a})\right] \leq V_t(\hat{a}) \end{aligned}$$

holds since both $V_t(\hat{a}) < 0$ and $V_{\bar{t}'}(a) \leq V_t(\hat{a})$ are negative. Hence,

$$U_t(b) < \max\{U_t(\hat{a}); U_t(a)\}$$

and, therefore, act b is not chosen.

If portfolio a or portfolio $\hat{a}$ is the first one chosen, then the term $V_{\bar{t}'}(b) = 0$ and the above argument applies as well.

Hence, at period $\bar{t}''$ when the next switch occurs, the investor chooses again a. Applying this argument inductively and noting that in each period of time, $V_t(\alpha; \hat{\alpha}) > 0$ can hold for at most one portfolio at a time implies that b is never chosen again after period $\bar{t}$. Hence, its limit frequency is almost surely 0.

In contrast, a and $\hat{a}$ must be chosen for an infinite number of periods each by an argument analogous to that in the proof of proposition 3.12. Moreover, since the similarity between these two assets is 0 and since the finite $V_{\bar{t}'}(a)$ does not influence the limit behavior, it follows that the frequencies with which $\hat{a}$ and a are chosen are determined analogously to the proof of proposition 3.2 and are given by

$$\frac{\pi(a)}{\pi(\hat{a})} = \frac{\mu_{\hat{a}} - \bar{u}}{\mu_a - \bar{u}}.$$

Suppose now that

$$s_{ab} + s_{\hat{a}b} < 1$$

holds. Assume that portfolio b is chosen only for a finite number of times. Denote the last period in which b is chosen by $\bar{t}$. As in the first case, it can be shown that a and $\hat{a}$ must be chosen infinitely often almost surely and that the difference of their cumulative utilities is almost surely

[65] A symmetric argument holds for $V_t(\hat{a}) \leq V_{\bar{t}'}(a)$.

bounded above and below. Now consider the difference between the cumulative utilities of b and $\hat{a}$:

$$\begin{aligned} U_t(\hat{a}) - U_t(b) &= [V_t(\hat{a}) - V_{\bar{t}}(b)](1 - s_{\hat{a}b}) - V_t(a)\, s_{ab} = \\ &= [V_t(\hat{a}) - V_t(a)]\, s_{ab} - V_{\bar{t}}(b)(1 - s_{\hat{a}b}) \\ &\quad + V_t(\hat{a})(1 - s_{\hat{a}b} - s_{ab}). \end{aligned}$$

Whereas $[V_t(\hat{a}) - V_t(a)]$ has the same limit properties as $U_t(\hat{a}) - U_t(a)$ and is, therefore, bounded above and below, $V_{\bar{t}}(b)$ is finite and

$$V_t(\hat{a}) \to -\infty$$

almost surely since $\hat{a}$ is chosen infinitely often and the expected increments of $V_t(\hat{a})$ are negative, $\mu_{\hat{a}} - \bar{u} < 0$. Combined with

$$s_{ab} + s_{\hat{a}b} < 1,$$

this implies that

$$U_t(\hat{a}) - U_t(b) \to -\infty$$

almost surely. In the same way, it can be shown that

$$U_t(a) - U_t(b) \to -\infty$$

But then, on almost each path, it would be possible to find a period $\tilde{t}$, such that

$$\begin{aligned} U_{\tilde{t}}(a) &< U_{\tilde{t}}(b) \\ U_{\tilde{t}}(\hat{a}) &< U_{\tilde{t}}(b) \end{aligned}$$

and still act b is not chosen. This, obviously, contradicts the case-based decision rule. Hence, act b must be chosen infinitely often on almost all paths.

Assuming that act $\hat{a}$ is chosen for only a finite number of times with $\bar{t}'$ being the last period in which $\hat{a}$ is chosen, whereas the other two portfolios are chosen infinitely often, would imply

$$\begin{aligned} U_t(\hat{a}) - U_t(b) &= [V_{\bar{t}'}(\hat{a}) - V_t(b)](1 - s_{\hat{a}b}) - V_t(a)\, s_{ab} = \\ &= [V_t(b) - V_t(a)]\, s_{ab} + V_{\bar{t}'}(\hat{a})(1 - s_{\hat{a}b}) \\ &\quad - V_t(b)(1 - s_{\hat{a}b} + s_{ab}). \end{aligned}$$

Hence,

$$U_t(\hat{a}) - U_t(b) \to \infty,$$

which is in contradiction with the case-based decision rule. The same argument applies to the case when act a is chosen for a finite number of times, whereas acts b and a are chosen infinitely often.

Now suppose that two acts are chosen for a finite number of periods each. Obviously, these cannot be acts b and $\hat{a}$, since then

$$U_t(a) \to -\infty,$$

whereas $U_t(\hat{a})$ remains finite, implying a contradiction to the case-based decision rule. In the same way, it cannot be the acts b and a. Hence, suppose that a and $\hat{a}$ are chosen for a finite number of times each. Then

$$U_t(\hat{a}) - U_t(b) = [V_t(\hat{a}) - V_t(b)](1 - s_{\hat{a}b}) - V_t(a)\, s_{ab}$$

and since $V_t(\hat{a})$ and $V_t(a)$ are finite, whereas $V(b)$ has negative expected increments, it follows that

$$U_t(\hat{a}) - U_t(b) \to \infty$$

and still act b is always chosen after some period $\bar{t}''$, which again contradicts the case-based decision rule. Hence, each of the three acts must be chosen for an infinite number of times on almost each path.

Now write the differences between the cumulative utilities as:

$$\begin{aligned}
\varepsilon_t(a;\hat{a}) &= V_t(a) - V_t(\hat{a}) + V_t(b)[s_{ab} - s_{\hat{a}b}] \\
\varepsilon_t(b;\hat{a}) &= (1 - s_{\hat{a}b})[V_t(b) - V_t(\hat{a})] + s_{ab}V_t(a) \\
\varepsilon_t(a;b) &= (1 - s_{ab})[V_t(a) - V_t(b)] + s_{\hat{a}b}V_t(\hat{a}).
\end{aligned}$$

At time t, at which the investor switches from $\hat{a}$ to a,

$$0 \le \varepsilon_t(a;\hat{a}) \le \bar{u} - u(\underline{\delta}_{\hat{a}})$$

holds. From t on, $\varepsilon_t(a;\hat{a})$ behaves as a random walk on the half line with negative expected increments, as long as a is chosen. Hence, the expected time until its return to 0 is almost surely finite and uniformly bounded above for all initial values on the interval $[0; \bar{u} - u(\underline{\delta}_{\hat{a}})]$. In the same way, if the investor switches from $\hat{a}$ to b,

$$0 \le \varepsilon_t(b;\hat{a}) \le (1 - s_{\hat{a}b})[\bar{u} - u(\underline{\delta}_{\hat{a}})]$$

and $\varepsilon_t(b;\hat{a})$ behaves as a random walk on the half line with negative expected increments, as long as b is chosen. Hence, the expected time until its return to 0 is almost surely finite and uniformly bounded above for initial values on the interval

$$[0; (1 - s_{\hat{a}b})[\bar{u} - u(\underline{\delta}_{\hat{a}})]].$$

Analogous arguments apply for the other two portfolios b and a. Hence, the argument of the proof of proposition 3.2 can be used to show that the differences ε_t are bounded above and below almost surely. It follows that

$$\begin{aligned}\lim_{t\to\infty}\frac{U_t(a)}{U_t(b)} &= 1 \qquad (3.15)\\ \lim_{t\to\infty}\frac{U_t(\hat{a})}{U_t(b)} &= 1\\ \lim_{t\to\infty}\frac{U_t(a)}{U_t(\hat{a})} &= 1\end{aligned}$$

holds on almost each path.

Because of $|C_t(\alpha;\hat{\alpha})| \to \infty$ on almost all paths for all $(\alpha;\hat{\alpha}) \in \{\hat{a};a;b\}$, it follows that

$$\lim_{t\to\infty}\frac{\sum_{\tau\in C_t(\alpha;\hat{\alpha})}[u_\tau(\alpha;\hat{\alpha})-\bar{u}]}{|C_t(\alpha;\hat{\alpha})|} = \mu_{\alpha;\hat{\alpha}} - \bar{u}$$

holds with probability 1 for all non-diversified portfolios. Hence, (3.15) can be rewritten as

$$\begin{aligned}\lim_{t\to\infty}\frac{|C_t(a)|(\mu_a-\bar{u})+s_{ab}|C_t(b)|(\mu_b-\bar{u})}{s_{ab}|C_t(a)|(\mu_a-\bar{u})+|C_t(b)|(\mu_b-\bar{u})+s_{\hat{a}b}|C_t(\hat{a})|(\mu_{\hat{a}}-\bar{u})} &= 1\\ \lim_{t\to\infty}\frac{|C_t(\hat{a})|(\mu_{\hat{a}}-\bar{u})+s_{\hat{a}b}|C_t(b)|(\mu_b-\bar{u})}{s_{ab}|C_t(a)|(\mu_a-\bar{u})+|C_t(b)|(\mu_b-\bar{u})+s_{\hat{a}b}|C_t(\hat{a})|(\mu_{\hat{a}}-\bar{u})} &= 1\\ \lim_{t\to\infty}\frac{|C_t(a)|(\mu_a-\bar{u})+s_{ab}|C_t(b)|(\mu_b-\bar{u})}{s_{\hat{a}b}|C_t(b)|(\mu_b-\bar{u})+|C_t(\hat{a})|(\mu_{\hat{a}}-\bar{u})} &= 1.\end{aligned}$$

Since $\frac{\pi(\alpha;\hat{\alpha})}{\pi(\alpha';\hat{\alpha}')} = \lim_{t\to\infty}\frac{|C_t(\alpha;\hat{\alpha})|}{|C_t(\alpha';\hat{\alpha}')|}$ by definition, it follows that

$$\begin{aligned}\frac{\pi(a)(\mu_a-\bar{u})+s_{ab}\pi(b)(\mu_b-\bar{u})}{s_{ab}\pi(a)(\mu_a-\bar{u})+\pi(b)(\mu_b-\bar{u})+s_{\hat{a}b}\pi(\hat{a})(\mu_{\hat{a}}-\bar{u})} &= 1\\ \frac{\pi(\hat{a})(\mu_{\hat{a}}-\bar{u})+s_{\hat{a}b}\pi(b)(\mu_b-\bar{u})}{s_{ab}\pi(a)(\mu_a-\bar{u})+\pi(b)(\mu_b-\bar{u})+s_{\hat{a}b}\pi(\hat{a})(\mu_{\hat{a}}-\bar{u})} &= 1\\ \frac{\pi(a)(\mu_a-\bar{u})+s_{ab}\pi(b)(\mu_b-\bar{u})}{s_{\hat{a}b}\pi(b)(\mu_b-\bar{u})+\pi(\hat{a})(\mu_{\hat{a}}-\bar{u})} &= 1.\end{aligned}$$

After simplifying and solving for $\frac{\pi(b)}{\pi(a)}$, $\frac{\pi(b)}{\pi(\hat{a})}$ and $\frac{\pi(a)}{\pi(\hat{a})}$, the relationships stated in the proposition obtains.■

Proof of proposition 3.14

First note that the similarity between 0 and 1 can be normalized to 0 without loss of generality, see Gilboa and Schmeidler (1997 (a)). To see this perform the following linear transformation of the similarity function:

$$s'(\alpha;\alpha') = \frac{s(\alpha;\alpha')-s(0;1)}{\max_{\alpha,\alpha'\in[0;1]}\{s(\alpha;\alpha')-s(0;1)\}}.$$

It is clear that the new similarity function s' takes on only non-negative values between 0 and 1 and ascribes a similarity of 0 to the pair $(0;1)$.

Since $U_t(\alpha) \geq U_t(\alpha')$, iff

$$\sum_{\tau=1}^{t} s(\alpha;\alpha'')[u_\tau(\alpha'') - \bar{u}_t] \geq \sum_{\tau=1}^{t} s(\alpha';\alpha'')[u_\tau(\alpha'') - \bar{u}_t]$$

it follows that a linear transformation of the similarity function does not change the preferences of an investor. Hence, using s' instead of s does not influence the behavior of the investor. Hence, it is assumed that

$$s(0;1) = 0$$

holds.

I first show that if the aspiration level is updated according to (3.3), only non-diversified portfolios are chosen from period $t = 2$ on. Indeed, in period $t = 2$

$$U_2(\bar{\alpha}) = V_2(\bar{\alpha}) = u_1(\bar{\alpha}) - \beta\bar{u}_1 - (1-\beta)u_1(\bar{\alpha}) = \beta[u_1(\bar{\alpha}) - \bar{u}_1] < 0$$

according to the assumption made. Hence, $\bar{u}_2 > X_2(\bar{\alpha})$. Moreover, (3.3) implies $\bar{u}_t \geq X_2(\bar{\alpha})$ as long as $\bar{\alpha}$ is not chosen again. To see this, note that the aspiration level is updated towards $X_t \geq X_2(\bar{\alpha}) = u_1(\bar{\alpha})$. Hence, starting with $\bar{u}_2 > X_2(\bar{\alpha})$, the aspiration level remains above $X_2(\bar{\alpha})$ as long as $\bar{\alpha}$ is not chosen again. Therefore, $V_t(\bar{\alpha}) \leq 0$ as long as $\alpha_\tau \neq \bar{\alpha}$ for all $t > \tau > 2$.

In period $t = 2$, the cumulative utility of an arbitrary act α is given by:

$$U_2(\alpha) = V_2(\bar{\alpha})\, s(\alpha;\bar{\alpha}) < 0,$$

and since $s(\alpha;\bar{\alpha})$ is strictly decreasing,

$$\arg\max_{\alpha\in[0;1]} U_2(\alpha) \in \{0;1\}.$$

Assume without loss of generality that $\alpha_2 = 0$, hence that

$$\min_{\alpha\in[0;1]} s(\alpha;\bar{\alpha}) = 0$$

As long as $\alpha_\tau = 0$ is chosen for all $t > \tau \geq 2$

$$U_t(\alpha) = s(\alpha;\bar{\alpha})\, V_t(\bar{\alpha}) + s(\alpha;0)\, V_t(0)$$

holds. If $V_t(0) > 0$, $\alpha = 0$ is chosen, since

$$\begin{aligned} U_t(\alpha) &= s(\alpha;\bar{\alpha})\, V_t(\bar{\alpha}) + s(\alpha;0)\, V_t(0) \leq \\ &\leq s(0;\bar{\alpha})\, V_t(\bar{\alpha}) + V_t(0) = U_t(0). \end{aligned}$$

If $V_t(0) < 0$ obtains, the function

$$U_t(\alpha) = s(\alpha;\bar{\alpha}) V_t(\bar{\alpha}) + s(\alpha;0) V_t(0)$$

becomes convex (since now $V_t(\bar{\alpha}) < 0$ and $V_t(0) < 0$ hold and s is concave). Therefore,

$$\arg\max_{\alpha\in[0;1]} U_t(\alpha) \in \{0;1\}.$$

Suppose that $\alpha = 1$ is chosen at some time $\bar{t}'$. As long as $\alpha_\tau \neq \bar{\alpha}$ for all $t > \tau \geq \bar{t}'$,

$$U_t(\alpha) = s(\alpha;\bar{\alpha}) V_t(\bar{\alpha}) + s(\alpha;0) V_t(0) + s(\alpha;1) V_t(1).$$

$V_{\bar{t}'}(0) < 0$, is implied by the usage of the case-based decision rule at $\bar{t}'$. Applying the same argument as above, $V_t(0) < 0$ as long as $\alpha = 0$ is not chosen again.

If $V_t(1) \leq 0$ holds at t, the cumulative utility function becomes convex and obtains its maximum at 0 or 1.

Consider, therefore, the case $V_t(1) > 0$. Since $V_t(1) \geq 0 > \max\{V_t(\bar{\alpha}); V_t(0)\}$, it follows that $X_t = X_t(1)$. Denote by $\tilde{t}$ the last period between $\bar{t}'$ and the current period t at which $V(1)$ was non-positive and still $\alpha = 1$ was selected:

$$\tilde{t} = \max\{\bar{t}'; \tau < t \mid V_\tau(1) \leq 0, \alpha_\tau = 1, V_{\tau+1}(1) > 0\}.$$

Since $V_{\tilde{t}}(1) \leq 0$, it follows that $X_{\tilde{t}}(1) - \bar{u}_{\tilde{t}} \leq 0$ holds. On the other hand, since $V_{\tilde{t}+1}(1) > 0$,

$$X_{\tilde{t}+1}(1) - \bar{u}_{\tilde{t}+1} > 0 > \max\{X_{\tilde{t}+1}(0) - \bar{u}_{\tilde{t}+1}; X_{\tilde{t}+1}(\bar{\alpha}) - \bar{u}_{\tilde{t}+1}\}.$$

Hence, $X_{\tilde{t}+1} = X_{\tilde{t}+1}(1)$ and, therefore,

$$\bar{u}_{\tilde{t}+1} = \beta\bar{u}_{\tilde{t}} + (1-\beta) X_{\tilde{t}+1}(1).$$

Since $X_{\tilde{t}+1}(1) - \bar{u}_{\tilde{t}+1} > 0$,

$$\bar{u}_{\tilde{t}+1} > \bar{u}_{\tilde{t}}$$

follows.

At time $\tilde{t}+2$, $V_{\tilde{t}+2}(1) > 0$ holds and, therefore, again $X_{\tilde{t}+2} = X_{\tilde{t}+2}(1) > \bar{u}_{\tilde{t}+2}$ and

$$\bar{u}_{\tilde{t}+2} = \beta\bar{u}_{\tilde{t}+1} + (1-\beta) X_{\tilde{t}+2}(1).$$

Hence,

$$\bar{u}_{\tilde{t}+2} > \bar{u}_{\tilde{t}+1}$$

obtains. Reasoning inductively, we obtain that $\bar{u}_t > \bar{u}_{\tilde{t}}$.

At time t, the cumulative utility of act $\alpha = 1$ is given by

$$U_t(1) = s(1;\bar{\alpha}) V_t(\bar{\alpha}) + V_t(1).$$

Hence, $\alpha = 1$ is chosen if

$$s(1;\bar{\alpha}) V_t(\bar{\alpha}) + V_t(1) \geq s(\alpha;\bar{\alpha}) V_t(\bar{\alpha}) + s(\alpha;0) V_t(0) + s(\alpha;1) V_t(1) \tag{3.16}$$

holds for all $\alpha \in [0;1]$. Rewrite (3.16) as

$$V_t(1)(1 - s(\alpha;1)) - s(\alpha;0) V_t(0) + [s(1;\bar{\alpha}) - s(\alpha;\bar{\alpha})] V_t(\bar{\alpha}) \geq 0$$

$$V_t(1)(1 - s(\alpha;1)) - s(\alpha;0)(\bar{t}' - 2)[X_{\tilde{t}}(0) - \bar{u}_{\tilde{t}}] + [s(1;\bar{\alpha}) - s(\alpha;\bar{\alpha})][u_1(\bar{\alpha}) - \bar{u}_{\tilde{t}}]$$

$$+ [\bar{u}_t - \bar{u}_{\tilde{t}}][s(\alpha;0)(\bar{t}' - 2) - s(1;\bar{\alpha}) + s(\alpha;\bar{\alpha})] \geq 0.$$

Consider the function $s(\alpha;0)(\bar{t}' - 2) + s(\alpha;\bar{\alpha})$ and note that since $s(\cdot;\cdot)$ is concave, it has a minimum at 0 or at 1. For $\alpha = 1$,

$$s(\alpha;0)(\bar{t}' - 2) + s(\alpha;\bar{\alpha}) = s(1;\bar{\alpha}),$$

whereas for $\alpha = 0$,

$$(\bar{t}' - 2) + s(0;\bar{\alpha}) > 1 + s(0;\bar{\alpha}) > s(1;\bar{\alpha}).$$

It follows that:

$$\min_{\alpha \in [0;1]} [\bar{u}_t - \bar{u}_{\tilde{t}}][s(\alpha;0)(\bar{t}' - 2) - s(1;\bar{\alpha}) + s(\alpha;\bar{\alpha})] = 0.$$

Therefore,

$$\begin{aligned} U_t(1) - U_t(\alpha) &\geq V_t(1)(1 - s(1;\alpha)) - s(\alpha;0) V_{\tilde{t}}(0) + [s(1;\bar{\alpha}) - s(\alpha;\bar{\alpha})] V_{\tilde{t}}(\bar{\alpha}) \\ &> V_{\tilde{t}}(1)(1 - s(\alpha;1)) - s(\alpha;0) V_{\tilde{t}}(0) + [s(1;\bar{\alpha}) - s(\alpha;\bar{\alpha})] V_{\tilde{t}}(\bar{\alpha}) \\ &= U_{\tilde{t}}(1) - U_{\tilde{t}}(\alpha) \geq 0 \end{aligned}$$

for each $\alpha \in [0;1]$, since at time $\tilde{t}$, $V_{\tilde{t}}(1) \leq 0$ holds and $\alpha_{\tilde{t}} = 1$. Hence, $\alpha_t = 1$ is chosen.

Analogously, if $\alpha_{t-1} = 0$ and $V_t(0) < 0$, the optimum is a corner solution, whereas if $V_t(0) \geq 0$ holds, the difference between the cumulative utilities of 0 and arbitrary act α becomes:

$$U_t(0) - U_t(\alpha) = V_t(0)(1 - s(\alpha;0)) - s(\alpha;1) V_t(1) + [s(0;\bar{\alpha}) - s(\alpha;\bar{\alpha})] V_t(\bar{\alpha}) \geq 0,$$

since $V_t(0)(1 - s(\alpha;0)) \geq 0$, $V_t(1) < 0$, $V_t(\bar{\alpha}) < 0$ and $s(0;\bar{\alpha}) - s(\alpha;\bar{\alpha}) < 0$ for all $\alpha \in [0;1]$ holds. Hence, for $V_t(0) \geq 0$, 0 is the optimal choice.

Reasoning by induction shows that

$$P\{\omega \in \Phi \mid \alpha_t \in \{0;1\} \text{ for each } t > 1\} = 1.$$

Given an $\varepsilon > 0$ define $\bar{C}_1$ and $\bar{C}_0$ to be the number of observations of utility realizations of portfolios $\alpha = 0$ and $\alpha = 1$, respectively, such that the average utilities of both portfolios are sufficiently close to their mean utilities with probability $(1-\varepsilon)^{\frac{1}{2}}$. Especially, for $\mu_0 \neq \mu_1$[66], let

$$\xi = \frac{\max\{\mu_0;\mu_1\} - \min\{\{\mu_0;\mu_1;u_1(\bar{\alpha})\} \setminus \max\{\mu_0;\mu_1\}\}}{3}$$

and

$$P\left\{\left|\frac{1}{C_1}\sum_{l=1}^{C_1} u_l(1) - \mu_1\right| \leq \xi\right\} \geq (1-\varepsilon)^{\frac{1}{2}}$$
$$P\left\{\left|\frac{1}{C_0}\sum_{l=1}^{C_0} u_l(0) - \mu_0\right| \leq \xi\right\} \geq (1-\varepsilon)^{\frac{1}{2}}$$

for any $C_1 \geq \bar{C}_1$ and any $C_0 \geq \bar{C}_0$ and denote by C the maximum of $\bar{C}_1$ and $\bar{C}_0$. Let $T_0 = 4C$ and set the initial aspiration level $\bar{u}_1$ so that:

$$\bar{u}_1 \geq \bar{u}_0 = \min\{u(\underline{\delta}_a);u(\underline{\delta}_b)\} + 2\left(\frac{1}{\beta}\right)^{T_0}\left(\max\{u(\bar{\delta}_a);u(\bar{\delta}_b)\} - \min\{u(\underline{\delta}_a);u(\underline{\delta}_b)\}\right).$$

Claims 7.1, 7.2. and 7.3 in Gilboa and Schmeidler (2001 (a), pp. 164-166) can then be used to establish that after T_0 periods, both acts $a=1$ and $a=0$ will be chosen at least C times each. It follows that with probability of at least $(1-\varepsilon)$ the average utilities of both portfolios 0 and 1 are ξ-close to their mean utilities[67]. Let $\mathcal{B}$ denote the set of paths on which this is satisfied, $P\{\mathcal{B}\} \geq (1-\varepsilon)$. Obviously, the events $\mathcal{B}$ and $\tilde{\Phi}$ are independent.

If the initial realization of $\bar{\alpha}$ does not exceed $\max\{\mu_1;\mu_0\}$, then claims 7.4-7.7 in Gilboa and Schmeidler (2001 (a), pp. 166-171) can be used to show that the better of the two acts 0 and 1 is chosen with frequency 1 on the set $\mathcal{B}$, hence

$$P\left\{\omega \in \Phi \mid \exists\pi\left(\arg\max_{\alpha\in\{0;1\}}\mu_\alpha\right) = 1\right\} \geq (1-\varepsilon)P\left(\tilde{\Phi}\right).$$

If, however $u_1(\bar{\alpha}) > \max\{\mu_1;\mu_0\}$, then the aspiration level $\bar{u}_t$ converges almost surely to $u_1(\bar{\alpha})$ on the set $\mathcal{B}$. Hence, on almost each path in $\mathcal{B}$, there is a period T such that

$$\bar{u}_t > \max\{\mu_1;\mu_0\} + \xi$$

for each $t > T$. It follows that both acts $a=0$ and $a=1$ are chosen infinitely often on each such path. An argument similar to the one applied in the proof of proposition 3.2 shows that $\pi(0)$ and $\pi(1)$ satisfy

$$\frac{\pi(0)}{\pi(1)} = \frac{\mu_1 - u_1(\bar{\alpha})}{\mu_0 - u_1(\bar{\alpha})}$$

[66] For $\mu_0 = \mu_1$, the proof is trivial.

[67] Although it is not assumed that the variables δ_a and δ_b are independent in a single period of time, they are independent accross time. Since the two sequences $u_l(1)$ and $u_m(0)$ observed by the decision maker have entries only for distinct time periods ($l \neq m$), it follows that the two events (the average utility of 1 being ξ-close to μ_1 and the average utility of 0 being ξ-close to μ_0) are independent.

on $\mathcal{B}\cap\Phi\backslash\tilde{\Phi}$. As has been shown above, all other acts are chosen with frequency 0. Since

$$P\left(\mathcal{B}\cap\Phi\backslash\tilde{\Phi}\right)\geq(1-\varepsilon)\left[1-P\left(\tilde{\Phi}\right)\right],$$

the result of the proposition obtains.■

Proof of proposition 3.15

At $t=2$, either

$$U_2(\bar{\alpha})=V_2(\bar{\alpha})=u_1(\bar{\alpha})-\beta\bar{u}_1-(1-\beta)\,u_1(\bar{\alpha})=\beta\left[u_1(\bar{\alpha})-\bar{u}_1\right]<0$$

or

$$U_2(\bar{\alpha})=V_2(\bar{\alpha})=u_1(\bar{\alpha})-X_2-h=-h<0$$

holds. Hence, $\bar{u}_2>X_2(\bar{\alpha})$. Moreover, analogous to the proof of proposition 3.14, it can be shown that (3.4) implies $\bar{u}_t\geq X_2(\bar{\alpha})$ as long as $\bar{\alpha}$ is not chosen again. To see this, note that the aspiration level is updated towards $X_t\geq X_2(\bar{\alpha})=u_1(\bar{\alpha})$ and is increased by h in some periods. Hence, starting with $\bar{u}_2>X_2(\bar{\alpha})$, the aspiration level remains above $X_2(\bar{\alpha})$ as long as $\bar{\alpha}$ is not chosen again. Therefore, $V_t(\bar{\alpha})\leq 0$ as long as $\alpha_\tau\neq\bar{\alpha}$ for all $t>\tau\geq 2$.

At $t=2$

$$U_2(\alpha)=V_2(\bar{\alpha})\,s(\alpha;\bar{\alpha})<0,$$

and since $s(\alpha;\bar{\alpha})$ is strictly decreasing,

$$\arg\max_{\alpha\in[0;1]}U_2(\alpha)\in\{0;1\}.$$

Assume, without loss of generality that $\alpha_2=0$, hence that

$$\min_{\alpha\in[0;1]}s(\alpha;\bar{\alpha})=0$$

As long as $\alpha_\tau=0$ is chosen for all $t>\tau\geq 2$

$$U_t(\alpha)=s(\alpha;\bar{\alpha})\,V_t(\bar{\alpha})+s(\alpha;0)\,V_t(0)$$

holds. If $V_t(0)>0$, $\alpha=0$ is chosen, since:

$$\begin{aligned}U_t(\alpha)&=s(\alpha;\bar{\alpha})\,V_t(\bar{\alpha})+s(\alpha;0)\,V_t(0)\leq\\&\leq s(0;\bar{\alpha})\,V_t(\bar{\alpha})+V_t(0)=U_t(0).\end{aligned}$$

If $V_t(0)<0$ obtains, the function

$$U_t(\alpha)=s(\alpha;\bar{\alpha})\,V_{\bar{t}}(\bar{\alpha})+s(\alpha;0)\,V_t(0)$$

becomes convex (since now $V_t(\bar{\alpha}) < 0$ and $V_t(0) < 0$ hold and s is concave). Therefore,

$$\arg\max_{\alpha\in[0;1]} U_t(\alpha) \in \{0; 1\}.$$

Suppose that $\alpha = 1$ is chosen at some time $\bar{t}'$. As long as $\alpha_\tau \neq \bar{\alpha}$ for all $t > \tau \geq \bar{t}'$,

$$U_t(\alpha) = s(\alpha; \bar{\alpha}) V_t(\bar{\alpha}) + s(\alpha; 0) V_t(0) + s(\alpha; 1) V_t(1).$$

$V_{\bar{t}'}(0) < 0$, is implied by the usage of the case-based decision rule at $\bar{t}'$. Applying the same argument as above, $V_t(0) < 0$ as long as $\alpha = 0$ is not chosen again.

If $V_t(1) \leq 0$ holds at t, the cumulative utility function becomes convex and obtains it maximum at 0 or 1.

Consider, therefore, the case $V_t(1) > 0$. Since $V_t(1) > 0 > \max\{V_t(\bar{\alpha}); V_t(0)\}$, it follows that $X_t = X_t(1)$. Hence, $\bar{u}_t \geq \bar{u}_{\tilde{t}}$ must hold, where

$$\tilde{t} = \max\{\bar{t}'; \tau < t \mid V_\tau(1) \leq 0, \alpha_\tau = 1, V_{\tau+1}(1) > 0\}.$$

by the argument stated in the proof of proposition 3.14. Note that the argument applies for the updating rule (3.4) as well, since the definition of $\tilde{t}$ implies that

$$\tilde{t} + 1, ...\bar{t}' \notin N.$$

To see this note that if $t \in N$, then $V_t(\alpha_{t-1}) < 0$ must hold and, hence, $t \notin \left\{\tilde{t} + 1, ...\bar{t}'\right\}$.

At time t, the cumulative utility of act $\alpha = 1$ is given by

$$U_t(1) = s(1; \bar{\alpha}) V_t(\bar{\alpha}) + V_t(1).$$

Hence, $\alpha = 1$ is chosen if

$$s(1; \bar{\alpha}) V_t(\bar{\alpha}) + V_t(1) \geq s(\alpha; \bar{\alpha}) V_t(\bar{\alpha}) + s(\alpha; 0) V_t(0) + s(\alpha; 1) V_t(1) \tag{3.17}$$

holds for all $\alpha \in [0; 1]$. Rewrite (3.17) as

$$\begin{aligned}
&V_t(1)(1 - s(\alpha; 1)) - s(\alpha; 0) V_t(0) + [s(1; \bar{\alpha}) - s(\alpha; \bar{\alpha})] V_t(\bar{\alpha}) \geq 0 \\
&V_t(1)(1 - s(\alpha; 1)) - s(\alpha; 0)(\bar{t}' - 2)[X_{\tilde{t}}(0) - \bar{u}_{\tilde{t}}] + [s(1; \bar{\alpha}) - s(\alpha; \bar{\alpha})][u_1(\bar{\alpha}) - \bar{u}_{\tilde{t}}] \\
&+ [\bar{u}_t - \bar{u}_{\tilde{t}}][s(\alpha; 0)(\bar{t}' - 2) - s(1; \bar{\alpha}) + s(\alpha; \bar{\alpha})] \geq 0.
\end{aligned}$$

Consider the function $s(\alpha; 0)(\bar{t}' - 2) + s(\alpha; \bar{\alpha})$ and note that since s is concave, it has a minimum at 0 or at 1. For $\alpha = 1$,

$$s(\alpha; 0)(\bar{t}' - 2) + s(\alpha; \bar{\alpha}) = s(1; \bar{\alpha}),$$

whereas for $\alpha = 0$,

$$(\bar{t}' - 2) + s(0; \bar{\alpha}) > 1 + s(0; \bar{\alpha}) > s(1; \bar{\alpha}).$$

It follows that:

$$\min_{\alpha\in[0;1]}\left[\bar{u}_t-\bar{u}_{\tilde{t}}\right]\left[s\left(\alpha;0\right)\left(\overline{t}'-2\right)-s\left(1;\bar{\alpha}\right)+s\left(\alpha;\bar{\alpha}\right)\right]=0.$$

Hence,

$$\begin{aligned} U_t\left(1\right)-U_t\left(\alpha\right) &\geq V_t\left(1\right)\left(1-s\left(\alpha;1\right)\right)-s\left(\alpha;0\right)V_{\tilde{t}}\left(0\right)+\left[s\left(1;\bar{\alpha}\right)-s\left(\alpha;\bar{\alpha}\right)\right]V_{\tilde{t}}\left(\bar{\alpha}\right)\\ &> V_{\tilde{t}}\left(1\right)\left(1-s\left(\alpha;1\right)\right)-s\left(\alpha;0\right)V_t\left(0\right)+\left[s\left(1;\bar{\alpha}\right)-s\left(\alpha;\bar{\alpha}\right)\right]V_t\left(\bar{\alpha}\right)\\ &= U_{\tilde{t}}\left(1\right)-U_{\tilde{t}}\left(\alpha\right)\geq 0\end{aligned}$$

for each $\alpha\in[0;1]$, since at time $\tilde{t}$, $V_{\tilde{t}}\left(1\right)\leq 0$ holds. Hence, $\alpha_t=1$ is chosen.

Analogously, if 0 is chosen at some time and $V_t\left(0\right)<0$, the optimum is a corner solution, whereas if $V_t\left(0\right)\geq 0$ holds, the difference between the cumulative utilities of 0 and an arbitrary act α becomes:

$$V_t\left(0\right)\left(1-s\left(\alpha;0\right)\right)-s\left(\alpha;1\right)V_t\left(1\right)+\left[s\left(0;\bar{\alpha}\right)-s\left(\alpha;\bar{\alpha}\right)\right]V_t\left(\bar{\alpha}\right)\geq 0,$$

since $V_t\left(0\right)\left(1-s\left(\alpha;0\right)\right)>0$, $V_t\left(1\right)<0$, $V_t\left(\bar{\alpha}\right)<0$ and $s\left(0;\bar{\alpha}\right)-s\left(\alpha;\bar{\alpha}\right)<0$ for all $\alpha\in[0;1]$ holds. Hence, for $V_t\left(0\right)\geq 0$, 0 is the optimal choice.

Reasoning by induction shows that

$$P\left\{\omega\in\Phi_1\mid\alpha_t\in\{0;1\}\text{ for each }t>1\right\}=1.$$

Suppose that both $\alpha=0$ and $\alpha=1$ are chosen infinitely often. Then claim 7.8 of Gilboa and Schmeidler (2001, p. 172-173) shows that for almost each $\omega\in S_2$,

$$\lim_{t\to\infty}\left(\bar{u}_t-X_t\right)=0$$

and

$$\lim_{t\to\infty}X_t\left(\alpha\right)-\mu=0$$

for each α chosen infinitely often. Therefore, if $X_{\tilde{t}}\left(\bar{\alpha}\right)=u_1\left(\bar{\alpha}\right)\leq\max\left\{\mu_0;\mu_1\right\}$, then there is a period T (depending on the path ω) such that for all but a sparse set of periods after T

$$\left|\bar{u}_t-\max\left\{\mu_0;\mu_1\right\}\right|<\zeta$$

for any initially chosen ζ on $\omega\in\tilde{\Phi}_1$. According to claim 7.9 in Gilboa and Schmeidler (2001, p. 173), acts 0 and 1 are indeed chosen in an infinite number of periods. Therefore, $X_t\left(0\right)\to\mu_0$ and $X_t\left(1\right)\to\mu_1$. Claims 7.4-7.7 in Gilboa and Schmeidler (2001, p. 166-170) then show that on paths ω such that

$$\lim_{t\to\infty}\left(\bar{u}_t-X_t\right)=0,$$

$$\pi\left(\arg\max_{\alpha\in[0;1]}\mu_\alpha\right)=1$$

almost surely obtains.

If, however, $u_1(\bar{\alpha})>\max\{\mu_0;\mu_1\}$, then $\bar{u}_t\to u_1(\bar{\alpha})$. Hence, there is a period T (depending on the path ω) such that

$$|\bar{u}_t-u_1(\bar{\alpha})|<\zeta$$

for each $t>T$ on $\omega\in\Phi_1\backslash\tilde{\Phi}_1$ except on a sparse set of periods for an arbitrary chosen ζ. Again, claim 7.9 of Gilboa and Schmeidler (2001, p. 173) assures that each the acts 0 and 1 are chosen an infinite number of times in the limit. But since now

$$\lim_{t\to\infty}\bar{u}_t=u_1(\bar{\alpha})>\max\{\mu_0;\mu_1\},$$

$\pi(0)>0$ and $\pi(1)>0$ obtain. $\pi(0)$ and $\pi(1)$ satisfy

$$\frac{\pi(0)}{\pi(1)}=\frac{\mu_1-u_1(\bar{\alpha})}{\mu_0-u_1(\bar{\alpha})},$$

as shown in Gilboa and Pazgal (2001).■

Proof of proposition 3.16

Since $\alpha_1=0$ and the aspiration level is raised by h in some periods,

$$U_{\bar{t}}(0)=V_{\bar{t}}(0)<0$$

for some finite $\bar{t}>0$. Since $s(0;\alpha)>0$ only for $\alpha\in\left(0;\frac{1}{l}\right)$, it follows that

$$U_{\bar{t}}(\alpha)=0>U_{\bar{t}}(\alpha')\text{ for all }\alpha\geq\frac{1}{l}\text{ and }\alpha'<\frac{1}{l}.$$

Therefore, $\alpha_{\bar{t}+1}=\frac{1}{l}$ is chosen by assumption 2. Note that

$$U_t(0)=V_t(0)<0$$

for all $t\geq\bar{t}$, such that $\alpha_\tau\neq0$ for all $\bar{t}\leq\tau<t$ by the argument in the proof of proposition 3.14. Since

$$U_{\hat{t}}\left(\frac{1}{l}\right)<0$$

obtains for a finite $\hat{t}>\bar{t}$,

$$U_{\hat{t}}(\alpha)=0>U_{\hat{t}}(\alpha')\text{ for all }\alpha\geq\frac{2}{l}\text{ and }\alpha'<\frac{2}{l}.$$

Hence $\alpha_{\hat{t}+1}=\frac{2}{l}$, etc.

Once each of the acts $\left\{0;\frac{1}{l};\frac{2}{l}...\frac{l-1}{l};1\right\}$ has been chosen at least once,

$$U_t(\alpha)\;=\;\sum_{i=0}^{l}V_t\left(\frac{i}{l}\right)s\left(\alpha;\frac{i}{l}\right)$$

$$= \quad V_t\left(\frac{k}{l}\right)s\left(\alpha;\frac{k}{l}\right)+V_t\left(\frac{k-1}{l}\right)s\left(\alpha;\frac{k-1}{l}\right),$$

for $\alpha \in \left[\frac{k-1}{l};\frac{k}{l}\right]$, since $s(\alpha;\alpha') > 0$ only for $\alpha' = \frac{k}{l}$ and $\alpha' = \frac{k-1}{l}$ out of $\left\{0;\frac{1}{l};\frac{2}{l}...\frac{l-1}{l};1\right\}$. By an argument analogous to the one used in the proof of proposition 3.15, $V_t\left(\frac{k}{l}\right) \geq 0$ can hold only for one $k \in \{0;...l\}$. If

$$V_t\left(\frac{k}{l}\right) > 0,$$

then $\alpha_t = \alpha_{t-1} = \frac{k}{l}$, since

$$V_t\left(\frac{k}{l}\right) > V_t\left(\frac{k}{l}\right)s\left(\alpha;\frac{k}{l}\right)+V_t\left(\frac{k-1}{l}\right)s\left(\alpha;\frac{k-1}{l}\right)$$

for all $\alpha \in \left[\frac{k-1}{l};\frac{k}{l}\right]$ and

$$V_t\left(\frac{k}{l}\right) > 0 \geq V_t\left(\frac{k'}{l}\right)s\left(\alpha;\frac{k'}{l}\right)+V_t\left(\frac{k'-1}{l}\right)s\left(\alpha;\frac{k'-1}{l}\right)$$

for all $k' \neq k$ and all $\alpha \in [0;1]$ hold. If

$$\max\left\{V_t\left(\frac{k}{l}\right);V_t\left(\frac{k-1}{l}\right)\right\} < 0,$$

then $U_t(\alpha)$ is convex, because $s(\alpha;\cdot)$ is concave. Hence,

$$\arg\max_{\alpha\in\left[\frac{k-1}{l};\frac{k}{l}\right]} U_t(\alpha) \in \left\{\frac{k}{l};\frac{k-1}{l}\right\}$$

and

$$\arg\max_{\alpha\in[0;1]} U_t(\alpha) \in \left\{0;\frac{1}{l};...\frac{k}{l};...1\right\} \text{ for every } t.$$

Therefore,

$$\lim_{t\to\infty}\bar{u}_t = \max_{\alpha}\left\{\mu_\alpha \mid \alpha \in \left\{0;\frac{1}{l};\frac{2}{l}...\frac{l-1}{l};1\right\}\right\}$$

obtains almost surely, see claims 7.4 – 7.7 in Gilboa and Schmeidler (2001, p. 166-170). Hence, only

$$\arg\max_{\alpha\in\left\{0;\frac{1}{l};\frac{2}{l}...1\right\}} \mu_\alpha$$

is satisficing in the limit and

$$\pi\left(\arg\max_{\alpha\in\left\{0;\frac{1}{l};\frac{2}{l}...1\right\}} \mu_\alpha\right) = 1$$

almost surely obtains on Φ_7, as shown in claim 7.8 and 7.9 in Gilboa and Schmeidler (2001, p. 172-174).■

Chapter 4. On the Definition and Existence of an Equilibrium in an OLG Economy with Case-Based Decisions

In the last chapter, the individual portfolio choice problem with case-based decisions was considered. It was found that investors acting in accordance with the case-based decision theory in general violate the predictions of the expected utility theory by holding undiversified portfolios, trading too much and not using arbitrage possibilities present in the market. However, these results were achieved by assuming that asset prices (and therefore returns) are exogenously given and identically distributed over time. In this chapter, this assumption is dropped and an asset market populated by case-based decision-makers is constructed.

The construction of a market populated by case-based decision-makers makes necessary the discussion of some methodological issues first, the aim being to define a notion of an equilibrium for a market populated by case-based investors and to guarantee that such an equilibrium exists, hence that the definition is meaningful.

The case-based decision theory has been applied in multiple settings, as for instance, in the consumer theory, Gilboa and Pazgal (2001), theory of voting, Aragones (1997), production theory, Jahnke, Chwolka and Simons (2001), social learning Blonski (1999), cooperation in games Pazgal (1997) and portfolio choice, see chapter 3 of this thesis. Still, up to now the applications of the case-based decision theory have been restricted to individual decisions, neglecting equilibrium considerations. Especially in financial markets, the assumption of an exogenously given and stable price process is very difficult to defend, both on an empirical and theoretical level. In a financial market populated with case-based decision-makers, past prices influence the portfolio choices and, therefore, future prices. Investigating the price process of such an economy might help us gain new insights about the way financial markets function when (some of) the investors make decisions based on their own or on other investors' experience. That is why a notion of an equilibrium of an economy populated with case-based decision-makers is necessary.

I make a first attempt to formulate such an equilibrium for a financial market with an exogenously fixed asset supply. Whereas the dividend payments of the assets are exogenously given, their prices are determined endogenously according to a market clearing condition. I consider an overlapping-generations economy in order to capture the insight that decision-makers who live only for one period are not able to learn much about the structure of the economy as a whole. It is only natural that in an overlapping generations economy, the knowledge about the profitability of assets is based on the experience of past generations. In the model suggested below, the investors are distributed on a continuum and differ only with respect to their aspiration levels. I, therefore, assume that each investor learns from his predecessors with the same aspiration level as his own. This provides an intuitive interpretation of a memory of a decision-maker in this context as experience of previous generations.

At the same time, the assumption about learning from previous generations makes the evaluation of an asset by the young investors (which is based on the consumption possibilities it bears for the old) implicitly dependent on its current price. This dependence determines the demand function for assets on an individual and on aggregate level. It is possible to show that an equilibrium point exists under quite general conditions on the utility function of the investors and the initial asset holdings in $t = 1$.

Since the old investors consume the returns of the asset, a higher price of an asset implies a higher utility realization observed by the old consumers and, therefore, a higher evaluation by the young investors. Hence, the demand for assets is in general increasing for relatively low prices and decreasing for high prices. Moreover, the demand for an asset of an individual investor may be very insensitive to price changes at low prices. If this feature is also present at the aggregate level, it is not possible to exclude equilibria with 0-asset prices. However, conditions can be identified for which degenerate equilibria cannot occur, or for which at least one non-degenerate equilibrium exists.

Since the evaluation of an asset by a young investor depends on its current price, price-dependent preferences emerge in the model. This kind of preferences is first introduced into the economic literature by Veblen (1899) for ”snob” goods, whose demand increases in price. A different interpretation of such preferences is given by Pollak (1977) and Martin (1986), who argue that price dependent preferences might be a sensible assumption in cases in which the quality of a product is not public knowledge, but only known to some of the consumers. Martin (1986) suggests to speak of price dependent *expectations*, instead. In this case, prices convey information about the quality of the product. Samuelson (1966) notes that price-dependent preferences are natural in monetary economies, in which the demand for money should depend on the price level. In the context of financial markets, price-dependent preferences are introduced into a portfolio choice problem by Allingham and Morishima (1973). They derive comparative statics results for the influence of a price change on the optimal portfolio.

Equilibrium results for economies with price dependent preferences are derived by Arrow and Hahn (1971, p. 129-131) for the case of finite number of consumers and by Greenberg, Shitowitz and Wieczorek (1979) for an economy with a continuum of consumers. The main assumption, which insures the existence (apart from the standard assumptions of Debreu (1959)) is the continuity of the utility function with respect to consumption and prices. The main features of such economies do not differ substantially from those of standard Arrow-Debreu economies, see Balasko (2003 (a)).

Financial economies with temporary equilibria, in which beliefs of investors depend on prices naturally exhibit price-dependent preferences. Grandmont (1982, p. 892) derives necessary conditions for the existence of a temporary equilibrium. In an economy with heterogenous agents, existence requires that their beliefs about future prices are not too different. Whereas Grandmont imposes an expected utility representation of preferences, Balasko (2003 (b)) constructs a temporary equilibrium with arbitrary preferences. He shows that a temporary equilibrium in his economy is equivalent to an equilibrium with price-dependent preferences in an Arrow-Debreu economy and shows existence. Nevertheless, he also has to impose a certain agreement on probabilities. Especially, in the case of von Neumann Morgenstern expected utility representation, the beliefs of each investor should assign positive probability to each state of nature, Balasko (2003 (b), p. 3). Page and Wooders (1999) derive existence results in financial markets with price-dependent preferences by imposing non-arbitrage conditions. Their assumptions on expectations are similar to those made by Grandmont (1982) and Balasko (2003 (b)).

The fact that all of the cited models are constructed in the Arrow-Debreu setting with assets representing vectors of state-contingent outcomes, as well as the restrictions imposed on the beliefs of consumers over the states of nature does not allow the direct application of these results to an economy populated by case-based decision-makers, even once the (implicit) price-dependence of preferences is demonstrated. Since beliefs and state-contingent outcomes are not well defined in such a model, different factors, such as the aspiration level of the investors and their memory play an important role for the existence of an equilibrium. Nevertheless, it is possible to draw a parallel between the findings in the literature and the existence result derived here.

The rest of the chapter is organized as follows: section 1 presents the model of an overlapping-generations economy populated by case-based decision-makers. Section 2 describes the decision-making process, whereas section 3 derives the individual demand for assets. In section 4, it is shown that the Walras' Law holds for the economy. Section 5 gives a definition of a temporary equilibrium. In section 6, the existence of an equilibrium is shown under quite general conditions. Section 7 provides an example of an economy with two assets, one of which is in fixed supply, the other one — in perfectly elastic supply. The conditions for existence of an equilibrium are then discussed for this example. Section 8 analyzes conditions under which equilibria with 0-prices can be excluded. A discussion of the results is provided in section 9. Section 10 concludes.

4.1 The Economy

I consider an economy evolving in discrete time and consisting of a continuum of investors uniformly distributed on the interval $[0; n]$. For each $i \in [0; n]$ and some constant $\bar{u}^0 \in \mathbb{R}$ denote by $\bar{u}^i = \bar{u}^0 + i$ the aspiration level of investor i. Hence, the mapping from investors to aspiration levels is one-to-one and one can identify the continuum of investors with the continuum of aspiration levels $[\bar{u}^0; \bar{u}^n]$. The aspiration level of the investors is then also uniformly distributed on $[\bar{u}^0; \bar{u}^n]$.

Each investor lives for two periods. The preferences of the investors are assumed to be such that they wish to consume only in the second period of their life. The preferences about the consumption in the second period are represented by a utility function $u(\cdot)$, which is identical for all consumers and independent of their aspiration level. I assume, as usual that u is strictly increasing and continuous in consumption in period two.

There is one consumption good in the economy. The initial endowment of the investors consists of one unit of the consumption good in the first period and is 0 in the second period.

There are K assets in the economy, which allow to transfer consumption over time. The supply of the risky asset $k \in \{1...K\}$ is fixed at $A_k > 0$. The payoff of one unit of the asset k in period t is a random variable δ_t^k, which is identically and independently distributed in every period on a closed and bounded set. $\min\left\{\delta^k\right\} \geq 0$ for all k is assumed.

4.2 Investment Decision of the Young Investors

The decision situation is described as a problem[68] to be solved, by choosing an act out of a given set. In the present context the problem can be formulated as: "Choose a portfolio of assets today to enable consumption tomorrow".

The decision of a young investor now consists in choosing a portfolio of k assets — $1...K$. For simplicity, I consider only the case, in which there are only k portfolios available for a single investor: he can invest his whole initial endowment in one asset only. Short sales are prohibited[69]. Denote by α_{t-1}^i the act, chosen by a young investor with an aspiration level $\bar{u}^i$ in period $t-1$. Then the values of α^i —

$$\mathfrak{A} = \{1...K\}$$

represent the set of acts available to the investors, when solving the problem formulated above.

The portfolio, chosen in period $(t-1)$ is held until t, when the old investors sell the risky asset they own to the young investors and consume the dividends of the asset and the revenues from the asset sales. Normalizing the price of the consumption good to 1, the indirect utility[70] from

[68] I assume here implicitly that the price vector does not characterize the decision problem. This assumption is dropped in section 6 of chapter 5, where a problem is identified with the price vector in the economy.

[69] Since the investors in the economy are infinitesimally small, it is plausible, that short sales are impossible for a single investor, because of high transaction costs and legal requirements. The assumption of no diversification can be justified partially by the results of the previous chapter, which show that among the investors with a concave similarity functions, only those with relatively low aspiration levels will be willing to hold diversified portfolios infinitely long. This assumption will be dropped in section 6 of the chapter 5, in which similarity considerations in a market environment will be discussed.

[70] I.e. conditioned on the alternative chosen at $(t-1)$.

consumption of an old investor in period t can be written, as[71]:

$$v_t\left(\alpha_{t-1}^i\right) = \quad u\left(\frac{p_t^k}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k}\right), \quad \text{if } \alpha_{t-1}^i = k \ ,$$

where p_t^k denotes the price of asset k at time t. In terms of the case-based decision theory $v_t\left(\alpha_{t-1}^i\right)$ denotes the utility realization of act α_{t-1}^i.

In the spirit of the case-based decision theory, I assume that the decision-makers have almost no information about the problem they are facing. They do not know the structure of the economy, nor the process of price formation. They have no information about possible prices and returns of the assets and their distribution. Their information consists of the problem formulation, the set of possible acts and their memory. Hence, they can only learn from experience of subjects, who have lived before them.

The memory is a vector of cases[72]:

$$(\alpha_{\tau-1}; v_\tau(\alpha_{\tau-1}))$$

of previous choices and achieved utilities. I assume that the memory of a young investor i at time t consists only of the $(m+1)$ last cases, realized by investors from previous generations with the same aspiration level[73] i:

$$M_t^i = \left(\left(\alpha_{t-1}^i; v_t\left(\alpha_{t-1}^i\right)\right); ... \left(\alpha_{t-m-1}^i; v_{t-m}\left(\alpha_{t-m-1}^i\right)\right)\right)$$

Hence, $m \in \{0; 1...t-1\}$ parameterizes the length of the memory (which is assumed to be equal for all investors). If $m = t-1$, the young investors remember all past cases from $t = 0$ on, if $m = 0$, the memory contains only the last case.

Note that since the memory of each investor at time t contains the last case $\left(\alpha_{t-1}^i; v_t\left(\alpha_{t-1}^i\right)\right)$, it in general depends on the price vector p_t.

Based on his memory at time t, the young investor i constructs the cumulative utility of each of

[71] If an investor has chosen k at $t-1$, he holds $\frac{1}{p_{t-1}^k}$ units of the risky asset in period t, from the sale of which he receives $\frac{p_t^k}{p_{t-1}^k}$. Since the dividends are distributed proportionally to the number of assets held, the investor receives $\frac{1}{p_{t-1}^k}$ of the dividend δ_t^k paid per unit.

[72] Since the problem formulated here is considered to remain unchanged in each period, a case consists only of an act chosen and utility obtained.

[73] This means that the investors do not observe all past choices and realizations, but only those of a given cohort of their predecessors. One possibility to relax this assumption is by introducing social learning, the possibility to learn from investors with different characteristics, as in Blonski (1999) and Krause (2003).

the acts:

$$U_t^i(k) = \sum_{\tau \in C_t^i(k)} \left[v_\tau(k) - \bar{u}^i \right],$$

where

$$C_t^i(k) = \left\{ t - m - 1 \leq \tau < t \mid \alpha_\tau^i = k \right\}$$

denotes the set of cases in the memory of investor i in which act k has been chosen. The sum over an empty set is assumed to be 0. Note that the cumulative utility of the same act in general differs among the investors, depending on the memory they have (past cases observed by them), but also on their aspiration level.

After determining the cumulative utilities of all acts, the decision-maker compares them and chooses the act with the highest cumulative utility, i.e.:

$$\alpha_t^i = \arg \max_{k \in \{1 \ldots K\}} \left\{ U_t^i(k) \right\}.$$

Note that the case-based decision theory does not require the knowledge of past prices and dividend payments[74]. The decision of an investor is only influenced by past cases and not directly by prices or by the probability distribution of returns. The influence of past and current prices is only indirect, through the observed utility realizations $v_t(\cdot)$. Nevertheless, it is obvious that current prices influence the choice of an investor through his evaluation of an act and not only through the budget constraint. Hence, price-dependent preferences emerge.

4.3 Individual Demand for Assets

Before proceeding to define a temporary equilibrium, I analyze the individual behavior in this economy. Consider a young investor i in period t whose memory is given by M_t^i. Suppose, first that the old investor i holds asset k in period t. This means that the cumulative utility of k for the young investor i can be written as:

$$U_t^i(k) = \sum_{\tau \in C_{t-1}^i(k)} \left[v_\tau(k) - \bar{u}^i \right] + u\left(\frac{p_t^k}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k} \right) - \bar{u}^i,$$

[74] Although the cumulative utility of act k can be written as:

$$U_t^i(k) = \sum_{\tau \in C_{t-1}^i(k)} \left[v_\tau(k) - \bar{u}^i \right] + u\left(\frac{p_t^k}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k} \right) - \bar{u}^i,$$

the investor need not know p_{t-1}^k, p_t^k and δ_t^k. It is sufficient for him to know the values of $u\left(\frac{p_t^k}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k} \right)$ and $v_\tau(k)$ in order to make a decision.

whereas the cumulative utility of an asset $\tilde{k} \neq k$ is:

$$U_t^i\left(\tilde{k}\right) = \sum_{\tau \in C_{t-1}^i\left(\tilde{k}\right)} \left[v_\tau\left(\tilde{k}\right) - \bar{u}^i\right].$$

Note that only the cumulative utility of the asset chosen by the direct predecessor of the young investor i depends on the price vector at t, whereas the cumulative utilities of the other acts are independent of current prices. Therefore, it is possible to identify a subset of acts $K' \subset \mathfrak{A}$, which have the highest cumulative utility among the acts not chosen by the investor i at time $(t-1)$:

$$K' = \left\{ \arg \max_{\left\{ \begin{array}{c} \tilde{k} \in 1...K \\ \tilde{k} \neq k \end{array} \right\}} U_t^i\left(\tilde{k}\right) \right\}. \tag{4.18}$$

It is then obvious that the young investor will either choose asset k or some of the assets $k' \in K'$. Take one such k'. Suppose that there exists a price of asset k, $p_t^{k*}(i) \geq 0$ such that[75]

$$U_t^i(k)\left(p_t^{k*}(i)\right) = \sum_{\tau \in C_{t-1}^i(k)} \left[v_\tau(k) - \bar{u}^i\right] + u\left(\frac{p_t^{k*}(i)}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k}\right) - \bar{u}^i = U_t^i(k'). \tag{4.19}$$

Then three cases are possible:

1. If $p_t^k > p_t^{k*}(i)$, then investor i chooses asset k ($\alpha_t^i = k$).
2. If $p_t^k < p_t^{k*}(i)$, then investor i chooses one of the assets $k' \in K'$ ($\alpha_t^i \in K'$).
3. If $p_t^k = p_t^{k*}(i)$, then investor i is indifferent between holding k and some of the assets of the set K' ($\alpha_t^i \in K' \cup \{k\}$).

Therefore, the demand for k of investor i at time t is given by the correspondence:

$$x_t^i(k) = \left\{ \begin{array}{cc} 0, & \text{if } 0 \leq p_t^k < p_t^{k*}(i) \\ \left\{\frac{1}{p_t^{k*}(i)}; 0\right\}, & \text{if } p_t^k = p_t^{k*}(i) \\ \frac{1}{p_t^k}, & \text{if } p_t^k > p_t^{k*}(i) \end{array} \right\}. \tag{4.20}$$

Of course, it is possible that no such price $p_t^{k*}(i) \geq 0$ exists. This is the case if

$$\sum_{\tau \in C_{t-1}^i(k)} \left[v_\tau(k) - \bar{u}^i\right] + u\left(\frac{\delta_t^k}{p_{t-1}^k}\right) - \bar{u}^i > U_t^i(k')$$

[75] If $p_t^{k*}(i)$ exists, then it will be unique, since $u(\cdot)$ and, thus, $U_t^i(k)$ are strictly increasing in p_t^k, whereas $U_t^i(k')$ is a constant.

for the given realization of δ_t^k and for each $k' \in K'$. Then

$$U_t^i(k) > U_t^i(k')$$

for each $p_t^k \geq 0$ and all $k' \in K'$, hence $\alpha_t^i = k$.

In this case, the demand of i for k is a function of p_t^k and is given by:

$$x_t^i(k) = \left\{ \begin{array}{c} \frac{1}{p_t^k} \text{ for all } p_t^k > 0 \\ \infty \text{ for } p_t^k = 0 \end{array} \right\}. \tag{4.21}$$

The demand $x_t^i(k)$ for asset k of an investor whose predecessor holds k at time t is illustrated in figures 4 and 5 for the two cases described above.

The opposite case, in which k' is the act chosen for any price, cannot occur if the utility function $u(\cdot)$ and the range of possible prices p_t^k are unbounded above. If, however, (as it will be shown to be in an equilibrium), p_t^k is bounded above, say by $\bar{p}^k$, and if $p_t^{k*}(i) > \bar{p}^k$, i.e.:

$$\sum_{\tau \in C_{t-1}^i(k)} \left[v_\tau(k) - \bar{u}^i\right] + u\left(\frac{\bar{p}^k}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k}\right) - \bar{u}^i < U_t^i(k')$$

for $k' \in K'$, then $\alpha_t^i \in K'$ for each price $p_t^k \in \left[0; \bar{p}^k\right]$ and the demand for k is 0.

Few comments are in place. First, note that the demand of an investor for an asset depends only on the relative prices. Increasing all prices by a certain factor λ (including, of course, the price of the consumption good in every period) leaves the consumption of the old investors and, therefore, the portfolio choice of the young investors unchanged. Furthermore, the proportion of the endowment which an investor whose predecessor holds k invests in k is 0 if p_t^k is sufficiently low and jumps to 1, once p_t^k exceeds the critical level $p_t^{k*}(i)$. Hence, the proportion of the endowment invested in k is monotonically increasing in the price of k. It is, therefore, only possible to induce the investor to substitute k by k' by *reducing* the price of k and *not* by *increasing* it. If the investor is ready to hold k, even at $p_t^k = 0$, or if he is not ready to buy k even if the price rises to $\bar{p}^k$, then his portfolio choice is absolutely insensitive to price changes. At the same time, the investment decision of an investor, whose predecessor holds k, is independent of the prices of the other assets $\left(p_t^1 ... p_t^{k-1}; p^{k+1} ... p_t^K\right)$. Hence, no substitution effects can be induced by changing these prices. Therefore, the investment decision of case-based investors is relatively insensitive to price changes. The absence of substitution effects will subsequently turn out to be the reason for the existence of equilibria with 0-asset prices.

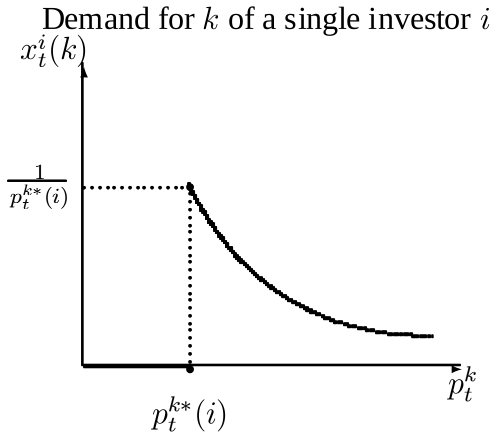


Figure 4

4.4 Walras' Law for an Economy with Case-Based Decision-Makers

It is possible to show that the Walras' Law holds for the economy populated by case-based decision-makers. In order to guarantee its validity it is, however, necessary to specify the property rights on assets which are not traded in some period and have an equilibrium price of 0, because else positive dividend payments might get lost for the economy as a whole. Suppose that an old investor i holds asset k in period t and the equilibrium price of this asset satisfies $p_t^k = 0$. Since the old investor i will not live at $(t+1)$ and will not be able to sell the asset (since at a price of 0 the demand for the asset must be 0 in an equilibrium), I assume that the young consumer i (with the same aspiration level as the old one who owns k) inherits the asset from him. The inheritance does not influence the budget constraint of the young consumer and, there-

Demand for k of a single investor i if $p_t^{k*}(i)$ does not exist

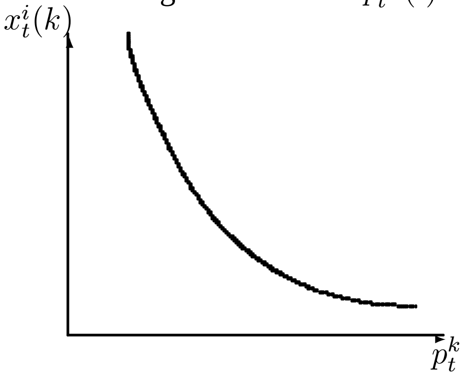


Figure 5

fore, he chooses an act from the set K' defined above. Since the investor has received the asset by chance and not by an explicit choice, I assume for simplicity of exposition that the returns of the inheritance are considered to be irrelevant for the computation of the cumulative utilities in the next period. It is not difficult to show that the characteristics of the value of demand function $d_t\left(p_t\right)$, which is defined and analyzed in section 6 remain unchanged and the existence of an equilibrium is still guaranteed, if these returns are also taken into account.

To prove the validity of the Walras' Law, write the budget constraints of the individual in the economy as follows:

$$\sum_{k=1}^{K} x_t^i\left(k\right) p_t^k \leq 1, \text{ for the young investors} \tag{4.22}$$

with $x_t^i(k) > 0$, iff $\alpha_t^i = k$, and

$$\sum_{k=1}^{K} \left(x_{t-1}^i(k) + g_{t-1}^i(k)\right)\left(p_t^k + \delta_t^k\right) \geq c_t^i, \text{ for the old investors} \tag{4.23}$$

where c_t^i denotes the consumption derived by the old investor i at time t and $g_{t-1}^i(k)$ denotes the inherited number of shares of asset k at $(t-1)$[76]. Both (4.22) and (4.23) hold with equality, since the whole initial endowment of the young consumers is invested and because of the strict monotonicity of $u(\cdot)$.

Now integrate the budget constraints over the continuum of investors and note that the assumption that all assets which are not demanded at $(t-1)$ are inherited implies that:

$$\int_{i=0}^{n} \left(\sum_{k=1}^{K} \left(x_{t-1}^i(k) + g_{t-1}^i(k)\right) p_t^k\right) di = \sum_{k=1}^{K} A_k p_t^k.$$

Combining this with (4.22) and (4.23), one obtains:

$$\int_{i=0}^{n} \sum_{k=1}^{K} x_t^i(k) p_t^k di + c_t = n + \sum_{k=1}^{K} A_k \left(p_t^k + \delta_t^k\right), \tag{4.24}$$

where $c_t = \int_{i=0}^{n} c_t^i di$ denotes the consumption of the old investors at time t. Since the l.h.s. of (4.24) represents the value of the demand, whereas the r.h.s. is the value of supply (including the dividend payments), it follows that the value of the excess demand is 0 for all possible price vectors $p_t = \left(p_t^1 ... p_t^K\right) \geq 0$.

Note, however that for some investors non-satiation might be violated at a price of 0. Especially, $x_t^i(k)\left(p_t^k = 0\right) = 0$ might obtain as has been shown in the derivation of the individual demand for assets. If this holds for all $i \in [0; n]$, then the aggregate supply of k exceeds the aggregate demand for k at the price $p_t^k = 0$. Hence, asset k is not desirable, see Varian (1992, p. 318). Therefore, its equilibrium price would be 0.

4.5 Definition of a Temporary Equilibrium

Having described the individual decision-making process in the economy, the notion of a market equilibrium at time t is now introduced. The property which allows to find a price vector $p_t = \left(p_t^1 ... p_t^K\right)$ such that the markets are equilibrated is, of course, the (indirect) dependence of $U_t^i(k)$ on p_t^k for all investors i, whose predecessors have chosen k in $t-1$. This property makes the

[76] Of course, $x_{t-1}^i(k) > 0$, iff $\alpha_{t-1}^i = k$ and $g_{t-1}^i(k) > 0$, iff $p_t^k = 0$;... $p_{t-h}^k = 0$; $p_{t-h-1}^k \neq 0$ and $\alpha_{t-h-1}^i = k$ for some $h > 1$ must hold.

demand of the young investors at least partially dependent on the price vector p_t, so that an equilibrium exists under quite general conditions. However, the fact that the demand function is not always sufficiently elastic with respect to prices leads to equilibria in which some of the assets are not demanded and their price is 0.

A temporary equilibrium at time t is defined by:

- portfolio choices of the young investors — $\alpha_t^i \in \mathfrak{A}$ for each $i \in [0;n]$;
- utility of consumption obtained by the old investors — $v_t\left(\alpha_{t-1}^i\right)$ for each $i \in [0;n]$;
- a price vector — $p_t = \left(p_t^1 ... p_t^K\right)$

such that following conditions are fulfilled:

1. Case-based decision-making of the young consumers:
$$\alpha_t^i \in \arg \max_{k \in \{1...K\}} \left\{U_t^i(k)\right\}$$
for all $i \in [0;n]$ at the equilibrium price vector p_t.

2. Indirect utility of the old consumers, derived from α_{t-1}^i
$$v_t\left(\alpha_{t-1}^i\right) = \left\{\ u\left(\frac{p_t^k}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k}\right), \quad \text{if } \alpha_{t-1}^i = k\ \right\}$$
for all $i \in [0;n]$ at the equilibrium price vector p_t.

3. The excess demand in each of the markets is 0 or negative. If the excess demand in one of the markets is strictly negative, then the price in the market is 0.
$$A_k = \int_{i=1}^{n} x_t^i(k)\, di, \text{ if } p_t^k \neq 0 \tag{4.25}$$
$$\int_{i=1}^{n} x_t^i(k)\, di = 0 < A_k, \text{ if } p_t^k = 0$$
for all $k \in \{1...K\}$ and
$$n = \sum_{k=1}^{K} \left(p_t^k \int_{i=1}^{n} x_t^i(k)\, di\right). \tag{4.26}$$

Remark 4.1 *(4.25) can be formulated as:*
$$A_k p_t^k = \int_{\left\{i:\alpha_t^i = k\right\}} di =: d_t^k \text{ for } \forall k \in \{1...K\}$$
where d_t^k denotes the mass of the young investors, who choose to hold k in period t. In other words, d_t^k represents the value of demand for k at time t, whenever $\int_{i=1}^{n} x_t^i(k)\, di \neq \infty$ at $p_t^k = 0$.

Note, however, that $p_t^k = 0$ cannot be an equilibrium price if $\int_{i=1}^{n} x_t^i(k)\,di = \infty$ at 0, since then the excess demand is positive. Therefore, for the purpose of showing the existence of an equilibrium it is enough to analyze the characteristics of the value of demand d_t^k and show that it crosses the value of supply at least once.

Remark 4.2 *Since the price of the consumption good is normalized to 1, (4.26) represents the clearing condition for the market for the consumption good. Note that n represents the initial supply of consumption good (apart from dividend payments). It is obtained by noting that (4.22) holds with equality and integrating the budget constraint (4.22) with respect to i. According to remark (4.25), (4.26) can be rewritten as:*

$$\sum_{k=1}^{K} d_t^k = n.$$

4.6 Existence of a Temporary Equilibrium

It can be shown that under mild conditions on the utility function and the initial holdings of the old consumers in $t = 1$, an equilibrium exists.

Following assumptions are made:

$(A1)$ The utility function $u(\cdot)$ is strictly increasing and continuous.

$(A2)$ At $t = 1$, the population of the old investors can be partitioned into a finite number of intervals such that all investors of the same interval hold the same asset.

$(A3)$ Suppose that at some time t, an interval of investors with identical memories is indifferent between two or more acts. Then the interval is partitioned into a finite number of subintervals such that the investors of the same interval choose the same act.

Note that $(A3)$ is not in contradiction to the assumptions of the case-based decision theory, but just specifies how decisions are made in case of indifference.

The idea of the proof is relatively simple. First, it can be shown (proposition 4.1) that condition $(A2)$ imposed on the initial holdings of the old investors at $t = 1$ also holds in all subsequent periods, as long as $(A1)$ and $(A3)$ are fulfilled. This allows to partition the young investors in each period into intervals such that all young investors in the same interval have identical memories. This structure allows to integrate the individual demand for assets on one such interval of young investors with identical memories to obtain the aggregate demand of the interval. Further, remark (4.25) states that it is not necessary to analyze the demand correspondence, but only the value of demand correspondence $d_t = \left(d_t^1 ... d_t^K\right)$. The properties of the utility function, as well as the interval structure described above imply that the value of demand for assets d_t is an upper hemicontinuous, non-empty, convex-valued correspondence, which maps the convex and closed set

$$\left[0; \frac{n}{A_1}\right] \times ... \times \left[0; \frac{n}{A_K}\right]$$

into $[0; n]^K$ for all possible memories (corollaries 4.1, 4.2, 4.3, 4.4, proposition 4.4). The value of supply $s_t = \left(A_1 p_t^1 ... A_K p_t^K\right)$ is a continuous function of the price vector, which also maps

$$\left[0; \frac{n}{A_1}\right] \times ... \left[0; \frac{n}{A_K}\right]$$

into $[0; n]^K$. A common point of d_t and s_t is an equilibrium at time t. With the help of Kakutani's fixed point theorem, it is shown that such a common point exists.

The following proposition follows from $(A1)$, $(A2)$ and $(A3)$ and will be proved in the course of the discussion.

Proposition 4.1 *Assume $(A1)$, $(A2)$ and $(A3)$. Then, for each $t \geq 1$, the population of the old investors can be partitioned into a finite number of intervals such that all the investors of the same interval hold the same asset.*

First, note that if in period $t = 1$ the initial asset holdings satisfy $(A2)$, then in the next period the population will again be (sub)divided into such intervals. Indeed, consider an interval $\left[\bar{u}^j; \bar{u}^l\right] \subset [\bar{u}^0; \bar{u}^n]$ such that $\alpha_0^i = k$ for all i with $\bar{u}^i \in \left[\bar{u}^j; \bar{u}^l\right]$. Fix a price[77] $p_0^k > 0$. Then in period $t = 1$, all individuals in this interval have the same memory and asses the cumulative utility of k as:

$$U_1^i(k) = u\left(\frac{p_1^k}{p_0^k} + \frac{\delta_1^k}{p_0^k}\right) - \bar{u}^i \text{ for all } \bar{u}^i \in \left[\bar{u}^j; \bar{u}^l\right].$$

Since $u(\cdot)$ is continuous and increasing in p_1^k, it follows that three cases are possible[78]:

1. Let $\hat{p}_1^k$ be such that[79]:

$$u\left(\frac{\hat{p}_1^k}{p_0^k} + \frac{\delta_1}{p_0^k}\right) - \bar{u}^l = 0,$$

If $p_1^k > \hat{p}_1^k$ holds, all investors in this interval choose k:

$$u\left(\frac{p_1^k}{p_0^k} + \frac{\delta_1^k}{p_0^k}\right) - \bar{u}^i > 0 \text{ for all } \bar{u}^i \in \left[\bar{u}^j; \bar{u}^l\right]$$

77 If $p_0^k = 0$ was an equilibrium price in $t = 0$, then the demand for k must have been 0 in $t = 0$, hence none of the old investors considers the returns of k to be relevant for his decision in $t = 1$.

78 Note that since the memory of the investors is empty at $t = 0$, the cumulative utilities of the assets other than k are 0.

79 Of course, $\hat{p}_1^k$ depends on the interval $\left[\bar{u}^j; \bar{u}^l\right]$. I neglect this dependence in the notation for convenience.

i.e. $\alpha_1^i = k$ for all $i \in [j;l]$.

2. Let $\check{p}_1^k$ be such that[80]:
$$u\left(\frac{\check{p}_1^k}{p_0^k} + \frac{\delta_1^k}{p_0^k}\right) - \bar{u}^j = 0.$$
If $p_1^k < \check{p}_1^k$ holds, none of the investors chooses k:
$$u\left(\frac{p_1^k}{p_0^k} + \frac{\delta_1^k}{p_0^k}\right) - \bar{u}^i < 0 \text{ for all } \bar{u}^i \in \left[\bar{u}^j; \bar{u}^l\right],$$
i.e. $\alpha_1^i \in K' = \{1...K\} \setminus \{k\}$ for all $i \in [j;l]$.

3. If $p_1^k \in \left[\check{p}_1^k; \hat{p}_1^k\right]$, then there is an aspiration level $\bar{u}^* \in \left[\bar{u}^j; \bar{u}^l\right]$,
$$\bar{u}^* = u\left(\frac{p_1^k}{p_0^k} + \frac{\delta_1}{p_0^k}\right)$$
such that the investors with an aspiration level higher than $\bar{u}^*$ choose an asset from the set K', whereas those with a lower aspiration level continue to choose k:
$$\alpha_1^i \in K', \text{ if } \bar{u}^i > \bar{u}^*$$
$$\alpha_1^i = k, \text{ if } \bar{u}^i < \bar{u}^*.$$

Of course, which of these cases will occur (or which are relevant), depends on the range of possible prices, as well as on the range of aspiration levels in the interval considered (e.g. if $\bar{u}^l$ is relatively low, all investors may want to hold k, even for $p_1^k = 0$, hence $\alpha_1^i = k$ obtains independently of the price). Nevertheless, $(A3)$ implies that the interval $\left[\bar{u}^j; \bar{u}^l\right]$, consisting of investors with identical memories, is divided into at most two K intervals, which also consist of investors with identical past choices and identical memory.

A similar argument holds also for any period t, by induction. Let $p_{t-1}^k > 0$. Consider an interval of investors $\left[\bar{u}^b; \bar{u}^c\right]$ with identical memory, such that $\alpha_{t-1}^i = k$ for all i in this interval. Consider the acts different from k. Note that the cumulative utility for investor i of each such act is given by:
$$U_t^i(k') = \sum_{\tau \in C_{t-1}^i(k')} v_\tau(k') - \left|C_{t-1}^i(k')\right| \bar{u}^i,$$
or is 0, (if $\left|C_{t-1}^i(k')\right| = 0$) and is thus linear and continuous in $\bar{u}^i$. Therefore, the investors, for whom k' is in the set K' of the assets with highest cumulative utility build a sub-interval of $\left[\bar{u}^b; \bar{u}^c\right]$. Consider one such (open) subinterval $\left(\bar{u}^j; \bar{u}^l\right) \subseteq \left[\bar{u}^b; \bar{u}^c\right]$ of investors with identical sets K'. $|K'| > 1$ can occur in two cases only: first, if K' consists only of acts k', with $\left|C_{t-1}^i(k')\right| =$

[80] Of course, $\check{p}_1^k$ depends on the interval $\left[\bar{u}^j; \bar{u}^l\right]$. I neglect this dependence in the notation for convenience.

0, second, if

$$\sum_{\tau\in C_{t-1}^{i}(k')} v_\tau(k') = \sum_{\tau\in C_{t-1}^{i}(k'')} v_\tau(k'')$$

and

$$\left|C_{t-1}^{i}(k')\right| = \left|C_{t-1}^{i}(k'')\right|$$

for all k' and k'' in K'. In the second case, obviously $\left(\bar{u}^j;\bar{u}^l\right) \equiv \left(\bar{u}^b;\bar{u}^c\right)$[81]. The analysis of the first case is analogous to the analysis for $t=1$, discussed above. As for the second case, $(A3)$ insures that showing the result of proposition 1 for the case $|K'|=1$ will allow to extend it to the case $|K'|>1$, as well. The different cases for an arbitrary period t and $|K'|=1$ are summarized in the following propositions 4.2 and 4.3:

Proposition 4.2 *Consider an interval of investors with aspiration levels $\left[\bar{u}^j;\bar{u}^l\right]\subset\left[\bar{u}^0;\bar{u}^n\right]$ with identical memories and identical sets K' such that $K'=\{k'\}$. Let $\alpha_{t-1}^{i}=k$ and suppose that*

$$\left(\left|C_{t-1}^{i}(k)\right|+1\right)-\left|C_{t-1}^{i}(k')\right|\neq 0$$

holds for all $i\in[j;l]$. Define $\hat{p}_t^k$ by:

$$\hat{p}_t^k = \min\left\{\begin{array}{c} p_t^k\in\mathbb{R}_0^+ \mid \sum\limits_{\tau\in C_{t-1}^{i}(k)} v_\tau(k) + u\left(\frac{p_t^k}{p_{t-1}^k}+\frac{\delta_t^k}{p_{t-1}^k}\right) - \left(\left|C_{t-1}^{i}(k)\right|+1\right)\bar{u}^i \\ \geq \sum\limits_{\tau\in C_{t-1}^{i}(k')} v_\tau(k') - \bar{u}^i\left|C_{t-1}^{i}(k')\right| \text{ for every } \bar{u}^i\in\left[\bar{u}^j;\bar{u}^l\right]\end{array}\right\}$$

and $\check{p}_t^k$ by:

$$\check{p}_t^k = \max\left\{\begin{array}{c} p_t^k\in\mathbb{R}_0^+ \mid \sum\limits_{\tau\in C_{t-1}^{i}(k)} v_\tau(k) + u\left(\frac{p_t^k}{p_{t-1}^k}+\frac{\delta_t^k}{p_{t-1}^k}\right) - \left(\left|C_{t-1}^{i}(k)\right|+1\right)\bar{u}^i \\ \leq \sum\limits_{\tau\in C_{t-1}^{i}(k')} v_\tau(k') - \bar{u}^i\left|C_{t-1}^{i}(k')\right| \text{ for every } \bar{u}^i\in\left[\bar{u}^j;\bar{u}^l\right]\end{array}\right\}$$

if a maximum exists and set $\check{p}_t^k=0$, else. Then the individual choices of the young investors in this interval are given by:

- $\alpha_t^i=k'$ *for all* $i\in[j;l]$ *if* $p_t^k\leq\check{p}_t^k$;
- $\alpha_t^i=k$ *for all* $i\in[j;l]$ *if* $p_t^k\geq\hat{p}_t^k$;
- *if* $\hat{p}_t^k>0$ *and* $p_t^k\in\left[\check{p}_t^k;\hat{p}_t^k\right]$, *then there exists a critical aspiration level* $\bar{u}^*\in\left[\bar{u}^j;\bar{u}^l\right]$ *such that*

$$\bar{u}^* = \frac{\sum\limits_{\tau\in C_{t-1}^{i}(k)} v_\tau(k) + u\left(\frac{p_t^k}{p_{t-1}^k}+\frac{\delta_t^k}{p_{t-1}^k}\right) - \sum\limits_{\tau\in C_{t-1}^{i}(k')} v_\tau(k')}{\left(\left|C_{t-1}^{i}(k)\right|+1\right)-\left|C_{t-1}^{i}(k')\right|} \tag{4.27}$$

[81] Of course, $|K'|>1$ can occur for a single investor, even if these two conditions are not satisfied. Since, however, a single investor has a mass 0, his decision (in case of indifference) does not influence the prices and the equilibrium allocation.

and

$$\alpha_t^i = k', \text{ if } \bar{u}^i \underset{(\leq)}{\geq} \bar{u}^*$$

$$\alpha_t^i = k, \text{ if } \bar{u}^i \underset{(\geq)}{\leq} \bar{u}^*,$$

if

$$\left(\left|C_{t-1}^i(k)\right| + 1\right) - \left|C_{t-1}^i(k')\right| \underset{(<)}{>} 0. \tag{4.28}$$

If (4.28) does not hold, then the denominator in (4.27) is 0 and u^* is not well defined. This case is considered in proposition 4.3.

The following corollary obtains:

Corollary 4.1 *Consider an interval of investors with aspiration levels* $\left[\bar{u}^j; \bar{u}^l\right] \subset \left[\bar{u}^0; \bar{u}^n\right]$ *and identical memories. Let* $\alpha_{t-1}^i = k$ *and the sets* $K' = \{k'\}$ *be identical for all investors in this interval. Suppose that*

$$\left(\left|C_{t-1}^i(k)\right| + 1\right) - \left|C_{t-1}^i(k')\right| \neq 0$$

holds for all $i \in [j; l]$*. The value of demand for* k *(or the mass of investors, who wish to hold* k*) of this interval,* $d_t^k\left(\left[\bar{u}^j; \bar{u}^l\right]\right)$*, is an increasing, continuous function of the price* p_t^k*. The function consists of at most three segments:*

- *for* $p_t^k > \hat{p}_t^k$,

$$d_t^k\left(\left[\bar{u}^j; \bar{u}^l\right]\right) = \bar{u}^l - \bar{u}^j = const,$$

- *for* $p_t^k < \check{p}_t^k$,

$$d_t^k\left(\left[\bar{u}^j; \bar{u}^l\right]\right) = 0 = const,$$

- *for* $p_t^k \in \left[\check{p}_t^k; \hat{p}_t^k\right]$, d_t^k *is strictly increasing in* p_t^k *and convex, concave or linear, if* $u(\cdot)$ *is convex, concave or linear, respectively.* $d_t^k\left(\left[\bar{u}^j; \bar{u}^l\right]\right)$ *is bounded above*[82] *by* $\left[\bar{u}^l - \bar{u}^j\right]$ *and below by* 0.

The result of the corollary is illustrated in figure 6 for a concave utility function $u(\cdot)$:

It is also possible to derive the demand for k' of these investors:

Corollary 4.2 *Consider an interval of investors with aspiration levels* $\left[\bar{u}^j; \bar{u}^l\right] \subset \left[\bar{u}^0; \bar{u}^n\right]$ *and identical memories. Let* $\alpha_{t-1}^i = k$ *and the sets* $K' = \{k'\}$ *be identical for all investors. Suppose that*

$$\left(\left|C_{t-1}^i(k)\right| + 1\right) - \left|C_{t-1}^i(k')\right| \neq 0$$

[82] The upper boundary results from the budget constraint of each investor.

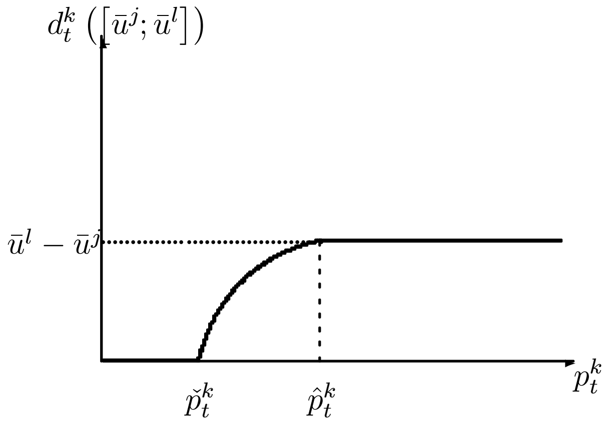


Figure 6

holds for all $i \in [j;l]$. *The value of demand for* k' *(or the mass of investors, who wish to hold* k'*) of this interval,* $d_t^{k'}\left(\left[\bar{u}^j;\bar{u}^l\right]\right)$, *is a decreasing, continuous function of the price* p_t^k, *bounded between* $\left[0;\left(\bar{u}^l-\bar{u}^j\right)\right]$.

Assume now that $\left(\left|C_{t-1}^i(k)\right|+1\right)-\left|C_{t-1}^i(k')\right|=0$.

Proposition 4.3 *Consider an interval of investors with aspiration levels* $\left[\bar{u}^j;\bar{u}^l\right]\subset\left[\bar{u}^0;\bar{u}^n\right]$ *and identical memories. Let* $\alpha_{t-1}^i=k$ *and the sets* $K'=\{k'\}$ *be identical for all investors. Suppose that*

$$\left(\left|C_{t-1}^i(k)\right|+1\right)-\left|C_{t-1}^i(k')\right|=0 \tag{4.29}$$

holds for all $i\in[j;l]$. *Define* $\tilde{p}_t^k$ *as*

$$\tilde{p}_t^k=\max\left\{\left\{p_t^k \mid \sum_{\tau\in C_{t-1}^i(k)} v_\tau(k)+u\left(\frac{p_t^k}{p_{t-1}^k}+\frac{\delta_t^k}{p_{t-1}^k}\right)=\sum_{\tau\in C_{t-1}^i(k')} v_\tau(k')\right\};0\right\}.$$

Then the individual choices of the young investors in this interval are given by:

- *if* $p_t^k<\tilde{p}_t^k$, *then* $\alpha_t^i=k'$ *for all* $i\in[j;l]$;
- *if* $p_t^k>\tilde{p}_t^k$, *then* $\alpha_t^i=k$ *for all* $i\in[j;l]$;
- *if* $p_t^k=\tilde{p}_t^k$, *then* $\alpha_t^i\in\{k;k'\}$ *for all* $i\in[j;l]$.

Note that in order to insure that proposition 4.1 holds, it must be that at $p_t^k = \tilde{p}_t^k$ the interval of investors is divided into two intervals (one of which possibly empty), one of the intervals choosing k, and the other one k', which is guaranteed by $(A3)$.

Propositions 4.2 and 4.3 complete the proof of proposition 4.1.

The following corollary is obtained directly from proposition 4.3:

Corollary 4.3 *Consider an interval of investors with aspiration levels $\left[\bar{u}^j;\bar{u}^l\right] \subset \left[\bar{u}^0;\bar{u}^n\right]$ and identical memories. Let $\alpha_{t-1}^i = k$ and the sets $K' = \{k'\}$ be identical for all investors. Suppose that*

$$\left(\left|C_{t-1}^i(k)\right| + 1\right) - \left|C_{t-1}^i(k')\right| = 0$$

for all $i \in [j;l]$. The value of demand for k of this interval, $d_t^k\left(\left[\bar{u}^j;\bar{u}^l\right]\right)$, is an upper hemicontinuous correspondence, which is non-empty closed- and convex-valued for each $p_t^k \geq 0$. It has the form:

- $d_t^k\left(\left[\bar{u}^j;\bar{u}^l\right]\right) = 0$ *for* $p_t^k < \tilde{p}_t^k$,
- $d_t^k\left(\left[\bar{u}^j;\bar{u}^l\right]\right) \in \left[0;\left(\bar{u}^l - \bar{u}^j\right)\right]$ *for* $p_t^k = \tilde{p}_t^k$
- $d_t^k\left(\left[\bar{u}^j;\bar{u}^l\right]\right) = \bar{u}^l - \bar{u}^j$ *for* $p_t^k > \tilde{p}_t^k$.

The value of demand is bounded above by $\left(\bar{u}^l - \bar{u}^j\right)$ and below by 0.

Corollary 4.3 is illustrated in figure 7.

The value of demand for k' of the interval $\left[\bar{u}^j;\bar{u}^l\right]$ then has the following properties:

Corollary 4.4 *Consider an interval of investors with aspiration levels $\left[\bar{u}^j;\bar{u}^l\right] \subset \left[\bar{u}^0;\bar{u}^n\right]$ and identical memories. Let $\alpha_{t-1}^i = k$ and the sets $K' = \{k'\}$ be identical for all investors. Suppose that*

$$\left(\left|C_{t-1}^i(k)\right| + 1\right) - \left|C_{t-1}^i(k')\right| = 0$$

for all $i \in [j;l]$. The value of demand for k' of this interval, $d_t^{k'}\left(\left[\bar{u}^j;\bar{u}^l\right]\right)$, is an upper hemicontinuous correspondence, which is non-empty, closed- and convex-valued for each $p_t^k \geq 0$. The value of demand is bounded above by $\left(\bar{u}^l - \bar{u}^j\right)$ and below by 0.

Since the investors whose predecessors do not hold k do not observe the returns of k, their decision does not depend on p_t^k. The value of demand of these investors for k is derived in the same way, as the value of demand for k' of the investors, whose predecessors hold k, hence it has the same properties, as stated in corollaries 4.2 and 4.4.

Corollaries 4.1, 4.2, 4.3 and 4.4 clarify how the value of demand correspondence for asset k at

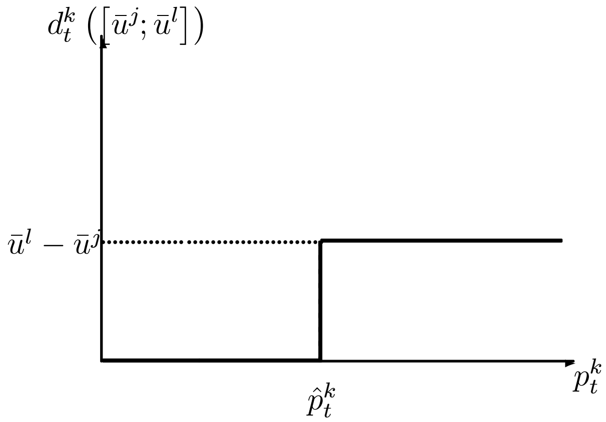


Figure 7

time t can be constructed for the whole population of investors in the economy. Suppose that at time t, the population of old investors can be partitioned into L intervals such that the investors in the same interval have identical memories and identical sets K'. Denote the value of demand for an asset k of one such interval by $d_t^{kl}(p_t)$[83] for $l = 1...L$. Then the value of demand for asset k is given by:

$$d_t^k(p_t) = \sum_{l=1}^{L} d_t^{kl}(p_t). \tag{4.30}$$

Proposition 4.4 *The value of demand for asset k, $d_t^k(p_t)$, is a correspondence, which maps the range of non-negative price vectors $p_t \in [0;\infty]^K$ into the range of possible values of demand $[0;n]$. It is upper hemicontinuous, non-empty, closed- and convex-valued for every price vector p_t. The value of demand for assets $d_t(p_t) = \left(d_t^1(p_t)...d_t^K(p_t)\right)$ is, therefore, also an upper hemicontinuous correspondence, which is non-empty, closed- and convex-valued for every price vector p_t and maps the non-negative price vectors $p_t \in [0;\infty]^K$ into the range of possible values of demand $[0;n]^K$.*

Now consider the value of supply function. By definition, it is a continuous function of p_t.

[83] Note that the value of demand $d_t^{kl}(p_t)$ depends only on one of the components of the price vector and is constant in its other components. Therefore, the characteristics of $d_t^{kl}(p_t)$ are identical with those found in corollaries 4.1, 4.2, 4.3 and 4.4.

According to remarks 4.1 and 4.2, the equilibrium price vector p_t must satisfy:

$$d_t^k = A_k p_t^k$$

for all $k \in \{1...K\}$ and

$$\sum_{k=1}^{K} d_t^k = n,$$

hence

$$\begin{aligned}\sum_{k=1}^{K} A_k p_t^k &= n, \text{ or} \\ p_t^k &\leq \frac{n}{A_k} \; \forall k = 1...K,\end{aligned}$$

since the prohibition of short sales implies $d_t^k \geq 0$. Hence, the value of demand and the value of supply functions need only be considered for a range of prices $p_t^k \in \left[0; \frac{n}{Ak}\right]$ for $k = 1...K$.

Proposition 4.5 *The correspondence*

$$\tilde{d}_t(p_t) = \left(\frac{d_t^1}{A_1}...\frac{d_t^K}{A_K}\right),$$

which maps $\left[0; \frac{n}{A_1}\right] \times ... \times \left[0; \frac{n}{A_K}\right] \rightarrow \left[0; \frac{n}{A_1}\right] \times ... \times \left[0; \frac{n}{A_K}\right]$ *has a fixed point.*

Corollary 4.5 *The correspondence* $\tilde{d}_t(p_t)$ *has at least one common point with* $\tilde{s}_t(p_t) = p_t$ *on the set* $\left[0; \frac{n}{A_1}\right] \times ... \times \left[0; \frac{n}{A_K}\right]$. *Hence* $d_t(p_t)$ *has at least one common point with* $s_t(p_t)$ *for* $p_t \in \left[0; \frac{n}{A_1}\right] \times ... \times \left[0; \frac{n}{A_K}\right]$.

Any common point of s_t and d_t is a temporary equilibrium of the economy. Indeed, if at this point the price is strictly positive, the usual equilibrium condition (demand equals supply) obtains by dividing both the value of demand and the value of supply for asset k by the price p_t^k:

$$A_k = \frac{d_t^k}{p_t^k} \text{ for each } k = 1...K,$$

If, however, $p_t^k = 0$, the intersection point represents a degenerate equilibrium, at which the excess demand for k is $-A < 0$. Hence, a temporary equilibrium exists at any time $t \geq 1$ and for any length of memory.

The results are summarized in the following proposition:

Proposition 4.6 *Let* $(A1)$, $(A2)$ *and* $(A3)$ *hold. Then for each length of memory* $m \in \{0...t-1\}$, *a temporary equilibrium of the economy exists in every period* $t \geq 1$.

4.7 The Case of Two Assets

It might be useful to look at an example in order to clarify the intuition of the proof. Suppose that there are two assets in the economy: a risky asset, denoted by a and a riskless asset b. The risky asset has a fixed supply A and is characterized by a dividend process δ_t, identically and independently distributed according to:

$$\delta_t = \left\{ \begin{array}{ll} \delta D & \text{with probability } q \\ 0 & \text{with probability } 1-q \end{array} \right\}.$$

Denote the price of the risky asset by p_t. The riskless asset is available in perfectly elastic supply at a price of 1 and has a return of $(1+r)$ per unit.

Note that in case of two assets assumption $(A3)$, which specifies how decisions are made in case of indifference, is not needed, since only one alternative is available, i.e. the set K' always consists of one act only.

The demand for the risky asset a of an investor whose predecessor holds a is described by (4.20) and (4.21) and illustrated in figures 4 and 5. The value of demand for a of an interval of such investors is an upper hemicontinuous correspondence, as illustrated in figures 6 and 7. Moreover, since the only investment alternative is b, the demand for b of each of these investors is 1, if their demand for a is 0 and vice versa.

Since the price of b remains constant over the time, the indirect utility achieved by holding b is $(1+r)$ in each period of time in which b is chosen. Consider a young investor i whose predecessor holds b at time t. Such an investor compares the cumulative utilities of a and b, given by:

$$U_t^i(a) = \sum_{\tau \in C_t^i(a)} \left[v_\tau(a) - \bar{u}^i\right] = \sum_{\tau \in C_{t-1}^i(a)} \left[v_\tau(a) - \bar{u}^i\right],$$
$$U_t^i(b) = \sum_{\tau \in C_t^i(b)} \left[u(1+r) - \bar{u}^i\right].$$

Since both terms do not depend on p_t, the decision of the young investor amounts to comparing two constants:

1. If $U_t^i(a) > U_t^i(b)$, then investor i chooses asset a ($\alpha_t^i = a$).

2. If $U_t^i(a) < U_t^i(b)$, then investor i chooses asset b ($\alpha_t^i = b$).

3. If $U_t^i(a) = U_t^i(b)$, then investor i is indifferent between holding a and b ($\alpha_t^i \in \{a; b\}$).

Hence, the demand of such an investor does not depend on the price of the risky asset p_t and is constant. The following two propositions illustrate the demand for a of an interval of investors, whose predecessors hold b:

Proposition 4.7 *Consider an interval of investors with aspiration levels $\left[\bar{u}^j; \bar{u}^l\right] \subset \left[\bar{u}^0; \bar{u}^n\right]$ and identical memories. Let $\alpha_{t-1}^i = b$ and suppose that*

$$\left|C_{t-1}^i(a)\right| - \left|C_{t-1}^i(b)\right| - 1 \neq 0$$

for all $i \in [j; l]$. Then

- *if*
$$\sum_{\tau \in C_{t-1}^i(a)} \left[v_\tau(a) - \bar{u}^i\right] \leq \left(\left|C_{t-1}^i(b)\right| + 1\right)\left(1 + r - \bar{u}^i\right) \text{ for each } \bar{u}^i \in \left[\bar{u}^j; \bar{u}^l\right], \tag{4.31}$$
then $\alpha_t^i = b$ for every $i \in [j; l]$.

- *if*
$$\sum_{\tau \in C_{t-1}^i(a)} \left[v_\tau(a) - \bar{u}^i\right] \geq \left(\left|C_{t-1}^i(b)\right| + 1\right)\left(1 + r - \bar{u}^i\right) \text{ for each } \bar{u}^i \in \left[\bar{u}^j; \bar{u}^l\right], \tag{4.32}$$
then $\alpha_t^i = a$ for every $i \in [j; l]$.

- *if neither (4.31), nor (4.32) are satisfied, then there is a critical aspiration level $\hat{u} \in \left(\bar{u}^j; \bar{u}^l\right)$, such that:*
$$\hat{u} = \frac{\sum_{\tau \in C_{t-1}^i(a)} \left[v_\tau(a)\right] - \left(\left|C_{t-1}^i(b)\right| + 1\right)(1 + r)}{\left(\left|C_{t-1}^i(a)\right| - \left|C_{t-1}^i(b)\right| - 1\right)}.$$
In this case:
$$\alpha_t^i = a, \text{ if } \bar{u}^i \underset{(\leq)}{\geq} \hat{u}$$
$$\alpha_t^i = b, \text{ if } \bar{u}^i \underset{(\geq)}{\leq} \hat{u},$$
if
$$\left|C_{t-1}^i(a)\right| - \left|C_{t-1}^i(b)\right| - 1 \underset{(>)}{<} 0.$$

The following corollary characterizes the value of demand of these investors:

Corollary 4.6 *Consider an interval of investors with aspiration levels $\left[\bar{u}^j; \bar{u}^l\right] \subset \left[\bar{u}^0; \bar{u}^n\right]$ with identical memories. Let $\alpha_{t-1}^i = b$ and suppose that*

$$\left|C_{t-1}^i(a)\right| - \left|C_{t-1}^i(b)\right| - 1 \neq 0 \tag{4.33}$$

for all $i \in [j;l]$. Then the value of demand for a of the interval $\left[\bar{u}^j;\bar{u}^l\right]$, $d_t\left(\left[\bar{u}^j;\bar{u}^l\right]\right)$, is a constant function of p_t and can obtain (depending on the memory and the aspiration levels $\left[\bar{u}^j;\bar{u}^l\right]$) only values between 0 and $\left(\bar{u}^l-\bar{u}^j\right)$.

Figure 8 a) illustrates the value of demand for the three possible cases described in proposition 4.7.

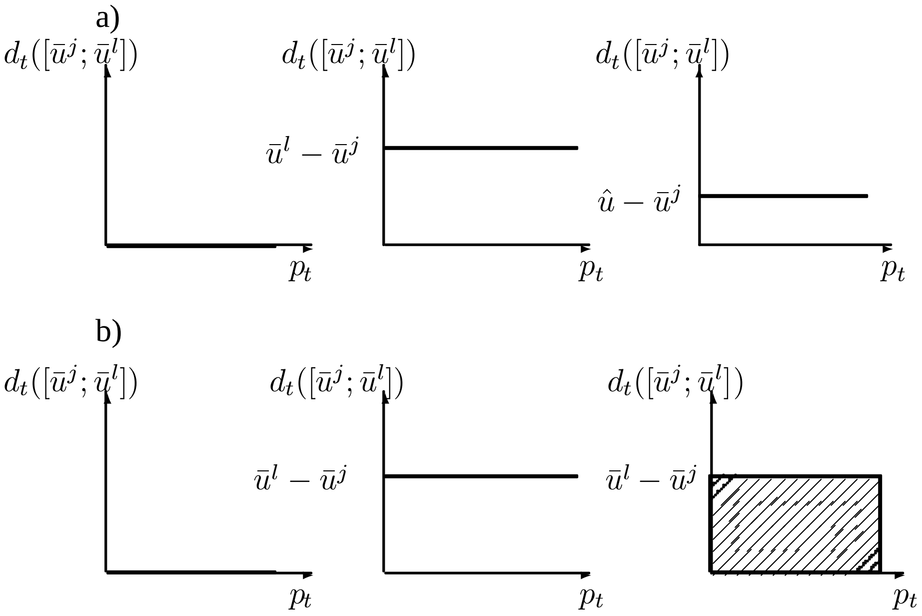


Figure 8

Now consider the case of the investors whose predecessors hold b and whose memory is such that (4.33) does not hold. It is evident (and therefore not proved) that the following result obtains:

Proposition 4.8 *Consider an interval of investors with aspiration levels $\left[\bar{u}^j;\bar{u}^l\right] \subset \left[\bar{u}^0;\bar{u}^n\right]$ and identical memories. Let $\alpha^i_{t-1}=b$ and suppose that*

$$\left|C^i_{t-1}(a)\right|-\left|C^i_{t-1}(b)\right|-1=0$$

for all $i\in[j;l]$. Then

- *if*
$$\sum_{\tau\in C^i_{t-1}(a)}\left[v_\tau(a)\right]>\left(\left|C^i_{t-1}(b)\right|+1\right)(1+r),$$
then $\alpha^i_t=a$ for every $i\in[j;l]$;

- *if*
$$\sum_{\tau \in C^i_{t-1}(a)} [v_\tau(a)] < \left(\left|C^i_{t-1}(b)\right| + 1\right)(1+r),$$
then $\alpha^i_t = b$ *for every* $i \in [j;l]$;

- *if*
$$\sum_{\tau \in C^i_{t-1}(a)} [v_\tau(a)] = \left(\left|C^i_{t-1}(b)\right| + 1\right)(1+r),$$
then $\alpha^i_t \in \{a;b\}$ *for every* $i \in [j;l]$.

Figure 8 b) illustrates the value of demand for these three cases. Its properties are characterized in the following corollary.

Corollary 4.7 *Consider an interval of investors with aspiration levels* $\left[\bar{u}^j;\bar{u}^l\right] \subset [\bar{u}^0;\bar{u}^n]$ *with identical memories. Let* $\alpha^i_{t-1} = b$ *and suppose that*
$$\left|C^i_{t-1}(a)\right| - \left|C^i_{t-1}(b)\right| - 1 = 0$$
for all $i \in [j;l]$*. Then the value of demand for* a *of the interval* $\left[\bar{u}^j;\bar{u}^l\right]$*,* $d_t\left(\left[\bar{u}^j;\bar{u}^l\right]\right)$*, is (in general) an upper hemicontinuous correspondence, which is independent of the price* p_t*, non-empty, closed- and convex-valued and can obtain values in the range* $\left[0;\bar{u}^l - \bar{u}^j\right]$.

In analogy to the case with K assets, $(A2)$ implies that each of the intervals into which the population is partitioned at time $(t-1)$ is divided into at most two new intervals at time t such that the investors in the same intervals have identical memories and make identical choices. In analogy to the case of K assets, it can be shown that $p_t \leq \frac{n}{A}$ must hold in equilibrium. Since the value of demand for a of each such interval is an upper hemicontinuous non-empty, closed- and convex-valued correspondence, the following proposition obtains:

Proposition 4.9 *The value of demand for asset* a*,* $d_t(p_t)$*, is a correspondence, which maps the range of non-negative prices* $p_t \in \left[0;\frac{n}{A}\right]$ *into the range of possible values of demand* $[0;n]$*. It is upper hemicontinuous, non-empty and closed- and convex-valued for every price* p_t*. Hence, for each length of memory* $m \in \{0;...t-1\}$*, a temporary equilibrium exists for each* $t \geq 1$.

The proposition is proved by applying the Kakutani fixed-point theorem and insures the existence of an equilibrium. The argument for two assets can be illustrated in a diagram, see figure 9.

Note that the uniqueness of the equilibrium is not guaranteed (similarly to the case of K assets) and that equilibria with $p_t = 0$ cannot be excluded in general. Moreover, the model allows for

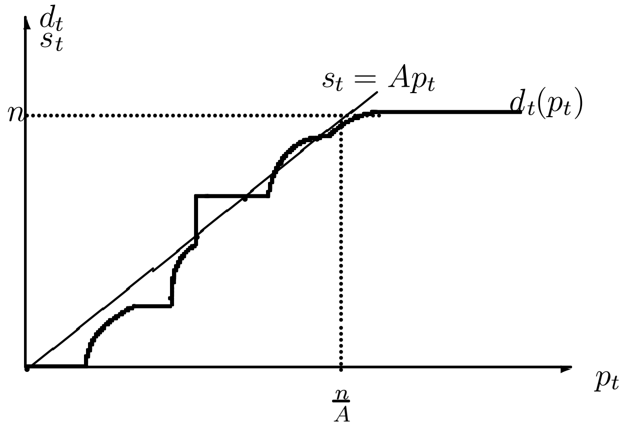


Figure 9

an equilibrium, in which the price of the risky asset is $p_t = \frac{n}{A}$ and the demand for the riskless asset is 0.

4.8 Conditions for a Non-Degenerate Equilibrium

The argument above makes clear that it is not possible to rule out equilibria in which the price of an asset is 0 and the demand for it is 0.

Nevertheless, it is possible to indicate conditions which guarantee that the equilibrium prices are strictly positive. Figure 10 captures the idea for the case of two assets.

The sufficient condition is that some positive mass of investors is ready to choose act k, even if the price of the asset falls from its highest possible value to 0 and the dividend of k is the lowest possible, i.e. $\min\left\{\delta^k\right\}$ in the period under consideration. To guarantee this, it is necessary that there is a positive mass of investors in the market, whose aspiration levels are below $u\left(\frac{A_k \min\left\{\delta^k\right\}}{n}\right)$, i.e. $\bar{u}^0 < u\left(\frac{A_k \min\left\{\delta^k\right\}}{n}\right)$, where $\frac{n}{A_k}$ is the highest possible equilibrium price of k. This is, however, not sufficient. To insure that the price p_t^k is positive at each time t, a pos-

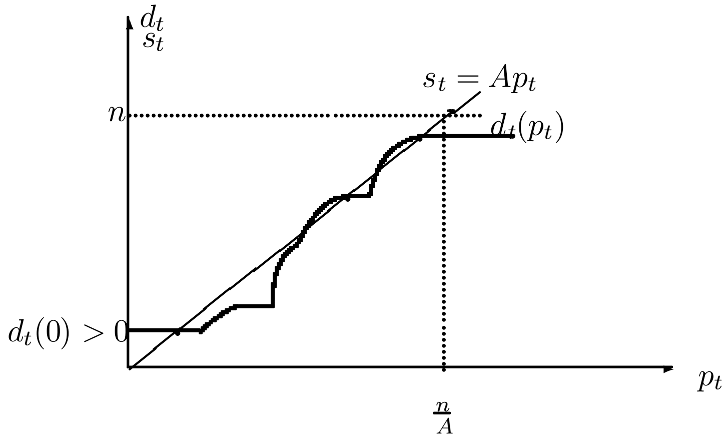


Figure 10

itive mass of the old investors with aspiration levels between $\bar{u}^0$ and $u\left(\frac{A_k \min\{\delta^k\}}{n}\right)$ must hold asset k in $t = 1$. Since the aspiration levels of these investors are very low, they will never want to switch to another asset, no matter how low the price of k may be, and how often it falls[84]. The demand of such an investor for k — x_t^k is illustrated in figure 5. These investors insure that there is a positive demand for asset k and, therefore that its price in equilibrium is positive.

The mathematical possibility to rule out this kind of equilibria, however, does not guarantee the applicability of the conditions defined above to economic models. Indeed, the condition above might require that $\bar{u}^0 < u(0)$, if $\min\{\delta^k\} = 0$. It is questionable, whether one can imagine consumers who would find it satisfactory to consume a zero-amount of the consumption good, or investors who would be satisfied with the opportunity to lose the whole amount of money invested. On the contrary, it is possible that zero equilibrium prices for assets with positive fundamental value help us gain new insights into the problem of emerging and bursting of bubbles in asset markets. If, however, $\min\{\delta^k\} > 0$, then the conditions excluding degenerate equilibria may still appear sensible: since investors derive utility from consumption and not directly

[84] In this case the cumulative utility of the alternative k will remain positive over the time, whereas an (untried) act k' will always have a cumulative utility of 0. Thus, k' will never be chosen.

from the returns of the assets, it is possible that some investors are satisfied with some small, but positive amount of consumption.

The severity of the conditions needed to exclude degenerate equilibria shows that it might be useful to state conditions under which at least one of the equilibria has non-zero prices. Such results will allow to construct economies which exhibit both non-degenerate equilibria and equilibria with 0-prices of some of the assets. On the one hand, this will enrich the model by introducing new effects, such as for instance zero-asset prices, bubbles and arbitrage possibilities. On the other hand, the assumptions imposed on the aspiration levels of the decision-makers will be more acceptable, than those needed to exclude degenerate equilibria at all.

The idea for stating such conditions is simple. The necessary condition for the existence of an equilibrium with a positive price of asset k in each period t for each length of memory m is that in each period of time there is an interval of investors holding k. Let this interval have aspiration levels $[\bar{u}^{j_k}; \bar{u}^{j_{k-1}}]$. Suppose that the length of this interval is λ. This would guarantee that the price of the asset k will never fall below $\frac{\lambda}{A_k}$, as long as the investors from this interval hold k. It is however not the price of the asset alone that determines the behavior of the investors, but its returns. Therefore, one has to insure that the investors with aspiration levels $[\bar{u}^{j_k}; \bar{u}^{j_{k-1}}]$ hold the asset even if its return is the lowest possible, i.e. its dividend is $\min\left\{\delta^k\right\}$ and its price falls by the greatest possible amount for this economy.

Suppose that the assets are enumerated from 1 to K, as follows: 1 is a riskless asset with $\delta_t^1 = r$ w.p. 1. Let $r > \min\left\{\delta^k\right\}$ for each $k \neq 1$ and $\min\left\{\delta^k\right\} > \min\left\{\delta^{k+1}\right\}$ for each $k \in \{2...K\}$. Now choose aspiration levels $\bar{u}^{j_1} > \bar{u}^{j_2} > ... > \bar{u}^{j_K} \in (\bar{u}^0; \bar{u}^n)$, such that:

$$\begin{array}{lll} (i) & u\left(\frac{\bar{u}^{j_1}-\bar{u}^{j_2}+rA_1}{\bar{u}^n-\bar{u}^{j_2}}\right) \geq \bar{u}^{j_1} \\ (ii) & u\left(\frac{\bar{u}^{j_2}-\bar{u}^{j_3}+\min\left\{\delta_t^2\right\}A_2}{\bar{u}^n-\bar{u}^{j_1}+\bar{u}^{j_2}-\bar{u}^{j_3}}\right) \geq \bar{u}^{j_2} \\ ... & ... \\ (K) & u\left(\frac{\bar{u}^{j_K}-\bar{u}^0+\min\left\{\delta^K\right\}A_K}{\bar{u}^n-\bar{u}^{j_1}+\bar{u}^{j_K}-\bar{u}^0}\right) \geq \bar{u}^{j_K} \end{array} . \tag{4.34}$$

Proposition 4.10 *Suppose that (4.34) has a solution $\bar{u}^{j_1} > \bar{u}^{j_2} > ... > \bar{u}^{j_K}$ with $\bar{u}^{j_k} \in (\bar{u}^0; \bar{u}^n)$ for all $k = 1...K$ and for the given $\bar{u}^0$, $\bar{u}^n$, r, $\min\left\{\delta^k\right\}$, $k \in \{2...K\}$. Let the initial endowments with assets of the old investors with aspiration levels between $[\bar{u}^0; \bar{u}^{j_1}]$ at $t = 1$ satisfy:*

$$\alpha_0^i = \left\{ \begin{array}{lll} K & \textit{if} & \bar{u}^i \in \left[\bar{u}^0; \bar{u}^{j_K}\right] \\ K-1 & \textit{if} & \bar{u}^i \in \left[\bar{u}^{j_K}; \bar{u}^{j_{K-1}}\right] \\ ... & ... & ... \\ 2 & \textit{if} & \bar{u}^i \in \left[\bar{u}^{j_3}; \bar{u}^{j_2}\right] \\ 1 & \textit{if} & \bar{u}^i \in \left[\bar{u}^{j_2}; \bar{u}^{j_1}\right] \end{array} \right\}$$

The initial endowments of the investors with aspiration levels $[\bar{u}^{j_1}; \bar{u}^n]$ can be chosen arbitrarily. Then, for each length of memory m, there exists in each period $t \geq 1$ a temporary equilibrium of the economy with prices $p_t = \left(p_t^1 ... p_t^K\right)$:

$$p_t^1 \geq \frac{\bar{u}^{j_1} - \bar{u}^{j_2}}{A_1}$$
$$p_t^2 \geq \frac{\bar{u}^{j_2} - \bar{u}^{j_3}}{A_2}$$
$$...$$
$$p_t^{K-1} \geq \frac{\bar{u}^{j_{K-1}} - \bar{u}^{j_K}}{A_{K-1}}$$
$$p_t^K \geq \frac{\bar{u}^{j_K} - \bar{u}^0}{A_K}$$

and allocations α_t^i such that:

$$\alpha_t^i = \left\{ \begin{array}{lll} K & \textit{if} & \bar{u}^i \in \left[\bar{u}^0; \bar{u}^{j_K}\right] \\ K-1 & \textit{if} & \bar{u}^i \in \left[\bar{u}^{j_K}; \bar{u}^{j_{K-1}}\right] \\ ... & ... & ... \\ 2 & \textit{if} & \bar{u}^i \in \left[\bar{u}^{j_3}; \bar{u}^{j_2}\right] \\ 1 & \textit{if} & \bar{u}^i \in \left[\bar{u}^{j_2}; \bar{u}^{j_1}\right] \end{array} \right\}.$$

Although the sufficient condition has been stated, it is not clear, whether there are values of the parameters for which it holds. For simplicity, consider the case in which the utility function is linear, $A_k = 1$ for all $k = 1...K$ and the intervals $[\bar{u}^{j_0}; \bar{u}^{j_K}]$; $[\bar{u}^{j_K}; \bar{u}^{j_{K-1}}]$... $[\bar{u}^{j_2}; \bar{u}^{j_1}]$ have equal length λ. With this assumption condition (4.34) can be written in terms of the parameters of the model and λ and becomes:

$$\frac{\lambda + \min\left\{\delta^k\right\}}{\bar{u}^n - \bar{u}^0 - \lambda\left(K-1\right)} \geq \bar{u}^0 + \lambda\left(K - (k-1)\right) \text{ for } k = 1...K. \tag{4.35}$$

It is easy to show that (4.35) is a quadratic inequality in λ with a positive coefficient in front of λ^2. Therefore, it will have a solution independently of the sign of its discriminants (one for each k). Moreover, the solutions will have the form $\lambda \in \left(-\infty; \check{\lambda}_k\right] \cup \left[\hat{\lambda}_k; +\infty\right)$ if the discriminant of inequality k is strictly positive. If the discriminant is non-positive, then every λ is a solution of the inequality. It remains, therefore, to state conditions insuring that some of the solutions lie in the interval $\left[0; \frac{\bar{u}^n - \bar{u}^0}{K}\right]$ of possible values of λ. It follows that at least one of the following two conditions must hold for every $k = 1...K$:

$$\frac{\min\left\{\delta^k\right\}}{\bar{u}^n - \bar{u}^0} > \bar{u}^0, \tag{4.36}$$

meaning that $\check{\lambda}_k > 0$ or

$$1 + \frac{K \min\left\{\delta^k\right\}}{\bar{u}^n - \bar{u}^0} > \bar{u}^n - \frac{\left(\bar{u}^n - \bar{u}^0\right)(k-1)}{K}, \tag{4.37}$$

meaning that $\hat{\lambda}_k < \frac{\bar{u}^n - \bar{u}^0}{K}$.

The interpretation of (4.36) is straightforward: it says that even at $p_t^k = 0$ and $\delta_t^k = \min\left\{\delta^k\right\}$, there are still investors with sufficiently low aspiration levels, who are ready to hold this asset. Thus, it is equivalent to the condition for non-existence of degenerate equilibria with zero-prices.

The second condition gives few insights into how the parameters of the model influence the existence of non-degenerate equilibria. The highest aspiration level in the economy $\bar{u}^n$ has a negative impact on the left-hand side and a positive one on the right-hand side of (4.37). Therefore, for higher $\bar{u}^n$ a non-degenerate equilibrium may fail to exist. For instance, in the two assets case with $\min\{\delta_t\} = 0$ and $\bar{u}^0 > 0$ only a low enough $\bar{u}^n$ can guarantee the existence of a non-degenerate equilibrium: (4.37) implies

$$\bar{u}^n \leq 2 - \bar{u}^0.$$

Since $\min\left\{\delta^k\right\}$ was defined as decreasing in k it follows further that the inequality is more likely to hold for smaller k, i.e. for assets with higher minimal dividends, than for those with lower minimal dividends.

To illustrate the result, consider again the case of two assets introduced in section 7. Suppose that $\bar{u}^n > (1+r) > 1 > \bar{u}^0$ and let $\bar{u}^a \in (\bar{u}^0; 1)$ be some aspiration level. Now endow the investors with aspiration levels between $[\bar{u}^0; \bar{u}^a]$ with asset a, investors with aspiration levels between $[\bar{u}^a; (1+r)]$ with asset b and let the portfolio holdings of the investors from the interval $[(1+r); \bar{u}^n]$ be arbitrary (but satisfy the interval condition imposed by assumption $(A2)$). Now note that if $\bar{u}^a$ is chosen in such a way that[85]

$$\bar{u}^a < \frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0},$$

then the investors with aspiration levels $[\bar{u}^0; \bar{u}^a]$ will always hold a, whereas the investors with

[85] Such a $\bar{u}^a$ always exists if, for instance, $\bar{u}^n - r < \frac{3}{2}$ and $\bar{u}^n < 2 + r - \bar{u}^0$. Note that in this case the condition that the intervals have equal length is not necessarily fulfilled.

aspiration levels $[\bar{u}^a; (1+r)]$ will always choose b. Hence, the price of a will satisfy:

$$p_t \geq \bar{u}^a - \bar{u}^0$$

in each period of time. Nevertheless, it is possible that an equilibrium with $p_t = 0$ exists. Especially, if

$$\bar{u}^0 > 0,$$

an equilibrium in which

$$\begin{aligned} p_t &= 0 \\ \alpha_t^i &= b \text{ for all } i \in [0; n] \end{aligned}$$

can obtain in any period t, such that $\alpha_{t-1}^i = a$ holds for all investors with aspiration levels $[(1+r); \bar{u}^n]$.

4.9 Discussion of the Results

Grandmont (1982) considers an economy with one consumption good and fiat money, which only has the function of storing value. He shows the possibility of an equilibrium in which money has a price of 0 and states sufficient conditions with respect to the expectations of the investors guaranteeing the existence of a monetary equilibrium (i.e. an equilibrium in which the price of money is positive). A key condition is that for each system of prices in period t (especially, even if the current price of money is 0) , the investors place a positive probability on the event that money has a positive price in the next period.

Although the expectations are not explicitly modelled in the case-based decision-theory (they are replaced by the constructs of aspiration level, memory and cumulative utility of an act), it seems that the similarities between the conditions imposed by Grandmont (1982) to guarantee the existence of a monetary equilibrium and the condition required for the existence of an equilibrium with positive asset prices are straightforward. In both cases the model should insure that the asset is held by some positive mass of investors even if the price of this asset is currently 0. In the model constructed by Grandmont (1982, p. 897), this is guaranteed by imposing requirements on the expectations of the investors, which should not be too sensitive to the current price, but also take into account past, possibly positive prices. With case-based decisions, the same result is achieved by imposing (at least for some investors) a relatively low aspiration level, which prevents the cumulative utility of an asset from becoming negative and, thus, insures that the investors will continue to hold an asset even if its price falls to 0. The sensitivity with respect to current prices (and current dividends) is, therefore, captured by the aspiration level of an investor and can be analyzed using the graph of the demand of a single investor for asset k. The individual elasticity of demand with respect to the current price is 0 in the interval $\left[0; p_t^{k*}(i)\right)$ and -1 in the interval $\left(p_t^{k*}(i); 0\right]$ (of course, provided that a positive $p_t^{k*}(i)$ exists), see figure 4. At the point $p_t^{k*}(i)$, however, the elasticity is not well defined. Whereas a small increase of the price towards $p_t^{k*}(i)$ has a 0-effect, a small decrease of the price towards $p_t^{k*}(i)$ causes the demand to fall from $\frac{1}{p_t^{k*}(i)} > 0$ to 0. Thus, at the point $p_t^{k*}(i)$ the demand of a single investor for k is infinitely sensitive to decreases of the price. The position of $p_t^{k*}(i)$ depends on the aspiration level of the investor considered. If

$$\left(\left|C_{t-1}^{i}(k)\right| + 1\right) - \left|C_{t-1}^{i}(k')\right| > 0$$

holds[86], then $p_t^{k*}(i)$ is an increasing function of the aspiration level of the investor. It is the position of the lowest $p_t^{k*}(i)$ in the economy that determines whether an equilibrium with a 0-price of k exists. If for a positive mass of investors $p_t^{k*}(i)$ does not exist (meaning that the price would have to be negative to insure that (4.19) holds), then $p_t^k = 0$ cannot occur in equilibrium.

The new result, however, is that even assets with real value can be traded at 0 prices in equilibrium. Indeed, the fiat money in Grandmont's model is only valuable if it is expected to have a positive price in the future, thus if it is accepted by the young investors in exchange for consumption goods. But the assets in the model presented here are real, in the sense that they represent production possibilities and yield real dividends, independently of their price. This shows that introducing case-based decision-makers as investors in an economy with real assets may cause their prices to fall to 0, not because they are inferior to other assets in the economy, but because their returns are considered to be unsatisfactory with respect to some set of aspiration levels at some time.

However, even if $p_t^k = 0$ holds at some time, the price of asset k need not remain 0 forever. If the investors become dissatisfied with other acts and find k to have the highest cumulative utility in some later period, their demand for k will become positive and its price will recover to a strictly positive value. This means, however that a financial market populated by case-based decision-makers is not informationally efficient. Indeed, $p_t^k = 0$ could not hold, should the information

[86] This condition is always satisfied, if the length of the memory is 1. It also holds in each period for an investor i, all of whose predecessors have held the same asset k.

that asset k might yield some strictly positive dividends in the future be implemented into the price.

Arbitrage opportunities can be present in a market also in cases, in which the price vector is strictly positive in all its components. Indeed, whereas a relatively low $\bar{u}^0$ prevents the prices from becoming 0, it also makes the investors with such low aspiration levels satisfied with the act they have initially chosen. Investors with low aspiration levels have, therefore, little or no incentives to switch between the acts. Hence, they might end up holding an asset which is strictly dominated by another one (w.r. to zero-order stochastic dominance). As long as the investors are satisfied with the payoffs of the act which their predecessors have chosen, they do not have reasons to experiment and acquire more information, which would allow them to profit by choosing the asset which bears higher returns in the mean. Even with infinite memory, investors with relatively low aspiration levels will not be able to learn the possible return realizations in order to make optimal decisions.

4.10 Conclusion

I have presented a model of an overlapping-generations economy, populated by case-based decision-makers. The choice to invest the initial endowment into a given asset is positively influenced by the price of this asset and negatively by the prices of the other assets in the economy. Therefore, the demand of the investors for an asset is non-monotonic in the price of the asset: increasing for relatively low prices and decreasing at high prices.

A temporary equilibrium of the economy exists under quite general conditions on continuity and strict monotonicity of the utility function and on the initial holdings of the old investors in $t = 1$. These conditions, combined with the continuity of the distribution of investors and the aspiration levels in the economy, are crucial for the existence, since they guarantee the upper hemicontinuity of the value of demand function and allow the application of Kakutani's fixed point theorem. The conditions which exclude 0-equilibrium prices are comparable with those of Grandmont (1982) and require that some positive mass of investors demands an asset when its price is 0. This requirement implies relatively low aspiration levels for some positive mass of investors, which contradicts the economic intuition that an investor can only be satisfied with a positive amount of consumption. Therefore, although possible from mathematical point of view, exclusion of equilibria with 0-prices may not be sensible in a model of economic behavior.

Furthermore, I believe that the possibility that a price of an asset which pays real dividends, falls to 0 in some period, but might recover later, can provide some explanations of the arising and bursting of bubbles, overreaction and underreaction observed in real financial markets and in laboratory settings. The presence of case-based decision-makers in the economy could further account for unused arbitrage possibilities, which are not always identified by case-based decision-makers (even if their memory is infinite) or for predictability in price movements, resulting from the dependence of the investment decisions on past information. An analysis of the dynamics of equilibrium prices can therefore provide useful insights about markets with adaptively learning non-Bayesian investors and allow to explain some of the paradoxes of the asset pricing, with which the economic research is confronted. The dynamic of asset prices in a market populated by case-based decision-makers will be topic of the next chapter of this thesis.

Appendix

Proof of proposition 4.2:

Write the cumulative utility of the act k as:

$$\begin{aligned} U_t^i(k) &= \sum_{\tau\in C_{t-1}^i(k)} \left[v_\tau(k) - \bar{u}^i\right] + u\left(\frac{p_t^k}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k}\right) - \bar{u}^i = \\ &= \underbrace{\sum_{\tau\in C_{t-1}^i(k)} v_\tau(k) + u\left(\frac{p_t^k}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k}\right)}_{\text{identical for the whole interval}} \underbrace{-\left(\left|C_{t-1}^i(k)\right| + 1\right)\cdot}_{\text{identical for the whole interval}} \underbrace{\bar{u}^i}_{\text{investor specific}} \end{aligned} \tag{4.38}$$

and those of k', as:

$$\begin{aligned} U_t^i(k') &= \sum_{\tau\in C_t^i(k')} \left[v_\tau(k') - \bar{u}^i\right] = \sum_{\tau\in C_{t-1}^i(k')} \left[v_\tau(k') - \bar{u}^i\right] = \\ &= \underbrace{\sum_{\tau\in C_{t-1}^i(k')} v_\tau(k') - \left|C_{t-1}^i(k')\right|\cdot}_{\text{identical for the whole interval}} \underbrace{\bar{u}^i}_{\text{investor specific}} . \end{aligned} \tag{4.39}$$

$\hat{p}_t^k$, as defined in the proposition, then denotes the lowest possible price of k, for which the cumulative utility of k exceeds the cumulative utility of k' for all investors, whereas $\check{p}_t^k$ is the highest possible price, for which the inverse relation holds. The continuity and the strict monotonicity of the utility function with respect to p_t^k insures that $\hat{p}_t^k$ and $\check{p}_t^k$ are well defined and unique. The three cases listed in the proposition emerge naturally, when comparing the cumulative utilities given by (4.38) and (4.39). (4.28) guarantees that the comparison between the cumulative utilities depends on the aspiration level of the investors. The definition of $\hat{p}_t^k$ and $\check{p}_t^k$ implies immediately that if $p_t^k \geq \hat{p}_t^k$, then act k is preferred by all investors, whereas $p_t^k \leq \check{p}_t^k$ implies that everyone chooses k'. If $p_t^k \in \left[\check{p}_t^k; \hat{p}_t^k\right]$, some of the investors will choose k and some k'. The critical aspiration level $\bar{u}^*$ (which, of course, depends on the price p_t^k) is determined by setting the two cumulative utilities equal and solving for $\bar{u}^i$. $(A1)$ insures that $\bar{u}^*$ is continuous in p_t^k and increasing (decreasing), if

$$\left(\left|C_{t-1}^i(k)\right| + 1\right) - \left|C_{t-1}^i(k')\right| \underset{(<)}{>} 0.$$

Therefore, the result of the proposition obtains.■

Proof of proposition 4.3:

Write the cumulative utilities of k and k' as in (4.38) and (4.39). (4.29) implies that the choice of the investors depends on the comparison

$$\sum_{\tau\in C_{t-1}^{i}(k)} v_{\tau}(k) + u\left(\frac{p_t^k}{p_{t-1}^k} + \frac{\delta_t^k}{p_{t-1}^k}\right) \begin{matrix}\geq\\<\end{matrix} \sum_{\tau\in C_{t-1}^{i}(k')} v_{\tau}(k'),$$

which depends only on the price p_t^k and leads to equality if $p_t^k = \tilde{p}_t^k$, with $\tilde{p}_t^k$ defined in the statement of the proposition.■

Proof of proposition 4.4:

The value of demand for k is defined in (4.30). Since a function which is continuous in p_t^k, constant in all other prices and defined for each p_t, is a special case of an upper hemicontinuous correspondence, which is non-empty, closed- and convex-valued for each p_t, the value of demand of all intervals $d_t^{kl}(p_t)$ satisfies the same conditions. Therefore, $d_t^k(p_t)$ is also non-empty, closed- and convex-valued for each p_t. Since there is only a mass of n investors, each of whom has 1 unit of the consumption good, the maximal value of demand for an asset k can be n. The prohibition of short sales insures that the value of demand does not fall below 0. It remains to show that $d_t^k(p_t)$ is upper hemicontinuous. By the definition of upper hemicontinuity it follows that for each $\bar{p}_t \in [0;\infty)^K$, and for each neighborhood $V\left(d_t^{kl}(\bar{p}_t)\right)$, there exists a neighborhood of $\bar{p}_t$, $N^l(\bar{p}_t)$, such that for every $p_t \in N^l(\bar{p}_t)$, $d_t^{kl}(p_t) \subseteq V\left(d_t^{kl}(\bar{p}_t)\right)$. Now, take an arbitrary neighborhood

$$V\left(d_t^k(\bar{p}_t)\right) \supset \left[\sum_{l=1}^{L}\min\left\{d_t^{kl}(\bar{p}_t)\right\}; \sum_{l=1}^{L}\max\left\{d_t^{kl}(\bar{p}_t)\right\}\right]$$

and define $N(\bar{p}_t) = \bigcap_{l=1}^{L} N^l(\bar{p}_t)$. Since for each l and any $V\left(d_t^{kl}(\bar{p}_t)\right)$

$$V\left(d_t^{kl}(\bar{p}_t)\right) \supseteq \left[\min\left\{d_t^{kl}(p_t)\right\}; \max\left\{d_t^{kl}(p_t)\right\}\right]$$

for each $p_t \in N(\bar{p}_t)$, it follows that

$$V\left(d_t^k(\bar{p}_t)\right) \supseteq \left[\sum_{l=1}^{L}\min\left\{d_t^{kl}(p_t)\right\}; \sum_{l=1}^{L}\max\left\{d_t^{kl}(p_t)\right\}\right]$$

for each $p_t \in N(\bar{p}_t)$. Therefore, $d_t^k(p_t)$ is upper hemicontinuous. Of course, the same line of reasoning holds for any $k \in 1...K$. Therefore, the correspondence $d_t(p_t) = \left(d_t^1(p_t)\,...d_t^K(p_t)\right)$ is also upper hemicontinuous.■

Proof of proposition 4.5:

An application of the Kakutani fixed-point theorem, see Mas-Collel, Whinston and Green (1995, p. 953). ■

Proof of proposition 4.7:

Write the cumulative utility of act a as:

$$U_t^i(a) = \sum_{\tau \in C_{t-1}^i(a)} \left[v_\tau(a) - \bar{u}^i \right] \tag{4.40}$$

and those of b as:

$$U_t^i(b) = \sum_{\tau \in C_{t-1}^i(b)} \left[1 + r - \bar{u}^i \right] + \left[1 + r - \bar{u}^i \right] = \left(\left| C_{t-1}^i(b) \right| + 1 \right) \left(1 + r - \bar{u}^i \right). \tag{4.41}$$

Neither of the cumulative utilities depends on the price of a, p_t. To make their decisions, the investors compare:

$$\sum_{\tau \in C_{t-1}^i(a)} \left[v_\tau(a) - \bar{u}^i \right] \gtreqless \left(\left| C_{t-1}^i(b) \right| + 1 \right) \left(1 + r - \bar{u}^i \right).$$

As long as

$$\left| C_{t-1}^i(a) \right| - \left| C_{t-1}^i(b) \right| - 1 \neq 0$$

is satisfied, $\hat{u}$ is well defined and represents the aspiration level of the investor, who is indifferent between a and b.■

Proof of proposition 4.10:

The aspiration levels and the initial holdings are chosen in such a way that none of the investors with aspiration levels between $[\bar{u}^0; \bar{u}^{j_1}]$ has an incentive to switch away from his initial portfolio, as long as none of the other investors in this interval changes his portfolio. The k^{th} line of (4.34) further implies that even if the investors with aspiration levels in the interval $[\bar{u}^{j_1}; \bar{u}^n]$ switch away from any asset k they might be holding, causing its price to fall from $\frac{\bar{u}^n - \bar{u}^{j_1} + \bar{u}^{j_k} - \bar{u}^0}{\bar{u}^{j_k} - \bar{u}^0}$ and its utility realization to become at most

$$u\left(\frac{\bar{u}^{j_k} - \bar{u}^0 + \delta_t^k A_k}{\bar{u}^n - \bar{u}^{j_1} + \bar{u}^{j_k} - \bar{u}^0} \right),$$

the investors with aspiration levels $[\bar{u}^{j_{k-1}}; \bar{u}^{j_k}]$ are still satisfied with the realized return from asset k even if its dividend is the lowest possible. Hence, the young investors with aspiration levels $[\bar{u}^{j_{k-1}}; \bar{u}^{j_k}]$ choose asset k in each period of time. Since they have a positive mass ($\bar{u}^{j_k} > \bar{u}^{j_{k-1}}$), there is always an equilibrium with a positive price of k. Since an analogous condition is imposed for each asset k, it follows that there is an equilibrium path, such that in each period of time the price vector is strictly positive, $p_t \gg 0$.■

Chapter 5. Asset Price Dynamics with Case-Based Decisions

In the last chapter, an equilibrium concept for an economy populated by case-based decision-makers has been proposed and studied. The existence of an equilibrium is insured for a wide class of utility functions and initial conditions, which makes the concept well applicable. In section 7 of the last chapter, an example of an economy with two assets has been presented. In this chapter, I study the dynamic of asset prices in an asset market populated by case-based decision-makers, based on this example.

The aim of this chapter is in the first place explorative. It analyzes what kind of behavior can emerge in a market with case-based investors and how this behavior influences asset prices. Moreover, by considering an economy with heterogenous consumers, who differ in their aspiration levels, it is not only possible to study how the aspiration level determines the behavior of a given investor, but also to analyze the interaction between the different aspiration levels and to indicate their influence on asset prices. The relationship which has already been found between the aspiration level and the investment behavior in an individual portfolio choice problem reappears in a market environment, but the results also depend on the interaction of prices and portfolio choices in equilibrium, which was not present in chapter 3.

Apart from the aspiration level, the memory of the individuals plays a crucial role for the dynamic. It is intuitively clear that relatively short memory does not allow investors to learn enough about the possible price and dividend realizations, so as to be able to form correct beliefs and make optimal choices. However, it is also questionable whether the ability to remember long sequences of realizations insures optimal behavior in the limit. Indeed, the results of chapter 3 have shown that endogenous memory[87] combined with a constant aspiration level leads to optimal behavior, only if the environment is stationary and the aspiration level is appropriately chosen. That is why the cases of both short memory (with only one case remembered) and long memory (all previous cases remembered) is considered in order to study the influence of memory on learning and on asset prices. Since the results show that even in the case of long memory,

[87] That is memory, consisting only of cases the decision maker has personally experienced.

optimal learning obtains only for investors with appropriately chosen aspiration levels, I drop the assumption of completely endogenous memory and examine whether the usage of hypothetical cases can insure optimal decisions in the limit. Again, the results depend on the prespecified value of the aspiration level.. Whereas optimal behavior emerges for relatively low aspiration levels, investors with high aspiration levels exhibit switching behavior even if they are allowed to use hypothetical cases.

Having once identified the characteristic features of asset prices in an economy populated by case-based decision-makers, the second aim of this chapter consists in comparing them to empirically observed phenomena and asking whether the presence of case-based investors in real asset markets could help explain such phenomena.

The empirical work in financial economics has found significant violations of the rational expectations hypothesis. Phenomena, such as bubbles, i.e. significant deviations of the prices from fundamental values, excessive volatility, predictability of asset returns and arbitrage possibilities are observed in real, as well as in experimental markets.

Indeed, the literature on overlapping generations models initiated by Samuelson (1958) and Diamond (1965) and further developed by Tirole (1982, 1985) shows that rational bubbles emerge as a stationary state in OLG models with population growth. Whereas in Tirole's model, bubbles cannot burst, Weil (1987) demonstrates that bubbles bursting with positive probability can also obtain in an OLG model, provided that the economy is not dynamically efficient. However, rational bubbles can only be positive (i.e. the price can become higher than the fundamental value, but never lower) and the question of how such bubbles get started is not answered in the literature[88]. Moreover, dynamic inefficiency is a necessary condition for emergence of rational bubbles. It requires that the population growth rate exceeds the interest rate in the economy. Nevertheless, the empirical evidence demonstrates that "irrational" bubbles are not rare in real and in experimental markets[89].

In the model presented in this chapter, rational bubbles are excluded by the assumption of no

[88] Indeed, if under rational expectations it were known that a bubble would start in some period t, then in period $(t-1)$ the investors would already start to invest in this asset in order to profit from the bubble at t. But then the bubble would already start at $(t-1)$. Hence, a rational bubble cannot have an initial date.

[89] Kindelberger (1978) gives a historical account on the most famous bubbles. Sunder (1995) reviews the experimental literature, whereas Camerer (1989) provides a review of the theory with experimental results. Recent results are provided by Hommes, Sonneman, Tuinstra and van de Helden (2002).

population growth. Nevertheless, prices may deviate significantly from the fundamental value. Small bubbles obtain in equilibrium. The one-period rise in the asset price, which is not conditioned either on dividend payments, or on changes in the fundamentals, is followed by a decrease in the price caused by a low dividend realization. The evolution of the economy is described by such two-state cycles in the presence of investors with relatively high aspiration levels, who switch among the available portfolios. The arising of a bubble, as well as its bursting, can, therefore, be ascribed to the dissatisfaction of investors with high aspiration levels with the returns achieved and their willingness to abandon the unprofitable asset and switch to a different one.

Whereas in a model without similarity considerations the price increase takes only one period, including similarity between problem-act pairs allows a bubble to develop over a longer period of time. The price may rise even in a period in which the dividend of the risky asset is low and this upward movement may last for few periods. The price drops in a single period and the bubble never reemerges again.

Excessive volatility is another phenomenon observed in asset markets, see Shiller (1981, 1990). Empirical findings show that the asset prices are too volatile to be explained by subsequent changes in the dividend payments. Roll (1984, 1989) demonstrates this on the examples of the orange juice future prices and the crash of 1987. Moreover, it is found that the frequency of trades is too high to be justified by changes in fundamentals or by profitability considerations, see Odean (1999) and Barber and Odean (2001 (a), 2001 (b)). Hence, the hypothesis of rational expectations fails to explain observed data.

Note that models with rational expectations do not in general predict constant asset prices. So, for instance, the consumption asset pricing model of Lucas (1978) is based on the assumption that consumers are infinitely lived and the process of dividends is a Markovian. Hence, dividends influence asset prices in two ways: first, through the changes in the value of the endowment of the consumer, who receives higher income, the higher the dividends paid to him; second, through the Markov process, which specifies the dependence of the dividend tomorrow on the dividend today. The model of Lucas, therefore, predicts that prices should change over time and the empirical models must control for such price changes explained by the model. However, these changes should be correlated with the dividend process, a result rejected by the data.

In contrast to the model of Lucas (1978), the model of overlapping generations used here does not predict any price changes over the time, given expected utility maximization combined with rational expectations. Indeed, if all investors are identical, the price should equal the fundamental value of the asset at any period of time, since the dividend process is i.i.d.[90]. Hence, any price volatility observed in this model is *excessive* compared to the bench-mark of rational expectations.

Moreover, similar to the results of Daniel, Hirshleifer and Subrahmanyam (1998) and Gervais and Odean (2001), who explain the excessive trades by overconfidence and the self-attribution bias, in a model with case-based decision-makers, those investors who have extremely high aspiration levels (i.e. who expect unrealistically high returns), trade too much. They generate excessive price volatility, which depends positively on their share in the economy. Moreover, these investors lose money, since they tend to buy at high prices and sell at low prices.

Predictability of asset returns is found in the market data, see De Bondt and Thaler (1985), Chopra, Lakonishok and Ritter (1992), Bernard (1992), Bernard and Thomas (1989,1990), Loughran and Ritter (1995). Their results show that factors like past performance, market-to-book ratios and capitalizations of stocks, as well as seasonality can help predict future returns. Especially, negative short run correlation of price movements for individual assets has been observed by Blume and Friend (1978), Lo and MacKinlay (1988) and Jegadeesh (1990).

This evidence violates the efficient market hypotheses, since it implies that information freely available in the market is not priced. The observed phenomena are, thus, usually associated with underreaction and overreaction to information. Since the investors do not react adequately to new information, the market later corrects the prices so that they coincide with the fundamental values. As such corrections take place slowly, they cause autocorrelation of returns in the short run.

In theoretical models, such effects are obtained by introducing noise traders, whose behavior depends on past prices. So, for instance, the presence of positive feedback traders in the market can lead to price bubbles, as De Long, Shleifer, Summers and Waldmann (1990 (b)) demonstrate. The representativeness bias (i.e. the fact that people interpret short sequences of observations as representative for the population) is used by Barberis, Shleifer and Vishny (1998) to explain

90 As already noted above, rational bubbles are excluded by the assumption of no population growth.

under- and overreaction in financial markets. The assumption about the existence of chartists (positive feedback traders) and fundamentalists (who ignore prices and trade only on signals about future returns) in the market generates positive autocorrelation in the short run and negative autocorrelation in the long run, see Cutler, Poterba and Summers (1990).

With case-based decision-makers, predictability of asset returns also obtains. The interpretation of the result is, however, somewhat different from those in the theoretical work cited above. In fact, no new information is needed in the market to induce autocorrelation of returns, no change of fundamentals occurs in the model. Instead, the fact that the decision of a case-based investor is influenced by his memory, hence, by past prices and returns, implies that the market prices also depend on past data and become predictable to some extent.

Arbitrage possibilities have been observed in experimental markets by Rietz (1998) and Oliven and Rietz (1995), who comment on how difficult it is to enforce arbitrage restrictions, even by explaining that they exist and encouraging market participants to use them. In real markets Rosenthal and Young (1990) present an example of two assets representing ownership of the same company which are traded at different prices with up to 30% difference. Similar examples are presented by Lamont and Thaler (2001) and in Shleifer (2000, Chapter 3).

De Long, Shleifer, Summers and Waldmann (1990 (a)), as well as Shleifer and Vishny (1997) show how noise traders can generate arbitrage possibilities in a theoretical model. If the rational arbitrageurs are fully invested or if they face the risk of dropping from the market before the asset price returns to the fundamental value, they will in general not be able to eliminate the arbitrage possibilities present in the market. Their model differs from the current one in two aspects: first, they consider two ex-ante identical assets with positive fundamental value and demonstrate that they can be traded at different prices; second, the noise traders in their model differ from the rational ones only by their misperceptions of the expectation of the returns of one of the assets. Still, the noise traders are able to form expectations about the realizations of the possible states of the nature, i.e. they are expected utility maximizers with biased beliefs. In contrast, in the model presented in this chapter, only case-based decision-makers are present, who do not form any beliefs about states and state-contingent payoffs. The assets in this model are not identical and the effect that an asset with positive fundamental value is not traded and has a price of 0 is an interesting feature of the current model, not present in the models cited above. Moreover, in chapter 6 it will be shown that this feature of the model does not disappear, even if expected utility maximizers, who believe that the asset has a positive fundamental value, are introduced into the market.

The no-arbitrage restriction belongs to the basics of the financial economics. To use arbitrage opportunities, the investor needn't even entertain beliefs about the probabilities of the states of nature. However, the investor has to know the possible states of nature and the payoff of each asset in each state. This knowledge which is usually presupposed in asset market models, is missing in a model with case-based decision-makers. Since case-based investors learn from past utility realizations of the assets and compare these realizations to an aspiration level, it is possible that they do not see arbitrage opportunities, even if they are present in the market. For instance, suppose that an asset with positive fundamental value has brought a low return in the last period. If the aspiration level of the decision-makers is relatively high, they will switch to a different asset, abandoning the one, whose return they find unsatisfactory. If every investor in the market behaves in this way, the asset in question will have a price of 0, although its fundamental value is positive. Hence, an arbitrage possibility emerges. As was shown in chapter 4, this is a typical feature of a market populated with case-based decision-makers.

Of course, the validity of the approach selected here is limited by the fact that *only* case-based decision-makers are present in the market. Should it turn out that the wealth share of the case-based decision-makers becomes 0 in the limit in presence of expected utility maximizers, the explanatory power of the results obtained would be at best marginal. This question will be analyzed in chapter 6, where it will be shown that case-based decision-makers *are indeed* able to survive in a financial market. The implications of their behavior for the price dynamic are therefore of interest for the financial literature.

At the same time, the simplifying assumption that only case-based decision-makers are present in the market renders the model tractable and allows to focus on the aspects of the case-based reasoning alone, thus, providing a bench-mark for the evolutionary model which is constructed in chapter 6.

The chapter is structured as follows. In section 1, I shortly repeat the market model introduced in the last chapter and the results derived there for the individual and aggregate demand for assets. In section 2, I use the concept of a market equilibrium in a context of case-based decision making specified in chapter 4 and the existence results derived there to define an equilibrium for the economy. Section 3 analyzes the evolution of the economy for the case of short memory and determines the long run distribution of asset prices, as well as the dynamics of asset holdings for different groups of investors. It is discussed how empirically observed phenomena such as bubbles, predictability of returns, excessive volatility and arbitrage possibilities can emerge in a market populated by case-based decision-makers. Section 4 considers the case of long memory. It identifies conditions under which investors with high aspiration level are able to learn to behave as expected utility maximizers in the limit. However, even with infinite memory, not all of the investors learn the real distribution of returns. Phenomena such as the equity premium puzzle can thus obtain in the limit even if the investors in the economy are risk-neutral. In section 5, I examine whether hypothetical reasoning leads to optimal decisions in a market environment. In section 6, a similarity function on the space of problem-act / price-portfolio pairs is introduced and the price dynamic studied for the cases of long and short memory. Section 7 concludes. The proofs of the results are stated in the appendix.

5.1 The Economy

Consider the economy, described in section 1 of chapter 4, consisting of a continuum of investors, uniformly distributed on the interval $[0; n]$. For each $i \in [0; n]$ and some constant $\bar{u}^0 \in \mathbb{R}_0^+$ denote by $\bar{u}^i = \bar{u}^0 + i$ the aspiration level of investor i. The aspiration level of the investors is then also uniformly distributed on $[\bar{u}^0; \bar{u}^n]$.

Each investor lives for two periods. The preferences of the investors are assumed to be such that they wish to consume only in the second period of their life. The preferences over consumption in the second period are represented by a utility function $u(\cdot)$, which is identical for all investors. $u(\cdot)$ is assumed to be strict monotonically increasing and continuous in consumption in period two with $u(0) > 0$. There is one consumption good in the economy. The initial endowment of the investors consists of one unit of the consumption good in the first period and is 0 in the second period.

I use the example from section 7 of chapter 4 to describe the possibilities for transferring consumption between two periods: a riskless asset, denoted by b and a risky asset with exogenous supply, denoted by a. The riskless asset is available in a perfectly elastic supply at a price of

1 in each period. It delivers $(1+r)$ units of consumption good in period t for each unit of the consumption good, stored in period $(t-1)$.

The supply of the risky asset is fixed at $A > 0$. The payoff of one unit of the asset in period t is:[91]

$$\delta_t = \left\{ \begin{array}{ll} \delta D & \text{with probability } q \\ 0 & \text{with probability } 1-q \end{array} \right\},$$

and is identically and independently distributed in each period[92]. The price of the risky asset at time t is denoted by p_t and determined endogenously in an equilibrium. New emissions are not considered, since I am interested in the behavior of prices on the secondary asset market only.

The decision of a young investor in terms of the case-based decision theory is described as a problem to be solved by choosing an act out of a given set. In the present context the problem can be formulated as: "Choose a portfolio of assets today to enable consumption tomorrow".

The decision of a young investor now consists in choosing a portfolio of two assets — the riskless asset b and the risky asset a. For the present, I consider only the case, in which there are only two portfolios available for a single investor: either the whole initial endowment is invested in a or in b. Short sales are prohibited. The case of allowed diversification is discussed in section 6 of this chapter.

Denote by α^i_{t-1} the act chosen by a young investor with an aspiration level $\bar{u}^i$ in period $(t-1)$. Then the values of α^i —

$$\mathfrak{A} = \{a; b\}$$

represent the set of acts available to an investor who solves the problem formulated above.

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91 The risky asset can be thought of as an asset of a firm with a given initial capital D, which does not depreciate and which is reinvested in each period. The production function for the consumption good is linear with a random coefficient κ:

$$y(D) = \kappa D$$

with

$$\begin{aligned} \kappa &= \delta, \text{ with probability } q \\ \kappa &= 0, \text{ with probability } 1-q, \end{aligned}$$

identically and inependently distributed in each period. Hence, the profits of the firm (normalizing the price of the consumption good to 1) are

$$D\delta, \text{ with probability } q$$
$$0, \text{ with probability } 1-q.$$

Assuming that the profit is paid out as dividends to the owners of the firm proportionally to the shares they own and that no reinvestment (no new emmissions) take place, we obtain the payoff structure of the risky asset.

92 q is interpreted as the objective probability of high returns, known to an external observer, but not to the investors in the economy. Hence, q will be irrelevant for the investors' decisions. However, the specification of q makes it possible to analyze the long run behavior of the economy.

Now, define the indirect utility of consumption by $v_t(\alpha_{t-1})$, the memory of an investor i by M_t^i and parameterize the length of memory by m, as in chapter 4. It is again assumed that the memory of an investor i consists of the cases experienced by the investors of the previous $(m+1)$ generations with the same aspiration level as i, $\bar{u}^i$. The cumulative utility of an asset $\alpha \in \{a; b\}$ is, therefore:

$$U_t^i(\alpha) = \sum_{\tau \in C_t^i(\alpha)} \left[v_\tau(\alpha) - \bar{u}^i \right]$$

with

$$C_t^i(\alpha) = \left\{ t - m - 1 \leq \tau < t \mid \alpha_\tau^i = \alpha \right\}$$

describing the set of periods (among the last $m+1$) in which the predecessors of investor i have chosen act α.

As usual, the investor chooses in each period the asset with the highest cumulative utility.

The individual and aggregate demand for assets, given the memory of the investors at time t have already been derived in sections 3 and 7 of chapter 4. It was shown that the individual demand for the risky asset in general depends non-monotonically on its current price, rising for relatively low prices and decreasing for high prices. Moreover, the individual demand for the risky asset can be very insensitive to price changes near 0. These characteristics of the individual demand determine the properties of the aggregate value of demand for the risky asset, which is increasing in its price and can be 0 for low prices. Moreover, as shown in section 7 of chapter 4, the value of demand for the risky asset is a correspondence, which maps the range of non-negative prices $p_t \in [0; \infty]$ into the range of possible values of demand $[0; n]$. It is upper hemicontinuous, non-empty, closed- and convex-valued for every price p_t. These properties insure the existence of a temporary equilibrium.

5.2 Temporary Equilibrium

In chapter 4, it was demonstrated that a temporary equilibrium exists under quite general conditions (assumptions $(A1)$ and $(A2)$). I now shortly restate the definition of a temporary equilibrium and the existence result for the economy at hand.

Definition: A temporary equilibrium at time t is defined by:

- portfolio choices for all young investors $\alpha_t^i \in \{a; b\}$ for each $i \in [0; n]$;
- utility of consumption derived by the old investors $v_t\left(\alpha_{t-1}^i\right)$ for each $i \in [0; n]$;
- a price of the asset a, p_t

such that following conditions are fulfilled:

1. Case-based decision-making of the young consumers:
$$\alpha_t^i = \left\{ \begin{array}{ll} a, & \text{if } U_t^i(a) \geq U_t^i(b) \\ b, & \text{if } U_t^i(a) \leq U_t^i(b) \end{array} \right\}$$
at the equilibrium price p_t.

2. Consumption-decision of the old consumers:
$$v_t\left(\alpha_{t-1}^i\right) = \left\{ \begin{array}{ll} u\left(\frac{p_t}{p_{t-1}} + \frac{\delta_t}{p_{t-1}}\right) & \text{if } \alpha_{t-1}^i = a \\ u(1+r) & \text{if } \alpha_{t-1}^i = b \end{array} \right\}$$

3. Market-clearing condition:
$$s_t =: Ap_t = \int\limits_{\left\{i:\alpha_t^i = a\right\}} di =: d_t$$
where d_t denotes the mass[93] of the young investors who choose to hold a in period t and s_t is the value of supply of a.

Following assumptions are needed to insure existence:

$(A1)$ The utility function $u(\cdot)$ is strictly increasing and continuous.

$(A2)$ At $t = 1$, the population of the old investors can be partitioned into a finite number of intervals such that all the investors of the same interval hold the same asset.

The following proposition insures the existence of an equilibrium in each period $t \geq 1$ for an arbitrary length of memory. Its proof was given in section 7 of chapter 4.

Proposition 5.1 *Assume that $(A1)$ and $(A2)$ hold. Then, for each length of memory $m \in \{0...t-1\}$, a temporary equilibrium of the economy exists in every period $t \geq 1$.*

[93] Or, in other words — the value of demand for a in period t. Remarks 4.1 and 4.2 in chapter 4 show the equivalence of
$$Ap_t = \int\limits_{\left\{i:\alpha_t^i = a\right\}} di$$
to the market clearing conditions in this economy.

The uniqueness of the equilibrium is, however, not guaranteed. The discussion of non-degeneracy of equilibria, see section 8 of chapter 4, applies here as well. Since I believe that the conditions excluding degenerate equilibria are too strong to be justified by economic intuition, I make assumptions on the aspiration levels that insure that at least one equilibrium with positive price of the risky asset exists in each period of time. Where appropriate, I also discuss the issue of multiplicity of equilibria.

5.3 Price-Dynamics with One-Period Memory

After introducing the notion and guaranteeing the existence of a temporary equilibrium, I proceed to investigate the dynamics of asset prices and asset holdings in the economy. Of course, the paths will depend on the initial distribution of the asset holdings, as well as on the assumptions about the range of aspiration levels of the investors. Another important factor is also the length of the investors' memory parameterized by m. If investors are only learning about the utility realizations of the last period, they are not likely to learn much about the economy and its structure. On the opposite, if they have the experience of all previous generations, then information will gradually accumulate in the economy and one would suppose that (given a stationary structure of dividends) in the limit the investors will be able to learn to act as expected utility maximizers, who know the true distribution of returns.

I consider only the two extreme cases — one-period memory and infinite memory. The analysis in general is quite complicated that is why I concentrate on examples, which, I think, provide some intuition into how a market, populated by case-based decision-makers evolves and in which aspects it differs from a market, in which only expected utility maximizers with rational expectations trade.

First, assume that the investors in the economy have a memory of length 1. This implies that the agents are myopic, they base their decisions only on the nearest past, without considering any information about the long run behavior of the economy. It might be that they are unable to extract information from past data, or that the data is not available to them. Limited time, capacity and resources can also be a reason for the agents to adapt simple heuristics based on small amounts of information. It may be also that the agents, not knowing the structure of the model, consider important for their decision only what happened in the recent past and thus do not try to obtain more information.

Now, consider the following example, which illustrates the price dynamics in an economy with short memory. I make the following assumptions:

$(A4)$ Let the utility functions of the investors be linear, i.e.:

$$u\left(\frac{p_t}{p_{t-1}}+\frac{\delta_t}{p_{t-1}}\right)=\frac{p_t}{p_{t-1}}+\frac{\delta_t}{p_{t-1}}$$
$$u(1+r)=1+r.$$

$(A5)$ Suppose that the initial holdings[94] of the old consumers at time $t=1$ be such that the investors $[0;\bar{u}^a-\bar{u}^0]$ hold the risky asset a, whereas those on the interval $[\bar{u}^a-\bar{u}^0;n]$ hold the riskless asset b for some $\bar{u}^a\in(\bar{u}^0;1)$.

$(A6)$ Fix the price of the risky asset in $t=0$ to be[95]

$$p_0=\frac{[\bar{u}^a-\bar{u}^0]}{A}=\left[\bar{u}^a-\bar{u}^0\right],$$

with $A=1$.

Assumption $(A4)$ about the linearity of the utility function is not crucial: it simplifies the computations and allows for explicit solutions, but does not influence the qualitative implications. $(A5)$ specifies the initial holdings in the economy in such a way that for all possible sequences of dividend realizations there is an equilibrium path on which the price of the risky asset remains strictly positive over the time. Although equilibria with $p_t=0$ still exist, I concentrate my analysis on non-degenerate equilibria. $(A6)$ fixes the price at $t=0$ to be the equilibrium price for the initial allocation specified in $(A5)$.

Given these assumptions, three possible cases can be distinguished which qualitatively change the dynamics of the price p_t: either the highest aspiration level in the economy is lower than $(1+r)$, or it lies in the interval between $(1+r)$ and the return of the risky asset, when it yields positive dividends, or it is higher than the return of the risky asset when the dividend is positive.

94 Since at $t=0$ the memory of all investors is empty, their choices can be set arbitrarily.

95 This would be the price of a in $t=0$ if the young consumers chose their asset holdings to be a for those with aspiration levels $\left[\bar{u}^0;\bar{u}^a\right]$ and b for those with aspiration levels $[\bar{u}^a;\bar{u}^n]$. Remember that the mass of the investors with aspiration levels in a given range $\left[\bar{u}^0;\bar{u}^a\right]$ is $\bar{u}^a-\bar{u}^0$, since the aspiration levels and the investors are distributed uniformly on an interval with a length n.

5.3.1 The Case of Low Aspiration Levels

Let

$$(1+r) \geq \bar{u}^n > 1 \geq \bar{u}^a > \bar{u}^0. \tag{5.42}$$

The specification of the aspiration levels is illustrated in figure 11 a). The following proposition obtains:

Proposition 5.2 *Assume $(A4)$, $(A5)$ and $(A6)$. Suppose as well that (5.42) holds. Then the allocation*

$$\alpha_t^i = a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right]$$
$$\alpha_t^i = b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; n\right]$$

and the price

$$p_t = \left[\bar{u}^a - \bar{u}^0\right]$$

for each $t \geq 1$ represent a stationary state of the economy.

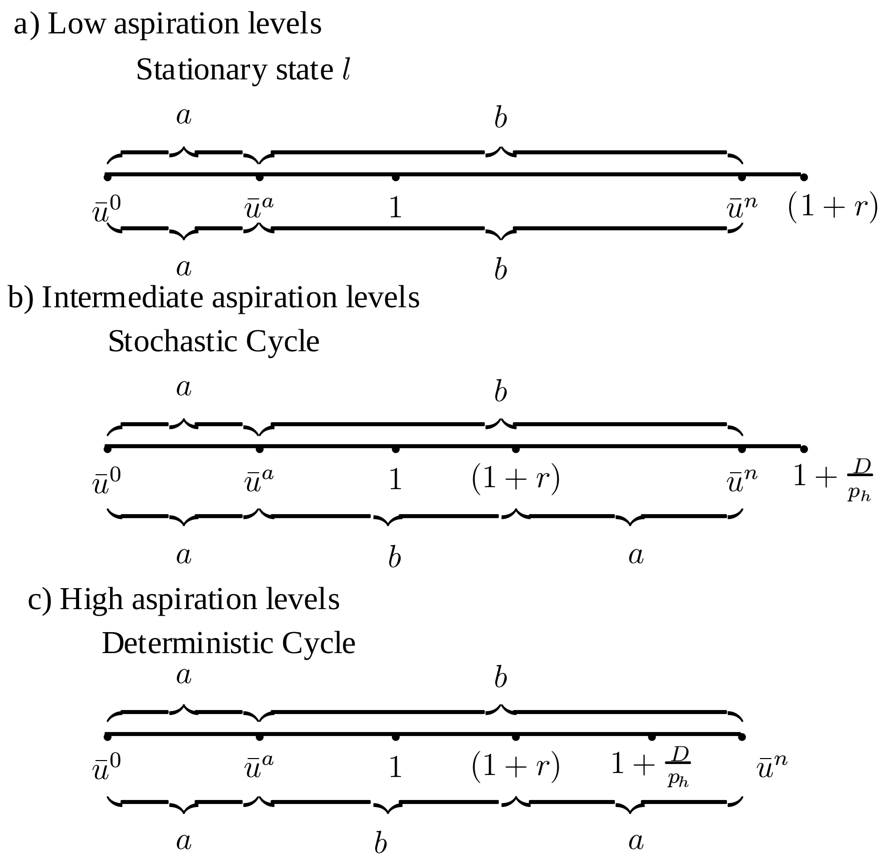

Figure 11

$p_h = [\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0]$

Although the utility function is assumed to be linear, p_t may differ from the fundamental value of a as estimated by a statistician who knows the dividend process of the asset. Computed in this way, the fundamental value of a is given by:

$$FV = \frac{q\delta D}{r}. \tag{5.43}$$

In general, the fundamental value need not coincide with the price of the asset $p_t = [\bar{u}^a - \bar{u}^0]$. If D is relatively high and $[\bar{u}^a - \bar{u}^0]$ relatively low, the asset might be undervalued and vice versa. This does not necessarily mean that riskless arbitrage is possible. Indeed, as long as $\frac{\delta D}{\bar{u}^a - \bar{u}^0} > r$, the price

$$p_t = \left[\bar{u}^a - \bar{u}^0\right]$$

does not allow for arbitrage, as long as the arbitrageurs live for two periods only.

Note, however, that if $\frac{\delta D}{\bar{u}^a - \bar{u}^0} < r$, then the riskless asset is always better than the risky one. In this case, arbitrage opportunities are present in the market, but not identified by the case-based decision-makers with short memory, who are not able to learn the possible return realizations in order to make optimal decisions.

This result illustrates that the notion of arbitrage crucially depends on the knowledge of the investors about the economy. If the investors do not know (or do not believe) that two securities are identical, or that one security is dominated by another one in each state of nature, then no arbitrage possibilities exist for them in the market, although they might be present from the point of view of an external observer, who knows (or believes to know) the structure of payoffs.

5.3.2 The Case of Intermediate Aspiration Levels

Let the dividend of the risky asset δD, the riskless interest rate r and the ranges of the aspiration levels $\bar{u}^0$ and $\bar{u}^n$ be such that:

$$1 + \frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} > \bar{u}^n > 1 + r > 1 > \bar{u}^a > \bar{u}^0. \tag{5.44}$$

Hence, the return of the risky asset is satisfactory for the investors with high aspiration levels $[(1+r)\,; \bar{u}^n]$ only if the dividend is positive. The specification of the parameters in this case is illustrated in figure 11 b). Assume further that

$$\frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} > \bar{u}^a \tag{5.45}$$

holds. (5.45) says that the investors with low aspiration levels, $[\bar{u}^0; \bar{u}^a]$, are ready to hold the risky asset, even if all of the investors with high aspiration levels, $[1+r; \bar{u}^n]$ switch to the riskless asset[96]. As was shown in section 8 of chapter 4, this condition insures the existence of an equilibrium path with positive price of the risky asset in each period of time.

5.3.2.1 Computation of the Equilibrium

(5.44) implies that the investors with high aspiration levels, $[1+r; \bar{u}^n]$ are neither satisfied with the returns of the riskless asset, nor with the return of a if its dividend is low and the price of a remains unchanged relative to the last period. Therefore, these investors will permanently switch between a and b, holding a as long as its dividend is high, switching to b in the first period in which the dividend of a becomes 0 and buying a in the following period, since the return of b is unsatisfactory for them. In contrast, the investors with aspiration levels lower than $(1+r)$ are always satisfied with the returns of their initially chosen portfolio and therefore never switch away from it. The following proposition obtains:

Proposition 5.3 *Assume $(A4)$, $(A5)$ and $(A6)$. Suppose as well that (5.44) and (5.45) hold. Then the economy evolves according to a Markov process with two states h and l such that:*

$$\begin{aligned} \alpha_h^i &= a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \cup \left[(1+r) - \bar{u}^0; n\right] \\ \alpha_h^i &= b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; (1+r) - \bar{u}^0\right] \\ p_h &= \left[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right] \end{aligned}$$

and

$$\begin{aligned} \alpha_l^i &= a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \\ \alpha_l^i &= b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; n\right] \\ p_l &= \left[\bar{u}^a - \bar{u}^0\right] \end{aligned}$$

and a transition matrix $\bar{P}$

$$\bar{P} = \left(\begin{array}{|c|c|c|} \hline & p_{t+1} = p_h & p_{t+1} = p_l \\ \hline p_t = p_h & q & 1-q \\ \hline p_t = p_l & 1 & 0 \\ \hline \end{array} \right).$$

5.3.2.2 Discussion of the Results

Whether condition (5.45) holds or not, in period $t = 1$, a price upward movement is observed which does not depend on the size of the dividends that asset a pays. Hence, the price increase cannot be attributed either to changes of fundamentals, nor to changes of the dividend payment. In contrast, the downward movement is conditioned on the asset paying a low dividend in the

[96] (5.44) and (5.45) hold simultaneously if, for instance, $\bar{u}^0 = 0$, $\bar{u}^n + \bar{u}^a < 2 + r$ and $\bar{u}^n > 1 + \delta D$ for $\delta D < (1+r)$.

second period. As a whole, the structure of asset-price movements reminds of small bubbles, which emerge without visible reasons (apart from the fact that some of the investors are dissatisfied with the safe technology) and then burst because of low dividend payments. Of course, if the asset continues to pay high dividends, the high price of the asset persists, until in some period t the dividend becomes low. As long as the probability $1-q$ of a low dividend is positive, the bubble bursts in finite time with probability one.

Note that the probability that the price rises in period $t+1$ if it was low (p_l, as defined in proposition 5.3) in t, is 1, whereas the probability that the price falls in $t+1$ if it was high (p_h, as defined in proposition 5.3) in t, is equal to the probability of a low dividend $(1-q)$. This means that asset prices are predictable to some extent. Especially, a rational external observer, who knows the model and can predict the behavior of the case-based decision-makers, can also predict the price p_{t+1} if p_t is low, because he can be sure that the price will rise in the next period. Still, he cannot predict the price movements in periods, in which the price of the asset is high.

The predictability of the asset price, however, does not mean that arbitrage is necessarily possible. Note that, if at time t the price is

$$p_t = [\bar{u}^a - \bar{u}^0],$$

then the payoff of 1 unit invested in a in $t+1$ is

$$\frac{[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0] + \delta D}{\bar{u}^a - \bar{u}^0} > 1 + r \text{ if } \delta_t = \delta D.$$

and

$$\frac{[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0]}{\bar{u}^a - \bar{u}^0} \text{ if } \delta_t = 0.$$

Whereas the payoff is always greater than the payoff of 1 unit invested in b if $\delta_t = \delta D$, this is not necessarily satisfied if the dividend is 0. Especially, it will be always smaller than $(1+r)$ if $\bar{u}^a > \frac{1}{1+r}$ holds. In this case, arbitrage is impossible.

To study the limit behavior of the economy and, especially, the frequencies with which the investors with relatively high aspiration levels hold asset a and asset b, it is useful to introduce the following notation. Let ω denote a typical equilibrium path characterizing the evolution of the economy. The computation of temporary equilibria has shown that only paths on which the two states h and l occur will play a role for the evolution of the economy. Hence, a typical path ω is a random sequence of h and l and can be written as

$$\omega = (\omega_t)_{t=0}^{\infty}$$

with $\omega_t \in \{h; l\}$. Let Φ denote the set of all such paths and Σ denote the σ-algebra on Φ. $C_t(h)$ and $C_t(l)$ describe the set of periods in which the economy is in state h and l, respectively (on a path ω, where this dependence is omitted for convenience in the notation):

$$\begin{aligned} C_t(h) &= \{\tau < t \mid \omega_\tau = h\} \\ C_t(l) &= \{\tau < t \mid \omega_\tau = l\}. \end{aligned}$$

Denote by π_h and π_l the limit frequencies states h and l:

$$\begin{aligned} \pi_h &= \lim_{t\to\infty} \frac{|C_t(h)|}{t} \\ \pi_l &= \lim_{t\to\infty} \frac{|C_t(l)|}{t} \end{aligned}$$

if these limits exist. Usually, these limits will depend on the path ω as well. However, the following proposition shows that for the economy at hand these frequencies are well defined and independent of the chosen path ω. Denote by P the probability distribution on $(S; \Sigma)$ induced by the ∞-fold of the transition matrix $\bar{P}$.

Proposition 5.4 *On P-almost all paths $\omega \in \Phi$ the limit frequencies π_h and π_l coincide with the invariant probability distribution of a Markov chain with transition matrix*[97]*:*

$$\bar{P} = \left(\begin{array}{|c|c|c|} \hline & p_{t+1} = p_h & p_{t+1} = p_l \\ \hline p_t = p_h & q & 1-q \\ \hline p_t = p_l & 1 & 0 \\ \hline \end{array} \right)$$

and can be computed to be:

$$\begin{aligned} \pi_h &= \frac{1}{2-q} \\ \pi_l &= \frac{1-q}{2-q}. \end{aligned}$$

Now, knowing the distribution of asset prices over time, the moments of the distribution can be determined. The mean of the asset price is:

$$\mu_a^1 = \bar{u}^a - \bar{u}^0 + \frac{[\bar{u}^n - (1+r)]}{(2-q)}. \tag{5.46}$$

It is possible to choose the parameters in such a way that the fundamental value of the asset (as defined in (5.43)) is equal to the mean asset price[98]. Indeed, this is the case if

$$D^* = \frac{r(2-q)(\bar{u}^a - \bar{u}^0) + r(\bar{u}^n - (1+r))}{(2-q)q\delta}.$$

[97] Since $P^2 = \begin{pmatrix} q^2 + 1 - q & q - q^2 \\ q & 1-q \end{pmatrix}$ has only positive entries (for non-degenerate probability distribution of dividends), π_h and π_l exist, see Lawler (1996, p.15).

[98] Since the price of the asset changes over time, whereas the fundamental value remains constant, the price will obviously differ from the fundamental value at least in some periods. Thus, we can ask, wether the fundamental value can still be the best estimator for the expected price in the long run.

However, the equality need not hold in general and it is also possible that the mean price of the asset in the long run remains above or below the fundamental value[99].

The price of the asset has a positive variance, given by:

$$\sigma_a^1 = \sqrt{1-q}\frac{[\bar{u}^n - (1+r)]}{(2-q)}. \tag{5.47}$$

The fluctuation of the price is neither due to changes in the fundamental value of the asset, nor to new information, nor even necessarily to changes in the dividend payments. Hence, the asset price exhibits extreme volatility, which can not be explained by the characteristics of the asset, but which is consistent with the empirical evidence on price volatility cited in the introduction.

The price fluctuation in the model can be explained by the decision-making process of the investors. The risk (as represented by σ_a^1) faced by the investors in the market consists of two parts: exogenous and endogenous. The exogenous risk is created by the random fluctuation of the dividends and depends negatively on the probability of high dividends q. The endogenous risk is caused by the ignorance of the investors and their reliance on the cumulative utility of an asset to predict returns. It depends positively on the mass of investors with high aspiration levels $[\bar{u}^n - (1+r)]$, who are not satisfied with the returns of the riskless asset and thus would hold the risky asset as long as its dividends are high, but sell it, once the dividend falls to 0. In chapter 3, I have shown that investors with relatively high aspiration levels usually trade too much in the sense of Odean (1999). The results derived here demonstrate that in a market environment increasing the number of case-based decision-makers with high aspiration levels leads to an increase of the risk faced by the economy, as well as to frequent change of asset holdings. By switching too often between the available portfolios, not acting on information, these investors not only cause the price of the risky asset to fluctuate without changes of the fundamental value but also lower their profits, since they buy when prices are high and sell at low prices. To understand the last point consider three cases. If the expected return of asset a at price p_h is higher than those of asset b, it immediately follows that the expected return of a is also higher than

99

$$D^*\delta > (\bar{u}^n - 1)\left(\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right),$$

as required in condition (5.44) holds, for instance if

$$r > (2-q)\, q\, (\bar{u}^n - 1)\,.$$

If

$$D^*\delta < (\bar{u}^n - 1)\left(\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right),$$

then $D > D^*$ always holds and the risky asset is undervalued, $\mu_a^1 < FV$.

those of b at price p_l. It follows that the rational choice of an investor should be to buy a if the observed price of a is p_l. Hence, by switching from a to b, the investors with aspiration levels higher than $(1+r)$ lower their returns compared to the situation in which they hold a in each period of time.

Alternatively, if the expected return of asset b is higher than those of a at a price p_l, it follows that the return of b is also higher than the return of a at a price p_h and therefore the switch from b to a lowers the expected returns of an investor.

In the case in which the return of b exceeds those of a at a price p_h but is lower than those of a at a price p_l, the investors with aspiration levels higher than $(1+r)$ always hold the asset with the lower expected return. Obviously, from an individual point of view, the optimal choice would be a, at p_l and b at p_h, hence the switching behavior lowers the expected return of the investors in this case, as well.

Of course, it might be that case-based reasoning still leads to higher expected returns than choosing a portfolio at random and holding it forever. Nevertheless, in an empirical study, observing a behavior similar to that of the case-based investors with relatively high aspiration levels in this model would lead to the conclusion that these investors lower their profits by switching too often among the available portfolios, compared to the situation in which the optimal portfolio is held.

The cycle determined in proposition 5.4 is not unique. In fact, in each period t, multiple equilibria exist. For instance, in $t=2$, with $\delta_2=0$, there is an equilibrium in which everyone switches to b and the price of a falls to 0. The price p_t can also crash in a period with high dividends. As long as

$$\bar{u}^0 > \frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0},$$

there is an equilibrium in $t=2$ (with $\delta_2=\delta D$) in which the price of the risky asset equals 0.

It is also not guaranteed that a cycle emerges from the first period on — the time needed to reach a state, from which on a cycle can evolve, depends on the initial allocation of the assets in the economy and on the parameters of the model. In this example, the cycle will start only in the first period in which $\delta_2=0$ holds if (5.45) is not satisfied.

The dynamics is further simplified by the assumption of only two assets in the economy. Should the number of assets exceed two, further equilibria would emerge. This is due to the short memory of the investors. Since the investors observe only one past utility realization, they assign a cumulative utility of 0 to those assets whose returns they do not observe. Hence, an investor who is dissatisfied with one of the acts, has to choose another one at random, which implies multiple equilibria even if assumption $(A3)$ from chapter 4 is fulfilled.

The assumption that the riskless asset is available in perfectly elastic supply at a price of 1 and, hence, its price and returns are fixed, does not influence the results qualitatively. Assuming a variable price of b also leads to a two-state stochastic cycle. Differently from the cycle described above, the price of b rises when the price of a falls and vice versa. Hence, the investors with high aspiration levels do not only create excessive volatility of the prices of the risky assets, but also cause the ex-ante riskless asset to exhibit volatile stochastic returns.

5.3.3 The Case of High Aspiration Levels

Now consider the third case, in which the highest aspiration level $\bar{u}^n$ exceeds the return of the risky asset even if the dividend is positive, i.e.:

$$\bar{u}^n > 1 + \frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} > 1 + r > 1 > \bar{u}^a > \bar{u}^0. \tag{5.48}$$

The position of the aspiration levels relative to the asset returns is illustrated in figure 11 c). To insure that degenerate equilibria exist, assume that (5.45) holds.

5.3.3.1 Computation of the Equilibrium

Differently from the case of intermediate aspiration levels, with high aspiration levels the investors with aspiration levels $[1+r;\bar{u}^n]$ are not satisfied with the returns of the risky asset, even if its dividend is high, i.e. $\delta_t = \delta D$. Therefore, they switch between the two possible portfolios in each period of time, holding a in odd periods and b in even periods. Hence, the economy evolves according to a deterministic cycle:

Proposition 5.5 *Assume $(A1)$, $(A4)$, $(A5)$ and $(A6)$. Suppose as well that (5.48) and (5.45) hold. Then the deterministic cycle with two states h and l such that:*

$$\alpha_h^i = a \textit{ for } i \in \left[0;\bar{u}^a - \bar{u}^0\right] \cup \left[(1+r) - \bar{u}^0; n\right]$$
$$\alpha_h^i = b \textit{ for } i \in \left[\bar{u}^a - \bar{u}^0; (1+r) - \bar{u}^0\right]$$
$$p_h = \left[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right]$$

and

$$\alpha_l^i = a \textit{ for } i \in \left[0;\bar{u}^a - \bar{u}^0\right]$$

$$\alpha_l^i = b \textit{ for } i \in \left[\bar{u}^a - \bar{u}^0; n\right]$$
$$p_l = \left[\bar{u}^a - \bar{u}^0\right].$$

describes the evolution of the economy. The state in period t is h if $t = 2k+1$ and l if $t = 2k$ with $k \in \mathbb{Z}_0^+$.

5.3.3.2 Discussion of the Results

Again, as in the case of intermediate aspiration levels, the economy evolves according to a cycle with two possible states. However, now the cycle is no longer stochastic — the high- and low-price-states alternate in each period so that the price-sequence is completely predictable for an external observer. Hence, the small bubbles observed are deterministic. The mean price of the asset is

$$\mu_a = \frac{1}{2}\left[\bar{u}^n - (1+r)\right] + \left[\bar{u}^a - \bar{u}^0\right],$$

whereas its variance is positive and equals

$$\sigma_a = \frac{1}{2}\left[\bar{u}^n - (1+r)\right].$$

One sees that the larger the mass of the investors with relatively high aspiration levels, the higher is the mean price of the asset, but also the higher is its volatility. Hence, as in the previous example, these traders create excessive risk in the economy by trading more, than it would be optimal for them and by possibly causing the risky asset to be overvalued.

Note that in both the second and the third case, the movements of the asset price are negatively correlated over the time. Indeed, it was shown that in the case of intermediate aspiration levels a price fall is followed by a price rise with certainty. In the case of high aspiration level, a period of rising prices is followed by a downward movement and vice versa. Such negative short run correlation is found in market data and Lo and MacKinlay (1988) observe that this correlation is higher for stocks with smaller capitalization. If we assume that the aspiration levels of the investors remain constant, but the dividend paid by the asset increases[100], it is easy to see that the correlation between the returns of the asset a is smaller in the case in which the aspiration levels are relatively low as compared to δD (the capitalization of the firm is large)[101] and rises as the capitalization diminishes, rendering the aspiration levels of the investor relatively high as

100 As observed in footnote 1 the value of the dividend δD can be used as a proxy for the capitalization of the firm D. Hence, the higher the dividend, the higher is the capitalization of the firm.

101 In the case of low aspiration levels (δD is relatively high) there is no correlation between the returns. As δD decreases (the case of intermediate aspiration levels), the correlation increases (upward movements follow downward movements).

compared to δD.

Another interesting finding is that in the case of high aspiration levels, the market-to-book ratio and the inverse of the price are good predictors of the future price movements. This was empirically observed by De Bondt and Thaler (1987), Fama and French (1992) and Lakonishok and Shleifer and Vishny (1994). In periods in which the state of the economy is h, the market-to-book ratio is high and the inverse of the price low, hence the expected price movement is a downward one. At state l the inverse relationship holds. De Bondt and Thaler (1987) interpret this phenomenon as an overreaction of the investors to past positive or negative earnings. In an economy populated by case-based decision-makers this predictability of the price movements is due to the high aspiration levels, which render the investors unsatisfied with the returns of both assets in the economy in any possible state. The investors with high aspiration levels are forced to change their holdings in every period of time, thus, creating forecastable price fluctuations in the market.

Of course, the cycles of the type described can only emerge if the memory of the decision-makers is particularly short, i.e. if they are unable to learn from the past and to realize that the environment in which they act is stable. It is, therefore, of interest to examine whether introducing long memory would help to smooth the price movements and allow agents to learn to choose the asset with the higher expected returns.

5.4 Price-Dynamics with Long Memory

Now assume that the agents can remember all cases of their predecessors, from $t = 0$ on. One may argue that the effort to remember so many cases goes beyond their bounded capacity to process information. In fact, the decision-maker i at time t need not remember the particular cases, but only the cumulative utilities computed by his predecessor at $(t-1)$. He then adds the net utility realization of α^i_{t-1}, $\left[v_t\left(\alpha^i_{t-1}\right) - \bar{u}^i\right]$ to the cumulative utility[102] $U^i_{t-1}\left(\alpha^i_{t-1}\right)$ and chooses the act with the maximal cumulative utility at t.

The analysis of an economy, in which the agents have infinite memory is very complicated in the case in which the aspiration levels are distributed on a continuum. If this assumption is

[102] This simplification is made possible by the assumption, that agents know only about cases of their predecessors with the same aspiration level.

relaxed, then the value of demand correspondence is not necessarily convex-valued. However, the convex-valuedness is indispensable for the existence of a fixed point of a correspondence in the Kakutani fixed-point theorem[103]. Hence, I construct an economy with three types of agents for which an equilibrium with a positive price of the risky asset exists in each period.

5.4.1 Investor Types

Assume, as in section 3 that assumptions $(A4)$, $(A5)$ and $(A6)$ hold. Suppose, however that the aspiration levels of the investors in the economy are such that the interval of investors $[1+r-\bar{u}^0; n]$ has an aspiration level $\bar{u}^3 \in (1+r; \bar{u}^n)$, the interval of investors $[\bar{u}^a - \bar{u}^0; 1+r-\bar{u}^0]$ has an aspiration level $\bar{u}^2 \in (\bar{u}^a; 1+r)$ and the investors $[0; \bar{u}^a - \bar{u}^0]$ have an aspiration level $\bar{u}^1 \in (\bar{u}^0; \bar{u}^a)$. Call these groups of investors type 3, type 2 and type 1, respectively. Hence, $(A5)$ implies that the initial holdings of the assets in $t = 0$ are such that the investors of type 3 and 2 hold b, whereas those of type 1 hold a. Assuming again that

$$1 + \frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} > \bar{u}^3 > 1 + r > \bar{u}^2 > 1 > \bar{u}^1 \tag{5.49}$$

and

$$\frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} > \bar{u}^1 \tag{5.50}$$

hold, it is easy to conclude that:

- the investors of type 1 will always hold a, since (5.50) insures that the return of a (even if all of the investors of type 3 switch to b in a single period and the dividend of a is 0) is higher than their aspiration level $\bar{u}^1$;
- the investors of type 2 will always hold b, since their aspiration level is smaller than the returns of b, $(1+r)$;
- the investors of type 3 will in general switch between a and b. Since the return of a when its dividend is positive exceeds their aspiration level, they will only switch from a to b in periods in which the dividend paid by a is 0.

The following proposition guarantees that in each period of time at least one equilibrium with

[103] In fact, relaxing this assumption in my model endangers only the existence of an equilibrium with a strictly positive price. The reason for this is the prohibition of short sales, which insures that the price of the risky asset cannot fall below 0.

$p_t \neq 0$ exists:

Proposition 5.6 *Assume* $(A4)$, $(A5)$ *and* $(A6)$. *Suppose as well that (5.49) and (5.50) hold. Then, in each period of time, (at least) one of the following two states, denoted by* h *and* l, *is an equilibrium of the economy:*

$$\begin{aligned} \alpha_h^i &= a \textit{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \cup \left[(1+r) - \bar{u}^0; n\right] \\ \alpha_h^i &= b \textit{ for } i \in \left[\bar{u}^a - \bar{u}^0; (1+r) - \bar{u}^0\right] \\ p_h &= \left[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right] \end{aligned}$$

and

$$\begin{aligned} \alpha_l^i &= a \textit{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \\ \alpha_l^i &= b \textit{ for } i \in \left[\bar{u}^a - \bar{u}^0; n\right] \\ p_l &= \left[\bar{u}^a - \bar{u}^0\right]. \end{aligned}$$

This proposition also allows a direct comparison between an economy with short and long memory. Since the allocations and equilibrium prices in proposition 5.3 and proposition 5.6 coincide, it is possible to derive conclusions about the influence of a long memory on the evolution of the economy.

Proposition 5.6 further clarifies that despite the constant choices of the investors of type 1 and 2, the asset holdings and the price of the asset a in general vary over time. Hence, the distribution of the returns of a depends on the behavior of type 3.

5.4.2 Price Dynamics — An Example

To enable the analysis of the economy in the long run, I make the following simplifying assumptions:

$(A7)$ Symmetry of returns: the net utility achieved by the investors of type 3 from a when it pays positive dividends is equal to the negative of the net utility achieved by the investors of type 3 when a pays zero dividend and type 3 still chooses a:

$$1 + \frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - \bar{u}^3 = \bar{u}^3 - 1 := c$$

$(A8)$ Once the investors of type 3 have switched from a to b, they hold b for $k \in \mathbb{Z}^+ \setminus \{0\}$ periods exactly[104]:

$$\frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - \bar{u}^3 := k\left(1 + r - \bar{u}^3\right)$$

104 Here, I implicitly assume that if an investor of type 3 is indifferent between a and b at time t, he chooses asset a.

$$k \in \mathbb{Z}^{+} \backslash \{0\}$$

These two assumptions allow a very simple representation of the cumulative utility of a for the investors of type 3 and imply following results:

Proposition 5.7 *Assume* $(A4)$, $(A5)$, $(A6)$, $(A7)$ *and* $(A8)$. *Let further (5.49) and (5.50) hold. The expected number of periods, during which the investors of type* 3 *hold asset* a *in a row is given by:*

$$E[t] = \left\{ \begin{array}{ll} \frac{1}{(1-2q)} & \text{for } q < \frac{1}{2} \\ \infty & \text{for } q \geq \frac{1}{2} \end{array} \right\}.$$

Denote by π_h and π_l the invariant probabilities of states h and l.

Corollary 5.1 *If* $q < \frac{1}{2}$, *then the economy will be in state* h *with*

$$\alpha_h^i = a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \cup \left[(1+r) - \bar{u}^0; n\right]$$
$$\alpha_h^i = b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; (1+r) - \bar{u}^0\right]$$
$$p_h = \left[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right]$$

during a fraction of time

$$\pi_h = \frac{1}{1 + k(1-2q)}$$

and in state l, *given by:*

$$\alpha_l^i = a, \text{ if } i \in \left[0; \bar{u}^a - \bar{u}^0\right]$$
$$\alpha_l^i = b, \text{ if } i \in \left[\bar{u}^a - \bar{u}^0; n\right]$$
$$p_l = \left[\bar{u}^a - \bar{u}^0\right].$$

during a fraction of time

$$\pi_l = \frac{k(1-2q)}{1 + k(1-2q)}$$

almost surely in the limit.

If $q \geq \frac{1}{2}$, *then the economy will be in state* h *during a fraction of time* $\pi_h = 1$ *almost surely in the limit.*

5.4.3 Discussion of the Results

Consider first the case $q < \frac{1}{2}$. Using corollary 5.1, the mean of p_t can be computed to be:

$$\mu_p^{\infty} = \bar{u}^a - \bar{u}^0 + \frac{1}{1 + k(1-2q)} \left(\bar{u}^n - (1+r)\right)$$

and its variance is:

$$\sigma_p^{\infty} = \frac{\sqrt{k(1-2q)}}{1 + k(1-2q)} \left(\bar{u}^n - (1+r)\right).$$

Now, it is possible to compare the moments of the distribution of the prices in an economy with long memory to the moments of distribution in an economy with short memory. It is easy to see that $\mu_p^\infty = \mu_p^1$, as computed in (5.46), if

$$k = \frac{1-q}{1-2q}$$

(the equality can only hold if $\frac{1-q}{1-2q} \in \mathbb{Z}^+$). The same condition implies that the variances of the two distributions are also equal: $\sigma_p^\infty = \sigma_p^1$, (as computed in (5.47)). In general, however, the moments of the two distributions do not coincide. Since a higher value of k means[105] that asset b is held for a longer period of time, once chosen by type 3, this implies that if

$$k < \frac{1-q}{1-2q},$$

$\mu_p^\infty > \mu_p^1$. At the same time the variance σ_p^∞ increases, as k assumes values smaller than $\frac{1-q}{1-2q}$. Since k decreases in $\bar{u}^3$, it follows that the mean price and the variance of a depend both positively on the highest aspiration level in the economy, as well as on the mass of the investors of type 3 (of course, as long, as type 3 still finds the positive dividend of a to be satisfactory).

It is also of interest to compare the results obtained in an economy with representative consumers to the results of an individual portfolio choice problem. In chapter 3, I have analyzed an individual portfolio choice problem of a case-based decision-maker with a constant aspiration level and long memory. Using the results of Gilboa and Pazgal (2001), it could be shown that if the returns of the available portfolios are exogenously given, bounded above and below and i.i.d. over the time, following statements hold:

1. If the aspiration level of the investor exceeds the highest achievable mean return, then the investor will switch infinitely often among all available acts. He will choose each act with a frequency which is inversely proportional to the difference between its mean and the aspiration level of the investor.

2. If the aspiration level of the investor is lower than the mean return of some of the available portfolios, then one of these portfolios will be chosen with frequency 1 almost surely in the limit.

The results for the economy with representative agents with long memory are similar. Note that,

[105] Note that once the parameters of the economy with infinite memory are defined, k is fixed. Of course, it is only meaningful to compare the economy with long memory to the economy with one-period memory if they have the same parameters. Still, $\bar{u}^3$ is only defined for the economy with infinite memory and can vary between $[(1+r)\,;\bar{u}^n]$. Since k depends negatively on $\bar{u}^3$, changing $\bar{u}^3$ (while keeping all other parameters constant) allows us to compare the distributions of the two economies for different values of k.

if $q < \frac{1}{2}$, the realized mean returns of an investor of type 3 are:

$$\mu_b^r = 1 + r \text{ for asset } b$$

and

$$\mu_a^r = \frac{1}{2}\frac{1}{\frac{1}{1-2q}}\left(\frac{1}{1-2q} - 1\right)\left(1 + \frac{\delta D}{(\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0)}\right) + \tag{5.51}$$
$$+\frac{1}{2}\frac{1}{\frac{1}{1-2q}}\left(\frac{1}{1-2q} - 1\right) + \frac{1}{\frac{1}{1-2q}}\left(\frac{\bar{u}^a - \bar{u}^0}{(\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0)}\right)$$

for asset a, where the first term represents the return during those periods, in which type 3 holds a and the dividend of a is positive, the second term represents the return in periods, in which type 3 holds a, but the dividend is 0, whereas the last term represents the returns of the periods, in which the dividend of a is 0 and the investors of type 3 switch from a to b. Using $(A7)$, $(A8)$ and (5.51), it is then easy to compute that

$$\mu_a^r - \bar{u}^3 = k\left(1 + r - \bar{u}^3\right)(1 - 2q),$$

and since

$$\mu_b^r - \bar{u}^3 = 1 + r - \bar{u}^3,$$

it follows that:

$$\frac{\mu_a^r - \bar{u}^3}{\mu_b^r - \bar{u}^3} = k(1 - 2q) = \frac{\pi_l}{\pi_h}.$$

Hence, in the limit, the frequencies π_h and π_l, with which type 3 chooses acts a and b, are indeed inversely proportional to the difference of the realized mean returns of the act considered (as observed by the investors of type 3) and the aspiration level $\bar{u}^3$.

Note further that if $q < \frac{1}{2}$, neither act is satisfactory for the investors of type 3 ex-ante. The maximal expected return of asset a, (if the investors of type 3 always hold a), is

$$1 + \frac{q\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} < 1 + \frac{\delta D}{2(\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0)} = \bar{u}^3,$$

where the equality follows from $(A7)$. Since $\bar{u}^3 > 1 + r$ holds, it follows that the mean returns of a and b are not satisfactory for the investors of type 3, which makes them switch between the acts infinitely often in the limit. Hence, result $2.$, which was shown for an individual portfolio choice problem holds in a market environment, as well.

As in the case of short memory, the investors of type 3 exhibit behavior similar to those of overconfident investors described by Odean (1999), since they expect higher returns than any asset in the economy can earn on average and, hence, switch too often between the available portfolios, reducing their earnings. The investors with high aspiration levels further increase the price fluctuations in the economy compared to the situation in which only investors with relatively low aspiration levels ($\bar{u}^1$ and $\bar{u}^2$) interact[106].

Now, consider the case in which $q \geq \frac{1}{2}$ holds. Since in this case there is a positive probability that the cumulative utility of a never falls beneath the cumulative utility of b for the investors of type 3, they hold a with frequency 1 almost surely in the limit. Hence, the mean of the asset price p_t is

$$\mu_p^\infty = p_h = \bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0$$

and its variance is 0[107]. Of course, these results qualitatively differ from the results for an economy with one-period memory. Whereas with one-period memory, the investors with high aspiration level are never satisfied and switch permanently between the acts, with long memory they learn to choose one of the acts in the limit.

Again, this result is compatible with the result obtained for a portfolio choice problems in which the price-process is exogenously given. If the investors of type 3 always hold a, its maximal achievable average return is given by:

$$1 + \frac{q\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0}.$$

Since now $q \geq \frac{1}{2}$, this average return exceeds $\bar{u}^3$ and makes a satisfactory in the limit, whereas b generates returns which are always lower than the aspiration level $\bar{u}^3$.

Note that if $q \geq \frac{1}{2}$, then asset a is the ex-ante optimal choice from the point of view of a rational risk-neutral expected utility maximizer, since its expected mean return at the equilibrium prices is higher than those of b. In this case, the investors of type 3 learn to behave as expected utility maximizers in the limit.

The learning process, however, does not spread over the whole economy. The investors of type 2 never learn that the act they choose is suboptimal at the equilibrium price. As in the case of one-period memory, their aspiration level is too low to give them an incentive to experiment and learn that expected returns of a are higher than those of b.

This last result provides a possible explanation of the equity premium puzzle, defined as aversion of investors to hold stocks, see Mehra and Prescott (1985). They find that, although the observed

[106] In an economy in which only type 1 and type 2 investors are present the price of a remains constant over time.

[107] This does not mean that the investors of type 3 never hold b, but the time during which b is held is negligible compared to the time during which a is held.

returns on the stock market have been substantially higher than the returns of bonds for long periods, investors prefer to hold bonds. The estimated levels of risk aversion needed to explain this phenomena are too high to be realistic.

In an economy populated by case-based decision-makers, the equity premium puzzle can obtain even for risk-neutral investors. Indeed, suppose that the mass of the investors of type 1 and type 3 is relatively small compared to the positive dividend of the asset δD (or alternatively to the capitalization of the firm D), whereas the mass of investors of type 2 is relatively large. This means that the long run mean price of a is relatively low. In this case, the mean returns of asset a exceed the mean returns of b (remember that $q > \frac{1}{2}$) in the long run. Nevertheless, a large mass of investors in the economy chooses the riskless asset in each period, because these investors are satisfied with the returns of b and are not apt to switch to another asset, with ”unknown” returns.

The validity of the qualitative results derived in the last section does not depend on the assumption of $(A4)$, $(A7)$ and $(A8)$. They have been useful in order to allow an analytically tractable discussion and the computation of the limit distributions. In fact, statements analogous to statements $1.$ and $2.$ can be formulated for an economy with an endogenously priced asset under much more general conditions. Remember that act b has been assumed to be unsatisfactory for type 3. If $u(1+r)$ exceeds $\bar{u}^3$, then the initial allocation of the economy is a stationary state.

$1'$. If the highest aspiration level in the economy $\bar{u}^3$ exceeds the maximal achievable mean utility of a, then the investors of type 3 switch infinitely often between the acts a and b. In the limit, they almost surely choose the acts a and b with frequencies π_h and π_l, respectively such that:

$$\frac{\pi_h}{\pi_l} = \frac{\bar{u}^3 - \mu_b^r}{\bar{u}^3 - \mu_a^r} = \frac{\bar{u}^3 - u(1+r)}{\bar{u}^3 - \mu_a^r}, \tag{5.52}$$

where μ_a^r and μ_b^r denote the *actual mean utility achieved* by choosing a, respectively b, *as observed* by the investors of type 3.

Note that the frequency π_h is only implicitly determined by (5.52), since μ_a^r now depends on the frequency, with which type 3 switches between a and b, hence on π_h.

2′. If the highest aspiration level in the economy, $\bar{u}^3$, is lower than the maximal achievable mean utility of a, then the investors of type 3 hold a with frequency 1 almost surely in the limit, i.e. $\pi_h = 1$ and $\pi_l = 0$ obtain.

The next section demonstrates that statements $1'$. and $2'$. hold when the distribution of the dividends has an absolute continuous part with respect to the Lebesgue measure on the real numbers. Moreover, this result is independent of the specific form of the utility function.

5.4.4 Price Dynamics for General Probability Distributions

Assume $(A1)$ and $(A2)$. Consider the economy consisting of three types of investors 1, 2 and 3 with aspiration levels $\bar{u}^1$, $\bar{u}^2$ and $\bar{u}^3$, respectively. Assume that $n = 1$. It is then convenient to define the mass of each type of investors by θ_i $(i = 1, 2, 3)$ with $\theta_3 = 1 - \theta_1 - \theta_2$. The initial holdings of the investors of types 1 and 2 are given by:

$$\begin{aligned} \alpha_0^1 &= \alpha_0^3 = a \qquad (5.53) \\ \alpha_0^2 &= b. \end{aligned}$$

Set $D = 1$ and suppose that the dividend payments of a are distributed identically and independently in each period of time according to a probability distribution Q with a support $\left[\underline{\delta}; \bar{\delta}\right]$. Let the aspiration levels of the three types of investors fulfill:

$$u\left(1 + \frac{\bar{\delta}}{1 - \theta_2}\right) > \bar{u}^3 > u\left(1 + r\right) > \bar{u}^2 > u\left(1 + \frac{\underline{\delta}}{1 - \theta_2}\right) > \bar{u}^1 \qquad (5.54)$$

and

$$\frac{\theta_1 + \underline{\delta}}{1 - \theta_2} > \bar{u}^1 \qquad (5.55)$$

Denote by $\tilde{\delta}$ the dividend payment which yields a 0-net utility to the investors of type 3, if the price of a is $(1 - \theta_2)$:

$$u\left(1 + \frac{\tilde{\delta}}{1 - \theta_2}\right) = \bar{u}^3.$$

Assume that the probability distribution Q has an absolute continuous part with respect to the Lebesgue measure μ^{Leb} on the real numbers, with a density g which is positive and bounded away from 0 on the interval $\left(\tilde{\delta} - \zeta; \tilde{\delta} + \zeta\right)$ for some $\zeta > 0$.

First note that the possible temporary equilibria are not changed by the assumption of a more general probability distribution.

Proposition 5.8 *Suppose that (5.53), (5.54) and (5.55) hold. Then, in each period of time, (at least) one of the following two states, denoted by h and l, is an equilibrium of the economy:*

$$\alpha_h^i = a \text{ for } i \in \{1;3\}$$
$$\alpha_h^i = b \text{ for } i = 2$$
$$p_h = [1 - \theta_2]$$

and

$$\alpha_l^i = a \text{ for } i = 1$$
$$\alpha_l^i = b \text{ for } i \in \{2;3\}$$
$$p_l = \theta_1.$$

The following two propositions assure the validity of statements $1'$. and $2'$. and generalize the results of section 4 of chapter 3 to the case of endogenous prices.

Proposition 5.9 *Let the probability distribution Q of the dividend payments has an absolute continuous part with respect to the Lebesgue measure μ^{Leb} on the real numbers with a density g such that*

$$g(\delta) \geq \phi > 0$$

for all $\delta \in \left(\tilde{\delta} - \zeta; \tilde{\delta} + \zeta\right)$, where $\tilde{\delta}$ is given by:

$$u\left(1 + \frac{\tilde{\delta}}{1 - \theta_2}\right) = \bar{u}^3.$$

Suppose further that

$$\int_{\underline{\delta}}^{\bar{\delta}} u\left(1 + \frac{\delta}{1 - \theta_2}\right) g(\delta)\, d\delta < \bar{u}^3$$

and

$$u(1 + r) < \bar{u}^3.$$

Then the state of the economy is h, if $\varepsilon_t > 0$ and l for $\varepsilon_t < 0$, where ε_t is given by:

$$\varepsilon_0 = 0$$

$$\varepsilon_t = \left\{ \begin{array}{lll} \varepsilon_{t-1} + u\left(1 + \frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(1 + \frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3 \geq 0 \\ \varepsilon_{t-1} + u\left(\frac{\theta_1 + \delta_t}{1-\theta_2}\right) - \bar{u}^3, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(1 + \frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3 < 0 \\ \varepsilon_{t-1} - u(1 + r) + \bar{u}^3, & \text{if } \varepsilon_{t-1} < 0. \end{array} \right\}$$

and $\delta_t \sim^{iid} Q$. For the Markov chain described by ε_t, there exists an invariant (finite) probability measure π on the set $\left[u\left(\frac{\theta_1 + \underline{\delta}}{1-\theta_2}\right) - \bar{u}^3; +\infty\right)$ such that

$$\pi_l = \pi\left[u\left(\frac{\theta_1 + \underline{\delta}}{1 - \theta_2}\right) - \bar{u}^3; 0\right)$$

and

$$\pi_h = \pi\,[0; +\infty)$$

almost surely describe the frequencies with which state h and state l occur in the limit, respectively. $\pi_h > 0$ and $\pi_l > 0$ hold. Moreover, there exists a limit mean utility of asset a, μ_a^r, as *observed by the investors of type 3, which satisfies*

$$\frac{\pi_h}{\pi_l} = \frac{\bar{u}^3 - u(1+r)}{\bar{u}^3 - \mu_a^r}.$$

The proposition states that analogously to the setting in which the utility realizations are identically and independently distributed, in a market setting relatively high aspiration levels lead to constant switching between the available acts. The frequency with which each of the acts is chosen is inversely proportional to the net-mean utility of the act as observed by the investors of type 3.

Proposition 5.10 *Suppose that*

$$u(1+r) < \bar{u}^3 < \int_{\underline{\delta}}^{\bar{\delta}} u\left(1 + \frac{\delta}{1-\theta_2}\right) g(\delta)\, d\delta.$$

Then the state of the economy is h for $\varepsilon_t > 0$ and l for $\varepsilon_t < 0$, where ε_t is defined as in proposition 5.9. The limit frequencies of the states h and l almost surely satisfy

$$\pi_h = 1$$
$$\pi_l = 0.$$

Hence, if the aspiration level of type 3 is chosen to lie between the highest achievable mean utilities of the best and the second-best act, these investors learn to choose the optimal portfolio in the limit.

The results of propositions 5.9 and 5.10 rely on the assumption that the memory of the investors is endogenous and that investors observe only past cases experienced by investors with the same aspiration level as their own. In the context of financial markets these two assumptions might seem too severe, since hypothetical cases can be easily constructed and included in the memory. The next section, therefore, discusses an example which illustrates the influence of hypothetical reasoning.

5.5 Hypothetical Cases

Assume that the economy consists of identical agents and $n = 1$ holds. The aspiration level of an investor is denoted by $\bar{u}$ and is assumed to be constant over time. Differently from the model presented above, it is assumed that asset b is also in fixed supply of 1 unit in each period of time. In each period, b pays a dividend of r per unit. The representative investor is allowed to choose among three investment alternatives: investing his whole initial endowment in a, in b, or in the market portfolio, which consists of one unit of a and one unit of b. The indirect utility achieved by choosing a or b is given by:

$$\begin{aligned} v_t(a) &= u\left(\frac{p_t^a + \delta_t}{p_{t-1}^a}\right) \\ v_t(b) &= u\left(\frac{p_t^b + r}{p_{t-1}^b}\right), \end{aligned}$$

respectively. p_t^a and p_t^b denote the price of a and b in period t. As in the previous section, the dividend payments of the risky asset δ are distributed according to a distribution Q on the interval $\left[\underline{\delta}; \bar{\delta}\right]$. $1 > \bar{\delta} > r > \underline{\delta} \geq 0$ is assumed. Note that the return of the market portfolio is given by[108]:

$$1 + \delta_t + r$$

in each period of time, hence its indirect utility is

$$v_t(MP) = u(1 + \delta_t + r).$$

The market portfolio dominates the other two investment opportunities independently of the dividend payment of the risky asset[109]. Hence, it is of interest to know, whether the case-based decision-makers are able to learn to make optimal choices in the limit, in spite of their limited knowledge of the economy.

5.5.1 Individual Portfolio Choice

Let α_t denote the portfolio choice of the representative investor at time t: $\alpha_t \in \{a; b; MP\}$, where MP stays for the investment into the market portfolio. The memory of the representative investor is constructed in the following way: in each period, the investor remembers the last $(m+1)$ choices of his predecessors and the utility realizations achieved from choosing these portfolios. Apart from that, if the market portfolio was not the act chosen at time

[108] The return of the market portfolio is computed as:

$$\begin{aligned} & \frac{p_{t-1}^a}{p_{t-1}^a + p_{t-1}^b} \frac{p_t^a + \delta_t}{p_{t-1}^a} + \frac{p_{t-1}^b}{p_{t-1}^a + p_{t-1}^b} \frac{p_t^b + r}{p_{t-1}^b} \\ =\ & 1 + r + \delta_t, \end{aligned}$$

since the market clearing condition of the market for consumption goods insures that $p_t^a + p_t^b = 1$ for each t.

[109] This dominance, however, is not just a property of the assets, but an equilibrium property in a model with a representative consumer. Indeed, if the representative investor in this model holds a, the price of a is 1 and the return of a is at most $1 + \delta_t$, which is less than the return of the market portfolio. If, however, the representative investor holds the market portfolio, then the return of a is at most $\frac{1+\delta_t}{p_{t-1}^a}$, which can exceed $1 + \delta_t + r$ for low enough p_{t-1}^a. Still, since everyone in the economy would be holding the market portfolio, there is no possibility to profit from the increase of the price of a, since it automatically leads to a decrease in the price of b.

$\tau \in \{t-m-1; t-m...; t-1\}$, the representative investor also observes the realization of the market portfolio at τ. Hence, for those periods in which MP was not chosen, the investor has two cases in his memory. The memory at time t can be, therefore, written as

$$M_t = ((\alpha_\tau; v_{\tau+1}(\alpha_\tau)); (MP; v_{\tau+1}(MP)))_{\tau=t-m-1}^{t-1}.$$

There is some redundancy in this definition: in periods in which MP was indeed the choice of the investor this case is listed twice in his memory. Still, this will create no problem when the cumulative utilities of the acts are computed. Given the memory of the investor M_t and his aspiration level $\bar{u}$, the cumulative utilities are determined as follows:

$$\begin{aligned} U_t(a) &= \sum_{\tau \in C_t(a)} [v_{\tau+1}(a) - \bar{u}] \\ U_t(b) &= \sum_{\tau \in C_t(b)} [v_{\tau+1}(b) - \bar{u}] \\ U_t(MP) &= \sum_{\tau=t-m}^{t} [v_\tau(MP) - \bar{u}], \end{aligned}$$

with

$$C_t^i(\alpha) = \{t-m-1 \leq \tau < t \mid \alpha_\tau = \alpha\}$$

for $\alpha \in \{a; b\}$. A sum over an empty set is assumed to equal 0.

5.5.2 Limit Behavior with Long Memory

Again, two cases have to be considered: the case of low aspiration level is given by:

$$\bar{u} < \int_{\underline{\delta}}^{\bar{\delta}} u(1+\delta+r)\, g(\delta)\, d\delta.$$

Hence, the aspiration level should be lower than the average utility achieved by holding the market portfolio in each period of time. Note that since the investor is observing the realizations of the market portfolio in each period of time, he learns that the market portfolio dominates the assets a and b, hence it is not possible that he chooses one of the assets a and b with frequency 1 in the limit. He, therefore, eventually chooses the market portfolio and, since the average utility achieved by the market portfolio exceeds his aspiration level, the expected time until the first switch to an alternative portfolio is ∞. But even if the investor switches from the market portfolio to a or to b, he eventually chooses the market portfolio again (since the returns of the market portfolio are always higher and with positive probability exceed his aspiration level). Hence, in the limit the market portfolio will be chosen with frequency 1.

Proposition 5.11 *Suppose that the aspiration level of the representative investor satisfies:*

$$\bar{u} < \int_{\underline{\delta}}^{\bar{\delta}} u\left(1+\delta+r\right) g\left(\delta\right) d\delta.$$

Denote by π_a, π_b and π_{MP} the frequencies with which the risky asset, the bond and the market portfolio are chosen in the limit, respectively.

$$\pi_{MP} = 1$$
$$\pi_a = \pi_b = 0$$

obtains almost surely in the limit.

The previous proposition shows that optimal behavior obtains even for relatively low aspiration levels, e.g. even for aspiration levels lower than $u\left(1+r\right)$, if the decision-maker includes the hypothetical case containing the return of the market portfolio into his memory in each period of time. Nevertheless, even with hypothetical reasoning optimal learning does not always obtain in the limit. Especially, if the aspiration level of the representative investor is relatively high,

$$\bar{u} > \int_{\underline{\delta}}^{\bar{\delta}} u\left(1+\delta+r\right) g\left(\delta\right) d\delta, \tag{5.56}$$

he is not satisfied with the average utility achieved from the market portfolio by holding it in each period of time. But condition (5.56) also implies that:

$$\bar{u} > \int_{\underline{\delta}}^{\bar{\delta}} u\left(1+\delta\right) g\left(\delta\right) d\delta$$

and $\bar{u} > u\left(1+r\right)$. Hence, since the investor finds all portfolios available unsatisfactory, he will switch among them infinitely often in the limit.

Proposition 5.12 *Suppose that the aspiration level satisfies:*

$$\bar{u} > \int_{\underline{\delta}}^{\bar{\delta}} u\left(1+\delta+r\right) g\left(\delta\right) d\delta.$$

The representative investor holds portfolios a and b during a strictly positive proportion of time with probability 1 in the limit.

The analysis of an economy in which the investors are allowed to learn the realizations of the market portfolio apart from the realizations of actually chosen portfolios in the past shows that this additional information improves the learning in the economy. Especially, for low aspiration levels, investors acquiring information about the market portfolio learn that it dominates the alternative investment opportunities and choose it eventually with frequency 1. This result contrasts the results with endogenous memory which show that investors with relatively low aspirations choose a possibly suboptimal portfolio in each period of time.

However, for high aspiration levels, the results do not differ qualitatively from those achieved with endogenous memory. The investor never learns to choose the optimal portfolio in the limit. Since he is dissatisfied with each one of the possible acts, he chooses the dominated ones with strictly positive frequency in the limit.

In this example, investors with low aspiration levels are able to learn to choose the optimal market portfolio in the limit, because the market portfolio weakly dominates the other two available portfolios. If no dominance relationship is present, learning need not obtain in the limit even if the returns of *all* available portfolios are observed in each period of time. To see this, consider again the case, in which only the two undiversified portfolios consisting of assets a and b are available and the supply of b is fixed. Suppose that the current choice of the investors is b and observe that the utility realization of b is $u(1+r)$, as long as the young investors choose b as well, whereas the utility realization of a is at most $u\left(\bar{\delta}\right)$, as long as none of the investors chooses a. Hence, if $\bar{\delta} < 1$, as assumed above, it follows that there is an equilibrium path on which the representative investor chooses b in each period of time, independently of his aspiration level. Equivalently, there is an equilibrium path, on which a is chosen in each period of time, whereas the utility realization of b is

$$v_t(b) = u(r) < u(1) \leq v_t(a)$$

in each period. It follows that optimal learning in a market with case-based decision-makers is not guaranteed, even if they observe all past utility realizations of all acts available.

This result contrasts the result obtained in an individual portfolio choice problem with identically and independently distributed returns. The reason for this difference stems from the fact that in a market environment, the return of a portfolio depends not only on the dividend payment, but also on its price. When constructing a hypothetical case, an individual takes the price as given, without taking into account price changes which would occur if everyone changed his behavior. Hence, hypothetical cases act as self-confirming prophecies: if everyone beliefs that the price of a will be 0 the price of a indeed remains 0, since at this price the utility realizations of a never exceed those of b. As a consequence, multiple equilibria emerge and it is à priori not clear which equilibrium path will be chosen. Overvaluation and undervaluation may occur in an equilibrium and persist for long periods of time.

The last example, in which the representative investor is allowed to choose only between the two undiversified portfolios, demonstrates further that learning might be more effective if hypothetical cases are not used. Indeed, in the case

$$\bar{u} \in \left(u\left(1+r\right); \int_{\underline{\delta}}^{\bar{\delta}} u\left(1+\delta\right) g\left(\delta\right) d\delta \right),$$

investors not using hypothetical cases learn with probability 1 to choose the better of the two portfolios a with frequency 1 in the limit. In contrast, when hypothetical cases are used, the probability that a is chosen with frequency 1 in the limit depends crucially on the ability of the investors to coordinate on the equilibrium in which a is chosen in the initial $\left\lceil \left(\frac{u(1+r)-u(\underline{\delta})}{u(1+\underline{\delta})-u(r)} \right) \right\rceil$ periods of time[110].

5.6 Similarity in Asset Markets

Up to know, it has been assumed that no similarity considerations influence the decisions in the economy. Due to this, it was necessary to restrict the choice set to a finite number of acts. In this section, similarity between problem - act pairs is introduced and the investors are allowed to choose among all diversified portfolios consisting of assets a and b.

In section 8 of chapter 3, similarity considerations were embedded into a portfolio choice problem. There, it was assumed that only similarity among portfolios is taken into account by the investor. However, the market situation might also influence the evaluation of different acts. Buying an asset in a market boom might be quite different from buying the same asset, when prices fall. The characteristics of a given decision situation are captured by the notion of a *problem*. In a financial market, asset prices seem to bear the most important information about the decision situation and, thus, will influence similarity perceptions[111].

In a model of case-based decision making in financial markets, these two aspects of similarity — similarity between problems and between acts — are captured in a single similarity function:

$$s\left(\left(\rho;\alpha\right);\left(\rho';\alpha'\right)\right),$$

which is to be interpreted as the degree of similarity of choosing act α in problem ρ to choosing

110 After a has been chosen for $\left\lceil \left(\frac{u(1+r)-u(\underline{\delta})}{u(1+\underline{\delta})-u(r)} \right) \right\rceil$ periods, the choice of b ceases to be an equilibrium, since

$$U_{t+1}\left(b\right) = tu\left(r\right) + u\left(1+r\right) < tu\left(1+\underline{\delta}\right) + u\left(\underline{\delta}\right) = U_{t+1}\left(a\right)$$

holds for all $t \geq \left\lceil \left(\frac{u(1+r)-u(\underline{\delta})}{u(1+\underline{\delta})-u(r)} \right) \right\rceil$.

111 The initial endowment of an investor (or, alternatively, his income at the beginning of the period) might also influence the perceived similarity between two decision situations. However, since the initial endowment remains constant in this model, it will not play any role, even if it were included into the similarity function.

act α' in problem ρ'. It has already been shown that the acts α can be situated on a metric space, depending on how similar they are perceived to one another. Since it seems to me that the major characteristic of a portfolio choice problem is represented by the prices in the economy, I propose to identify each problem with a price vector $(p_1...p_K)$ and to represent a problem - act pair in $\Delta^{K-1} \times \Delta^{K-1}$. As in chapter 3, I will again use the Euclidean distance between such points as a measure of similarity.

5.6.1 OLG-Model with Two Types of Investors

Consider a simplified version of the economy described in section 1. The length of the continuum of investors is assumed to be 1. There are two types of investors, $i \in \{1; 2\}$, with constant aspiration levels $\bar{u}^1$ and $\bar{u}^2$, respectively. The shares of these two types are denoted by $\theta_1 \in (0; 1)$ and $\theta_2 = 1 - \theta_1$ and remain constant over the time.

Set $D = 1$ and suppose that the dividend paid by the risky asset is identically and independently distributed according to a probability distribution Q on the interval $\left[\underline{\delta}; \overline{\delta}\right]$. Let $g(\cdot)$ denote the density of the distribution Q. The supply of the risky asset a is fixed at $A = 1$. The riskless asset is available in perfectly elastic supply and delivers a return of $(1 + r)$ per unit invested. In contrast to the model introduced in section 1, diversification is allowed. Still, no short sales are possible.

$(A1)$, i.e. the continuity and strict monotonicity of the utility function $u(\cdot)$, is assumed throughout this section.

Let α_t^i denote the act chosen at time t by an investor of type $i \in \{1; 2\}$ and identify α_t^i by the share of the initial endowment invested in a $(\alpha_t^i \in [0; 1])$. p_t denotes the price of asset a at time t. The similarity function of an investor is defined for such pairs $(p; \alpha)$, i.e. the investor considers only the price of the risky asset as relevant for the description of the market situation.

Since short sales are not allowed and since the initial endowment of the economy is fixed at 1, it follows that the price of the asset a, p_t, can only take on values between $[0; 1]$. Since the portfolio share of a can also vary between $[0; 1]$, it follows that the $(p; \alpha)$-pairs can be represented on a square with side length one. The Euclidean distance on this square can, therefore, be taken as a measure of similarity. Therefore, assume that

$$s((p;\alpha);(p';\alpha')) = f(\|(p;\alpha)-(p';\alpha')\|),$$

where $f(\cdot)$ is strictly decreasing. $s(\cdot)$ is assumed not to depend on the type of investors and

$$s((p;\alpha);(p;\alpha)) = 1.$$

Assume that the memory of the investors is endogenous, i.e. they can only remember cases $(p;\alpha;u(\alpha))$ that really occurred in the economy. Moreover, each investor of type $i \in \{1;2\}$ can only observe past cases experienced by investors of his own type, i.e. cases of the type $(p_t;\alpha_t^i;u(\alpha_t^i))$. Let m parameterize the length of memory, as above. Since in period $t=0$ the memory of the investors is empty, let $\alpha_0 \in (0;1)$ be the act chosen[112] (at random) in period 0 by both types and let $p_0 = \alpha_0$ be the equilibrium price at $t=0$.

5.6.2 Equilibrium Paths

Given the initial allocation α_0 and the initial price $p_0 = \alpha_0$, an equilibrium path of the economy is defined as a vector of asset prices $(p_t^*)_{t=0,1,\ldots}$ and a vector of portfolios $(\alpha_t^{1*};\alpha_t^{2*})_{t=0,1,\ldots}$ chosen by the young investors at t (with $\alpha_0^{1*}=\alpha_0$, $\alpha_0^{2*}=\alpha_0$, $p_0^*=p_0$), such that:

(i) young investors make case-based decisions in each period:

$$\alpha_t^{i*} \in \arg\max_{\alpha\in[0;1]} U_t^i(\alpha) =$$
$$= \arg\max_{\alpha\in[0;1]} \sum_{\tau=t-m-1}^{t-1} s\left((p_\tau;\alpha_\tau^i);(p_t;\alpha)\right)\cdot$$
$$\cdot\left[u\left(\left(\frac{p_{\tau+1}+\delta_{\tau+1}}{p_\tau}\right)\alpha_\tau^i+(1+r)\left(1-\alpha_\tau^i\right)\right)-\bar{u}^i\right]$$

and

(ii) the market for the risky asset is cleared in each period: either $p_t^*>0$ and satisfies

$$\frac{\alpha_t^{1*}(p_t^*)+\alpha_t^{2*}(p_t^*)}{p_t^*}=1$$

or

$$p_t^*=0 \text{ and } \alpha_t^{1*}(0)+\alpha_t^{2*}(0)=0.$$

I will not discuss the question of existence of an equilibrium path in general. For a concave similarity function and for the cases of one-period and long memory, it will be shown that equi-

[112] The results for $\alpha_0=1$ and $\alpha_0=0$ are qualitatively the same, the interesting case is, however, the one of a diversified initial portfolio.

librium paths exist, by studying the price dynamics. Note that the market clearing condition allows for degenerate equilibria, in which no one holds asset a and its price falls to 0. Since it has been shown that case-based decision-makers with concave similarity functions do not diversify if their aspiration level is relatively high, see section 8 of chapter 3, it is natural to expect that such degenerate equilibria occur for high values of $\bar{u}^1$ and $\bar{u}^2$. That is why I assume that the aspiration level of the investors of type 1 is sufficiently low so that they never switch from their initially chosen portfolio. This insures the existence of an equilibrium path with a positive price of the risky asset in each period of time.

$(A9)$ Suppose that $\bar{u}^1 < u\left(\frac{\theta_1\alpha_0+\underline{\delta}}{1-\theta_1(1-\alpha_0)}\alpha_0 + (1-\alpha_0)(1+r)\right)$.

$(A9)$ insures that even if all investors of type 2 hold $\alpha = 1$ at some time t, whereas the investors of type 1 hold α_0 (hence $p_t = 1 - \theta_1(1-\alpha_0)$), all investors of type 2 switch to $\alpha = 0$ at $(t+1)$, causing the price to fall to $p_{t+1} = \theta_1\alpha_0$ and if the dividend of the risky asset is $\underline{\delta}$ at $(t+1)$, the investors of type 1 are still satisfied by the return of their portfolio α_0. Given this condition on $\bar{u}^1$, the investors of type 1 will hold α_0 forever, no matter how long their memory is and independently of the price and dividend realizations and of the portfolio choices of type 2.

To avoid the discussion of multiple cases, I assume

$$1 > \bar{\delta} > r > \alpha_0 r > \underline{\delta} \geq 0.$$

5.6.3 Price Dynamics with One-Period Memory

Consider first the case of one-period memory, hence, the investors only remember the last case observed. Since the aspiration level of type 1 is fixed in such a way that they never switch from their initially chosen portfolio, only the aspiration level of type 2 needs to be considered. If this aspiration level is relatively low, then type 2 is always satisfied with the return of his initial portfolio α_0, given that everyone in the economy continues to hold α_0. Therefore, the following proposition obtains:

Proposition 5.13 *Assume $(A9)$ and let*

$$\bar{u}^2 < u\left(\left(1+\frac{\underline{\delta}}{p_0}\right)\alpha_0 + (1-\alpha_0)(1+r)\right).$$

Then, there is an equilibrium path on which $\alpha_t^{1*} = \alpha_0$, $\alpha_t^{2*} = \alpha_0$ and $p_t^* = p_0$ for each $t = 0, 1, \ldots$. Hence, $(\alpha^1 = \alpha_0; \alpha^2 = \alpha_0; p = p_0)$ is a stationary state of the economy.

Now, let the aspiration level be such that the portfolio α_0 is not satisficing for type 2 if the risky asset pays a dividend lower than $\tilde{\delta} \in \left(\underline{\delta}; \bar{\delta}\right)$ even if the price of a remains unchanged. Hence, let[113]

$$\bar{u}^2 = u\left(\left(1 + \frac{\tilde{\delta}}{p_0}\right)\alpha_0 + (1 - \alpha_0)(1 + r)\right)$$

for some $\tilde{\delta} \in \left(\underline{\delta}; \bar{\delta}\right)$. As long as the utility from the return of the riskless asset exceeds $\bar{u}^2$, the state in which the investors of type 2 hold portfolio $\alpha^2 = 0$ in each period is a stationary state of the economy.

Proposition 5.14 *Assume* $(A9)$ *and let*

$$\bar{u}^2 \in \left(u\left(\left(1 + \frac{\underline{\delta}}{p_0}\right)\alpha_0 + (1 - \alpha_0)(1 + r)\right); u(1 + r)\right).$$

Then, on almost all paths of dividend realizations $\tilde{\omega} = (\delta_1; \delta_2 ... \delta_t ...)$*, there is an equilibrium path, such that* $\alpha_t^{1*} = \alpha_0$*,* $\alpha_t^{2*} = 0$ *and* $p_t^* = \theta_1 \alpha_0$ *for all* $t \geq \bar{t}(\tilde{\omega})$*, for some* $\bar{t}(\tilde{\omega})$*.* $(\alpha^1 = \alpha_0; \alpha^2 = 0; p = \theta_1 \alpha_0)$ *is, thus, a stationary state of the economy.*

Since the proof of this proposition demonstrates how a bubble can endogenously emerge and burst in an economy populated by case-based decision-makers, I include part of it into the main text.

Proof of proposition 5.14

Since $(A9)$ guarantees that the investors of type 1 never switch away from their initially chosen portfolio, only the behavior of the investors of type 2 needs to be considered.

Note that for $\delta \geq \tilde{\delta}$

$$u\left(\left(1 + \frac{\delta}{p_0}\right)\alpha_0 + (1 - \alpha_0)(1 + r)\right) = u(1 + \delta + (1 - \alpha_0) r) \geq \bar{u},$$

hence, the return of the investors of type 2 is satisfactory for them if the young investors continue to hold α_0 and, therefore, by the argument of proposition 5.13, there is an equilibrium, such that $\alpha_t^{2*} = \alpha_0$ and $p_t^* = p_0$ for all t such that $\delta_\tau \geq \tilde{\delta}$ for all $\tau \leq t$. Let $t' = \min\left\{t \mid \delta_t < \tilde{\delta}\right\}$. t' is finite on almost all paths of dividend realizations $\tilde{\omega}$, but its value depends on the chosen path[114].

[113] Observe that since

$$u\left(\left(1 + \frac{\delta}{p_0}\right)\alpha_0 + (1 - \alpha_0)(1 + r)\right) = u(1 + \delta + (1 - \alpha_0) r) \leq \bar{u} < u(1 + r)$$

for $\delta < \tilde{\delta}$, it follows that

$$1 + \delta + (1 - \alpha_0) r < 1 + r$$

and therefore that for each $\delta < \tilde{\delta}$

$$\delta < \alpha_0 r < r.$$

Hence, for $\tilde{\delta} \in \left[\underline{\delta}; \bar{\delta}\right]$ to hold, $\underline{\delta} < \alpha_0 r$ must be satisfied. If this assumption is violated, no such $\tilde{\delta}$ exists.

[114] Similarly, all period numbers introduced hereafter depend on the realized dividend path $\tilde{\omega}$. I neglect this depen-

In period t', the utility realization of α_0 is at most $u\,(1 + \delta_{t'} + (1 - \alpha_0)\,r) < \bar{u}^2$ if the portfolio holdings remain unchanged. Therefore, the cumulative utility of α_0 is negative for the investors of type 2. Since the similarity function is decreasing in the distance between two portfolios for a given price p, it follows that the investors of type 2, who take the price as given, choose the portfolio furthest away from α_0. Hence $\alpha^2_{t'} = 1$ if $\alpha_0 < \frac{1}{2}$ and $\alpha^2_{t'} = 0$ if $\alpha_0 > \frac{1}{2}$.

Suppose first that $\alpha_0 \geq \frac{1}{2}$. Then, $p^*_{t'} = \theta_1 \alpha_0$ is the equilibrium price corresponding to $\alpha^{2*}_{t'} = 0$ and one easily checks that $\alpha^2 = 0$ indeed maximizes the cumulative utility of type 2 in this case. Once the portfolio consisting only of bonds has been chosen, the utility realization becomes $u\,(1 + r)$ in each period, independently of the price p_t. Since $u\,(1 + r) > \bar{u}$, it follows that the state $(\alpha^1 = \alpha_0; \alpha^2 = 0; p = \theta_1 \alpha_0)$ is stationary.

Now consider the case $\alpha_0 < \frac{1}{2}$. Given $p_{t'}$, the investors of type 2 choose the portfolio which is furthest away from α_0, i.e. $\alpha = 1$. However, if $\alpha_{t'} = 1$ is chosen, the price $p_{t'}$ rises to $\theta_1 \alpha_0 + (1 - \theta_1)$ and the utility achieved by type 2 increases to

$$u\left(\frac{\theta_1 \alpha_0 + (1 - \theta_1) + \delta_{t'}}{p_0} \alpha_0 + (1 - \alpha_0)\,(1 + r)\right).$$

If this is still smaller than $\bar{u}^2$, then the cumulative utility is indeed maximized at $\alpha^2 = 1$, given $p_{t'} = \theta_1 \alpha_0 + (1 - \theta_1)$.

However, if

$$u\left(\frac{\theta_1 \alpha_0 + (1 - \theta_1) + \delta_{t'}}{p_0} \alpha_0 + (1 - \alpha_0)\,(1 + r)\right) > \bar{u}^2, \tag{5.57}$$

then the cumulative utility of α_0 is positive at $p_{t'} = \theta_1 \alpha_0 + (1 - \theta_1)$ and, therefore, $\alpha^2 = 1$ is not optimal given $p_{t'} = \theta_1 \alpha_0 + (1 - \theta_1)$. Should this be the case, choose $\alpha^{2*}_{t'}$ in such a way that

$$u\left(\frac{p^*_{t'} + \delta_{t'}}{p_0} \alpha_0 + (1 - \alpha_0)\,(1 + r)\right) = \bar{u}^2,$$

where $p^{*'}_t$ clears the market, given that $\alpha^{2*}_{t'}$ is chosen by type 2, whereas type 1 still holds α_0:

$$p^*_{t'} = \theta_1 \alpha_0 + (1 - \theta_1)\,\alpha^{2*}_{t'}.$$

Since $u\,(\cdot)$ is continuous and strictly increasing, such portfolio and equilibrium price exist by the intermediate value theorem and are unique. Note further that $1 > \alpha^{2*}_{t'} > \alpha_0$ and

$$\theta_1 \alpha_0 + (1 - \theta_1) > p^*_{t'} > p_0$$

must hold. Moreover, the cumulative utility of α_0 given $p^*_{t'}$ is

$$U^2_{t'}\,(\alpha_0) = u\,(p^*_{t'} + \delta_{t'} + (1 - \alpha_0)\,(1 + r)) - \bar{u}^2 = 0 = U^2_{t'}\,(\alpha)$$

dence in the notation for convenience.

for all $\alpha \in [0;1]$. Hence, at $p^*_{t'}$, the investors of type 2 are indifferent among all available portfolios and, therefore, $\alpha^{2*}_{t'}$ is an optimal choice.

Again, two cases can occur: either $\alpha^{2*}_{t'} > \frac{1}{2}$ and the investors of type 2 switch to $\alpha^{2*}_{t''} = 0$ at time $t'' = \min\left\{t > t' \mid \delta_t < \tilde{\delta}\right\}$, as shown above, or $\alpha^{2*}_{t'} < \frac{1}{2}$ holds. In the latter case, construct $\alpha^{2*}_{t''}$ in the same manner as $\alpha^{2*}_{t'}$. Again, $1 > \alpha^{2*}_{t''} > \alpha^{2*}_{t'} > \alpha_0$ must hold. Repeat the same procedure n times as long as $\alpha^{2*}_{t^n} < \frac{1}{2}$ holds. Now note that since

$$u\left(\frac{p^*_{t^k} + \delta_{t^k}}{p^*_{t^{k-1}}}\alpha^{2*}_{t^{k-1}} + (1 - \alpha^*_{t^{k-1}})(1+r)\right) = \bar{u}^2$$

and

$$p^*_{t^{k-1}} = \theta_1\alpha_0 + (1-\theta_1)\,\alpha^{2*}_{t^{k-1}},$$

it follows that the price at time t^k is given by

$$p_{t^k} = p_{t^{k-1}}\frac{\varpi - (1+r)}{p_{t^{k-1}} - \theta_1\alpha_0}p_{t^{k-1}}(1-\theta_1) + (1+r)\,p_{t^{k-1}} - \delta_{t^k} \tag{5.58}$$

for all $k = 1...n$, where $\varpi = u^{-1}(\bar{u}^2)$ denotes the return which yields a utility exactly equal to the aspiration level of the investors of type 2.

Note that

$$\alpha^{*2}_{t^n}\left(\frac{p^*_{t^n} + \underline{\delta}}{p^*_{t^n}}\right) + \left(1 - \alpha^{*2}_{t^n}\right)(1+r) < \alpha_0\left(\frac{p_0 + \underline{\delta}}{p_0}\right) + (1-\alpha_0)(1+r),$$

is equivalent to

$$\begin{aligned} & r\left(\alpha_0 - \alpha^{*2}_{t^n}\right) + \frac{\underline{\delta}\alpha^{*2}_{t^n}}{p^*_{t^n}} - \underline{\delta} \\ = \;& \left(\alpha_0 - \alpha^{*2}_{t^n}\right)\left[\frac{r - \underline{\delta}\theta_1}{p^*_{t^n}}\right] < 0, \end{aligned}$$

which is satisfied, since $\alpha_0 < \alpha^{*2}_{t^n}$ and $r > \underline{\delta} > \underline{\delta}\theta_1$ hold. Hence, for arbitrary high shares $\alpha^{*2}_{t^n}$, there are still values of δ_t, for which the portfolio choice is considered unsatisfactory.

It has to be shown that the sequence defined recursively by (5.58) satisfies

$$p^*_{t^n} \geq \frac{1 - \theta_1 + 2\theta_1\alpha_0}{2} = \theta_1\alpha_0 + \frac{1}{2}(1-\theta_1)$$

after a finite number of periods t^n, hence after a finite number of iterations $(n-1)$. Here, the critical value of $p^*_{t^n}$ is computed as the price necessary to render $\alpha^{*2}_{t^n} \geq \frac{1}{2}$. The demonstration of this is deferred to the appendix. But once this value of $p^*_{t^n}$ is reached,

$$\alpha^{2*}_{t^n} = \frac{p^*_{t^n} - \theta_1\alpha_0}{1-\theta_1} \geq \frac{\frac{2\theta_1(1-\theta_1)\alpha_0 + (1-\theta_1)}{2} - \theta_1\alpha_0}{1-\theta_1} = \frac{1}{2}$$

obtains and from the next period, $\bar{t}(\tilde{\omega})$

$$\bar{t}(\tilde{\omega}) = \min\left\{t > t^n \mid u\left(\alpha^{*2}_{t^n}\left(\frac{p^*_{t^n} + \delta_t}{p^*_{t^n}}\right) + \left(1 - \alpha^{*2}_{t^n}\right)(1+r)\right) < \bar{u}^2\right\}$$

in which a sufficiently low dividend realization obtains, the investors of type 2 switch to asset b and hold it forever. Observing that $\bar{t}(\tilde{\omega})$ is finite with probability 1, completes the proof of the proposition.■

It is obvious from the proof of proposition 5.14 that the price of a rises during the $(n-1)$ iterations if $\alpha_0 < \frac{1}{2}$ and condition (5.57) holds. Moreover, it rises in those periods in which the dividend paid by the risky asset is 0. Imagine, therefore, that the risky asset has a fundamental value of 0 (either $\delta = 0$ or $q = 0$). In this case, the case-based decision-makers holding a small initial share of the risky asset steadily increase the share of their wealth invested in a, until it exceeds $\frac{1}{2}$. Hence, they cause a bubble. At the time when the critical value of p_{t^n} is reached, the bubble bursts and never reemerges again.

If the aspiration level of type 2 exceeds $u(1+r)$, the economy starts to evolve in a cycle:

Proposition 5.15 *Let* $\bar{u}^2 \in \left(u(1+r)\,; u\left(1+\frac{\bar{\delta}}{1-\theta_1(1-\alpha_0)}\right)\right)$. *Then on almost all paths of dividend realizations* $\tilde{\omega}$, *there is a time* $\bar{t}(\tilde{\omega})$, *such that for all* $t \geq \bar{t}(\tilde{\omega})$ *the economy evolves according to a stochastic cycle with two states:*

$$\begin{array}{lll} h, \text{ with} & \alpha_h^1 = \alpha_0,\, \alpha_h^2 = 1 \text{ and} & p_h = 1-\theta_1(1-\alpha_0) \\ & \text{and} & \\ l, \text{ with} & \alpha_h^1 = \alpha_0,\, \alpha_l^2 = 0 \text{ and} & p_l = \theta_1\alpha_0. \end{array}$$

Define $\hat{\delta}$ *as*

$$\bar{u}^2 = u\left(1+\frac{\hat{\delta}}{1-\theta_1(1-\alpha_0)}\right)$$

and let q *denote the probability of a dividend payment higher than* $\hat{\delta}$, *according to* Q*:*

$$q = \int_{\hat{\delta}}^{\bar{\delta}} g(\delta)\, d\delta$$

The frequencies with which the two states h *and* l *occur almost surely satisfy:*

$$\pi_h = \frac{1}{2-q}$$
$$\pi_l = \frac{1-q}{2-q}.$$

If the aspiration level is set even higher, so that even $u\left(1+\frac{\bar{\delta}}{1-\theta_1(1-\alpha_0)}\right)$ is not satisficing, then the investors of type 2 switch between the two corner portfolios in each period:

Proposition 5.16 *Let* $\bar{u}^2 > u\left(1+\frac{\bar{\delta}}{1-\theta_1(1-\alpha_0)}\right)$. *Then, on almost all paths of dividend realizations* $\tilde{\omega}$, *there is a time* $\bar{t}(\tilde{\omega})$ *such that for all* $t \geq \bar{t}(\tilde{\omega})$ *the economy evolves in a deterministic cycle of period* 2 *with two states* h *and* l, *as described in proposition 5.15.*

The results of this section show that investors with short memory and a strictly decreasing similarity function diversify only for a finite number of periods, unless their aspiration level is relatively low. Note that to prove this, the assumption of a concave similarity function was not necessary. This is due to the fact that with one period memory only one utility realization at a time is observed. Since the similarity function obtains its maximum for identical problem- act pairs, the investor either retains his initially chosen portfolio (given a utility realization exceeding his aspirations) or chooses one of the corner portfolios, since they are most dissimilar to the initial one. It is possible to show that these results still hold with long memory, as long as the similarity function is concave.

5.6.4 Price Dynamics with Long Memory

Now assume that the investors can remember the whole history of the economy from time 0 on. Suppose that the similarity function of the investors is concave[115].

The result that in an economy with two types of agents only investors with relatively low aspiration level hold diversified portfolios, holds here as well. The introduction of a long memory further allows to consider learning effects. An investor who can only remember the last case realized is not able to learn much about the possible dividend and price realizations. In contrast, making observations for a long time might allow the investors to gather enough information so as to be able to choose the optimal portfolio from the point of view of the standard theory in the limit.

Denote by $\mu(\alpha \mid p)$ the expected utility from holding portfolio $\alpha \in [0;1]$ at time t, given that the price of α remains constant at $p = p_t = p_{t+1}$:

$$\mu(\alpha \mid p) = \int_{\underline{\delta}}^{\bar{\delta}} u\left(\left(1+\frac{\delta}{p}\right)\alpha + (1-\alpha)(1+r)\right).$$

[115] Since $s = f(\|(p;\alpha) - (p';\alpha')\|)$, for s to be concave, it is necessary that the decreasing function f is not too convex. To see this, denote the Euclidean distance functional by e and note that

$$s'' = f''(e')^2 + e''f'.$$

Since $e'' > 0$ and $f' < 0$,

$$-\frac{e''f'}{(e')^2} > 0$$

and $s'' < 0$ holds, as long as

$$f'' < -\frac{e''f'}{(e')^2}.$$

For instance, for $p_t = p_{t+1} = p_0$,

$$\mu(\alpha_0 \mid p_0) = \int_{\underline{\delta}}^{\bar{\delta}} u(1 + \delta + (1 - \alpha_0) r) g(\delta) d\delta$$

obtains. To avoid the discussion of multiple cases, assume that the following inequality holds:

$$\mu(\alpha_0 \mid p_0) < u(1 + r) < \mu(\alpha = 1 \mid p = 1 - \theta_1 (1 - \alpha_0)). \tag{5.59}$$

Note that as long as the investors of type 2 hold α_0 the price of α remains $p_0 = \alpha_0$. If type 2 holds $\alpha = 0$, $p = \theta_1 \alpha_0$ obtains and in the case that the choice of type 2 is $\alpha = 1$, $p = 1 - \theta_1 (1 - \alpha_0)$ is the equilibrium price of the risky asset. Since the investors of type 1 are constructed in such a way that they hold α_0 in each period, independently of how the economy evolves, the analysis concentrates on the behavior of the investors of type 2, who will determine the evolution of the asset price. Note that since now their memory consists of all observed cases, in the long run the mean of the observed utility realizations of an act determines its evaluation. Since, however, the behavior of type 2 has an influence on the market price, this mean utility shall be constructed for the respective equilibrium price which obtains, given the portfolio chosen by type 2. Should an expected utility of a portfolio be satisfactory at a constant equilibrium price, then the expected time for which this portfolio is held is infinity. Alternatively, if the expected utility of a portfolio lies below $\bar{u}^2$, then the investors of type 2 switch away from this portfolio in finite time. The first result is that the investors of type 2 only consider the utility realizations of three portfolios: α_0, $\alpha = 1$ and $\alpha = 0$. The inequality (5.59), therefore, assumes one possible ordering of the expected utilities of these three portfolios in order to avoid considering multiple cases.

Proposition 5.17 *Suppose that the probability distribution Q on $[\underline{\delta}; \bar{\delta}]$ has a density function which is continuous with respect to the Lebesgue measure and strictly bounded away from 0 on the interval $[\hat{\delta} - \zeta; \hat{\delta} + \zeta]$ for some $\zeta > 0$ and $\hat{\delta}$ such that*

$$u\left(\frac{1 - \theta_1 (1 - \alpha_0) + \hat{\delta}}{1 - \theta_1 (1 - \alpha_0)}\right) = \bar{u}^2.$$

1. If $\bar{u}^2 < \mu(\alpha_0 \mid p_0)$, the expected time during which the investors of type 2 hold α_0 is infinite.

2. If $\bar{u}^2 \in (\mu(\alpha_0 \mid p_0); u(1 + r))$, the investors of type 2 hold either $\alpha = 1$ or $\alpha = 0$ with frequency 1 almost surely in the limit.

3. If $\bar{u}^2 \in (u(1 + r); \mu(1 \mid 1 - \theta_1 (1 - \alpha_0)))$, the investors of type 2 hold $\alpha = 1$ with

frequency 1 almost surely in the limit.

4. If $\bar{u}^2 > \mu(1 \mid 1-\theta_1(1-\alpha_0))$, the investors of type 2 hold $\alpha = 1$ and $\alpha = 0$ with strictly positive frequencies almost surely in the limit, whereas the frequencies of all other acts are 0. The frequencies with which the investors of type 2 hold $\alpha = 1$ and $\alpha = 0$ are given by π_1 and π_0, respectively and satisfy:
$$\frac{\pi_1}{\pi_0} = \frac{u(1+r) - \bar{u}^2}{\mu_1^r - \bar{u}^2},$$
where μ_1^r denotes the actual mean utility derived by holding asset a as observed by the investors of type 2.

Comparing proposition 3.12 to proposition 5.17, one easily sees the analogy: if the aspiration level of the investors of type 2 is relatively low, the initially chosen portfolio is considered satisfactory. Hence, the initial allocation and price prevail infinitely long in expectations. If the investors of type 2 consider α_0 as unsatisfactory, they sooner or later switch to an undiversified portfolio and never diversify again, due to the concavity of their similarity function. Now, they have to choose between the two undiversified portfolios. If at least one of these portfolios is found to be satisfactory, then it is held forever. If, however the expected utility of none of these portfolios exceeds the aspiration level $\bar{u}^2$, then the investors of type 2 switch infinitely often between them, causing the price of a to fluctuate in a stochastic way.

Note that if the aspiration level of the investors of type 2 is appropriately chosen, i.e. if
$$\bar{u}^2 \in (u(1+r); \mu(1 \mid 1-\theta_1(1-\alpha_0))), \tag{5.60}$$
these investors learn to choose the best among the three acts α_0, $\alpha = 1$ and $\alpha = 0$, namely $\alpha = 1$ in the limit. Still, their choice might not be optimal from the point of view of an expected utility maximizer, since they only observe realizations of at most three portfolios.

Since the dynamic of the economy is predetermined solely by the behavior of the investors of type 2, it can easily be derived from proposition 5.17. The following corollary obtains:

Corollary 5.2 *Suppose that the probability distribution Q on $[\underline{\delta}; \overline{\delta}]$ has a density function which is strictly bounded away from 0 on the interval $\left[\hat{\delta} - \zeta; \hat{\delta} + \zeta\right]$ for some $\zeta > 0$ and $\hat{\delta}$ such that $u\left(\frac{1-\theta_1(1-\alpha_0)+\hat{\delta}}{1-\theta_1(1-\alpha_0)}\right) = \bar{u}^2$.*

1. Let $\bar{u}^2 < \mu(\alpha_0 \mid p_0)$. Then, the expected time which the economy spends in the state

$(\alpha^1 = \alpha_0; \alpha^2 = \alpha_0; p = p_0)$ is infinite.

2. Let $\bar{u}^2 \in (\mu(\alpha_0 \mid p_0); u(1+r))$. Then, with probability 1 in the limit, the economy remains either in state $(\alpha^1 = \alpha_0; \alpha^2 = 0; p = \theta_1 \alpha_0)$ with frequency 1 or in state $(\alpha^1 = \alpha_0; \alpha^2 = 1; p = 1 - \theta_1(1 - \alpha_0))$ with frequency 1.

3. Let $\bar{u}^2 \in (u(1+r); \mu(1 \mid 1 - \theta_1(1 - \alpha_0)))$. Then, with probability 1 in the limit, the economy remains in state $(\alpha^1 = \alpha_0; \alpha^2 = 1; p = 1 - \theta_1(1 - \alpha_0))$ with frequency 1.

4. Let $\bar{u}^2 > \mu(1 \mid 1 - \theta_1(1 - \alpha_0))$. Then, in the limit, the economy almost surely evolves according to a stochastic cycle with two states h and l, as described in proposition 5.15. The frequencies of these states satisfy:
$$\frac{\pi_h}{\pi_l} = \frac{u(1+r) - \bar{u}^2}{\mu_1^r - \bar{u}^2}.$$

The results are similar to those derived for an economy in which investors do not take similarity between acts and problems into account and in which diversification is not allowed. Investors with low aspiration levels induce stable prices and portfolio allocations. Nevertheless, the portfolios held by the investors in a stationary state need not coincide with the optimal portfolio in an economy with a representative investor, implying that the case-based decision-makers do not make optimal decisions, given the market price.

If at least some of the investors in the economy have a relatively high aspiration level, then the economy evolves according to a cycle with two states — a low-price and a high-price state. Moreover, as in the model without similarity perception, the fluctuation of the price has a greater amplitude, the higher the value of $1 - \theta_1$, i.e. the mass of the investors of type 2 in the economy.

5.7 Conclusion

In this chapter, I have analyzed the price dynamics in an economy populated by case-based decision-makers. If the investors have a short memory, the position of the highest aspiration level relative to the highest possible return of the risky asset determines the dynamics of prices and asset holdings. If the highest aspiration level is relatively low, the price of the risky asset and the holdings of the investors remain constant over the time. Still, the price of the asset might deviate significantly from its fundamental value and arbitrage possibilities may be present in the market.

Higher aspiration levels induce cycles, which may be stochastic or deterministic. The risky asset exhibits excess volatility, which depends positively on the mass of investors with high aspiration levels in the economy. The behavior of these investors is very similar to the behavior of ”overconfident” investors, described by Odean (1999): they trade too much, lowering their own profits and increasing the variance of prices. Since investors base their decisions on past information, price movements are forecastable to some extent and exhibit negative correlation in the short run.

For the case of long memory, I consider an economy with three types of investors. If the highest achievable mean utility of the risky asset exceeds both the highest aspiration level in the economy and the return of the riskless asset, then the investors with high aspiration level are able to learn this and hold the risky asset forever in the limit. However, if the mean utilities of both assets are unsatisfactory relative to the highest aspiration level, a two-state cycle emerges as in the case of short memory. This results show that even a long memory (and infinite sampling of the available acts) does not guarantee optimality of the decisions in the long run. Whereas investors with high aspiration levels fail to learn because they are dissatisfied with the returns of both assets, investors with low aspiration levels have no incentive to experiment with new acts and, thus, make suboptimal choices forever. This last fact may help explain the equity-premium puzzle observed in the financial markets, even if the investors are risk-neutral.

The consequences of the usage of hypothetical cases are not unequivocal. If one of the acts dominates all other available portfolios, then learning the utility realizations of the dominant act combined with a relatively low aspiration level leads to optimal behavior in the limit. In contrast, high aspiration levels imply that the dominated acts are chosen with strictly positive frequencies in the limit.

If no dominance relationship is present among the available acts, hypothetical reasoning might even worsen the limit choice compared to the case of completely endogenous memory.

Allowing for diversification and introducing a similarity function on problem - act / price - portfolio pairs does not change the results qualitatively. Diversified portfolios are chosen only by investors whose aspiration levels are relatively low. Investors with high aspiration levels switch to an undiversified portfolio in finite time and never diversify again. The price dynamic obtained is, therefore, very similar to the one without similarity considerations. It is, however, shown that in this case a bubble (increase of the price of the risky asset over few periods) can emerge and burst in finite time, never reappearing again.

A major criticism of the model presented in this chapter is that the economy consists only of case-based investors and that, (given the OLG structure of the model), the initial endowment does not change over time, hence, it is independent of the previous returns. These two criticisms will be addressed in the next chapter, where an economy with both case-based decision-makers and expected utility maximizers is studied. Differently from the descriptive approach taken up to now, the purpose of the last chapter is partly normative. There, the issue of evolution of decision rules (such as expected utility maximization and case-based decision-making) is examined, the question being whether case-based decision-makers can survive in a financial market in the presence of expected utility maximizers. This question is tightly connected to the issue of price dynamic analyzed in this chapter. Should it be found that the share of wealth owned by the case-based decision-makers shrinks to 0, as time evolves, the influence of case-based investors on prices and returns would become negligible. Prices and returns would behave as under rational expectations. The claim that the presence of case-based decision-makers can explain empirically observed phenomena such as bubbles, predictability of returns or arbitrage possibilities would be, therefore, unfounded. I turn to the analysis of these issues in the next chapter of this thesis.

Appendix

Proof of proposition 5.2:

From the assumptions made about the parameters, for investors with $\bar{u}^i \in [\bar{u}^a; \bar{u}^n]$ and $\alpha_0^i = b$, the cumulative utilities satisfy:

$$U_1^i(b) = 1 + r - \bar{u}^i > 0 = U_1^i(a).$$

Hence, $\alpha_1^i = b$ for $i \in [\bar{u}^a - \bar{u}^0; \bar{u}^n - \bar{u}^0]$. By induction, suppose that the investors with $\bar{u}^i \in [\bar{u}^a; \bar{u}^n]$ choose b in some period t and consider the decision of the young investors in period $(t+1)$. Since

$$U_{t+1}^i(b) = 1 + r - \bar{u}^i > 0 = U_{t+1}^i(a)$$

for all $\bar{u}^i \in [\bar{u}^a; \bar{u}^n]$, it follows that $\alpha_t^i = b$ obtains in each period of time for $i \in [\bar{u}^a - \bar{u}^0; \bar{u}^n - \bar{u}^0]$.

Now consider the investors with $\bar{u}^i \in [\bar{u}^0; \bar{u}^a]$ and $\alpha_0^i = a$. In $t = 1$, the cumulative utilities they observe satisfy:

$$U_1^i(a) = 1 + \frac{\delta_t}{\bar{u}^a - \bar{u}^0} - \bar{u}^i \geq 1 - \bar{u}^i > 0 = U_1^i(b),$$

as long as $p_1 = \bar{u}^a - \bar{u}^0$, hence as long as $\alpha_1^i = a$ for $i \in [0; \bar{u}^a - \bar{u}^0]$. By induction, $U_t^i(a) > U_t^i(b)$ holds for each $t \geq 1$ if $\alpha_{t-1}^i = a$ for all $i \in [0; \bar{u}^a - \bar{u}^0]$. Hence, in each $t \geq 1$, there is a temporary equilibrium in which

$$\begin{aligned}
\alpha_t^i &= a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \\
\alpha_t^i &= b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; \bar{u}^n - \bar{u}^0\right] \\
p_t &= \bar{u}^a - \bar{u}^0.
\end{aligned}$$

Hence, the initial state of the economy is stationary.■

Proof of proposition 5.3:

Equilibrium in $t = 1$

In $t = 1$, the investors with aspiration levels $\bar{u}^i \in [\bar{u}^a; (1+r)]$ observe cumulative utilities:

$$U_1^i(b) = 1 + r - \bar{u}^i > 0 = U_1^i(a)$$

and choose $\alpha_1^i = b$. The investors with aspiration levels $\bar{u}^i \in [(1+r); \bar{u}^n]$ observe cumulative utilities:

$$U_1^i(b) = 1 + r - \bar{u}^i < 0 = U_1^i(a)$$

and choose $\alpha_1^i = a$, regardless of the price p_t. Since their mass is $[\bar{u}^n - (1+r)]$, as long as the investors with aspiration levels $\bar{u}^i \in [\bar{u}^0; \bar{u}^a]$ choose $\alpha_1^i = a$, the cumulative utilities observed by these investors are

$$U_1(a) = \frac{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0 + \delta_t}{\bar{u}^a - \bar{u}^0} - \bar{u}^i > 1 - \bar{u}^i > 0 = U_1^i(b),$$

by assumption (5.44). Hence, the investor on the interval $[0; \bar{u}^a - \bar{u}^0]$ indeed choose $\alpha_1^i = a$ in equilibrium. Hence, in $t = 1$, state h[116]:

$$\begin{aligned}
p_1 &= p_h = \left[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right], \\
\alpha_1^i &= a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \cup \left[(1+r) - \bar{u}^0; n\right], \\
\alpha_1^i &= a \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; (1+r)\right]
\end{aligned}$$

is an equilibrium.

Equilibrium in $t = 2$

In $t = 2$, two cases must be considered:

- $\delta_2 = \delta D$

  The young investors with aspiration levels $[\bar{u}^a; (1+r)]$ observe cumulative utilities:

  $$U_2^i(b) = 1 + r - \bar{u}^i > 0 = U_2^i(a)$$

  and choose $\alpha_2^i = b$. As long as the investors with aspiration levels $\bar{u}^i \in [\bar{u}^0; \bar{u}^a] \cup [(1+r); \bar{u}^n]$ choose $\alpha_2^i = a$, they observe cumulative utilities:

  $$U_2^i(a) = 1 + \frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - \bar{u}^i > 0 = U_2^i(b),$$

  where the inequality holds by assumption (5.44). Hence, it is indeed optimal for them to choose a. It follows that in $t = 2$ with $\delta_2 = \delta D$, there is an equilibrium, in which state h obtains:

  $$\begin{aligned}
  p_2 = p_h &= \left[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right], \\
  \alpha_2^i = a \text{ for } i &\in \left[0; \bar{u}^a - \bar{u}^0\right] \cup \left[(1+r) - \bar{u}^0; n\right], \\
  \alpha_2^i = b \text{ for } i &\in \left[\bar{u}^a - \bar{u}^0; (1+r) - \bar{u}^0\right].
  \end{aligned}$$

- $\delta_2 = 0$

  The young investors with aspiration levels $[\bar{u}^a; (1+r)]$ observe cumulative utilities:

  $$U_2^i(b) = 1 + r - \bar{u}^i > 0 = U_2^i(a)$$

[116] Since the investors on the boundaries of the intervals are indifferent between holding a and b, I include them in both sets. This, of course, does not influence the price of a, since each investor has a mass of 0.

and choose $\alpha_2^i = b$.

Even if all of the young consumers with aspiration levels $[\bar{u}^0; \bar{u}^a] \cup [(1+r); \bar{u}^n]$ choose a, the cumulative utilities they observe are given by:

$$U_2^i(a) = 1 - \bar{u}^i$$
$$U_2^i(b) = 0.$$

Whereas $U_2^i(a) > U_2^i(b)$ holds for all $i \in [0; \bar{u}^a - \bar{u}^0]$, it is obviously violated for $i \in [(1+r) - \bar{u}^0; n]$, because of condition (5.44). Hence, $\alpha_2^i = b$ for all $i \in [(1+r) - \bar{u}^0; n]$. The cumulative utilities observed by the investors $i \in [0; \bar{u}^a - \bar{u}^0]$ then become:

$$U_2^i(a) = \frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - \bar{u}^i > 0 = U_2^i(b),$$

according to assumption (5.45) and these investors choose $\alpha_2^i = a$. Hence, in $t = 2$ with $\delta_2 = 0$, the economy returns to state l with

$$p_2 = p_l = \left[\bar{u}^a - \bar{u}^0\right] = p_0$$
$$\alpha_2^i = a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right],$$
$$\alpha_2^i = b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; n\right].$$

It follows by induction that if in period t the economy is in state l, the state in $(t+1)$ is h with probability 1. If in period t the state is h, then in $(t+1)$ the economy moves to state l if the dividend realization is 0, hence, with probability $(1-q)$ and stays in state h if the dividend realization is δD, or with probability q. The economy, therefore, evolves according to a Markov process with two states h and l and a transition matrix:

$$\bar{P} = \left(\begin{array}{|c|c|c|} \hline & p_{t+1} = p_h & p_{t+1} = p_l \\ \hline p_t = p_h & q & 1-q \\ \hline p_t = p_l & 1 & 0 \\ \hline \end{array}\right). \blacksquare$$

Proof of proposition 5.4:

In the proof of proposition 5.2 it was shown that the price process is a Markov process with a transition matrix:

$$\bar{P} = \left(\begin{array}{|c|c|c|} \hline & p_{t+1} = p_h & p_{t+1} = p_l \\ \hline p_t = p_h & q & 1-q \\ \hline p_t = p_l & 1 & 0 \\ \hline \end{array}\right).$$

The invariant probability distribution of this Markov process can now computed to be:

$$\begin{pmatrix} \pi_h \\ \pi_l \end{pmatrix} = \begin{pmatrix} \pi_h \\ \pi_l \end{pmatrix}' \bar{P},$$

which simplifies to

$$q\pi_h = (2 - \pi_h).$$

It follows that the invariant probabilities satisfy:

$$\pi_h = \frac{1}{2-q}$$
$$\pi_l = \frac{1-q}{2-q}.$$

These probabilities are obviously strictly positive for $q \in (0;1)$ and, therefore, the Markov chain described by $\bar{P}$ is positive recurrent. Since any positive recurrent chain on a countable space is also positive Harris recurrent, see Meyn and Tweedie (1996, p. 208), it follows that the Law of Large Numbers applies for this chain. Hence, let ι_h denote the indicator function for state h:

$$\iota_h(t) = \left\{ \begin{array}{ll} 1, & \text{if the state of the economy is } h \\ 0, & \text{if the state of the economy is } l \end{array} \right\}.$$

According to theorem 17.1.7 in Meyn and Tweedie (1996, p. 425),

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \iota_h(\tau) = \int \iota_h(\tau)\, d\pi = \pi_h$$

holds almost surely for any initial distribution over the states h and l. Since $\frac{1}{t}\sum_{\tau=1}^{t} \iota_h(\tau)$ describes the mean time up to period t that the economy spends in state h, it follows that the frequency of state h equals π_h almost surely in the limit. Analogous arguments show that the frequency of state l equals π_l on almost each path $\omega \in \Phi$.■

Proof of proposition 5.5:

Equilibrium in $t = 1$

In $t = 1$, the investors with aspiration levels $\bar{u}^i \in [\bar{u}^a; (1+r)]$ observe cumulative utilities:

$$U_1^i(b) = 1 + r - \bar{u}^i > 0 = U_1^i(a)$$

and choose $\alpha_1^i = b$. The investors with aspiration levels $\bar{u}^i \in [(1+r); \bar{u}^n]$ observe cumulative utilities:

$$U_1^i(b) = 1 + r - \bar{u}^i < 0 = U_1^i(a)$$

and choose $\alpha_1^i = a$, regardless of the price p_1. Since their mass is $[\bar{u}^n - (1+r)]$, as long as the investors with aspiration levels $\bar{u}^i \in [\bar{u}^0; \bar{u}^a]$ choose $\alpha_1^i = a$, the cumulative utilities observed by these investors are

$$U_1(a) = \frac{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0 + \delta_t}{\bar{u}^a - \bar{u}^0} - \bar{u}^i > 1 - \bar{u}^i > 0 = U_1^i(b),$$

by assumption (5.48). Hence, the investors on the interval $[0; \bar{u}^a - \bar{u}^0]$ indeed choose $\alpha_1^i = a$ in equilibrium. It follows that in $t=1$, state h:

$$\begin{aligned}
p_1 &= p_h = \left[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right], \\
\alpha_1^i &= a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \cup \left[(1+r) - \bar{u}^0; n\right], \\
\alpha_1^i &= b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; (1+r) - \bar{u}^0\right]
\end{aligned}$$

is an equilibrium.

Equilibrium in $t=2$

In period $t=2$, two cases are possible: either the dividend of the risky asset is positive or 0.

- $\delta_2 = 0$

  The young investors with aspiration levels $[\bar{u}^a; (1+r)]$ observe cumulative utilities:

  $$U_2^i(b) = 1 + r - \bar{u}^i > 0 = U_2^i(a)$$

  and choose $\alpha_2^i = b$.

  Even if all of the young consumers with aspiration levels $[\bar{u}^0; \bar{u}^a] \cup [(1+r); \bar{u}^n]$ choose a, the cumulative utilities they observe are given by:

  $$U_2^i(a) = 1 - \bar{u}^i$$
  $$U_2^i(b) = 0.$$

  Whereas $U_2^i(a) > U_2^i(b)$ holds for all $i \in [0; \bar{u}^a - \bar{u}^0]$, it is obviously violated for $i \in [(1+r) - \bar{u}^0; n]$, because of condition (5.44). Hence, $\alpha_2^i = b$ for all $i \in [(1+r) - \bar{u}^0; n]$. The cumulative utilities observed by the investors $i \in [0; \bar{u}^a - \bar{u}^0]$ then become:

  $$U_2^i(a) = \frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - \bar{u}^i > 0 = U_2^i(b),$$

  according to assumption (5.45) and these investors choose $\alpha_2^i = a$. Hence, in $t=2$ with $\delta_2 = 0$, the economy returns to state l with

  $$p_2 = p_l = \left[\bar{u}^a - \bar{u}^0\right] = p_0,$$
  $$\alpha_2^i = a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right],$$
  $$\alpha_2^i = b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; n\right].$$

- $\delta_2 = \delta D$

  The young investors with aspiration levels $[\bar{u}^a; (1+r)]$ observe cumulative utilities:

  $$U_2^i(b) = 1 + r - \bar{u}^i > 0 = U_2^i(a)$$

and choose $\alpha_2^i = b$.

Even if all of the young consumers with aspiration levels $[\bar{u}^0; \bar{u}^a] \cup [(1+r); \bar{u}^n]$ choose a, the cumulative utilities they observe are given by:

$$U_2(a) = 1 + \frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - \bar{u}^i.$$

Since

$$1 + \frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - \bar{u}^n < 0,$$

it follows that there is a subinterval of investors with aspiration levels between $[(1+r); \bar{u}^n]$ for whom

$$U_2^i(a) < U_2^i(b)$$

holds and who, therefore, choose $\alpha_2^i = b$.

I will show that if (5.45) holds, there is an equilibrium, in which all investors with aspiration levels $[(1+r); \bar{u}^n]$ choose $\alpha_2^i = b$. Indeed, suppose to the contrary that if all investors from $[(1+r) - \bar{u}^0; n]$ choose $\alpha_2^i = b$, for some of them

$$U_2^i(a) = \frac{\bar{u}^a - \bar{u}^0 + \delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - \bar{u}^i > 0 = U_2^i(b)$$

holds. This implies that

$$\frac{\bar{u}^a - \bar{u}^0 + \delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - (1+r) > 0,$$

since $(1+r)$ is the lowest aspiration level in this interval. Hence,

$$\frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} > (1+r) - \frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0}. \tag{5.61}$$

On the other hand, the first inequality of condition (5.48) implies that

$$\frac{\delta D}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} < \bar{u}^n - 1, \tag{5.62}$$

whereas condition (5.45) requires

$$\bar{u}^a - \bar{u}^0 > \bar{u}^a \left(\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right)$$

and since $\bar{u}^a - \bar{u}^0 \leq \bar{u}^a$ for $\bar{u}^0 \geq 0$, this means that

$$\left(\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right) \leq 1,$$

or

$$2 + r - \bar{u}^n - \bar{u}^a + \bar{u}^0 \geq 0$$

must hold.

Now compare the left hand sides of (5.61) and (5.62). It is easy to see that since

$$\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0 > 0,$$

$$(1+r)-\frac{\bar{u}^a-\bar{u}^0}{\bar{u}^n-(1+r)+\bar{u}^a-\bar{u}^0}\geq\bar{u}^n-1$$

is equivalent to

$$\left[\bar{u}^n-(1+r)\right]\left[2+r-\bar{u}^n-\bar{u}^a+\bar{u}^0\right]\geq 0,$$

which is always satisfied, as long as (5.48) and (5.45) hold. Hence, δD cannot satisfy (5.61) and (5.62) simultaneously and, therefore,

$$\frac{\bar{u}^a-\bar{u}^0+\delta D}{\bar{u}^n-(1+r)+\bar{u}^a-\bar{u}^0}\leq(1+r)$$

obtains. But then

$$U_2^i(a)=\frac{\bar{u}^a-\bar{u}^0+\delta D}{\bar{u}^n-(1+r)+\bar{u}^a-\bar{u}^0}-\bar{u}^i\leq 0=U_2^i(b)$$

holds for all $\bar{u}^i\in[(1+r);\bar{u}^n]$. Hence, it is optimal for the investors $i\in[(1+r)-\bar{u}^0;n]$ to choose $\alpha_2^i=b$, whereas $\alpha_2^i=a$ for all $i\in[0;\bar{u}^a-\bar{u}^0]$ holds. The equilibrium price is then given by $p_2=p_l=[\bar{u}^a-\bar{u}^0]$.

Since the state of the economy in $t=2$ coincides with the state in $t=0$, independently of the dividend payment, it follows by induction that the two states h and l defined in the proposition indeed determine a deterministic cycle of the economy.■

Proof of proposition 5.6:

Since the investors of type 1 and type 2 do not change their asset holdings over time, there are three cases to consider:

1. Let $\alpha_{t-1}^3=a$ and $\delta_t=\delta D$. From $\alpha_{t-1}^3=a$, it follows that:

$$U_{t-1}^3(a)\geq U_{t-1}^3(b).$$

From $\delta_t=\delta D$, it follows that the return of a if $\alpha_t^3=\alpha$, is

$$1+\frac{\delta D}{\bar{u}^n-(1+r)+\bar{u}^a-\bar{u}^0}>\bar{u}^3,$$

by assumption (5.49). It follows that if $\alpha_t^3=a$, then

$$U_t^3(a)=U_{t-1}^3(a)+1+\frac{\delta D}{\bar{u}^n-(1+r)+\bar{u}^a-\bar{u}^0}-\bar{u}^3>U_{t-1}^3(a)\geq U_{t-1}^3(b)=U_t^3(b).$$

Hence, there exists an equilibrium in which:

$$\alpha_t^i=\alpha_h^i=a \text{ for } i\in\left[0;\bar{u}^a-\bar{u}^0\right]\cup\left[(1+r)-\bar{u}^0;n\right]$$

$$\alpha_t^i=\alpha_h^i=b \text{ for } i\in\left[\bar{u}^a-\bar{u}^0;(1+r)-\bar{u}^0\right]$$

$$p_t=p_h=\left[\bar{u}^n-(1+r)+\bar{u}^a-\bar{u}^0\right].$$

2. Let $\alpha_{t-1}^3 = a$ and $\delta_t = 0$. From $\alpha_{t-1}^3 = a$, it follows that:
$$U_{t-1}^3(a) \geq U_{t-1}^3(b).$$
From $\delta_t = 0$, it follows that the return of a if $\alpha_t^3 = a$, is
$$1 < \bar{u}^3,$$
by assumption (5.49).

– If
$$U_t^3(a) = U_{t-1}^3(a) + 1 - \bar{u}^3 > U_{t-1}^3(b) = U_t^3(b),$$
then $\alpha_t^3 = a$ has the maximal cumulative utility for the investors of type 3 and is chosen again. Hence,
$$\alpha_t^i = \alpha_h^i = a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \cup \left[(1+r) - \bar{u}^0; n\right]$$
$$\alpha_t^i = \alpha_h^i = b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; (1+r) - \bar{u}^0\right]$$
$$p_t = p_h = \left[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right]$$
is an equilibrium.

– If
$$U_t^3(a) = U_{t-1}^3(a) + 1 - \bar{u}^3 = U_{t-1}^3(b) = U_t^3(b),$$
then a and b have the same cumulative utilities. Hence,
$$\alpha_t^i = \alpha_h^i = a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right] \cup \left[(1+r) - \bar{u}^0; n\right]$$
$$\alpha_t^i = \alpha_h^i = b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; (1+r) - \bar{u}^0\right]$$
$$p_t = p_h = \left[\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0\right]$$
is again an equilibrium, but
$$\alpha_l^i = a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right]$$
$$\alpha_l^i = b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; n\right]$$
$$p_l = \left[\bar{u}^a - \bar{u}^0\right]$$
is also an equilibrium, since for $\alpha_t^3 = b$, the return of a becomes
$$\frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} < 1 < \bar{u}^3$$
and the cumulative utility of a is smaller than those of b for type 3.

– If
$$U_t^3(a) = U_{t-1}^3(a) + 1 - \bar{u}^3 < U_{t-1}^3(b) = U_t^3(b),$$
then the cumulative utility of a is smaller than those of b for type 3 even if they choose a at t. Hence, $\alpha_t^3 = b$. The return of a then becomes:
$$\frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} < 1 < \bar{u}^3$$
and since
$$U_t^3(a) = U_{t-1}^3(a) + \frac{\bar{u}^a - \bar{u}^0}{\bar{u}^n - (1+r) + \bar{u}^a - \bar{u}^0} - \bar{u}^3 < U_{t-1}^3(b) = U_t^3(b),$$
it follows that
$$\alpha_l^i = a \text{ for } i \in \left[0; \bar{u}^a - \bar{u}^0\right]$$

$$\alpha_l^i = b \text{ for } i \in \left[\bar{u}^a - \bar{u}^0; n\right]$$
$$p_l = \left[\bar{u}^a - \bar{u}^0\right]$$

is an equilibrium at time t.

3. Let $\alpha_{t-1}^3 = b$. This means that
$$U_{t-1}^3(a) \leq U_{t-1}^3(b).$$
   - If
   $$U_t^3(b) = U_{t-1}^3(b) + 1 + r - \bar{u}^3 > U_{t-1}^3(a) = U_t^3(a),$$
   then $\alpha_t^3 = b$ is the choice of type 3 and the equilibrium is state l.
   - If
   $$U_t^3(b) = U_{t-1}^3(b) + 1 + r - \bar{u}^3 = U_{t-1}^3(a) = U_t^3(a),$$
   then type 3 is indifferent between a and b and both states h and l are equilibria.
   - If
   $$U_t^3(b) = U_{t-1}^3(b) + 1 + r - \bar{u}^3 < U_{t-1}^3(a) = U_t^3(a),$$
   then $\alpha_t^3 = a$ is the choice of type 3 and the equilibrium is state h.■

Proof of proposition 5.7:

Consider the Markov chain given by:

$$\begin{aligned}
\varepsilon_1 &= \bar{u}^3 - (1+r) \\
\varepsilon_{t+1} &= \varepsilon_t + c, \text{ w.p. } q \text{ if } \varepsilon_t \geq 0 \\
\varepsilon_{t+1} &= \varepsilon_t - c, \text{ w.p. } 1-q \text{ if } \varepsilon_t \geq c \\
\varepsilon_{t+1} &= \varepsilon_t - k\left(\bar{u}^3 - (1+r)\right), \text{ w.p. } 1-q \text{ if } 0 \leq \varepsilon_t < c \\
\varepsilon_{t+1} &= \varepsilon_t + \bar{u}^3 - (1+r), \text{ w.p. } 1 \text{ if } \varepsilon_t < 0.
\end{aligned}$$

Note that
$$\varepsilon_t = U_t^3(a) - U_t^3(b),$$
where it is implicitly assumed that the investors of type 3 hold asset a when indifferent between the two acts. Since
$$\varepsilon_1 = \bar{u}^3 - (1+r) > 0,$$
at $t = 1$, $\alpha_1^3 = a$ is chosen. a is held as long by the investors of type 3 as $\varepsilon_t \geq 0$ holds. Note that once $\varepsilon_t = \bar{u}^3 - (1+r)$ obtains, with probability $(1-q)$ in the next period the cumulative

utility of a becomes

$$U_{t+1}^{3}(a)=\bar{u}^{3}-(1+r)-k\left(\bar{u}^{3}-(1+r)\right)=(k-1)\left(\bar{u}^{3}-(1+r)\right),$$

since

$$\bar{u}^{3}-(1+r)<\bar{u}^{3}-1=c$$

by assumption $(A7)$. It follows that $\alpha_{t+1}^{3}=b$ and b is held by the investors of type 3 for exactly $(k-1)$ periods. Consider, therefore, period $t+k$ and note that $\varepsilon_{t+k}=0$ and, therefore, $\alpha_{t+k}^{3}=a$. As long as $\varepsilon_{\tau}\geq 0$ holds, a is chosen by the investors of type 3. When $\varepsilon_{\tau}<0$ obtains for the first time

$$\varepsilon_{\tau}=-k\left(\bar{u}^{3}-(1+r)\right),$$

hence b is chosen and held for exactly k periods in a row. At $\tau+k$, $\varepsilon_{\tau+k}=0$ obtains again and the same process repeats.

Whereas, $(A8)$ guarantees that b is held for axactly k periods in a row every time it is chosen by the investors of type 3, the time during which a is held is random[117]. In fact, denote by $\tilde{\varepsilon}_{t}$ the following process

$$\begin{aligned}\tilde{\varepsilon}_{t+k}&=\varepsilon_{t+k}=0\\ \tilde{\varepsilon}_{\tau}&=\tilde{\varepsilon}_{\tau-1}+c\text{ w.p. }q\\ \tilde{\varepsilon}_{\tau}&=\tilde{\varepsilon}_{\tau-1}-c\text{ w.p. }1-q.\end{aligned}$$

$\tilde{\varepsilon}_{t}$ is a random walk with a step length c. Moreover, as long as $\tilde{\varepsilon}_{\tau}\geq 0$ holds, $\tilde{\varepsilon}_{\tau}=\varepsilon_{\tau}$ holds, whereas in the first period at which $\tilde{\varepsilon}_{\tau}=-c$,

$$\varepsilon_{\tau}=-k\left(\bar{u}^{3}-(1+r)\right)$$

obtains. It follows that the expected time during which a is held is equal to the expected time needed by $\tilde{\varepsilon}$ to reach $-c$ for the first time starting at 0.

Now, I compute the expected time during which a is held in a row denoted by t. The analysis above shows that

$$E[t]=E[\min\{\tau\mid\tilde{\varepsilon}_{\tau}=-c\}]$$

It is a well-known result of the probability theory that the generating function of the time that a

117 Although it has been shown that b is held only for $(k-1)$ periods in a row after the first switch of the investors of type 3 from a to b at time $(t+1)$, this is an exception which only occurs once. Note, as well that because of $\alpha_{0}^{3}=b$, b is chosen during exactly k periods up to time $(t+k)$.

simple random walk needs to reach the point -1 for the first time is given by:

$$F_{-1}(s) = \left[\frac{1-(1-4q(1-q)s^2)^{\frac{1}{2}}}{2qs}\right],$$

see Grimmet and Stirzaker (1994, p.145). Differentiating $F_{-1}(s)$ with respect to s and taking the derivative at $s=1$, gives the expected value of the time the random walk needs to reach -1 for the first time:

$$F_{-1}^{'}(s=1) = \frac{1-\sqrt{(1-2q)^2}}{2q\sqrt{(1-2q)^2}}, \tag{5.63}$$

see Grimmet and Stirzaker (1994, p. 130). Since the time needed by $\tilde{\varepsilon}_\tau$ with a step-length of c to reach $-c$ for the first time is equal to the time needed for a simple random walk to reach -1 for the first time, the expected value of t is given by (5.63) if $\Pr\{t=\infty\}=0$. It remains to state the condition, for which $\Pr\{t=\infty\}=0$ holds. Using corollary (6) in Grimmet and Stirzaker (1994, p. 144) and the reflection principle, one finds that the probability that the random walk ever visits the negative part of the real axis is:

$$\min\left\{1;\frac{1-q}{q}\right\} = \left\{\begin{array}{ll} 1 & \text{if } q \leq \frac{1}{2} \\ \frac{1-q}{q} < 1 & \text{if } q > \frac{1}{2} \end{array}\right\}.$$

Hence, the probability that the random walk never visits the negative part of the real axis (and therefore never reaches $-c$) is positive if $q > \frac{1}{2}$ and in this case $E[t] = \infty$. If $q \leq \frac{1}{2}$, then the probability to reach $-c$ in finite time is 1 and the expected time until the first such visit is:

$$E[t] = \left\{\begin{array}{ll} \frac{1}{(1-2q)}, & \text{if } q < \frac{1}{2} \\ \infty, & \text{if } q = \frac{1}{2} \end{array}\right\}. \blacksquare$$

Proof of corollary 5.1:

Consider again the Markov chain given by:

$$\begin{aligned}
\varepsilon_1 &= \bar{u}^3 - (1+r) \\
\varepsilon_{t+1} &= \varepsilon_t + c, \text{ w.p. } q \text{ if } \varepsilon_t \geq 0 \\
\varepsilon_{t+1} &= \varepsilon_t - c, \text{ w.p. } 1-q \text{ if } \varepsilon_t \geq c \\
\varepsilon_{t+1} &= \varepsilon_t - k\left(\bar{u}^3 - (1+r)\right), \text{ w.p. } 1-q \text{ if } 0 \leq \varepsilon_t < c \\
\varepsilon_{t+1} &= \varepsilon_t + \bar{u}^3 - (1+r), \text{ w.p. } 1 \text{ if } \varepsilon_t < 0.
\end{aligned}$$

As in the proof of proposition 5.7,

$$\varepsilon_t = U_t^3(a) - U_t^3(b)$$

and ε_t evolves on the countable space

$$\Psi = \left\{-k\left(\bar{u}^3 - 1 + r\right); -(k-1)\left(\bar{u}^3 - 1 + r\right) \ldots - \left(\bar{u}^3 - 1 + r\right); 0; c; 2c\ldots\right\}.$$

The state h coincides with $\varepsilon_t \geq 0$, whereas the state l is represented by $\varepsilon_t < 0$. The chain is irreducible, since for each x and $y \in \Psi$ there is a positive probability to reach x starting from y and y, starting from x. To see this, consider the following cases:

1. $x \geq y \geq 0$. To reach x from y, the chain must make $\frac{x-y}{c}$ steps upwards in a row, which happens with probability $q^{\frac{x-y}{c}} > 0$. To reach y starting from x, the chain must make $\frac{x-y}{c}$ steps downwards in a row, which happens with probability $(1-q)^{\frac{x-y}{c}} > 0$.

2. $x \geq 0 > y$. To reach x starting from y, the chain must first reach 0, which happens with probability 1 and then make $\frac{x}{c}$ steps upward in a row, which happens with probability $q^{\frac{x}{c}} > 0$. To reach y starting from x, the system must first reach 0, hence make $\frac{x}{c}$ steps downwards in a row, which happens with probability of at least $q^{\frac{x}{c}} > 0$ and then make $\frac{k\left(\bar{u}^3-1-r\right)-y}{\bar{u}^3-1-r}$ upward steps in a row, which happens with probability 1.

3. $0 > x \geq y$. To reach x starting from y, the chain must make $\frac{x-y}{\bar{u}^3-1-r}$ upward steps in a row, which happens with probability 1. To reach y starting from x, the chain must first reach 0, which happens with probability 1 in $\frac{-x}{\bar{u}^3-1-r}$ steps, must then jump down to $-k\left(\bar{u}^3-1+r\right)$, which happens with probability $(1-q)$ and then make $\frac{k\left(\bar{u}^3-1-r\right)-y}{\bar{u}^3-1-r}$ upward steps in a row, which also happens with probability 1.

In all these cases, states x and y are connected and, therefore, the Markov chain is irreducible.

Suppose first that $q < \frac{1}{2}$. As observed in the proof of proposition 5.7, ε_t restricted to $[0;\infty)$ behaves as a random walk on the half-line as described by the process $\tilde{\varepsilon}_t$. Hence, according to Meyn and Tweedie (1996, pp. 184-185) 0 is reached with probability 1 starting from each $x \geq 0$ if $q < \frac{1}{2}$. On the other hand, starting from $x < 0$, 0 is reached in at most k steps with probability 1. Hence, 0 is reached with probability 1 from any initial state and is, therefore, a recurrent state. But from the properties of irreducible Markov chains on countable spaces, we know that if one state is recurrent, then so are all states of the chain, see Meyn and Tweedie (1996, p. 182). Moreover, since the Markov chain defined by ε_t contains an accessible atom, 0, it follows that the chain is positive recurrent, see theorem 10.2.1 in Meyn and Tweedie (1996, p. 242). Hence, there exist positive probabilities π_h and π_l describing the invariant probabilities with which $\varepsilon_t \geq 0$ and $\varepsilon_t < 0$ obtain, respectively. Since the economy is in states h for $\varepsilon_t \geq 0$

and in state l for $\varepsilon_t < 0$, π_h and π_l equal the frequencies of these states. Furthermore, the Law of Large Numbers applies, see theorem 17.1.7 in Meyn and Tweedie (1996, p. 425) and in the limit, π_h and π_l almost surely equal the mean time during which the investors of type 3 hold a and b, respectively. Hence, it follows from the results of proposition 5.7 that

$$\frac{\pi_h}{\pi_l} = \frac{1}{k\,(1-2q)}$$

holds almost surely in the limit. Combined with $\pi_h + \pi_l = 1$, this implies

$$\pi_h = \frac{1}{1 + k\,(1-2q)}$$

and

$$\pi_l = \frac{k\,(1-2q)}{1 + k\,(1-2q)}.$$

For $q \leq \frac{1}{2}$, according to the proof of proposition 5.7, there is a positive probability that starting from 0, $\tilde{\varepsilon}_t$ and, hence, also ε_t never crosses 0 from above. Since b is held for exactly k periods in a row, in order to spend a positive proportion of time taking on values below 0, ε_t must cross 0 from above for an infinite number of times. However, this event has a probability of 0. Hence, $\varepsilon_t < 0$ obtains only for a finite number of periods. It follows that the mean proportion of time the economy spends in state h, is 1.■

Proof of proposition 5.8:

Since the investors of type 1 and type 2 do not change their holdings over time, there are three cases to consider:

1. Let $\alpha_{t-1}^3 = a$ and $\delta_t \geq \tilde{\delta}$. From $\alpha_{t-1}^3 = a$ it follows that:

$$U_{t-1}^3\,(a) \geq U_{t-1}^3\,(b)\,.$$

From $\delta_t \geq \tilde{\delta}$, it follows by assumption (5.54) that the return of a is satisfactory for type 3 if $\alpha_t^3 = \alpha$:

$$u\left(1 + \frac{\delta_t}{1-\theta_2}\right) \geq \bar{u}^3.$$

Hence if $\alpha_t^3 = a$,

$$U_t^3\,(a) = U_{t-1}^3\,(a) + u\left(1 + \frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3 \geq U_{t-1}^3\,(a) \geq U_{t-1}^3\,(b) = U_t^3\,(b)$$

holds and therefore, there exists an equilibrium in which:

$$\alpha_t^i = \alpha_h^i = a \text{ for } i \in \{1;3\}$$

$$\alpha_t^i = \alpha_h^i = b \text{ for } i = 2$$

$$p_t = p_h = 1 - \theta_2.$$

2. Let $\alpha^3_{t-1} = a$ and $\delta_t < \tilde{\delta}$. From $\alpha^3_{t-1} = a,$ it follows that:

$$U^3_{t-1}(a) \geq U^3_{t-1}(b).$$

From $\delta_t < \tilde{\delta}$, it follows by assumption (5.49) that the utility derived from a is

$$u\left(1 + \frac{\delta_t}{1-\theta_2}\right) < \bar{u}^3$$

if $\alpha^3_t = a$.

– If

$$U^3_t(a) = U^3_{t-1}(a) + u\left(1 + \frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3 > U^3_{t-1}(b) = U^3_t(b),$$

then a has the maximal cumulative utility for the investors of type 3 and is chosen again, $\alpha^3_t = a$. Hence,

$$\alpha^i_t = \alpha^i_h = a \text{ for } i \in \{1;3\}$$
$$\alpha^i_t = \alpha^i_h = b \text{ for } i = 2$$
$$p_t = p_h = 1 - \theta_2$$

is an equilibrium.

– If

$$U^3_t(a) = U^3_{t-1}(a) + u\left(1 + \frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3 = U^3_{t-1}(b) = U^3_t(b),$$

then a and b have the same cumulative utilities for the young investors of type 3. Hence,

$$\alpha^i_t = \alpha^i_h = a \text{ for } i \in \{1;3\}$$
$$\alpha^i_t = \alpha^i_h = b \text{ for } i = 2$$
$$p_t = p_h = 1 - \theta_2.$$

is again an equilibrium, but

$$\alpha^i_t = \alpha^i_l = a \text{ for } i = 1$$
$$\alpha^i_t = \alpha^i_l = b \text{ for } i \in \{2;3\}$$
$$p_l = \theta_1$$

is also an equilibrium. To see this, note that the return of a is

$$u\left(\frac{\theta_1 + \delta_t}{1-\theta_2}\right) < u\left(1 + \frac{\delta_t}{1-\theta_2}\right) < \bar{u}^3$$

if $\alpha^3_t = b$. In this case, the cumulative utility of a is smaller than those of b for type 3.

– If

$$U^3_t(a) = U^3_{t-1}(a) + u\left(1 + \frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3 < U^3_{t-1}(b) = U^3_t(b),$$

then the cumulative utility of a is smaller than those of b for the investors of type 3 even if they choose a at t. Hence, $\alpha^3_t = b$ and the utility realization of a is

$$u\left(\frac{\theta_1 + \delta_t}{1-\theta_2}\right) < \bar{u}^3.$$

Since then

$$U_t^3(a) = U_{t-1}^3(a) + u\left(\frac{\theta_1+\delta_t}{1-\theta_2}\right) - \bar{u}^3 < U_{t-1}^3(b) = U_t^3(b),$$

it follows that

$$\begin{aligned} \alpha_t^i = \alpha_l^i = a \text{ for } i = 1 \\ \alpha_t^i = \alpha_l^i = b \text{ for } i \in \{2;3\} \\ p_l = \theta_1 \end{aligned}$$

is an equilibrium at time t.

3. Let $\alpha_{t-1}^3 = b$. This means that

$$U_{t-1}^3(a) \leq U_{t-1}^3(b).$$

– If
$$U_t^3(b) = U_{t-1}^3(b) + u(1+r) - \bar{u}^3 > U_{t-1}^3(a) = U_t^3(a),$$
then $\alpha_t^3 = b$ is the choice of type 3 and the equilibrium is state l.

– If
$$U_t^3(b) = U_{t-1}^3(b) + u(1+r) - \bar{u}^3 = U_{t-1}^3(a) = U_t^3(a),$$
then type 3 is indifferent between a and b and both states h and l are equilibria.

– If
$$U_t^3(b) = U_{t-1}^3(b) + 1 + r - \bar{u}^3 < U_{t-1}^3(a) = U_t^3(a),$$
then $\alpha_t^3 = a$ is the choice of type 3 and the equilibrium is state h.■

Proof of proposition 5.9:

Since ε_t is defined as:

$$\begin{aligned} \varepsilon_0 &= 0 \\ \varepsilon_t &= \left\{ \begin{array}{ll} \varepsilon_{t-1} + u\left(1+\frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(1+\frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3 \geq 0 \\ \varepsilon_{t-1} + u\left(\frac{\theta_1+\delta_t}{1-\theta_2}\right) - \bar{u}^3, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(1+\frac{\delta_t}{1-\theta_2}\right) - \bar{u}^3 < 0 \\ \varepsilon_{t-1} - u(1+r) + \bar{u}^3, & \text{if } \varepsilon_{t-1} < 0. \end{array} \right\}, \end{aligned}$$

ε_t, $t \geq 1$, describes the evolution of the difference between the cumulative utilities of a and b for the investors of type 3:

$$\varepsilon_t = U_t^3(a) - U_t^3(b)$$

and it is obvious that ε_t is a Markov chain, since δ_t is identically and independently distributed according to Q. Moreover, ε_t evolves on

$$\Psi' = \left[u\left(\frac{\theta_1+\underline{\delta}}{1-\theta_2}\right) - \bar{u}^3; +\infty\right),$$

since the greatest amount by which the cumulative utility of b can exceed the cumulative utility of a is $\bar{u}^3 - u\left(\frac{\theta_1+\underline{\delta}}{1-\theta_2}\right)$, whereas the cumulative utility of a can become very large, if $\delta > \tilde{\delta}$ occurs for a long period of time. Denote by P the transition probability kernel of ε_t. The idea of the proof consists in showing that ε_t is a stationary process with an invariant probability measure π, as defined in the statement of the proposition. Since for positive ε_t the investors of type 3 choose asset a, whereas for negative ε_t, they choose b, the frequency with which a and b are chosen in the limit coincide with

$$\pi\,[0;+\infty)$$

and

$$\pi\left[u\left(\frac{\theta_1+\delta}{1-\theta_2}\right)-\bar{u}^3;0\right),$$

respectively.

Denote by G the interval $[0;\bar{u}^3-u\,(1+r)]$. The following Lemma shows that the set G is a small set, i.e. that there exists a measure ν on the set

$$\Psi'=\left[u\left(\frac{\theta_1+\delta}{1-\theta_2}\right)-\bar{u}^3;+\infty\right)$$

such that

$$P^K\,(\varepsilon;F)\geq\nu\,(F)$$

for any set $F\in\Psi'$ and any $\varepsilon\in G$, where $P^K\,(\varepsilon;F)$ denotes the probability to reach a set F starting from ε in K steps, see Meyn and Tweedie (1996, p. 111).

Lemma 5.1 *The set* $G=[0;\bar{u}^3-u\,(1+r)]$ *is small.*

Proof of lemma 5.1:

The assumption about the probability distribution of δ and the continuity of the utility function $u\,(\cdot)$ implies that the net utility realizations

$$\tilde{u}=u\left(1+\frac{\delta}{1-\theta_2}\right)-\bar{u}^3$$

of a (as long as its cumulative utility remains positive) are distributed according to a probability distribution Q', such that Q' has an absolutely continuous part with respect to the Lebesgue measure on the real numbers. Moreover, there is a number ζ', such that the density of $\tilde{u}$ g' is bounded away from 0 on an interval $(-\zeta';\zeta')$ for some ζ' satisfying $\bar{u}^3-u\,(1+r)>\zeta'>0$, i.e.:

$$g'\,(\tilde{u})\geq\phi'>0$$

for all $\tilde{u}\in(-\zeta';\zeta')$ and for some ϕ'.

Divide the set G into K sets, $G_1...G_K$ with length less than $\frac{\zeta'}{2}$. Fix an $\varepsilon \in G_i$ and suppose that $F \subset G_j$. Now, for each $0 < \xi < \frac{\zeta'}{2}$, there is a positive probability $P_{\xi-}$ that

$$\varepsilon_{t+1} \in \left(\varepsilon_t + \frac{\zeta'}{2} - \xi; \varepsilon_t + \frac{\zeta'}{2}\right)$$

and a positive probability P_{ξ^+} that

$$\varepsilon_{t+1} \in \left(\varepsilon_t - \frac{\zeta'}{2}; \varepsilon_t - \frac{\zeta'}{2} + \xi\right)$$

Moreover, because of the assumptions made on the probability distribution Q, these probabilities are bounded away from 0:

$$\begin{aligned} P_{\xi^+} &\geq \phi'\xi \\ P_{\xi^-} &\geq \phi'\xi. \end{aligned}$$

Now choose ξ such that $\xi(K-1) \leq \frac{\zeta'}{2}$ holds. It follows that after $(K-1)$ steps the process ε_t will be at a distance of at most ζ' away from the set G_j, of which F is a subset with probability of at least

$$[\phi'\xi]^{K-1}.$$

Therefore, at step K, there is a positive probability of at least:

$$P(\varepsilon_{t+K-1}; F) = P(\tilde{u} \in (F - \varepsilon_{t+K-1})) = \int_{F-\varepsilon_{t+K-1}} g'(\tilde{u})\, d\tilde{u} \geq \phi'\mu^{Leb}(F).$$

Hence, the probability that set $F \subset G_j$ is reached after K steps starting at some ε_t is at least

$$P^K(\varepsilon_t; F) \geq [\phi'\xi]^{K-1} \phi'\mu^{Leb}(F) =: \nu(F),$$

where ν is absolutely continuous with respect to the Lebesgue measure on the interval G. The probability that a set $F \subset G$ which is not a subset of any G_j is reached in K steps fulfills

$$P^K(\varepsilon_t; F) = \sum_{i=1}^{K} P^K(\varepsilon_t; F_i) \geq [\phi'\xi]^{K-1} \phi' \sum_{i=1}^{K} \mu^{Leb}(F_i) = [\phi'\xi]^{K-1} \phi'\mu^{Leb}(F),$$

where $\cup_{i=1}^{K} F_i = F$ and $F_i \subset G_i$, i.e. F_i is a partition of F into sets each of which is a (possibly empty) subset of some G_i. Since each set outside G is reached with a non-negative probability starting from G, it follows that the set G is a small set and the measure $\nu(F)$ is defined as

$$\begin{aligned} \nu(F) &= [\phi'\xi]^{K-1} \phi'\mu^{Leb}(F),\ F \subset G \\ \nu(F) &= 0, \text{ else.} \end{aligned}$$

Moreover, according to proposition 5.5.3 in Meyn and Tweedie (1996, p. 127), since each small set is a petite set, G is a petite set.■

The next Lemma demonstrates that the Markov chain defined by ε_t is φ-irreducible. φ-irreducibility is an analogue to the concept of irreducibility of Markov chains on countable sets, defined for Markov chains on general sets. It defines a measure φ, which assigns a strictly positive value only to subsets of the set Ψ' which are reached with strictly positive probability from every initial point ε_t, see Meyn and Tweedie (1996, p. 91).

Lemma 5.2 *Let φ be defined as the Lebesgue measure on the set $[0; \bar{u}^3 - u\,(1+r)]$ and be 0 elsewhere. Then the Markov chain ε is φ-irreducible.*

Proof of lemma 5.2:

Obviously, φ assigns a positive probability only to subsets of the interval G. The statement of the lemma is therefore true if it can be shown that each of the subsets of this interval is reached with positive probability from any initial point. Since it has been shown that starting from any point in the interval G, any subset of G is reached with positive probability, it remains to demonstrate that starting outside the interval $[0; \bar{u}^3 - u\,(1+r)]$, a subset of this interval is reached with positive probability. Consider two cases: if $\varepsilon_t < 0$, then ε_t grows by $\bar{u}^3 - u\,(1+r)$ in each period, until $\varepsilon_{t+k} \geq 0$ obtains for the first time. But at time $(t+k)$ $\varepsilon_{t+k} \in [0; \bar{u}^3 - u\,(1+r)]$, hence the interval G is reached with probability 1, starting from a negative ε_t. If $\varepsilon_t > \bar{u}^3 - u\,(1+r)$ holds, then there is a positive probability that the next $\left\lceil 2\frac{\varepsilon_t}{\zeta'} \right\rceil$ steps are negative with realizations between $\left(-\frac{\zeta'}{2}; -\frac{\eta\zeta'}{2}\right)$ with $\left\lceil 2\frac{\varepsilon_t}{\zeta'} \right\rceil \frac{\eta\zeta'}{2} < \bar{u}^3 - u\,(1+r)$, (hence, $\eta < 1$) and, therefore, the subset $\left[0; \left\lceil 2\frac{\varepsilon_t}{\zeta'} \right\rceil \frac{\eta\zeta'}{2}\right]$ of G is reached with strictly positive probability in finite time from any initially chosen ε_t. Therefore, the Markov chain is φ-irreducible.

Since φ is finite, it follows according to proposition 4.2.2 in Meyn and Tweedie (1996, p. 92) that there exists a probability measure ψ on Ψ', which assigns a probability of 0 to a subset F of Ψ' if and only if

$$\psi\left(\varepsilon \mid \sum_{n=1}^{\infty} P^n(\varepsilon; F) > 0\right) = 0.$$

ψ is absolutely continuous with respect to φ, hence if $\varphi(F) > 0$, then $\psi(F) > 0$ holds as well. Denote by $\mathcal{B}(\Psi')$ the Borel σ-algebra on Ψ'. Let $\mathcal{B}^+(\Psi')$ denote the subset $\mathcal{B}(\Psi')$, whose elements are assigned a strictly positive probability according to ψ:

$$\mathcal{B}^+(\Psi') = \{F \in \mathcal{B}(\Psi') \mid \psi(F) > 0\}.$$

Note that the petite set G satisfies $G \in \mathcal{B}^+(\Psi')$.■

Part (ii) of theorem 10.4.10 in Meyn and Tweedie (1996, p. 254) combines the notion of petite set and irreducibility of a Markov chain with the notion of positive recurrency, which assures the existence of an invariant probability distribution π, as required in proposition 5.9:

Proposition 5.18 *Suppose that a Markov chain is ψ-irreducible. Let τ_G denote the first hitting time of the set G. The chain is positive recurrent, if for some petite set $G \in \mathcal{B}^+(\Psi')$*

$$\sup_{\varepsilon \in G} E_\varepsilon [\tau_G] < \infty.$$

Proof of proposition 5.18:

See Meyn and Tweedie (1996, p. 254).■

It has already been shown that the chain defined by ε_t is ψ-irreducible and that G is a petite set with $\psi(G) > 0$ (since $\varphi(G) > 0$). It remains, therefore, to show that the expected hitting time of the set G, starting from G, is bounded from above. To demonstrate this note that the process ε_t, constrained to its positive part, is a random walk on a half line with negative expected increment. Proposition 11.4.1 in Meyn and Tweedie (1996, p. 278) demonstrates that for such a random walk all compact sets are regular. A regular set F' has the property that

$$\sup_{\varepsilon \in F'} E_\varepsilon [\tau_F] < \infty$$

for all $F \in \mathcal{B}^+(\Psi')$, see Meyn and Tweedie (1996, p. 263). Since $G \in \mathcal{B}^+(\Psi')$, it follows that

$$\sup_{\varepsilon \in G} E_\varepsilon [\tau_G] < \infty$$

holds for the process ε_t reduced to a random walk on the half line. Moreover, since all compact sets are regular, it follows that

$$\sup_{\varepsilon \in F} E_\varepsilon [\tau_G] < \infty$$

holds for all compact sets $F \subset [0; +\infty)$.

Now, consider the unconstrained process ε_t. There are two possibilities: either ε_t remains non-negative forever and in this case it behaves like a random walk on the half line and, therefore, G is regular, or ε_t eventually becomes negative. If $\varepsilon_t < 0$ at some t, then the expected time in which ε reaches G is at most

$$\frac{\bar{u}^3 - u\left(\frac{\theta_1 + \delta}{1-\theta_2}\right)}{\bar{u}^3 - u(1+r)},$$

which is finite. Therefore, the expected time that the process needs to reach the set G starting from set G is bounded from above. But then the condition of proposition 5.18 is satisfied and the Markov chain defined by ε_t is positive recurrent. Hence, there exists an invariant probability measure π for the process ε_t, see Theorem 10.0.1 in Meyn and Tweedie (1996, p. 238).

Moreover, since

$$\sup_{\varepsilon \in \Psi'} E_{\varepsilon} \left[\tau_G \right] < \infty$$

holds, it follows that the process described by ε_t is a positive Harris chain[118].

It now remains to show that π_h and π_l as defined in the statement of the proposition are positive and satisfy

$$\frac{\pi_h}{\pi_l} = \frac{\bar{u}^3 - u(1+r)}{\bar{u}^3 - \mu_a^r}.$$

Note that according to the Strong Law of Large Numbers, the cumulative utility of a if it is chosen for an infinite number of times by the investors of type 3 satisfies:

$$\lim_{t \to \infty} U_t^3(a) = -\infty,$$

since the mean utility of a is lower than the aspiration level $\bar{u}^3$. Analogously, if b is chosen for an infinite number of periods,

$$\lim_{t \to \infty} U_t^3(b) = -\infty$$

obtains, since $u(1+r) < \bar{u}^3$ by assumption. Therefore, as in the proofs of propositions 3.1 and 3.2 in chapter 3, the case-based decision rule implies that on almost each path of dividend realizations, both portfolios will be held infinitely often by the investors of type 3.

Now consider the difference $U_t^3(a) - U_t^3(b) = \varepsilon_t$. It can be shown that ε_t remains bounded above on almost each path of dividend realizations. At times at which a is chosen ε_t never falls below 0, since this would contradict choosing the act with the highest cumulative utility in each period. Suppose, therefore that there is a sequence of periods t', t''..., such that $\varepsilon_{t'}$, $\varepsilon_{t''}$... grows to infinity. In other words, suppose that for each $N > 0$ there is a k such that $\varepsilon_{t^n} > N$ for all $n > k$. Since $U_t(a)$ has negative expected increments, it follows (as shown above) that b is chosen infinitely many times on almost each path of dividend realizations. But each time that b is chosen, the difference ε_t falls below 0. If $\varepsilon_{t^n} > N$, the time needed to return to the origin is at least

$$\frac{N}{\bar{u}^3 - u\left(\left(\frac{1-\theta_2+\delta}{1-\theta_2}\right)\right)},$$

which grows to infinity, as ε_{t^n} becomes very large. However, since the positive part of ε_t is a random walk on the half line, it follows from proposition 11.4.1 in Meyn and Tweedie (1996, p. 278) that the set G is regular for this process and, therefore, it is reached in finite expected time from each point in G. Moreover, the expected stopping time is uniformly bounded above by a

118 See Meyn and Tweedie (1996, p. 207) for a definition of a Harris chain.

number

$$\tilde{N} = \sup_{\varepsilon \in G} E_\varepsilon \left[\tau_G\right] < \infty.$$

The Law of Large Numbers then implies that for each $\kappa > 0$ on almost each path of dividend realizations, there is a period $\mathcal{K}$ such that

$$\frac{\sum_{i=1}^{n} \tau_{G_i}}{n} \leq \tilde{N} + \kappa$$

for all $n \geq \mathcal{K}$. On the other hand, the assumption that $\varepsilon_{t^n} \to \infty$ implies that there is a time $\mathcal{K}'$ such that $\tau_{G_i} > \tilde{N} + \kappa$ for all $i \geq \mathcal{K}'$. It is therefore always possible to choose n large enough, so that

$$\frac{\sum_{i=1}^{n} \tau_{G_i}}{n} > \tilde{N} + \kappa,$$

a contradiction. Hence, almost each sequence $\varepsilon_{t'}$, $\varepsilon_{t''}$... (where t', t''... denote periods at which a is chosen) is bounded above and below.

Analogously, at times at which b is chosen, ε_t assumes a minimal value of $u\left(\frac{\theta_1+\delta}{1-\theta_2}\right) - \bar{u}^3$ and increases in each period of time by an amount

$$\bar{u}^3 - u\left(1+r\right).$$

However, ε_t cannot exceed $\bar{u}^3 - u\left(1+r\right)$, since this would again be in contradiction with the case-based decision rule. Hence, ε_t is bounded on almost all paths of dividend realizations.

But then it follows that

$$\lim_{t\to\infty} \frac{U_t^3\left(a\right)}{U_t^3\left(b\right)} = \lim_{t\to\infty} \frac{U_t^3\left(a\right)}{U_t^3\left(a\right) - \varepsilon_t} = 1 \tag{5.64}$$

holds with probability 1 in the limit. For a given set $\left[u\left(\frac{\theta_1+\delta}{1-\theta_2}\right) - \bar{u}^3; 0\right)$, the invariant probability π_l describes the mean time that the Markov chain defined by ε_t spends in this set between its visits to another set, $[0; +\infty)$, see theorem 10.4.9 in Meyn and Tweedie (1996, p. 253). Note that a is chosen in periods in which $\varepsilon_t \geq 0$ holds, whereas b is chosen in periods in which $\varepsilon_t < 0$ holds. Now define a function $\iota_h : \Psi' \to \{0;1\}$ with

$$\iota_h\left(x\right) = \left\{ \begin{array}{ll} 1 \text{ if} & x \in [0; +\infty) \\ 0 \text{ if} & x \in \left[u\left(\frac{\theta_1+\delta}{1-\theta_2}\right) - \bar{u}^3; 0\right) \end{array} \right\}.$$

It is clear that $\iota_h \in L_1\left(\Psi'; \mathcal{B}\left(\Psi'\right); \pi\right)$, hence that ι_h has a finite expectation with respect to π on Ψ'. Moreover, it has been shown above that the process described by ε is positive Harris recurrent. Therefore, theorem 17.1.7 in Meyn and Tweedie (1996, p. 425) implies that

$$\lim_{t\to\infty} \frac{1}{t} \sum_{\tau=1}^{t} \iota_h\left(\varepsilon_\tau\right) = \int \iota_h d\pi$$

holds almost surely for any initial probability distribution. Note that $\frac{1}{t}\sum_{\tau=1}^{t}\iota_h(\varepsilon_\tau)$ represents the mean time that the system spends in state h. By the definition of $\iota_h(\varepsilon_\tau)$, it follows that

$$\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\iota_h(\varepsilon_\tau)=\pi_h$$

almost surely. Hence, the frequency with which the investors of type 3 choose a in the limit equals π_h on almost all paths of dividend realizations. It follows that

$$\lim_{t\to\infty}\frac{|C_t^3(a)|}{|C_t^3(b)|}=\frac{\pi_h}{\pi_l}$$

holds with probability 1 as well. Substituting this into (5.64) implies:

$$\begin{aligned}\lim_{t\to\infty}\frac{U_t^3(a)}{U_t^3(b)} &= \lim_{t\to\infty}\frac{|C_t^3(a)|}{|C_t^3(b)|}\frac{\sum_{\tau\in C_t^3(a)}\frac{v_\tau(a)}{|C_t^3(a)|}-\bar{u}^3}{u(1+r)-\bar{u}^3}= \\ &= \frac{\pi_h}{\pi_l}\frac{\lim_{t\to\infty}\sum_{\tau\in C_t^3(a)}\frac{v_\tau(a)}{|C_t^3(a)|}-\bar{u}^3}{u(1+r)-\bar{u}^3}=1.\end{aligned}$$

It follows that the mean utility of a, as observed by the investors of type 3 converges with probability 1 to a number μ_a^r, with

$$\mu_a^r=\lim_{t\to\infty}\sum_{\tau\in C_t^3(a)}\frac{v_\tau(a)}{|C_t^3(a)|}.$$

Hence,

$$\frac{\pi_h}{\pi_l}=\frac{\bar{u}^3-u(1+r)}{\bar{u}^3-\mu_a^r}.\blacksquare$$

Proof of proposition 5.10:

Consider the difference between the cumulative utilities of a and b restricted to its positive part:

$$\begin{aligned}\tilde{\varepsilon}_t &= \varepsilon_t, \text{ if } \varepsilon_t\geq 0\\ \tilde{\varepsilon}_t &= 0, \text{ else.}\end{aligned}$$

It is easily seen that $\tilde{\varepsilon}_t$ is a random walk on the half line with positive expected increments, since

$$\bar{u}^3<\int_{\underline{\delta}}^{\bar{\delta}}u\left(1+\frac{\delta}{1-\theta_2}\right)g(\delta)\,d\delta$$

holds by assumption. By proposition 9.5.1 in Meyn and Tweedie (1996, p. 278), $\tilde{\varepsilon}_t$ is transient. Hence, for each state and especially for $\tilde{\varepsilon}_t=0$, the expected number of visits to this state is finite, implying that the probability of an infinite number of visits to 0 is 0. Furthermore, since it is assumed that

$$u(1+r)<\bar{u}^3$$

holds, it follows that once $\varepsilon_t < 0$ obtains, b is held only for a maximum of

$$\left\lceil \frac{u\left(\frac{\theta_1+\delta}{1-\theta_2}\right) - \bar{u}^3}{u\left(1+r\right) - \bar{u}^3} \right\rceil$$

periods. Hence, $\varepsilon_{t+k} > 0$ occurs in finite time k. But then the transience of $\tilde{\varepsilon}_t$ implies that the decision-maker will switch to b only for a finite number of times with probability 1. Hence, on almost all paths of dividend realizations, b is held only for a finite number of periods. It follows that the limit frequency of state h is 1, whereas those of state l is 0.■

Proof of proposition 5.11:

Denote by $\varepsilon_t\left(a;b\right)$, $\varepsilon_t\left(a;MP\right)$ and $\varepsilon_t\left(b;MP\right)$ the differences between the cumulative utilities of a and b, a and the market portfolio and b and the market portfolio, respectively:

$$\varepsilon_t\left(\alpha;\alpha'\right) = U_t\left(\alpha\right) - U_t\left(\alpha'\right)$$

for $\alpha, \alpha' \in \{a; b; MP\}$ and $\alpha \neq \alpha'$. First, I show that the market portfolio is almost surely chosen in finite time. In a second step, I demonstrate that, once chosen, the market portfolio is held infinitely long in expectation. The Strong Law of Large Numbers then ascertains that the market portfolio is held with frequency 1 almost surely in the limit.

Denote by

$$\begin{aligned} \tilde{\varepsilon}_t\left(\alpha;\alpha'\right) &= \varepsilon_t\left(\alpha;\alpha'\right), \text{ if } \varepsilon_t\left(\alpha;\alpha'\right) \geq 0 \\ \tilde{\varepsilon}_t\left(\alpha;\alpha'\right) &= 0, \text{ if } \varepsilon_t\left(\alpha;\alpha'\right) < 0. \end{aligned}$$

Suppose that a is currently chosen. In such periods $\tilde{\varepsilon}_t\left(a;MP\right)$ and $\tilde{\varepsilon}_t\left(b;MP\right)$ behave as random walks on the half line with negative expected increments

$$\int_{\underline{\delta}}^{\bar{\delta}} \left[u\left(1+\delta\right) - u\left(1+\delta+r\right)\right] g\left(\delta\right) d\delta < 0$$

and

$$\bar{u} - \int_{\underline{\delta}}^{\bar{\delta}} u\left(1+\delta+r\right) g\left(\delta\right) d\delta < 0,$$

respectively. For such random walks on the half line all compact sets are regular, see proposition 11.4.1 in Meyn and Tweedie (1996, p. 278). Moreover, by proposition 4.3.1 in Meyn and Tweedie (1996, p. 96), the point 0 is an atom of a random walk on a half line with negative expected increment. Therefore, the chains defined by $\tilde{\varepsilon}_t\left(a;MP\right)$ and $\tilde{\varepsilon}_t\left(b;MP\right)$ are φ-irreducible with

$$\varphi\left(0;\infty\right) = 0$$

$$\varphi(0) = 1.$$

Hence, the set $\{0\}$ has a positive measure under the maximal irreducibility measure ψ of these two chains. Therefore, starting from any positive $\tilde{\varepsilon}_t(a; MP) > 0$ and $\tilde{\varepsilon}_t(b; MP) > 0$, the random walk on the half line reaches 0 in finite expected time. Since $\varepsilon_t(a; MP) = 0$ and $\varepsilon_t(b; MP) = 0$ obtain with probability 0, it follows that $\varepsilon_{t+k}(a; MP) < 0$ and $\varepsilon_{t+k}(b; MP) < 0$ obtain almost surely in finite time k, (where k in general depends on the path of dividend realizations chosen). But

$$\varepsilon_{t+k}(a; MP) < 0$$

implies that the market portfolio has a larger cumulative utility at time $(t+k)$ than the portfolio consisting of risky assets only. Hence, with probability 1, the representative investor abandons a in finite time.

Suppose first that $\varepsilon_t(a; b)$ has positive expected increments,

$$\int_{\underline{\delta}}^{\bar{\delta}} u(1+\delta) g(\delta) d\delta - u(1+r) > 0.$$

If at time $(t+k)$ the representative investors chooses b, then, by an argument similar to the one presented above, both $\tilde{\varepsilon}_t(a; b) > 0$ and $\tilde{\varepsilon}_t(b; MP) < 0$ obtain in finite time with probability 1. Therefore, in finite time the investor switches away from b and chooses the market portfolio or the risky asset. It remains to show that almost surely he will choose the risky asset and the riskless bond only for a finite number of times.

Indeed, suppose that a is chosen an infinite number of times and note that it cannot be that a is always abandoned for b, since $\varepsilon_t(a; b)$ has positive expected increments, if a is chosen and remains constant when b or MP is chosen. Hence, starting from 0, $\tilde{\varepsilon}_t(a; b)$ will return to 0 only a finite number of times in expectation and, therefore, with probability 1, the representative investor will switch from a to b only a finite number of times. Hence, since a is chosen infinitely often and since $\tilde{\varepsilon}_t(a; MP)$ becomes 0 in finite time, it follows that the representative investor switches to the market portfolio infinitely often. Note, however that once the market portfolio is chosen $\tilde{\varepsilon}_t(a; MP)$ and $\tilde{\varepsilon}_t(b; MP)$ have negative expected increments. Therefore, the probability that the cumulative utility of the market portfolio falls below the cumulative utility of the assets a and b is less than 1. It follows that the probability that $U_t(MP)$ falls below the cumulative utility of a for an infinite number of times is 0. But then, the decision-maker will switch only a finite number of times between a and the market portfolio, a contradiction. Hence, almost surely a is chosen only for a finite number of periods.

Alternatively, if b is chosen infinitely often, it follows that (since a can be chosen for a finite number of times on almost each path of dividend realizations, as shown above) the representative investor switches infinitely often between b and the market portfolio. Again, once he switches to the market portfolio, there is a positive probability that the cumulative utility of the market portfolio never falls below that of a and b. Hence, the probability that the cumulative utility of the market portfolio falls below the cumulative utility of a and b for an infinite number of times is 0 and, therefore, the decision-maker can only switch between b and the market portfolio for a finite number of times. The assumption that b is chosen an infinite number of times, therefore, also leads to a contradiction. It follows that both a and b are chosen for a finite number of times on almost all paths of dividend realizations. Therefore, the market portfolio must be chosen for an infinite number of times on almost each path of dividend realizations. Hence,

$$\begin{aligned} \pi_{MP} &= 1 \\ \pi_a &= \pi_b = 0 \end{aligned}$$

obtains.

The argument for the case

$$\int_{\underline{\delta}}^{\bar{\delta}} u\left(1+\delta\right) g\left(\delta\right) d\delta - u\left(1+r\right) < 0$$

is analogous and the proof is obtained from the two preceding paragraphs by replacing a by b and vice versa.■

Proof of proposition 5.12:

First note that none of the acts a and b can be chosen only for a finite number of times, since this would imply that the cumulative utility of the acts chosen infinitely often converges to $-\infty$ with probability 1, whereas the cumulative utility of the act chosen only for a finite number of times remains finite. This contradicts the case-based decision rule, which prescribes choosing the act with the maximal cumulative utility in each period. It follows that acts a and b are chosen infinitely often on almost each dividend path.

Now consider the differences between the cumulative utilities of the acts $\varepsilon_t\left(a;b\right)$, $\varepsilon_t\left(a;MP\right)$ and $\varepsilon_t\left(b;MP\right)$. Six cases are possible.

1. Suppose that at time t the representative investor switches from portfolio a to b. This means that the difference between the cumulative utilities of a and b satisfies:

$$\varepsilon_{t-1}(a;b) \geq 0$$
$$\varepsilon_t(a;b) \in [u(\underline{\delta}) - \bar{u}; 0].$$

Since from t on, b is the act chosen, $\varepsilon_t(a;b)$ behaves like a random walk on the negative half line with positive expected increment (since the cumulative utility of a remains unchanged, whereas the cumulative utility of b falls in expectation). Consider as above the process $\varepsilon_t(a;b)$ constrained to the negative half line and note that, by an argument similar to the one used in the proof of proposition 5.11, starting from the interval $[u(\underline{\delta}) - \bar{u}; 0]$, it will reach 0 in finite time with probability 1. Therefore, in finite time the difference between the cumulative utilities of a and b becomes positive again and hence, the representative investor switches either to MP or to a. Moreover, since the random walk on a half line has the property that all compact intervals are regular, it follows that the expected time needed for $\varepsilon_t(a;b)$ to become positive again, starting from $[u(\underline{\delta}) - \bar{u}; 0]$ is uniformly bounded above[119]. Hence, the time for which b is held is also uniformly bounded from above in expectation.

2. Suppose that at time t, the representative investor switches from portfolio b to a. This means that the difference between the cumulative utilities of a and b satisfies:

$$\varepsilon_{t-1}(a;b) \leq 0$$
$$\varepsilon_t(a;b) \in [\bar{u} - u(r); 0].$$

An argument analogous to those presented in case 1 shows that the time during which a is held is uniformly bounded above in expectation.

3. Suppose that at time t, the representative investor switches from portfolio a to MP. This means that the difference between the cumulative utilities of a and MP satisfies:

$$\varepsilon_{t-1}(a;MP) \geq 0$$
$$\varepsilon_t(a;MP) \in \left[\min_{\delta \in [\underline{\delta};\bar{\delta}]} \left\{u\left(\frac{1}{2} + \delta\right) - u(1 + \delta + r)\right\}; 0\right].$$

[119] Of course, if $\varepsilon_t(b;MP) < 0$ and $\varepsilon_t(a;MP) < 0$ obtain before $\varepsilon_t(a;b)$ becomes positive again, the investor will switch to MP even at an earlier period. This does not contradict the conclusion that the expected time during which b is held in a row is bounded from above.

Since the cumulative utility of the market portfolio behaves like a random walk with negative expected increment as long as the market portfolio is chosen, it follows that $\varepsilon_t(a; MP)$ is a random walk on the negative half line with positive expected increment. Hence, starting from

$$\left[\min_{\delta\in[\underline{\delta};\bar{\delta}]} u\left(\frac{1}{2}+\delta\right) - u(1+\delta+r);0\right],$$

it becomes positive in finite time with probability one. The expected time for $\varepsilon_t(a; MP)$ to become positive is moreover uniformly bounded above for all starting points lying in this interval.

4. Suppose that at time t, the representative investor switches from portfolio b to MP. This means that the difference between the cumulative utilities of b and MP satisfies:

$$\varepsilon_{t-1}(b; MP) \geq 0$$
$$\varepsilon_t(b; MP) \in \left[u\left(\frac{1}{2}+r\right) - u\left(1+\bar{\delta}+r\right);0\right].$$

An argument analogous to the one presented in case 3 shows that the time during which MP is held is uniformly bounded above in expectation.

5. Suppose that at time t, the representative investor switches from the market portfolio to a. This means that the difference between the cumulative utilities of a and MP satisfies:

$$\varepsilon_{t-1}(a; MP) \leq 0$$
$$\varepsilon_t(a; MP) \in [0; \bar{u} - u(1+\underline{\delta}+r)].$$

Since, however, the market portfolio always yields returns which are strictly higher than those of a, it follows that the difference between the cumulative utilities of a and MP strictly increases in each period of time and therefore in at most

$$\left\lceil \frac{\bar{u} - u(1+\underline{\delta}+r)}{\min_{\delta\in[\underline{\delta};\bar{\delta}]}\{u(1+\delta+r) - u(1+\delta)\}} \right\rceil$$

periods $\varepsilon_{t-1}(a; MP) \leq 0$ obtains again. Hence, as in the previous cases, the expected time during which the investors hold a is finite and uniformly bounded above.

6. Suppose that at time t the representative investor switches from the market portfolio to b. This means that the difference between the cumulative utilities of b and MP satisfies:

$$\varepsilon_{t-1}(b; MP) \leq 0$$
$$\varepsilon_t(b; MP) \in [0; \bar{u} - u(1+\underline{\delta}+r)].$$

Since the difference $\varepsilon_t(b;MP)$ decreases in each period of time by at least $u(1+r) - u(1+\underline{\delta}+r)$, which is positive with probability 1, it follows that almost surely $\varepsilon_t(b;MP) \leq 0$ obtains again in finite time. This happens in at most

$$\left\lceil \frac{\bar{u} - u(1+\underline{\delta}+r)}{u(1+r) - u(1+\underline{\delta}+r)} \right\rceil,$$

periods, if $\underline{\delta} > 0$ holds. If $\underline{\delta} = 0$, we can again use the property of a random walk with negative expected increment to demonstrate that the expected time needed for $\varepsilon_t(b;MP) \leq 0$ to obtain is finite and uniformly bounded above.

The discussion of the six possible cases shows that the expected time during which a single act is chosen in a row is finite and uniformly bounded from above. In analogy to the proof of proposition 5.10, it can be, therefore, shown that every sequence $\varepsilon_{t^k}(\alpha;\alpha')$, $\alpha \neq \alpha'$, α, $\alpha' \in \{a;b;MP\}$ is almost surely bounded. Hence, for each $\alpha \neq \alpha'$, the limit of the cumulative utilities satisfies:

$$\lim_{t\to\infty} \frac{U_t(\alpha)}{U_t(\alpha')} = \lim_{t\to\infty} \frac{U_t(\alpha)}{U_t(\alpha) + \varepsilon_t(\alpha;\alpha')} = 1.$$

Since the realization of the market portfolio is observed in each period of time, the Strong Law of Large Numbers implies that the cumulative utility of the market portfolio almost surely satisfies:

$$\lim_{t\to\infty} \frac{U_t(MP)}{t} = \int_{\underline{\delta}}^{\bar{\delta}} u(1+\delta+r)\, g(\delta)\, d\delta - \bar{u}.$$

It follows, therefore that

$$\lim_{t\to\infty} \frac{U_t(a)}{U_t(MP)} = \lim_{t\to\infty} \frac{|C_t(a)|}{t} \frac{\left(\frac{\sum_{\tau\in C_t(a)} u_\tau(a)}{|C_t(a)|} - \bar{u}\right)}{\int_{\underline{\delta}}^{\bar{\delta}} u(1+\delta+r)\, g(\delta)\, d\delta - \bar{u}} = 1.$$

Since $u_\tau(a)$ can obtain values between $\left[u(\underline{\delta}); u\left(1+\bar{\delta}\right)\right]$, it follows that $\frac{\sum_{\tau\in C_t(a)} u_\tau(a)}{|C_t(a)|}$ is also bounded between $\left[u(\underline{\delta}); u\left(1+\bar{\delta}\right)\right]$ on each possible path of dividend realizations. It follows that the quotient $\frac{|C_t(a)|}{t}$ must be bounded away from 0 on almost each dividend path, in order to guarantee that the limit is indeed 1. Hence, the mean proportion of time during which asset a is chosen in the limit is strictly positive on almost each path of dividend realizations. Analogous reasoning shows that the mean proportion of time during which asset b is chosen is also strictly positive in the limit with probability 1.■

Proof of proposition 5.13:

The aspiration level of the investors of type 1 has been chosen low enough, so as to guarantee that they never switch away from their initial choice. Hence, only the behavior of the investors with high aspiration level has to be considered. Note that

$$u\left(\left(\frac{p_0+\underline{\delta}}{p_0}\right)\alpha_0+(1-\alpha_0)\,r\right)=u\left(1+\underline{\delta}+(1-\alpha_0)\,r\right)$$

is the utility realization of portfolio α_0, given that the dividend of the risky asset is 0 (the lowest possible) and still all young investors choose α_0. Since for all $\delta_1\in\left[\underline{\delta};\bar{\delta}\right]$,

$$\bar{u}^2<u\left(1+\underline{\delta}+(1-\alpha_0)\,r\right)\leq u\left(1+\delta_1+(1-\alpha_0)\,r\right)$$

holds by the strict monotonicity of $u(\cdot)$, type 2 observes a cumulative utility of an act α given by:

$$\begin{aligned}U_1^2(\alpha) &= s\left((p_0;\alpha_0);(p_{t+1};\alpha_{t+1})\right)\cdot\\ &\quad\cdot\left[u\left(\left(\frac{p_1+\delta_1}{p_0}\right)\alpha_0+(1+r)(1-\alpha_0)\right)-\bar{u}^2\right].\end{aligned}$$

If $\alpha_1^{2*}=\alpha_0^{2*}$, then $p_1=p_0$ and therefore $U_1^2(\alpha_0)>U_1^2(\alpha)$ for all $\alpha\neq\alpha_0$. Therefore if every young investor chooses α_0, α_0^{2*} is indeed the optimal choice of type 2. Hence, in equilibrium $p_1=p_0$ obtains for each $\delta_1\in\left[\underline{\delta};\bar{\delta}\right]$.

By induction, the same result holds for each period of time t, hence $(\alpha_0;\alpha_0;p_0)$ is a stationary state of the economy.■

Proof of proposition 5.14 (continued from the main text):

To show that

$$p_{t^n}^*\geq\frac{1-\theta_1+2\theta_1\alpha_0}{2}=\theta_1\alpha_0+\frac{1}{2}(1-\theta_1)$$

obtains almost surely in finite time, first compute the difference between two subsequent members of the sequence $p_{t^k}^*$:

$$p_{t^k}^*-p_{t^{k-1}}^*=\frac{p_{t^{k-1}}^*(1-\theta_1)}{p_{t^{k-1}}^*-\theta_1\alpha_0}\left[\varpi-(1+r)+\left(r-\frac{\delta_{t^k}}{p_{t^{k-1}}^*}\right)\alpha_{t^{k-1}}\right],\tag{5.65}$$

where ϖ denotes the return of a portfolio which yields utility exactly equal to $\bar{u}^2$. Note that

$$\frac{p_{t^{k-1}}^*(1-\theta_1)}{p_{t^{k-1}}^*-\theta_1\alpha_0}>1-\theta_1,$$

whereas

$$\frac{1-\theta_1+2\theta_1\alpha_0}{2}-\theta_1\alpha_0=\frac{1-\theta_1}{2}$$

is the least amount by which p_{t^k} should grow to obtain a value higher than $\frac{1-\theta_1+2\theta_1\alpha_0}{2}$.

Note that

$$\varpi-(1+r)+\left(r-\frac{\delta_{t^k}}{p_{t^{k-1}}^*}\right)\alpha_{t^{k-1}}-\left[\varpi-(1-\alpha_0)(1+r)-\alpha_0-\delta_{t'}\right]$$

$$= \varpi - \left(1 - \alpha^{2*}_{t^{k-1}}\right)(1+r) - \left(\frac{p^*_{t^{k-1}} + \delta_{t^k}}{p^*_{t^{k-1}}}\right)\alpha^{2*}_{t^{k-1}} - \left[\varpi - (1-\alpha_0)(1+r) - \alpha_0 - \delta_{t'}\right]$$
$$= r\left(\alpha^{2*}_{t'} - \alpha_0\right) + \frac{\delta_{t^k}}{p^*_{t^{k-1}}}\alpha^{2*}_{t^{k-1}} - \delta_{t'}$$

and that by the choice of t',

$$\varpi - (1-\alpha_0)(1+r) - \alpha_0 - \delta_{t'} > 0$$

holds. Hence,

$$\varpi - (1+r) + \left(r - \frac{\delta_{t^k}}{p^*_{t^{k-1}}}\right)\alpha_{t^{k-1}} > r\left(\alpha^{2*}_{t'} - \alpha_0\right) > 0$$

if

$$\frac{\delta_{t^k}}{p^*_{t^{k-1}}}\alpha^{2*}_{t^{k-1}} - \delta_{t'} \geq 0.$$

Now choose z in such a way that

$$\frac{1}{2z} = r\left(\alpha^{2*}_{t'} - \alpha_0\right).$$

Obviously, after at most z (not necessarily sequential) periods in which $\delta_{t^k} \leq \frac{\delta_{t'}}{\alpha^{2*}_{t^{k-1}}} p^*_{t^{k-1}}$ obtains, the difference between $p^*_{t^k}$ and p_0 would exceed $\frac{1-\theta_1}{2}$ and, therefore, $p^*_{t^k} \geq \frac{1-\theta_1+2\theta_1\alpha_0}{2}$ would obtain.

Let t^z denote the last z^{th} period in which $\delta_{t^k} \leq \delta_{t'}$ obtains. It follows that there exists a t^n such that

$$t^n = \min\left\{t^k \leq t^z \mid p^*_{t^k} - p_0 \geq \frac{1-\theta_1}{2}\right\}$$

and, therefore,

$$\begin{aligned} p^*_{t^n} &\geq \frac{1-\theta_1+2\theta_1\alpha_0}{2} \\ \alpha^{2*}_{t^n} &\geq \frac{1}{2} \end{aligned}$$

obtains. On the other hand, since $\frac{p^*_{t^{k-1}}}{\alpha^{2*}_{t^{k-1}}} > 1$, the probability that the number of periods in which $\delta_{t^k} \leq \frac{\delta_{t'}}{\alpha^{2*}_{t^{k-1}}} p^*_{t^{k-1}}$ obtains is less than z is 0 on the set of sample paths of dividend realizations. Hence, with probability 1, t^z and, hence also t^n is finite.■

Proof of proposition 5.15:

It has been shown above that for $u\left(1 + \underline{\delta} + (1-\alpha_0) r\right) < \bar{u}^2$ the investors of type 2 switch either to $\alpha^{2*}_t = 0$, implying that the price $p^*_t = \theta_1\alpha_0$, or to $\alpha^{2*}_t = 1$, implying $p^*_t = 1 - \theta_1(1-\alpha_0)$ in some finite period $\bar{t}(\tilde{\omega})$ as derived in the proof of proposition 5.14. It remains to show that after the first time, in which $\alpha^{2*}_t = 0$ or $\alpha^{2*}_t = 1$ obtains, a cycle emerges.

Consider first the case of $\alpha_t^{2*} = 0$ and $p_t^* = \theta_1\alpha_0$. Since the last (and only) case observed now by type 2 is $(\theta_1\alpha_0; 0; u(1+r))$ and since $u(1+r) < \bar{u}$, it follows that the optimal act at p_{t+1} is $\alpha_{t+1}^2 = 1$. But if $\alpha_{t+1}^2 = 1$, the price becomes $p_{t+1}^* = 1 - \theta_1(1-\alpha_0)$ and since the similarity function is strictly decreasing in the Euclidean distance between problem-act pairs, it is easily seen that $\alpha_{t+1}^{2*} = 1$ is indeed the optimal choice.

Alternatively, if $\alpha_t^{2*} = 1$ and $p_t^* = 1 - \theta_1(1-\alpha_0)$ hold and the dividend realization at time $(t+1)$ is $\delta_{t+1} < \hat{\delta}$, then the investors of type 2 observe a utility realization of at most

$$u\left(1 + \frac{\delta_{t+1}}{1-\theta_1(1-\alpha_0)}\right) < \bar{u}^2.$$

Note that such a $\hat{\delta} \in [\underline{\delta}; \bar{\delta}]$ exists, since the return of the portfolio $\alpha = 1$ if the dividend realization is $\underline{\delta}$ and the investors of type 2 hold $\alpha = 1$ is at most

$$u\left(1 + \frac{\underline{\delta}}{1-\theta_1(1-\alpha_0)}\right).$$

But

$$u\left(1 + \frac{\underline{\delta}}{1-\theta_1(1-\alpha_0)}\right) < u(1 + \underline{\delta} + (1-\alpha_0)r)$$

is equivalent to

$$\underline{\delta}\theta_1 < r,$$

which is always satisfied under the assumption made.

Hence, if $\delta_{t+1} < \hat{\delta}$, the investors of type 2 are unsatisfied with $\alpha = 1$ even at the highest price that might obtain in period $(t+1)$. From the fact that the similarity function is strictly decreasing in the Euclidean distance, it follows that the investors of type 2 choose $\alpha_{t+1}^{2*} = 0$. The equilibrium price is computed as $p_{t+1}^* = \theta_1\alpha_0$ and it is easily verified that at p_{t+1}^* $\alpha_{t+1}^{2*} = 0$ is indeed optimal.

If at time $(t+1)$ at which the investors of type 2 hold the risky asset, its dividend realization is higher than $\hat{\delta}$, then the utility they obtain exceeds $\bar{u}^2$ as long as the price remains unchanged at

$$p_t^* = 1 - \theta_1(1-\alpha_0).$$

But in this case the investors of type 2 are satisfied with $\alpha = 1$ and

$$\arg\max_{\alpha\in[0;1]} U_{t+1}^2(\alpha) = 1$$

holds, since the similarity function obtains its maximum if the problem-act pairs are identical. Therefore $\alpha_{t+1}^{2*} = 1$ and $p_{t+1}^{2*} = 1 - \theta_1(1-\alpha_0)$ obtains in an equilibrium.

The argument above shows that the evolution of prices and portfolio choices follows a Markov process with a transition matrix:

$$\bar{P} = \left(\begin{array}{|c|c|c|} \hline & p^*_{t+1} = 1 - \theta_1 (1 - \alpha_0) & p^*_{t+1} = \theta_1 \alpha_0 \\ \hline p^*_t = 1 - \theta_1 (1 - \alpha_0) & q & 1 - q \\ \hline p^*_t = \theta_1 \alpha_0 & 1 & 0 \\ \hline \end{array} \right).$$

The invariant probability distribution of the states $p_h = 1 - \theta_1 (1 - \alpha_0)$; $\alpha^1_h = \alpha_0$; $\alpha^2_h = 1$ and $p_l = \theta_1 \alpha_0$; $\alpha^1_l = \alpha_0$; $\alpha^2_l = 0$ is computed in the same way as in the proof of proposition 5.4.■

Proof of proposition 5.16:

It follows from the proof of proposition 5.15 that a cycle with two states h and l emerges after a finite number of periods $t (\tilde{\omega})$. Moreover, the investors of type 2 switch to $\alpha^{2*}_{t+1} = 0$, if the last period price satisfies $p^*_t = p_h$ and the dividend is lower than $\hat{\delta}$, causing the price to fall to $p^*_{t+1} = p_l$. Conversely, given that the last period price of a is low ($p^*_t = p_l$, $\alpha^{2*}_t = 0$), the investors of type 2 are not satisfied with $u (1 + r)$ and switch to $\alpha^{2*}_{t+1} = 1$, causing the price to rise to p_h.

It remains only to consider periods t, such that $p^*_t = p_h$, $\alpha^{2*}_t = 1$ and $\delta_{t+1} \geq \hat{\delta}$ hold. Since $u \left(1 + \frac{\bar{\delta}}{p_h} \right) < \bar{u}^2$, it follows that the cumulative utility of $\alpha = 1$ is negative at $(t + 1)$ for the investors of type 2 and, thus, the optimal act at any price $p_t \leq p_h$ is $\alpha^{2*}_t = 0$. But for $\alpha^{2*}_t = 0$, the price $p_t = p_l < p_h$ must hold and, therefore, $\alpha^{2*}_t = 0$ and $p^*_t = p_l$ obtain as equilibrium at time t.

To summarize, the investors of type 2 choose $\alpha^{2*}_t = 1$ in each period t, such that $\alpha^{2*}_{t-1} = 0$ and they choose $\alpha^{2*}_t = 0$ in each period t, such that $\alpha^{2*}_{t-1} = 1$. Therefore, the result of the proposition obtains.■

Proof of proposition 5.17:

1. If $\bar{u}^2 < \mu (\alpha_0 \mid p_0)$, then the cumulative utility of α_0 for the investors of type 2 is given by:

   $$U^2_t (\alpha_0) = \sum_{\tau=1}^{t} \left[v_\tau (\alpha_0) - \bar{u}^2 \right],$$

   as long as they hold α_0. Since

   $$E [v_t (\alpha_0)] = \mu (\alpha_0 \mid p_0) > \bar{u}^2$$

   $U^2_t (\alpha_0)$ behaves as a random walk on $\mathbb{R}$ with positive expected increment. According to theorem 9.5.1 in Main and Tweedie (1996, p. 228) such random walks are transient, hence the expected time until their first return to 0 is infinite.

2. If $\bar{u}^2 \in (\mu(\alpha_0 \mid p_0); u(1+r))$, then the process
$$\tilde{U}_t^2(\alpha_0) = \left\{ \begin{array}{ll} U_t^2(\alpha_0), & \text{if } U_t^2(\alpha_0) \geq 0 \\ 0, & \text{else} \end{array} \right\}$$
describes the cumulative utility of α_0 for the investors of type 2 as long as it is non-negative. $\tilde{U}_t^2(\alpha_0)$ is a random walk on $\mathbb{R}_0^+$, but with negative expected increments, since now
$$\mu(\alpha_0 \mid p_0) < \bar{u}^2.$$
Since for such random walks all compact sets are regular, see proposition 11.4.1 in Meyn and Tweedie (1996, p. 278), it follows that
$$\tilde{U}_t^2(\alpha_0) = 0$$
obtains in finite time with probability 1. Therefore, since the distribution Q is continuous, it follows that
$$U_t^2(\alpha_0) < 0$$
obtains almost surely in finite time.
Once the cumulative utility of α_0 has become negative, apply the proof of proposition 5.15 to show that the investors of type 2 will choose either $\alpha = 0$ or $\alpha = 1$ in finite time. Note that this result can be applied, since the portfolios $\alpha_{t^k}^{2*}$, $t^k \leq t^{n-1}$, constructed in this proof[120] have a cumulative utility of 0 and, therefore, do not influence the evaluation of any of the acts available, whereas the cumulative utility of the last chosen diversified portfolio $\alpha_{t^n}^{2*}$ is negative. Once $\alpha = 1$ or $\alpha = 0$ has been chosen for the first time, its cumulative utility behaves as a random walk with positive expected increments, since
$$\bar{u}^2 < u(1+r) < \mu(1 \mid 1 - \theta_1(1-\alpha_0)).$$
Therefore, it remains positive infinitely long in expectations. Hence, the expected time during which the investors of type 2 hold $\alpha = 1$ or $\alpha = 0$ is infinity.
Note, further that the cumulative utility of any portfolio α as observed by the investors of

[120] In the case of long memory, however, the time periods t^k will not denote the *subsequent* periods in which the dividend realization is lower than $\tilde{\delta}$, but those periods in which $\delta_{t^k} < \tilde{\delta}$ and
$$U_{t^k}\left(\alpha_{t^k-1}^{2*}\right) + u\left(\alpha_{t^k-1}^{2*}(1+\delta_{t^k}) + \left(1-\alpha_{t^k-1}^{2*}\right)(1+r)\right) - \bar{u}^2 < 0,$$
whereas $U_{t^k-1}\left(\alpha_{t^k-1}^{2*}\right) \geq 0$ holds. The portfolio $\alpha_{t^k}^{2*}$ (and, hence, the price $p_{t^k}^*$) are then chosen in such a way that
$$U_{t^k}\left(\alpha_{t^k-1}^{2*}\right) + u\left(\frac{p_{t^k}^* + \delta_{t^k}}{p_{t^k-1}^*}\alpha_{t^k-1}^{2*} + \left(1-\alpha_{t^k-1}^{2*}\right)(1+r)\right) - \bar{u}^2 = 0$$
Hence, $U_{t^k}(\alpha) = 0$ for each $\alpha \in [0;1]$ and, therefore, the choices till time t^k do not influence the evaluation of the available portfolios.

type 2 is given by:

$$U_t^2(\alpha) = s((p_t;\alpha);((1-\theta_1(1-\alpha_0);1)))\, V_t^2(1) + s((p_t;\alpha);((\theta_1\alpha_0;0)))\, V_t^2(0)$$
$$+ s\left((p_t;\alpha);\left(\left(p_{t^n}^*;\alpha_{t^n}^{2*}\right)\right)\right) V_t^2\left(\alpha_{t^n}^{2*}\right),$$

where the notation from the proof of proposition 3.12 is used and the upper index 2 refers to the investors of type 2. Now, if exactly one of the numbers $V_t^2(1)$, $V_t^2(0)$ and $V_t^2(\alpha_{t^n}^{2*})$ is positive, the act 1, 0 or $\alpha_{t^n}^{2*}$ will be chosen, respectively. The rule of cumulative utility maximization precludes the case that two of these numbers are positive at some t[121]. But if all of them are negative, then $U_t^2(\alpha)$ becomes a convex function in α, since the similarity function is concave. Therefore, a corner solution obtains in each period of time. Hence, either $\alpha = 1$ or $\alpha = 0$ are chosen.

On all paths, on which the cumulative utility of $\alpha = 1$ never falls below the cumulative utility of $\alpha = 0$, the frequency of $\alpha = 1$ is 1. If, however the investors of type 2 switch to $\alpha = 0$ at some time T, then

$$u(1+r) > \bar{u}^2$$

implies that the cumulative utility $\alpha = 0$ exceeds the cumulative utility of $\alpha = 1$ for each $t > T$. Hence, $\alpha = 0$ is chosen in each period afterwards. On these paths, the limit frequency of $\alpha = 0$ is, therefore, equal to 1.

3. If $\bar{u}^2 \in (u(1+r);\mu(1 \mid 1-\theta_1(1-\alpha_0)))$, then the investors of type 2 switch to $\alpha = 1$ or to $\alpha = 0$ in finite time, as shown in part 2 of this proof. If $\alpha = 1$ has been chosen, then its cumulative utility behaves like a random walk with positive expected increments and, therefore, $\alpha = 1$ is held infinitely long in expectation. If $\alpha = 0$ has been chosen, then it will be only held for a finite time, since its return is considered unsatisfactory. Moreover, since the similarity function is concave, once the cumulative utilities of $\alpha = 0$, $\alpha_{t^n}^{2*}$ and, hence, also of $\alpha = 1$ have become negative, the optimal act will be a corner solution, as shown in part 2 of this proof. Therefore, a corner solution obtains and $\alpha = 1$ will be chosen in finite time and then held forever in expectation.

   Even, if the investors of type 2 should switch to $\alpha = 0$ at some time, the cumulative utility of this portfolio would become lower than the cumulative utility of $\alpha = 1$ in finite time. But since the events that the cumulative utility of $\alpha = 1$ becomes negative have a probability

[121] The proof of this statement is by induction and is analogous to the argument stated in the proof of proposition 3.12 in chapter 3.

lower than 1 and are independent, the probability of the event that the cumulative utility of $\alpha = 1$ falls below 0 infinitely often is 0. Hence, in the limit, type 2 indeed holds $\alpha = 1$ with frequency 1.

4. Now, let $\bar{u}^2 > \mu (1 \mid 1 - \theta_1 (1 - \alpha_0))$. In this case the investors of type 2 again switch to one of the corner acts in finite time. As in the previous cases, the concavity of the similarity function implies that only corner acts will be chosen in each period of time. However, in this case neither $\alpha = 0$, nor $\alpha = 1$ are considered satisfactory in expectations. Denote by ε_t the following process:

$$\varepsilon_0 = \frac{V_t^2 \left(\alpha_{t^n}^{2*}\right) \left[s \left((p_h; 1) ; \left(\left(p_{t^n}^*; \alpha_{t^n}^{2*}\right)\right)\right) - s \left((p_l; 0) ; \left(\left(p_{t^n}^*; \alpha_{t^n}^{2*}\right)\right)\right)\right]}{1 - s} =: \bar{V}$$

$$\varepsilon_t = \left\{ \begin{array}{ll} \varepsilon_{t-1} + u\left(1 + \frac{\delta_t}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(1 + \frac{\delta_t}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2 > 0 \\ \varepsilon_{t-1} + u\left(\frac{\theta_1\alpha_0 + \delta_t}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(\frac{\theta_1\alpha_0 + \delta_t}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2 < 0 \\ \varepsilon_{t-1} + u(1+r) - \bar{u}^2, & \text{if } \varepsilon_{t-1} < 0. \end{array} \right\}.$$

Note that $(1 - s)\, \varepsilon_t$, where

$$s = s\left((p = 1 - \theta_1 (1 - \alpha_0) ; \alpha = 1) ; (p = \theta_1 \alpha_0 ; \alpha = 0) \right) \in [0; 1),$$

describes the evolution of the difference between the cumulative utilities of a and b for the investors of type 2. To see this, write

$$\begin{aligned} \varepsilon_t = U_t^2 (1) - U_t^2 (0) &= V_t^2 (1) + sV_t^2 (0) - V_t^2 (0) - sV_t^2 (1) + \bar{V} (1 - s) = \\ &= \left[V_t^2 (1) - V_t^2 (0) \right] (1 - s) + \bar{V} (1 - s). \end{aligned}$$

By the same arguments as those used in the proof of proposition 5.9 it can be shown that ε_t is a ψ-irreducible Markov chain on

$$\Psi'' = \left[u\left(\frac{\theta_1 \alpha_0 + \underline{\delta}}{1 - \theta_1 (1 - \alpha_0)} \right) - \bar{u}^2; +\infty \right)$$

with an invariant (finite) probability measure π such that

$$\lim_{t \to \infty} \frac{|C_t^2 (1)|}{|C_t^2 (0)|} = \frac{\pi_1}{\pi_0},$$

almost surely holds, where $\pi_1 = \pi [0; +\infty)$ and $\pi_0 = \left[u\left(\frac{\theta_1\alpha_0 + \underline{\delta}}{1-\theta_1(1-\alpha_0)} \right) - \bar{u}^2; 0 \right)$ denote the limit frequencies with which acts a and b, respectively, are chosen by the investors of type 2.

In analogy to the proof of proposition 5.9 it can be show that $V_t^2 (1) - V_t^2 (0)$ must be bounded on almost each path. Hence, $U_t^2 (1) - U_t^2 (0)$ is also bounded with probability 1 and, therefore,

$$\lim_{t\to\infty}\frac{U_t^2(1)}{U_t^2(0)}=1$$

almost surely holds. It follows that

$$\lim_{t\to\infty}\frac{V_t^2(1)+sV_t^2(0)+V_t^2(\alpha_{t^n}^{2*})\,s\left((p_h;1);((p_{t^n}^*;\alpha_{t^n}^{2*}))\right)}{sV_t^2(1)+V_t^2(0)+V_t^2(\alpha_{t^n}^{2*})\,s\left((p_h;1);((p_{t^n}^*;\alpha_{t^n}^{2*}))\right)}=1.$$

Since $V_t^2(\alpha_{t^n}^{2*})$ is finite, it does not influence the limit, hence

$$\lim_{t\to\infty}\frac{\left|C_t^2(1)\right|\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}+s\left|C_t^2(0)\right|\frac{\sum_{\tau\in C_t^2(0)}\left(u(1+r)-\bar{u}^2\right)}{\left|C_t^2(0)\right|}}{s\left|C_t^2(1)\right|\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}+\left|C_t^2(0)\right|\frac{\sum_{\tau\in C_t^2(0)}\left(u(1+r)-\bar{u}^2\right)}{\left|C_t^2(0)\right|}}=1$$

$$\lim_{t\to\infty}\frac{\frac{\pi_h}{\pi_l}\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}+s\left(u(1+r)-\bar{u}^2\right)}{s\frac{\pi_h}{\pi_l}\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}+\left(u(1+r)-\bar{u}^2\right)}=1.$$

This implies

$$\frac{\pi_1}{\pi_0}\lim_{t\to\infty}\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}=\left(u(1+r)-\bar{u}^2\right),$$

with probability 1. Hence, there exists a μ_1^r such that

$$\mu_1^r=\lim_{t\to\infty}\frac{\sum_{\tau\in C_t^2(1)}v_\tau(1)}{\left|C_t^2(1)\right|}$$

holds with probability 1 and

$$\frac{\pi_1}{\pi_0}=\frac{u(1+r)-\bar{u}^2}{\mu_1^r-\bar{u}^2}$$

almost surely obtains.■

Chapter 6. Fitness and Survival of Case-Based Decisions — An Evolutionary Approach

In chapters 3 and 5 of this thesis, I have analyzed the behavior of case-based decision-makers who face a portfolio choice problem with exogenous prices or act in a market environment. The primary aim was a descriptive one, but I have also claimed that this behavior could explain empirically observed phenomena which seem to be inconsistent with the hypothesis of expected utility maximization combined with correct or rational expectations. In an individual portfolio problem, the case-based decision-makers fail in general to choose the optimal portfolio at the equilibrium price. Their portfolios are often underdiversified and high aspiration levels cause them to trade too much and lose money.

In a market context, equilibria with zero asset prices cannot be excluded in general and case-based decision-makers are not always able to learn to choose an optimal portfolio. The price dynamics in a market populated by case-based decision-makers exhibits patterns which can help explain phenomena such as arbitrage possibilities, excess volatility, bubbles and predictability of asset returns. However, if the aspiration level is chosen appropriately and the memory of the investors is long, case-based decision-makers learn to choose the optimal portfolio in the limit.

The explanatory power of the case-based decision theory depends crucially on the question of whether case-based decision-makers can survive in a financial market. Indeed, consider an economy with different types of investors whose shares in the population evolve over time. If it turns out that investors of a certain type (say, case-based decision-makers), acquire systematically lower profits compared to investors of another type (e.g., expected utility maximizers), then the wealth share of the case-based investors will become insignificant in the limit. Hence, these investors will not be able to influence the market process in the long-run. Whatever explanations have been found based on the premise that such investors exist in the market, they will be invalidated by the market evolution.

In this chapter, I address one small part of this question: whether case-based decision-makers can survive in a market in the presence of expected utility maximizers. Gilboa and Schmeidler (1996) show that if a case-based decision-maker adapts his aspiration level in a manner that is both ambitious and realistic, he eventually learn to imitate an expected utility maximizer with rational expectations provided that the environment is stationary. However, to the best of my knowledge, there are no results evaluating the relative performance of case-based decision-makers in a market environment. Therefore, a description of the evolution of such an economy is of interest.

Indeed, Matsui (2001) shows that each case-based decision-maker can be represented as an expected utility maximizer with beliefs over some state space. The state space is, however, different for expected utility maximizers and case-based decision-makers. Whereas this representation is still intuitive in an individual decision-making problem, in which the states of nature are viewed as subjective, its intuition seems to be lost if a market is considered in which the states of nature are exogenously determined (say by a dividend process) and in which investors with different beliefs can be naturally compared, only if their beliefs comprise the same state-space.

As usual, in the model presented in this chapter, case-based decision-makers base their decisions solely on what they have observed in the past. They evaluate an act according to its previous performance as recorded by their memory. The achieved returns are evaluated by comparing them to an aspiration level. The act that has the highest cumulative utility is then chosen. In contrast, the expected utility maximizers choose an act based on their (subjective) expectations about the asset returns in the next period. They prefer the act with the highest expected utility given their beliefs.

The proportions of the two types of investors and, therefore, their wealth shares evolve according to the relative success of both groups. The higher the returns achieved by a type of investors, the more off-springs they have, or, alternatively, the higher the share of the initial endowment they receive in the future. This endogenizes the initial endowment of the investors and allows to address the issue of the relative performance of these two strategies.

Two questions are of main interest:

First, whether expected utility maximizers can always outperform case-based decision-makers, i.e. whether case-based decision-makers are able to survive in such an environment;

Second, whether the effects observed in a market populated solely by case-based decision-makers also transfer to a market in which expected utility maximizers are present.

Indeed, even if it were found that case-based decision-makers can survive in presence of expected utility maximizers, it could be still the case that they survive by simply imitating the behavior of the latter. The answer to the second question is, therefore, crucial for the issue of the explanatory power of the case-based decision theory.

To analyze these issues, the chapter is organized as follows: section 1 surveys shortly the literature on evolutionary finance. Section 2 presents a description of the economy. In section 3, the evolutionary dynamic of investor types is introduced. In section 4, the evolutionary dynamic is analyzed for the case of risk-neutral expected utility maximizers in order to ask the question whether case-based decision-makers can survive in a financial market. Section 5 addresses the influence of case-based investors on prices. Sections 6 and 7 generalize some of the results from section 4 to the case of expected utility maximizers with a utility function exhibiting constant relative risk-aversion (CRRA). Section 8 concludes. The proofs of all propositions are stated in the appendix.

6.1 Survey of the Literature

In the last years, the problems of evolution in financial markets have been gaining attention in the economic literature. As a starting point serves the common view formulated by Friedman (1953) that markets select for rational traders with correct beliefs. At a first glance, the statement resembles a tautology: should noise traders enter the market, they would make losses, since they would be buying at high prices and selling at low prices. Therefore, rational traders would be able to make profits at their costs. As time evolves, the wealth of the noise traders would shrink to 0 and the market would be dominated by rational traders, who would determine the prices in equilibrium.

Thorough analysis of this question, however, leads to ambiguous results. One of the approaches chosen by De Long, Shleifer, Summers and Waldmann (1990, 1991) models explicitly the behavior of rational, as well as of noise traders. In an overlapping generations model they show that if the misperceptions of the noise traders lead them to choose a riskier portfolio than the one chosen by the rational traders, then the noise traders dominate the market by achieving higher expected returns than traders with correct beliefs. The reason is that rational traders engaging in arbitrage face the risk that they will have to leave the market at a time at which the mispricing caused by the noise traders aggravates. This effect is called noise trader risk. Whereas De Long, Shleifer, Summers and Waldmann (1990) obtain their results by comparative static, the latter article, De Long, Shleifer, Summers and Waldmann (1991), models an evolutionary financial market. The drawback of this evolutionary model consists in the assumption that asset prices are independent of the behavior of the market participants and, especially, of the strategy of the noise traders.

In a similar setting, but assuming a non-competitive market for assets, Palomino (1996) shows that noise traders can dominate the market even if the evolutionary selection accounts for the disutilities of risk-bearing. His results rely on an imitation dynamic based not on relative payoffs, but on a mean-variance function of the difference of returns of rational and noise traders.

These results opened a discussion on the criteria of investment strategies for which a market selects. Several studies on this issue, see Blume and Easley (1992), Hens and Schenk-Hoppé (2001), Evstigneev, Hens and Schenk-Hoppé (2002), show that the most successful strategy consists in maximizing logarithmic expected utility with correct beliefs. Since the logarithmic function has the property to maximize the growth of wealth, investors with such utility functions accumulate the whole market wealth over time and drive other types of investors to extinction. Evstigneev, Hens and Schenk-Hoppé (2002, 2003) further demonstrate that the strategy of expected utility maximizers with correct beliefs and logarithmic utility function is the sole globally stable strategy in a financial market.

Should a logarithmic utility maximizer be absent from the market, Blume and Easley (1992) show that the market selects for patient investors if relative risk-aversion is controlled for and for the investors with relative risk-aversion close to 1 if the discount factors are controlled for. Therefore, traders with systematically wrong beliefs survive only if they are more patient or if their coefficient of relative risk-aversion is close to 1, implying a utility function close to the logarithmic one. The influence of risk-aversion for survival is, hence, not unequivocal.

Correct beliefs can also be used as a selection criterium. Especially, in markets with perfect foresight, agents with beliefs closest to the truth perform best, as shown by Blume and Easley

(2001) and Sandroni (2000).

Whereas the major part of the literature searches for the best strategy, there is still little research into how different investment rules perform relative to each other. One such issue is addressed by Sciubba (2001), who analyzes the relative performance of the CAPM rule, as compared to logarithmic utility maximization with correct beliefs and mean-variance utility maximization. She shows that CAPM-traders vanish, whereas those maximizing a mean-variance utility imitate the logarithmic utility maximizers and, therefore, survive.

Given that selection criteria have been identified and studied, it seems that the question about the relative performance of two strategies is easy to answer by examining which of these strategies is superior according to a given criterium. This is, however, not the case for the two investment rules analyzed in this paper. The reason is that the models described above make strong assumptions about the available portfolio rules and the beliefs of the investors as well as on the market structure.

Most of the work cited, e.g., Blume and Easley (1992, 2001), Hens and Schenk-Hoppé (2001), Evstigneev, Hens and Schenk-Hoppé (2002), assumes that the assets in the economy are short-lived, thus, ignoring the influence of capital gains on the market selection. In this chapter, a long-lived asset is modelled.

Evstigneev, Hens and Schenk-Hoppé (2003) derive their results for a market with long-lived assets, but they assume, like Blume and Easley (1992) that the investment rules are simple, i.e. the share of wealth invested in a given asset remains constant over time, or that the investment rule does not depend on current prices. None of these assumptions holds, neither for the strategy of the case-based decision-makers, nor for those of the expected utility maximizers, in this model.

The results of Sandroni (2000) and of Blume and Easley (2001) are based on the assumption of perfect foresight, which cannot be fulfilled for case-based decision-makers in general and is not satisfied for the expected utility maximizers in this model, either.

6.2 The Economy

The structure of the model is essentially the same as the one presented in chapter 5, except for the introduction of expected utility maximizers. Consider an economy, consisting of a continuum of investors uniformly distributed on the interval $[0; 1]$. The economy evolves in discrete time $t = 0, 1....$ In period t, a proportion e_t of the investors are expected utility maximizers, whereas the rest, $c_t = 1 - e_t$, are case-based decision-makers. No population growth is considered.

The model has an overlapping generations structure. Each investor lives for two periods. The preferences of the investors are assumed to be such that they wish to consume only in the second period of their life. The preferences about the consumption in the second period are represented by a linear utility function $u(x) = x$, which is identical for all consumers. Hence, risk-neutrality is assumed. There is one consumption good in the economy with a price normalized to 1. The initial endowment of the investors consists of one unit of the consumption good in the first period and is 0 in the second period of their life.

There are two possible ways to transfer consumption between two periods: either using a riskless asset b, or investing in a risky asset a. The riskless asset is available in a perfectly elastic supply at a price of 1 in each period. It delivers $(1 + r)$ units of consumption good in period t for each unit of the consumption good invested in period $(t - 1)$.

The supply of the risky asset a is fixed at $A = 1$. The payoff of one unit of the asset in period t is:

$$\delta_t = \left\{ \begin{array}{ll} \delta & \text{with probability } q \\ 0 & \text{with probability } 1 - q \end{array} \right\}$$

and is identically and independently distributed in each period[122]. Let p_t denote the price of a in period t. New emissions are not considered, since I am interested in the behavior of prices on the secondary asset market only. I assume that the payoffs satisfy $1 > \delta > r > 0$.

Short sales are not permitted. Without loss of generality, it is also assumed that each single investor can invest in one of the assets only, i.e. diversification is allowed for the mass of investors

[122] q is interpreted as the objective probability of high returns, known to an external observer, but not necessarily to the investors in the economy, especially to the case-based decision makers.

of a given kind, but not for a single investor[123]. Therefore, the set of available acts reduces to:

$$\alpha_t^i \in \{a; b\},$$

with $i \in \{eu; cb\}$, where eu and cb identify the expected utility maximizers and the case-based decision-makers, respectively.

Given the act chosen by an investor of type i at time $(t-1)$, his indirect utility from consumption at time t can be written as:

$$v_t\left(\alpha_{t-1}^i\right) = \left\{ \begin{array}{ll} \frac{p_t}{p_{t-1}} + \frac{\delta_t}{p_{t-1}}, & \text{if } \alpha_{t-1}^i = a \\ 1+r, & \text{if } \alpha_{t-1}^i = b \end{array} \right\}. \tag{6.66}$$

Note that the utility derived from the choice of a depends not only on the dividend of the risky asset, but also on the price of a at time t, therefore on the decisions of the young investors at time t[124].

6.2.1 Information and Individual Decisions

The individual decision-making process predetermines the evolution of asset prices as well as of the shares of different investor types in the economy.

6.2.1.1 Case-Based Decision-Makers

First consider the case-based decision-makers. Their description of the situation contains the statement of the problem they have to solve: "Invest your initial endowment in one of the two assets, a or b to enable consumption tomorrow", as well as the set of available acts:

$$\alpha^{cb} \in \{a; b\}.$$

Unlike expected utility maximizers, case-based decision-makers do not use information about

[123] For expected utility maximizers with a linear utility function, this assumption can be made without loss of generality. In chapter 3, it has been shown that if the case-based decision makers in this model are allowed to diversify and if they have a similarity function on pairs of portfolios which is decreasing in the distance between two portfolios, then only case-based decision makers with relatively low aspiration level will diversify. Case-based decision makers with high aspiration levels will choose one of the non-diversified portfolios in each period. In section 4 of this chapter, it is shown, that case-based decision makers with low aspiration level ($\bar{u} < 1$) have no influence on prices in a stationary state. Therefore, I concentrate on the evolutionary dynamic aspiration levels exceeding 1. If these investors hold the same diversified portfolio in each period of time, then the analysis of section 4 applies, since the portfolio held by the case-based investors is less risky than the one chosen by the expected utility maximizers when $e \to 1$. If, however the case-based investors switch away from the initially held diversified portfolio in some period of time, they never diversify thereafter. Hence, the assumption of no diversification can be made without loss of generality, since it does not influence the limit results.

[124] This is due to the fact that the risky asset is long-lived. With a short-lived asset, the indirect utility of a would be independent of the current price p_t:

$$v_t(a) = \frac{\delta_t}{p_{t-1}}.$$

possible states of nature, state-contingent outcomes and their probability distribution. Therefore, they can only base their decisions on the experience of previous generations about which they are informed. This information is contained in their memory. Since in this model the problem of the case-based decision-makers is assumed to remain identical over time, an act chosen and a utility realization observed (indexed by the time) are sufficient to describe a case:

$$(\alpha_{t-1}; v_t(\alpha_{t-1})).$$

The memory of a given case-based decision-maker can, therefore, be represented by:

$$M_t^i = \left(\left(\alpha_{t-1}^i; v_t\left(\alpha_{t-1}^i\right)\right); \ldots \left(\alpha_{t-m-1}^i; v_{t-m}\left(\alpha_{t-m-1}^i\right)\right)\right)$$

where m parameterizes the length of memory, i.e. the number of cases remembered. I assume that the memory consists only of acts chosen and utilities realized by case-based decision-makers who lived in previous generations. The experience of expected utility maximizers is not taken into account.

Moreover, it is assumed that each investor can remember only one case per period. This assumption is important, because there might be periods in which some of the case-based decision-makers choose a, whereas others choose b. For simplicity, the case of one-period memory, i.e. $m = 0$ is analyzed[125]. Denote by γ_t^{cb} the proportion of case-based decision-makers which have chosen act a at time t. Now consider the memory of the young case-based decision-makers at time $(t + 1)$: since a proportion of γ_t^{cb} of the case-based decision-makers hold a in period t, I assume that the same proportion of the young case based decision-makers, or a mass of $\gamma_t^{cb} c_{t+1}$, will have memory

$$M_{t+1} = ((a; v_{t+1}(a)))$$

at time $(t + 1)$. In the same way, since a proportion of $\left(1 - \gamma_t^{cb}\right)$ of the case-based decision-makers hold b in period t, a mass of $\left(1 - \gamma_t^{cb}\right) c_{t+1}$ of the young case-based decision-makers will have memory

$$M_{t+1} = ((b; v_{t+1}(b)))$$

at time $(t + 1)$. This way of assigning memory has two advantages: first, it allows to capture the experience of all individual investors from the past period in the right proportions; second, this rule partitions the continuum of case-based decision-makers into at most two intervals, with

[125] Case-based decision makers with one-period memory behave in a very naive fashion, since they base their decision on one observation only. However, if it were found that they can survive in the presence of expected utility maximizers, one could argue that case-based decision makers with larger experience will also be able to survive in an asset market.

the property that all investors in a given interval have the same memory. In chapter 4, this was shown to be one of the properties guaranteeing the existence of an equilibrium in an economy with case-based decision-makers.

The aspiration level is denoted by $\bar{u}$ and is assumed to be identical for all case-based decision-makers in the economy and constant over time. However, since the memory might differ among the investors, the cumulative utilities will also differ in general. In this simple setting, the cumulative utilities can be written as follows:

$$U_t(\alpha) = \left\{ \begin{array}{l} [v_t(\alpha) - \bar{u}], \text{ if } M_t = ((\alpha; v_t(\alpha))) \\ 0, \text{ else.} \end{array} \right\}.$$

In each period, a case-based decision-maker compares the cumulative utilities of the two acts a and b available to him and chooses the one with the higher cumulative utility.

Given the indirect utility of consumption, (as determined above in (6.66)), the decision of a single case-based decision-maker takes the form:

$$\alpha_t^{cb} = \left\{ \begin{array}{lllll} a, & & \text{if } M_t = (a; v_t(a)) & \text{and} & v_t(a) \geq \bar{u} \\ & \text{or} & \text{if } M_t = (b; v_t(b)) & \text{and} & v_t(b) \leq \bar{u} \\ b, & & \text{if } M_t = (b; v_t(b)) & \text{and} & v_t(b) \geq \bar{u} \\ & \text{or} & \text{if } M_t = (a; v_t(a)) & \text{and} & v_t(a) \leq \bar{u} \end{array} \right\}$$

Note that, since the indirect utility of b is constant, the comparison $v_t(b) \gtreqless \bar{u}$ depends only on the parameters $\bar{u}$ and r and does not reverse over time. In contrast, the indirect utility of a and hence the comparison $v_t(a) \gtreqless \bar{u}$ depends on p_t.

Denote by $\tilde{p}_t$ the price of a, for which an investor with memory $(a; v_t(a))$ is indifferent between a and b:

$$\tilde{p}_t : \frac{\tilde{p}_t}{p_{t-1}} + \frac{\delta_t}{p_{t-1}} - \bar{u} = 0.$$

As long as $\tilde{p}_t > 0$, three possible cases can occur:

1. If $p_t > \tilde{p}_t$, then the investor chooses asset a ($\alpha_t^{cb} = a$).
2. If $p_t < \tilde{p}_t$, then the investor chooses asset b ($\alpha_t^{cb} = b$).
3. If $p_t = \tilde{p}_t$, then the investor is indifferent between holding a and b ($\alpha_t^{cb} \in \{a; b\}$).

Using this last argument, it is straightforward to aggregate over the population of case-based decision-makers to obtain the share of those holding asset a, $\gamma_t^{cb}(p_t)$. Since r and $\bar{u}$ are constants, three cases are relevant: $1 + r \lesseqgtr \bar{u}$:

1. $1 + r < \bar{u}$:

$$\gamma_t^{cb}(p_t) = \left\{ \begin{array}{ll} 1, & \text{if } \frac{p_t+\delta_t}{p_{t-1}} > \bar{u} \\ \left[1 - \gamma_{t-1}^{cb}; 1\right], & \text{if } \frac{p_t+\delta_t}{p_{t-1}} = \bar{u} \\ 1 - \gamma_{t-1}^{cb}, & \text{if } \frac{p_t+\delta_t}{p_{t-1}} < \bar{u} \end{array} \right\};$$

2. $1 + r = \bar{u}$:

$$\gamma_t^{cb}(p_t) = \left\{ \begin{array}{ll} \left[\gamma_{t-1}^{cb}; 1\right], & \text{if } \frac{p_t+\delta_t}{p_{t-1}} > \bar{u} \\ [0; 1], & \text{if } \frac{p_t+\delta_t}{p_{t-1}} = \bar{u} \\ \left[0; 1 - \gamma_{t-1}^{cb}\right], & \text{if } \frac{p_t+\delta_t}{p_{t-1}} < \bar{u} \end{array} \right\};$$

3. $1 + r > \bar{u}$:

$$\gamma_t^{cb}(p_t) = \left\{ \begin{array}{ll} \gamma_{t-1}^{cb}, & \text{if } \frac{p_t+\delta_t}{p_{t-1}} > \bar{u} \\ \left[0; \gamma_{t-1}^{cb}\right], & \text{if } \frac{p_t+\delta_t}{p_{t-1}} = \bar{u} \\ 0, & \text{if } \frac{p_t+\delta_t}{p_{t-1}} < \bar{u} \end{array} \right\}.$$

Figure 12 gives an illustration of $\gamma_t^{cb}(p_t)$ for these three cases. Note that $\gamma_t^{cb}(p_t)$ is a non-empty, closed- and convex-valued, upper hemicontinuous correspondence.

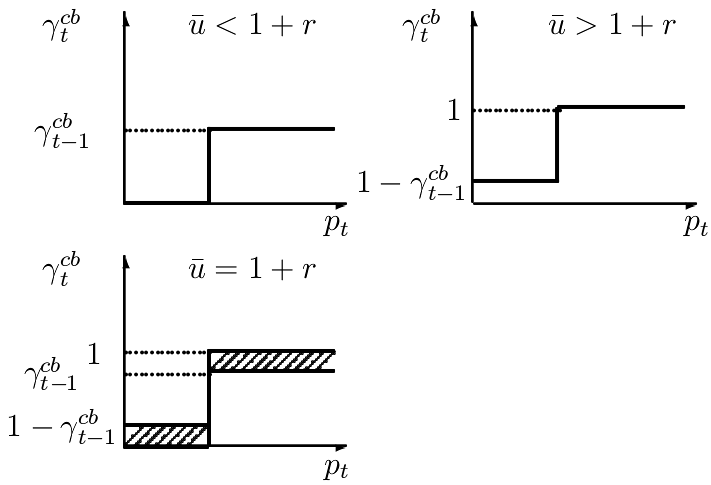

Figure 12

As the figure shows, the proportion of case-based decision-makers willing to hold a is increasing in the price p_t. This is the crucial difference between case-based decision-makers and expected utility maximizers, who prefer to hold asset a only if its price *does not exceed* some critical level.

6.2.1.2 Expected Utility Maximizers

Now I turn to the description of the expected utility maximizers. I assume that expected utility maximizers have expectations about the state-contingent payments of each of the assets. However, these expectations are not necessarily rational. Even if an expected utility maximizer is informed about the correct distribution of the dividends of the risky asset and of the returns of the safe technology, it is not clear that he will be able to predict the influence of the case-based decision-makers on the asset prices. To do so, an expected utility maximizer would also have to take into account the constitution of the population, the case-based decision-making process, as well as the influence the evolution of types has on prices and returns in the economy. I assume that expected utility maximizers neglect these issues. They act boundedly rational, taking into account the information about the correct distribution of dividends and the correct interest rate, but building their expectations about the price as if the economy consisted only of expected utility maximizers, identical to themselves.

The expected utility maximizers compute the fundamental value of the risky asset p^{eu} as the discounted value of the expected future dividends:

$$p^{eu} = \frac{q\delta}{r}$$

and perceive p^{eu} to be the "true" price of the risky asset[126]. Therefore, they hold a if $p_t < p^{eu}$, b if $p_t > p^{eu}$ and are indifferent between the two assets at the critical price p^{eu}. Differently from the case-based decision-makers, expected utility maximizers, therefore, hold the risky asset only if its price is relatively low.

Aggregating the demand of all expected utility maximizers, it is possible to derive the share of those willing to hold a, γ_t^{eu}, depending on p_t:

$$\gamma_t^{eu}(p_t) = \left\{ \begin{array}{ll} 1, & \text{if } p_t < p^{eu} \\ [0;1], & \text{if } p_t = p^{eu} \\ 0, & \text{if } p_t > p^{eu} \end{array} \right\}.$$

Like $\gamma_t^{cb}(p_t)$, $\gamma_t^{eu}(p_t)$ is a non-empty, closed- and convex-valued, upper hemicontinuous corre-

126 The fundamental value p^{eu} corresponds to the price under rational expectations if only risk-neutral expected utility maximizers are present in the market. Although expected utility maximizers do not need to have rational expectations in general, their expectations are rational in the limit when $e \to 1$.

spondence. The value of demand for a of the whole population is obtained as:

$$d_t(p_t) = e_t \gamma_t^{eu}(p_t) + (1 - e_t)\gamma_t^{cb}(p_t).$$

It is also a correspondence, which has the characteristics stated above and maps the interval $[0;1]$ into $[0;1]$.

6.2.2 Temporary Equilibrium

The state of the economy is determined by a quadruple of endogenous variables $\left(e_t; \gamma_t^{eu}; \gamma_t^{cb}; p_t\right)$ and by the random dividend payment δ_t. Assume first that the share of expected utility maximizers in the population at time t, e_t, is given.

A temporary equilibrium at time t for a given e_t is defined by:

- portfolio choices for all young investors described by the proportion of case-based decision-makers choosing a, γ_t^{cb}, and the proportion of expected utility maximizers choosing a, γ_t^{eu}, and
- a price of the asset a — $p_t^*(e_t)$,

such that following conditions are fulfilled:

1. the value of demand for a of the young case-based decision-makers satisfies:

$$\gamma_t^{*cb} = \gamma_t^{cb}(p_t^*(e_t)),$$

2. the value of demand for a of the young expected utility maximizers satisfies:

$$\gamma_t^{*eu} = \gamma_t^{eu}(p_t^*(e_t));$$

   where $p_t^*(e_t)$ is the equilibrium price for the given e_t and

3. the market for the risky asset is cleared: either $p_t^*(e_t) > 0$ satisfies

$$\frac{\gamma_t^{eu}(p_t^*(e_t))\, e_t + \gamma_t^{cb}(p_t^*(e_t))\,(1 - e_t)}{p_t^*(e_t)} = A \equiv 1,$$

   or

$$p_t^*(e_t) = 0 \text{ and } \gamma_t^{eu}(0)\, e_t + \gamma_t^{cb}(0)\,(1 - e_t) = 0.$$

Since the value of demand $d_t(p_t)$ is a closed- and convex-valued, non-empty and upper hemicontinuous correspondence, which, according to the budget constraints and to the short sales constraint, is bounded between $[0;1]$, the following corollary of proposition 4.9 in chapter 4 obtains:

Corollary 6.1 *Given the proportions of the case-based decision-makers and the expected utility maximizers in the population e_t and c_t such that $e_t + c_t = 1$, a temporary equilibrium exists in each period $t \geq 1$.*

Whereas in a market populated only by case-based decision-makers, it is in general not possible to prevent the price of the risky asset in an equilibrium from falling to 0, such a situation cannot occur in a market with expected utility maximizers. 0-equilibrium prices are caused by the fact that the demand of the case-based decision-makers can be very insensitive to price changes at low prices. Therefore, very low aspiration levels are needed to exclude degenerate equilibria. Expected utility maximizers, on the contrary, are ready to buy the risky asset, if its price is near 0. Therefore, $\gamma_t^{eu}(0) > 0$ always holds, as long as $p^{eu} > 0$. The following proposition obtains:

Proposition 6.1 *As long as $p^{eu} > 0$ and $e_t > 0$ hold, $p_t^*(e_t) > 0$ obtains in a temporary equilibrium.*

Hence, one of the effects caused by case-based investors, namely that some of the assets with strictly positive fundamental value can have 0 prices in an equilibrium, is eliminated by introducing some small positive amount of expected utility maximizers into the economy, who believe that the risky asset has a positive fundamental value. Hence, degeneracy of equilibria is not robust in this respect[127].

6.3 The Evolution of Investor Types

After insuring that an equilibrium exists in each period of time, I now introduce the selection dynamic. I measure the fitness of a given type of investors by the actual average returns they achieve relative to the average returns of the society as a whole. This gives rise to a replicator dynamic, in which the type of investors who perform better on average grows. Note that since each investor is born with the same initial endowment of 1 unit of the consumption good, the share of a type of investors can be identified with the total income of the investors of this type.

[127] However, see section 5 for an example that $e_t^* = 0$ and $p_t^* = 0$ can obtain in a temporary equilibrium even if the initial value of e is positive.

This raises the question of whether this kind of evolution is to be interpreted as internal or external[128].

Internal evolution is used to describe situations in which the number of investors following a given strategy remains unchanged and the selection is based on wealth changes. Strategies which are not successful lose money and the most successful strategy accumulates the whole wealth of the economy in the limit. Such is the evolution of wealth shares described in the analysis of Hens and Schenk-Hoppé (2001), Evstigneev, Hens, Schenk-Hoppé, (2002, 2003), as well as in Blume and Easley (1992, 2001).

An overlapping generations structure does not allow for a natural wealth dynamic to arise as in the models above. Therefore, I work with a replicator dynamic, which describes external evolution. In this case, the wealth of each investor remains unchanged, but the number of investors of a given type changes depending on the success of their strategy. It is, as if those who use more successful strategies found more imitators than those using less profitable strategies. In the limit all of the individuals follow the most successful strategy, driving suboptimal strategies to extinction. This idea is one of the basics of the evolutionary game theory. The replicator dynamic has been applied by Alós-Ferrer and Ania (2003) to analyze the asset market game introduced by Shapley and Shubik (1977). Since the assets in their model are short-lived, it is possible to apply the standard replicator dynamic used in the evolutionary game theory without changes. With long-lived assets, the replicator dynamic has to be modified so as to take into account the capital gains caused by price changes. The replicator dynamic introduced in the next paragraph, therefore, exhibits properties which differ from those of the replicator dynamic usually adopted in the evolutionary game theory.

As was explained above, in the current model, the number of investors following a given strategy coincides with the wealth share invested in this strategy. Therefore, the interpretations in terms of internal and external evolution are both feasible in this context[129].

[128] I thank Thorsten Hens for pointing out this aspect to me.

[129] An alternative interpretation of this model proposed by Thorsten Hens would be to think of infinitely living but myopic investors who reinvest their wealth in each period, but optimize for one period ahead only. In this case, the evolution of the wealth *shares* of these investors would coincide with the replicator dynamic introduced below. Note that this alternative assumption would not change the price expectations and the behavior of the expected utility maximizers as long as their utility function exhibits constant relative risk-aversion. The behavior of the case-based decision makers would also remain unchanged as long as the similarity between the problems encountered is not influenced by the wealth available.

6.3.1 Replicator Dynamic

Differently from the usual approach in evolutionary game theory, the replicator dynamic in this model is applied not to the portfolio strategy chosen by an individual, but to the "meta"-strategies used by the two types of investors, hence, to the performance of case-based decision-making versus expected utility maximization[130].

One can think of the "game" associated with the replicator dynamic introduced below in the following way: individuals out of an infinite population are allowed to choose one of the two pure strategies "case-based decision-maker" and "expected utility maximizer". The payoffs are then determined according to the equilibrium growth of wealth of the investors using each strategy. Blume and Easley (1992) demonstrate that this structure indeed gives rise to a replicator dynamic in a model with infinitely living investors and short-lived assets, see Friedman (1998, p. 25, footnote 7). Therefore, the fitness of the strategies is measured not according to the utility obtained, but according to the realized returns. Higher realized returns mean higher wealth share for the particular type of investors in the economy and, therefore, a greater influence on market processes[131]. If it were found that given this selection dynamic, case-based decision-makers' wealth share decreases to 0 almost surely, then the influence of this type of investors on market prices would vanish with time. Whatever effects then have been observed in an economy populated solely by case-based decision-makers, these effects would disappear.

The following replicator dynamic is introduced, following Weibull (1995, pp. 124-125)[132].

First, note that since investors from the same type might choose to hold different assets in a given period of time, it is necessary to compute the average performance of each type of investors. Therefore, denote by

$$\tilde{v}_t^{eu} = \gamma_{t-1}^{eu} v_t(a) + \left[1 - \gamma_{t-1}^{eu}\right] v_t(b) \tag{6.67}$$

[130] This approach is, therefore, similar to the indirect evolutionary approach initiated by Güth and Yaari (1992). In their setup, the genetic phenotype describes a decision rule for choosing a strategy in a game. The solution of the game, computed in accordance with the proportions in which these phenotypes are present, determines the payoffs and, hence, the evolution of decision rules (not of strategies) in the population. The present model differs, however, from the work of Yaari and Güth by the fact that instead by a game, the payoffs are determined by a market.

[131] This property, which follows from the market clearing condition in this model, need not hold in general. See for instance Kogan, Ross, Wang and Westerfield (2003) for a model, in which noise traders can influence the price process, even though their share converges to 0 in the limit.

[132] In this model the length of period is assumed to be 1 and the growth rate of the population is 0. This correspondes to $\tau = 1$ and $\beta = 0$ in the overlapping generations model presented by Weibull (1995).

$$\begin{aligned}
\tilde{v}_t^{cb} &= \gamma_{t-1}^{cb} v_t(a) + \left[1-\gamma_{t-1}^{cb}\right] v_t(b) \\
\tilde{v}_t &= \left[\gamma_{t-1}^{eu} e_{t-1} + (1-e_{t-1})\,\gamma_{t-1}^{cb}\right] v_t(a) + \left[1-\gamma_{t-1}^{eu} e_{t-1} - \gamma_{t-1}^{cb}(1-e_{t-1})\right] v_t(b)
\end{aligned}$$

the average returns achieved by an expected utility maximizer, by a case-based decision-maker and by the whole society at time t, respectively. I assume, as usual for the replicator dynamic that the type of investors who performs better than the average have increasing share in the population, whereas the share of the worse performing type shrinks. Since the population remains constant over the time, the dynamics can be described by the change of the variables e_t and c_t:

$$\begin{aligned}
e_t &= \frac{\tilde{v}_t^{eu}}{\tilde{v}_t} e_{t-1} \qquad (6.68)\\
c_t &= \frac{\tilde{v}_t^{cb}}{\tilde{v}_t} c_{t-1}.
\end{aligned}$$

Since

$$\begin{aligned}
e_t + c_t &= \frac{\tilde{v}_t^{eu}}{\tilde{v}_t} e_{t-1} + \frac{\tilde{v}_t^{cb}}{\tilde{v}_t} c_{t-1} = \\
&= \frac{\left[\gamma_{t-1}^{eu} v_t(a) + \left[1-\gamma_{t-1}^{eu}\right] v_t(b)\right] e_{t-1} + \left[\gamma_{t-1}^{cb} v_t(a) + \left[1-\gamma_{t-1}^{cb}\right] v_t(b)\right] c_{t-1}}{\left[\gamma_{t-1}^{eu} e_{t-1} + (1-e_{t-1})\,\gamma_{t-1}^{cb}\right] v_t(a) + \left[1-\gamma_{t-1}^{eu} e_{t-1} - \gamma_{t-1}^{cb}(1-e_{t-1})\right] v_t(b)} = \\
&= 1
\end{aligned}$$

holds, the condition that the population does not grow as a whole is satisfied. Therefore, the evolution of e_t is enough to determine the dynamic of the system. Using (6.66), the equilibrium share of expected utility maximizers from (6.68) can be written as:

$$e_t^* = \frac{\left[\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}} \gamma_{t-1}^{eu} + (1+r)\left(1-\gamma_{t-1}^{eu}\right)\right] e_{t-1}}{\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}} \left(\gamma_{t-1}^{eu} e_{t-1} + \gamma_{t-1}^{cb}(1-e_{t-1})\right) + (1+r)\left(1-\left(\gamma_{t-1}^{eu} e_{t-1} + \gamma_{t-1}^{cb}(1-e_{t-1})\right)\right)}. \qquad (6.69)$$

Note that the numerator represents the wealth of the expected utility maximizers (i.e. the value of the portfolio held by the old expected utility maximizers) at time t, whereas the denominator corresponds to the wealth of the whole society at t. Hence, the proportion of investors following a decision rule at time t is equal to the relative share of wealth held by these investors. I, therefore, claim that the replicator dynamic defined by (6.68) can indeed be interpreted as a relative wealth dynamic in the sense of Blume and Easley (1992, 2001), Hens and Schenk-Hoppé (2001) and Evstigneev, Hens and Schenk-Hoppé (2002, 2003).

The price of the risky asset p_{t-1}, however, might become 0 in a period, in which $e_{t-1} = 0$. Therefore, e_t^* is not always well defined. Nevertheless, a limit of e_t^* can be computed for this case. Should $p_{t-1} = 0$ be an equilibrium price, only expected utility maximizers would be (potentially) willing to hold a at time $(t-1)$, hence $p_{t-1} = \gamma_{t-1}^{eu} e_{t-1}$ in an equilibrium with a positive share of expected utility maximizers. Moreover, since $p^{eu} > 0$, $\gamma_{t-1}^{eu} = 1$ will hold for prices near 0. Substituting in (6.69) one obtains:

$$\lim_{e_{t-1}\to 0} e_t^* = \lim_{e_{t-1}\to 0} \frac{\frac{p_t^*(e_t^*)+\delta_t}{e_{t-1}} e_{t-1}}{\frac{p_t^*(e_t^*)+\delta_t}{e_{t-1}} e_{t-1} + (1+r)(1-e_{t-1})} = \frac{p_t^*(e_t^*)+\delta_t}{p_t^*(e_t^*)+\delta_t+1+r},$$

which is well defined. This means, especially, that starting with $e_t = 0$, the mass of expected utility maximizers may become strictly positive if expected utility maximizers hold an asset with positive fundamental value the price of which is 0.

Alternatively, if the initial mass of the case-based decision-makers is 0, then it remains 0 in all subsequent periods. Indeed, let $c_{t-1} = 1 - e_{t-1} = 0$. Then

$$\begin{aligned} e_t^* &= \frac{\left[\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}} \gamma_{t-1}^{eu} + (1+r)\left(1-\gamma_{t-1}^{eu}\right)\right] e_{t-1}}{\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}} \left(\gamma_{t-1}^{eu} e_{t-1} + \gamma_{t-1}^{cb}(1-e_{t-1})\right) + (1+r)\left(1-\left(\gamma_{t-1}^{eu} e_{t-1} + \gamma_{t-1}^{cb}(1-e_{t-1})\right)\right)} \\ &= \frac{\left[\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}} \gamma_{t-1}^{eu} + (1+r)\left(1-\gamma_{t-1}^{eu}\right)\right] e_{t-1}}{\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}} \gamma_{t-1}^{eu} e_{t-1} + (1+r)\left(1-\gamma_{t-1}^{eu} e_{t-1}\right)} = 1. \end{aligned}$$

This is due to the fact that the expected utility maximizers perceive the risky asset as valuable, preventing its price from falling to 0. Hence, if case-based decision-makers have an aggregate initial endowment of 0, they are not able to invest and to achieve returns in the future.

The replicator dynamic introduced above describes the evolution of wealth for the two types of investors. Nevertheless, even if the only fitness criterium considered is the wealth growth, (6.68) is not the only way to characterize the selection process. The choice of the replicator dynamic is made here for computational simplicity: its linearity allows for explicit solutions in a model with two strategies.

The question arises, whether the introduction of an alternative dynamic would change the results significantly. It seems that the stability results presented below are robust in this respect. Weibull (1995, p. 88) and Nachbar (1990, p. 78) show that a stable state under a regular replicator dynamic is a Nash equilibrium. Like the regular[133] replicator dynamic, (6.68) also selects a Nash-equilibrium of the "meta"-game described above[134]. Especially, $e = 1$ is a stable state of

[133] See Weibull (1995, p. 141) for a definition of a regular dynamic. In contrast to the replicator dynamic presented in this chapter, a regular dynamic has the property that starting from an interior point of the simplex, the system remains forever in its interior, whereas starting at a vertice, the system never leaves it. Crucial for this property is the assumption of Lipschitz continuity of the replicator dynamic. Since the replicator dynamic in this model is not Lipschitz continuous at $e = 0$, it violates this invariance property.

[134] This is due to the fact that Lipschitz continuity is violated only at $e = 0$ and that this point is never a Nash

the system only if it is a Nash equilibrium of this ”meta”-game.

Consider the class of evolutionary dynamics which are payoff-monotone, i.e. the growth of the share of a strategy in the population is increasing in the payoff of the strategy. This class contains the replicator dynamic as a special case. However, if there are only two available strategies, this class also includes the sign-preserving dynamic, i.e. the shares of the strategies performing better than the average grow and vice versa, as well as the positive correlation dynamic, i.e. the share of at least one of the strategies performing better than the average grows. Friedman (1998, p. 25) gives a classification, whereas Weibull (1995, pp. 149-152) provides a description and a characterization of these types of evolutionary dynamics.

Moreover, as Friedman (1998, p. 40) points out, for the case of two strategies, the class of payoff-monotonic dynamics also includes the best-reply dynamic, see Gilboa and Matsui (1991) and Fudenberg and Levine (1997), as well as fictitious play, see Brown (1951) and Fudenberg and Levine (1997).

Weibull (1995, pp. 147-148) further shows that every continuous-time pay-off monotone regular dynamic exhibits the property that the set of stable states contains only Nash equilibria. The same result is derived by Nachbar (1990, p. 78) for the discrete-time version.

Since the main focus of this paper is on the stability of the state $e = 1$, in which only expected utility maximizers are present in the market, and since the replicator dynamic (6.68) is Lipschitz continuous (and therefore regular) at $e = 1$[135], it seems that these results can be carried over to the model at hand. Especially, every selection dynamic, which is monotone in the growth of wealth and regular at $e = 1$ will exhibit the same stability properties at $e = 1$, as the replicator dynamic (6.68).

6.3.2 Temporary Equilibrium with Replicator Dynamic

The equilibrium share of expected utility maximizers is only implicitly determined by (6.69), since it depends on p_t^*, which on its turn depends on e_t^*. Therefore, it is necessary to introduce a new equilibrium concept in which the shares of the two types in the population are determined

equilibrium of the ”meta”-game.

135 See the derivation of the explicit form of the replicator dynamic at $e = 1$ in the proofs of propositions 6.5, 6.6, 6.8 and 6.10. It is easily seen that the replicator dynamic is continuously differentiable at $e = 1$ and, therefore, Lipschitz continuous.

endogenously.

Given $\left(e_{t-1};\gamma_{t-1}^{eu};\gamma_{t-1}^{cb};p_{t-1}\right)$, a temporary equilibrium with replicator dynamic at time t is defined as a vector: $\left(e_t^*;\gamma_t^{*eu};\gamma_t^{*cb};p_t^*\right)$, such that:

(i) $\gamma_t^{*eu}=\gamma_t^{eu}\left(p_t^*\right)$

(ii) $\gamma_t^{*cb}=\gamma_t^{cb}\left(p_t^*\right)$

(iii) $p_t^*\left(e_t^*\right)$ clears the market for the risky asset given e_t^*;

(iv) e_t^* is determined by the replicator dynamic:

$$e_t^*=\frac{\left[\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}}\gamma_{t-1}^{eu}+(1+r)\left(1-\gamma_{t-1}^{eu}\right)\right]e_{t-1}}{\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}}\left(\gamma_{t-1}^{eu}e_{t-1}+\gamma_{t-1}^{cb}\left(1-e_{t-1}\right)\right)+(1+r)\left(1-\left(\gamma_{t-1}^{eu}e_{t-1}+\gamma_{t-1}^{cb}\left(1-e_{t-1}\right)\right)\right)}.$$

Corollary 6.1 does not guarantee the existence of such an equilibrium. Nevertheless, it is possible to show that such an equilibrium exists in each period as long as the initial state $\left(e_{t-1};\gamma_{t-1}^{eu};\gamma_{t-1}^{cb};p_{t-1}\right)$ is an equilibrium[136]. The evolution of the system is therefore well defined.

Proposition 6.2 *Suppose that* $\left(e_{t-1};\gamma_{t-1}^{eu};\gamma_{t-1}^{cb};p_{t-1}\right)$ *is such that*

$$p_{t-1}=e_{t-1}\gamma_{t-1}^{eu}\left(p_{t-1}\right)+\left(1-e_{t-1}\right)\gamma_{t-1}^{cb}\left(p_{t-1}\right).$$

Given such $\left(e_{t-1};\gamma_{t-1}^{eu};\gamma_{t-1}^{cb};p_{t-1}\right)$*, a temporary equilibrium with replicator dynamic at time* t *exists.*

6.4 Analysis of the Dynamic

The definition of a temporary equilibrium with replicator dynamic, together with the dividend process determine the evolution of the system. I first discuss the stationary states.

6.4.1 Stationary States

Since I am interested in the evolution of the investor types in the market, I define a stationary state as a state in which $e_t^*=const$ for all t, whereas portfolio holdings and prices might but need not be constant. Nevertheless, it is possible to show that only steady states in which all four state variables are constant can occur.

Proposition 6.3 (i) $e=1$ *is a stationary state.*

(ii) $e=0$ *is a stationary state if* $\bar{u}<1$ *and* $\gamma_0^{cb}>0$ *hold.*

[136] In fact, the condition stated in proposition 6.2 is weaker. It only requires that at $(t-1)$ a temporary equilibrium for the given share e_{t-1} of the expected utility maximizers obtains.

Since in these two stationary states the whole population consists of one type of investors only, the mean return of this type equals the mean return of the population. Therefore, the proportions of the two types do not change over time. Note that in the case $e = 1$, the price of the risky asset is stable over time and satisfies:

$$p_t^* = \gamma_t^{eu} = \min\{p^{eu}; 1\} \text{ for all } t \geq 0.$$

Therefore, as long as the no-short-sales condition is not binding, the asset price coincides with the fundamental value of the asset and a rational expectations equilibrium emerges.

For the case $e = 0$, the price remains constant over the time and satisfies

$$p_t^* = \gamma_0^{cb}.$$

Obviously, the price need not coincide with the fundamental value p^{eu}. In this case, arbitrage opportunities might remain unused in the market.

Proposition 6.3 shows that only a relatively low aspiration level allows the case-based decision-makers to keep their mass at 1 in the market. Should their aspiration level exceed this benchmark, then there would be periods in which the case-based decision-makers refuse to hold a even at a price of 0. In such periods, expected utility maximizers would be able to acquire a for free. Should now the dividend of a be strictly positive in the next period, the mass of expected utility maximizers would become positive. Hence, $e = 0$ would not be stationary in this case.

The trivial stationary states derived in proposition 6.3 describe situations in which only one of the two types of investors dominates the market. One can also identify stationary states in which both types of traders coexist. Of course, in order to gain the same mean return in each period, both types must hold the same portfolio in every period. Furthermore, to insure that case-based decision-makers do not change their portfolio over time, their aspiration level should be relatively low:

$$\bar{u} < \frac{p^* + \min\{\delta_t\}}{p^*} = 1,$$

i.e. it should not exceed the return of the risky asset in periods in which the dividend payment is 0 and its price remains unchanged compared to the last period of time.

Proposition 6.4 *Let* $\bar{u} < 1$.

1. *If* $p^{eu} > 1$, *then each* $e \in [0; 1]$ *is a stationary state, provided that the portfolios held and*

the price of a *fulfil:*

$$\begin{aligned}\gamma^{cb} &= 1\\ \gamma^{eu} &= 1\\ p^* &= 1.\end{aligned}$$

2. *If* $p^{eu} \leq 1$*, then each* $e \in [0;1]$ *is a stationary state, provided that the portfolios held and the price of* a *fulfil:*

$$\begin{aligned}\gamma^{cb} &= p^{eu}\\ \gamma^{eu} &= p^{eu}\\ p^* &= p^{eu}.\end{aligned}$$

Note that as long as $p^{eu} \leq 1$ holds, the price in the stationary state coincides with the price under rational expectations. Moreover, it is not possible to distinguish between case-based decision-makers and expected utility maximizers. Both types of investors hold the same optimal portfolio at the equilibrium price. By imitating the expected utility maximizers, case-based decision-makers with relatively low aspiration levels are, thus, able to survive in a financial market. However, they cannot influence prices and it is not possible to empirically reject the hypothesis of rational expectations and expected utility maximization in such a market. It is, therefore, interesting whether a positive share of case-based decision-makers can survive if the portfolio strategies of expected utility maximizers and case-based decision-makers differ.

6.4.2 Stability of $e = 1$

It has been shown that case-based decision-makers with an aspiration level lower than 1 can survive in a market without influencing prices. Now I shall look at the dynamics of the system for case-based decision-makers with aspiration levels higher than 1.

If the aspiration level of the case-based decision-makers is relatively high, their behavior might influence prices. The price dynamic in a market populated only by case-based decision-makers has been discussed in chapter 5. Transferring the results obtained in chapter 5 to the context of the current model, it is easy to derive the dynamic of prices and asset holdings for the case $e = 0$. If $1 + r > \bar{u} > 1$, $p_t^* = 0$ in each period holds and all investors hold b in every period.

For relatively high aspiration levels, $1 + \delta > \bar{u} > 1 + r$, the price process is a stochastic cycle with two states: $p_h = 1$ and $p_l = 0$. The Markov transition matrix describing this process is given by:

	$p_{t+1} = p_h$	$p_{t+1} = p_l$
$p_t = p_h$	q	$1 - q$
$p_t = p_l$	1	0

.

Since, the evaluation of the risky asset by the expected utility maximizers is constant over time, it is intuitively clear that similar patterns can be expected to emerge in an economy in which both types of investors are present. Indeed, proposition 6.1 shows that in the presence of expected utility maximizers the price in state l would be strictly positive. Furthermore, as long as $p^{eu} < 1$ holds, the price p_h would be lower than 1. Nevertheless, it is possible that cycles similar to those described in section 3 of chapter 5 emerge in an economy in which both case-based decision-makers and expected utility maximizers are present.

In the current model, the magnitude of these cycles depends positive on the mass of case-based decision-makers in the economy. Therefore, such cycles can persist only if a positive mass of case-based decision-makers survives. Hence, it is necessary to examine the stability of the stationary state $e = 1$. Only if this stationary state is not stable, the case-based decision-makers will be able to survive and influence the prices in the market.

The discussion of the results for asset markets without expected utility maximizers in chapter 5 shows that the dynamic of the system crucially depends on the aspiration level of the case-based decision-makers. It has already been shown that for $\bar{u} < 1$ stationary states with a positive mass of case-based decision-makers are possible. Moreover, the value of e in these stationary states is undetermined and can vary between $[0; 1]$. Hence, the further discussion concentrates on the case in which such stationary states do not occur, i.e. on $\bar{u} > 1$. Two cases have to be considered: $\bar{u} \in (1 + r; 1 + \delta)$, referred to as the case of high aspiration levels and $\bar{u} \in (1; 1 + r)$, the case of low aspiration levels.

For $\bar{u} \in (1; 1 + r)$, it is easy to see that the return of b is satisfactory for the case-based decision-makers, whereas the return of a, given that the dividend is 0 and the price of a remains unchanged or falls, is not satisfactory. Even if all case-based decision-makers hold a in period 0, the return of a will fall below their aspiration level almost surely in finite time. Hence, they will switch to b and hold it forever.

For $\bar{u} \in (1 + r; 1 + \delta)$, the return of a is considered satisfactory when the dividend is high and the price of a weakly increases, whereas the return of b and the return of a if its dividend is low, are regarded as unsatisfactory. If the initial endowment with assets in period 1 is identical for all case-based decision-makers, these investors switch infinitely often between a and b. The case-based decision-makers hold a as long as its dividend is high, switch to b in the first period of low dividends t' and choose again a in period $t'+1$, since $(1+r) < \bar{u}$ holds.

6.4.2.1 The Case of High Aspiration Levels

Consider first the case of high aspiration level:

$$\bar{u} \in (1+r; 1+\delta)$$

and assume that the fundamental value p^{eu} exceeds 1.

Under these two assumptions, the dynamics of prices and asset holdings can be described as follows:

$$\begin{aligned}
\gamma_t^{*eu} &= 1 \text{ for each } t; \\
\gamma_t^{*cb} &= \left\{ \begin{array}{ll} 1, \text{ if} & \gamma_{t-1}^{cb} = 1 \text{ and } \delta_t = \delta \text{ or} \\ & \gamma_{t-1}^{cb} = 0 \\ 0, \text{ if} & \gamma_{t-1}^{cb} = 1 \text{ and } \delta_t = 0 \end{array} \right\}; \\
p_t^* &= \left\{ \begin{array}{ll} 1, \text{ if} & \gamma_{t-1}^{cb} = 1 \text{ and } \delta_t = \delta \text{ or} \\ & \gamma_{t-1}^{cb} = 0 \\ e_t^*, \text{ if} & \gamma_{t-1}^{cb} = 1 \text{ and } \delta_t = 0 \end{array} \right\}.
\end{aligned}$$

Note that the average returns of the two types of investors are identical in periods in which both types hold a. Therefore, the population shares remain unchanged:

$$e_t^* = e_{t-1}^*, \text{ if } \gamma_{t-1}^{*cb} = 1.$$

Hence, these periods do not influence the dynamic of population shares. It is, therefore, sufficient to analyze how e_t changes in periods in which the holdings of both types of investors differ.

Proposition 6.5 *Let the aspiration level satisfy*

$$\bar{u} \in (1+r; 1+\delta)$$

and suppose that $p^{eu} \geq 1$ holds.

1. *If $p^{eu} < \frac{1+\delta}{1+r}$, then there exists an $\tilde{e} \in \left(\frac{1}{1+r}; 1\right)$, such that e_t^* is a submartingale as long as $e_{t-1}^* < \tilde{e}$ and a supermartingale as long as $e_{t-1}^* \geq \tilde{e}$.*

2. *If* $p^{eu} \geq \frac{1+\delta}{1+r}$*, then* e_t^* *is a submartingale for all* $e_{t-1}^* \in [0;1]$*. The case-based decision-makers disappear with probability* 1.

First note that the condition that $p^{eu} \gtreqqless \frac{1+\delta}{1+r}$ is equivalent to $q \gtreqqless \frac{(1+\delta)r}{(1+r)\delta}$. Since $\frac{(1+\delta)r}{(1+r)\delta} \in (0;1)$, both cases $1.$ and $2.$ are possible. Proposition 6.5 has, therefore, a simple interpretation: if the initial share of expected utility maximizers in the market is relatively small, then in expectation their share increases:

$$E\left[e_t^* \mid e_{t-1}^* < \tilde{e}\right] \geq e_{t-1}^*.$$

If, however, their share is relatively large, then the behavior of the system depends on the probability of high dividends. If this probability is relatively low, then the share of expected utility maximizers decreases in expectations. This means that the stationary state $e = 1$ is not stable, in the sense that there is a positive probability that the replicator dynamics does not converge to it. Neither the case-based decision-makers, nor the expected utility maximizers will vanish with probability 1 in this case. Alternatively, if the probability of high dividends is large, then the case-based decision-makers vanish on almost all paths of dividend realizations.

The intuition behind this result is simple: the replicator dynamic of e_t^* is concave in the returns of the expected utility maximizers. Therefore, it selects for the less risky strategy, given that the expected returns of two strategies are identical. In those periods in which the case-based decision-makers hold b, their portfolio is less risky than the portfolio of the expected utility maximizers, who hold a. Moreover, if we let the share of the case-based decision-makers go to 0 and assume $p^{eu} = 1$, the expected returns of both portfolios become identical and the replicator dynamic selects for the less risky one, hence, for the one held by the case-based decision-makers. By continuity, the same result holds in some surrounding of $e = 1$ and in some surrounding of $p^{eu} = 1$ and, therefore, as long as p^{eu} is not very large, a positive share of case-based decision-makers survives with positive probability in the limit.

If, however, p^{eu} exceeds $\frac{1+\delta}{1+r}$, the excess return of the expected utility maximizers is sufficiently high to compensate for the higher risk of their portfolios. In this case, they accumulate the whole market wealth with probability 1 in the limit.

Note that higher values of p^{eu} correspond to higher values of q, ceteris paribus. The probability of high dividends has two effects on the evolutionary dynamic. On the one hand, higher q implies higher expected returns of the risky asset and, therefore, higher profits for the investors holding a, i.e. for the expected utility maximizers. On the other hand, higher values of q cause the case-based decision-makers to switch less frequently between the two undiversified portfolios and to hold the risky asset during a larger share of time, hence to behave in a less risk-averse manner[137]. These two effects work in the same direction, making the strategy of the expected utility maximizers more successful.

This result should not be surprising. The literature on evolutionary financial markets cited in the introduction shows that correct beliefs alone do not guarantee survival. The form of the utility function is crucial for the ability of an investor to accumulate wealth. Only for the logarithmic utility function does the selection criterium used by the wealth dynamic coincide with the target of expected utility maximization with correct beliefs. Hence, a proper degree of risk-aversion combined with correct beliefs is needed for survival. Therefore, expected utility maximizers with correct beliefs in the limit can be ”outperformed” according to the replicator dynamic by investors who do not have correct beliefs, but behave as if they were risk-averse.

Note that $e = 1$ is a Nash-equilibrium of the ”meta”-game described in section 3 only in the case in which it is stable under the replicator dynamic. Indeed, since the return of the risky asset is uncertain, a rational player would maximize the expected growth of wealth, or the expected value of the replicator dynamic given the present share of expected utility maximizers in the market so as to determine whether to behave as an expected utility maximizer or a case-based decision-maker. But this is exactly what has been computed in the proof of proposition 6.5. For $p^{eu} \geq \frac{1+\delta}{1+r}$, the strategy ”expected utility maximizer” is indeed a best response if everyone else plays ”expected utility maximizer”, hence $e = 1$ is a Nash equilibrium. However, for $p^{eu} < \frac{1+\delta}{1+r}$, the best-response to the strategy combination in which everyone plays ”expected utility maximizer” is ”case-based decision-maker”, therefore, $e = 1$ is not a Nash equilibrium and cannot be a stable state under a regular replicator dynamic.

Since at $e = 1$, the replicator dynamic (6.69) is Lipschitz continuous, it follows that every payoff-monotone replicator dynamic will exhibit the same stability properties at $e = 1$. The result is therefore robust with respect to the choice of an evolutionary dynamic. Analogous arguments hold also for the stability results obtained below.

137 See chapter 5, proposition 5.4 for the derivation of the limit frequencies with which the case-based decision makers hold asset a, respectively b.

A result similar to the one of proposition 6.5 can be derived for lower fundamental values of the risky asset. For $p^{eu} \in \left(\frac{1}{2}; 1\right)$, the following proposition obtains:

Proposition 6.6 *Let the aspiration level satisfy*

$$\bar{u} \in (1 + r; 1 + \delta).$$

Then there is a critical value $\tilde{p}^{eu} \in \left(\frac{1}{2}; 1\right)$ such that $E\left[e^*_{t+2} \mid e^*_t\right] < e^*_t$ holds for

$$e^*_t \in \left[\max\left\{p^{eu}; 1 - p^{eu} + \frac{rp^{eu^2}}{1+r}\right\}; 1\right)$$

if $p^{eu} > \tilde{p}^{eu}$.

Hence, for lower fundamental values, the result that there is a positive probability that the case-based decision-makers do not disappear also obtains. Near $e = 1$, the share of expected utility maximizers falls in expectation and, therefore, the state $e = 1$ is not stable.

For lower values of p^{eu}, (especially lower than $\frac{1}{2}$), the results are not clear. Whereas the expected share of expected utility maximizers decreases in periods in which the case-based decision-makers hold a, ($E\left[e^*_{t+2} \mid e^*_t, \gamma^{*cb}_t = 1\right] < e^*_t$ always holds near $e = 1$), their share increases in expectation in periods in which the case-based decision-makers hold b,

$$E\left[e^*_{t+2} \mid e^*_t, \gamma^{*cb}_t = 0\right] > e^*_t$$

as long as e_t is sufficiently close to 1. It is, a priori, not obvious which of these two effects will dominate.

Nevertheless, it is intuitively clear that for sufficiently low fundamental values of the risky asset, the case-based decision-makers disappear with probability 1. Indeed, imagine that $\delta = 0$ so that $p^{eu} = 0$ holds, hence, the risky asset never pays a positive dividend. In this case, the case-based decision-makers who hold a strictly dominated asset with positive frequency (and a portfolio identical to the portfolio of the expected utility maximizers, else) disappear with probability 1 in the limit. By continuity, this result holds in some surrounding of $p^{eu} = 0$ ($\delta = 0$) and, therefore, case-based decision-makers with high aspiration level cannot survive for low fundamental values of the risky asset.

To summarize, if the fundamental value of the risky asset is neither too high, nor too low, there is a positive probability that the case-based decision-makers do not disappear from the market. This result can be made even stronger:

Proposition 6.7 *Suppose that e_t^* is a supermartingale on some interval $[\tilde{e}; 1]$. Then*
$$\Pr\{e_t^* \to 1\} = 0.$$

The share of case-based decision-makers, thus, remains almost surely positive as long as it can be shown that e_t^* is a supermartingale near 1. This result can be interpreted in terms of the definition of survival and dominance introduced by Blume and Easley (1992). In their terminology, survival requires that the share of an investor type, say of case-based decision-makers, fulfills:
$$\Pr\left\{\limsup_{t\to\infty} c_t > 0\right\} = 1, \tag{6.70}$$
whereas the case-based decision-makers dominate the market if
$$\Pr\left\{\liminf_{t\to\infty} c_t > 0\right\} = 1 \tag{6.71}$$
is satisfied. Note that proposition 6.7 implies that both (6.70) and (6.71) are fulfilled as long as e_t^* is a supermartingale on some interval $[\tilde{e}; 1]$.

6.4.2.2 The Case of Low Aspiration Levels

Now suppose that the case-based decision-makers have an aspiration level which satisfies
$$1 < \bar{u} < 1 + r,$$
implying that the case-based decision-makers hold b in each period of time. For any fundamental value of the risky asset satisfying
$$\frac{1+\delta}{1+r} > p^{eu} > 0,$$
it can be shown that the share of expected utility maximizers is a submartingale near $e = 0$ and supermartingale near $e = 1$. Therefore, case-based decision-makers with low aspiration level need not vanish from the market even if their aspiration level is relatively low.

Proposition 6.8 *Suppose that the aspiration level satisfies*
$$1 < \bar{u} < 1 + r.$$
Let $p^{eu} \geq 1$ hold.

1. *If $p^{eu} < \frac{1+\delta}{1+r}$, then there is an $\hat{e} \in (0; 1)$ such that e_t^* is a supermartingale on $[\hat{e}; 1]$ and a submartingale on $[0; \hat{e})$.*

2. *If $p^{eu} \geq \frac{1+\delta}{1+r}$, then e_t^* is a submartingale on $[0; 1]$. The case-based decision-makers disappear with probability 1.*

Note that with low aspiration levels the case-based decision-makers survive for exactly the same values of q which were found in proposition 6.5. Although in the case of low aspiration level, q influences the selection only by increasing the average return of the expected utility maximizers and not through the less risk-averse behavior on the side of the case-based decision-makers, in the limit when c_t becomes very small, the condition for the survival of the case-based decision-makers is identical in both cases.

However, the cut-off values $\tilde{e}$ (as defined in proposition 6.5) and $\hat{e}$ from proposition 6.8 reflect the fact that the strategy of the case-based decision-makers is riskier in the case of high aspiration level. Therefore, the case-based decision-makers with low aspiration level are likely to survive in a higher proportion than case-based investors with high aspiration level. The following relationship between $\tilde{e}$ and $\hat{e}$ holds:

Proposition 6.9 *$\tilde{e}$, as defined in proposition 6.5 and $\hat{e}$ from proposition 6.8 satisfy:*

$$\tilde{e} > \hat{e}.$$

For fundamental values lower than 1 a result analogous to the result of proposition 6.6 applies. Since, however, with low aspiration levels the portfolio chosen by the case-based decision makers is less risky, they are able to survive for a larger range of parameter values.

Proposition 6.10 *Suppose that the aspiration level satisfies*

$$1 < \bar{u} < 1 + r.$$

Let $p^{eu} \in (0;1)$ hold. Then e_t^* is a supermartingale on the interval $\left[\max\left\{p^{eu}; 1 - p^{eu} + \frac{rp^{eu2}}{1+r}\right\}; 1\right]$.

The result of proposition 6.7 applies in this case as well, implying that the share of case-based decision-makers remains positive with probability 1 as long as e_t^* is a supermartingale in some interval $[\hat{e}; 1]$.

6.5 Asset Prices in the Presence of Case-Based Decision-Makers

Up to now, it has been shown that the case-based decision-makers are able to survive in a strictly positive proportion in the presence of expected utility maximizers. This section analyzes the effect of case-based reasoning on asset prices.

Consider first the case of high aspiration levels. When the fundamental value of the risky asset is smaller than 1, the case-based decision-makers can influence its price and cause bubbles, excessive volatility and predictability of returns as long as their share exceeds $\max\{p^{eu}; 1 -$

$p^{eu}\}$[138]. Indeed, since the case-based decision-makers switch between a and b infinitely often, the price of a fluctuates depending on the share of case-based decision-makers and on their behavior and exhibits excessive volatility[139]. Moreover, the returns of a are predictable. Especially, if $p^*_{t-1} = \gamma^{*eu}_t e^*_{t-1}$, meaning that in a certain period only expected utility maximizers hold a, an external observer could predict that the price of a in the next period will (weakly) rise, since the young case-based decision-makers will buy a in period t independently of the dividend paid by the risky asset.

Case-based decision-makers can cause a bubble to emerge and to persist in the market for several periods. Suppose, for instance, that the share of expected utility maximizers is lower than $(1 - p^{eu})$ at some time t and that case-based decision-makers hold a in period t. Then the equilibrium price of a is given by

$$p^*_t = (1 - e^*_t) > p^{eu}.$$

Moreover, if the case-based decision-makers achieve a high dividend, i.e. $\delta_{t+1} = \delta$, then their return will exceed those of the expected utility maximizers and $e^*_{t+1} < e^*_t$ holds in equilibrium in period $(t+1)$. Furthermore, since everyone of the young case-based decision-makers wishes to hold a at $(t+1)$,

$$p^*_{t+1} = \left(1 - e^*_{t+1}\right) > p^*_t > p^{eu}$$

obtain in equilibrium. Hence, the price increases above the fundamental value for several periods as long as the dividend of the risky asset remains positive. In the first period t' such that $\delta_{t'} = 0$, the bubble bursts, since the case-based decision-makers switch to b and their share in the population decreases. Moreover, the price of the risky asset might even fall below the fundamental value p^{eu}. Simple computations show that this happens if

$$(1 - e^*_{t'}) > \frac{(p^{eu} - 1)^2}{p^{eu}(1 + r)},$$

hence, if the bubble has lasted sufficiently long to decrease substantially the share of the expected utility maximizers.

The phenomenon described above might seem to lead to the following problem: suppose that the case-based decision-makers achieved a higher average return (by holding the risky asset)

[138] If this condition is satisfied, the wealth share of the expected utility maximizers is not sufficiently high to prevent deviations of the price of the risky asset from the fundamental value.

[139] In an overlapping generations model with constant initial endowments and no population growth the price of the risky asset should remain constant over the time, given rational expectations and expected utility maximization. As was shown above, stable prices obtain for $e = 1$.

than the expected utility maximizers. Then their share in the population of the young generation would rise. Since the young case-based investors would also prefer to buy the risky asset, their growing share would drive the price of the asset upwards and, therefore, increase the return of their "parents" further, which on its turn would increase the share of young case-based decision-makers, and so on. This would mean that expected utility maximizers could be driven out of the market in a single period if their evaluation of the risky asset is below 1. It turns out that this cannot happen[140].

Indeed, consider the case in which $\gamma_t^{*cb} = 1$ and $e_t^* < 1 - p^{eu}$ hold and, therefore, $\gamma_t^{*eu} = 0$ (since at the price $p_t^* = (1 - e_t^*)$ the expected utility maximizers are not ready to hold a). Let further the dividend in the next period be high: $\delta_{t+1} = \delta$. Since

$$1 + \frac{\delta}{p_t^*} > 1 + \delta > \bar{u},$$

the young case-based decision-makers in period $(t + 1)$ also wish to hold a[141]. Therefore, the return of a at $(t + 1)$ becomes:

$$\frac{1 - e_{t+1}^* + \delta}{p_t^*} = \frac{1 - e_{t+1}^* + \delta}{1 - e_t^*} = \frac{c_{t+1}^* + \delta}{c_t^*},$$

with $c_t^* = 1 - e_t^*$. Then the equilibrium share $c_{t+1}^* = 1 - e_{t+1}^*$ can be determined according to the equation:

$$c_{t+1}^* = \frac{\frac{c_{t+1}^*+\delta}{c_t^*} c_t^*}{\frac{c_{t+1}^*+\delta}{c_t^*} c_t^* + (1 + r)(1 - c_t^*)} = \frac{c_{t+1}^* + \delta}{c_{t+1}^* + \delta + (1 + r)(1 - c_t^*)}. \tag{6.72}$$

It can be shown that (6.72) has a single solution $c_{t+1}^* < 1$ and, therefore, the share of the case-based decision-makers does not rise to 1:

Lemma 6.1 *Equation (6.72) has only one non-negative root $c_{t+1}^* < 1$. Furthermore, $c_{t+1}^* > c_t$ holds.*

There is, however, a case in which the expected utility maximizers disappear from the market, at least for some finite number of periods. To see how this can happen, consider the case of low aspiration levels. Since the aspiration level satisfies

$$1 < \bar{u} < 1 + r,$$

140 I would like to thank to Hans Haller for encouraging me to pursue this issue.

141 To guarantee that the case-based decision makers will still be willing to hold a given the return $\frac{c_{t+1}^*+\delta}{c_t^*}$, it is sufficient to show that $c_{t+1}^* > c_t^*$ obtains in equilibrium. This is demonstrated in the proof of lemma 6.1.

$\gamma_t^{*cb} = 0$ holds for each t. Then, to equilibrate the market for the risky asset,

$$\begin{aligned}\gamma_t^{*eu} &= \min\{e_t^*; p^{eu}\} \\ p_t^* &= \min\{e_t^*; p^{eu}\}\end{aligned}$$

must hold. Suppose that the share of the expected utility maximizers is relatively small so that the price of the risky asset is lower than its fundamental value ($e_t^* < p^{eu}$) and let the next period dividend be low, $\delta_{t+1} = 0$. The equilibrium share of the expected utility maximizers is now given by the solution of the equation:

$$e_{t+1}^* = \frac{\frac{e_{t+1}^*}{e_t^*} e_t^*}{\frac{e_{t+1}^*}{e_t^*} e_t + (1+r)(1-e_t^*)} = \frac{e_{t+1}^*}{e_{t+1}^* + (1+r)(1-e_t^*)}. \tag{6.73}$$

(6.73) has two solutions: $e_{t+1}^* = 0$ and $e_{t+1}^{**} = e_t^*(1+r) - r$, which is always smaller than e_t^*. However, if the initial share of the expected utility maximizers is relatively small, i.e. if

$$e_t^* < \frac{r}{1+r},$$

$e_{t+1}^{**} < 0$ and $e_{t+1}^* = 0$ obtains in equilibrium.

The expected utility maximizers can vanish if they hold the risky asset, hoping that it is valuable, but if there are not enough of their type to prevent its price from falling when the dividend of the asset is low. This effect is similar to the noise trader risk identified by De Long, Shleifer, Summers and Waldmann (1990). Although the expected utility maximizers do not have rational expectations in this model, they suffer from an undervaluation of the risky asset caused by the case-based decision-makers. If, furthermore, the returns of the expected utility maximizers are relatively low compared to those of the population as a whole, then the share of the case-based decision-makers increases causing the undervaluation of the risky asset to become even more severe.

However, the expected utility maximizers do not disappear forever. In the next period, the mass of the expected utility maximizers is determined by the following equation:

$$e_{t+2}^* = \frac{e_{t+2}^* + \delta_{t+2}}{e_{t+2}^* + \delta_{t+2} + (1+r)}.$$

Note that the expected utility maximizers do not regain a positive mass in the next period if $\delta_{t+2} = 0$. Indeed, in this case their share satisfies:

$$e_{t+2}^* = \frac{e_{t+2}^*}{e_{t+2}^* + (1+r)}$$

and it is easily seen that the sole non-negative solution of this equation is $e_{t+2}^* = 0$. Nevertheless, the mass of the expected utility maximizers becomes positive again in the first period in which the dividend of the risky asset becomes positive, since then

$$e_{t+2}^* = \frac{e_{t+2}^* + \delta}{e_{t+2}^* + \delta + (1+r)}$$

holds. This equation has a unique strictly positive solution between 0 and 1.

Note that this result does not contradict proposition 6.1, which only guarantees a positive equilibrium price of the risky asset as long as e_t^* is positive. The effect arises, because of the dependence of the replicator dynamic on the price of the risky asset and, therefore, indirectly on e_t itself. It shows that even in markets in which expected utility maximizers are a priori present, the price of an asset with positive fundamental value may fall to 0 and remain so for few periods. The price recovers almost surely in finite time and the share of expected utility maximizers becomes positive again.

The results of this section imply that some of the phenomena empirically observed in financial markets could be attributed to the presence of case-based decision-makers in the economy. However, the emergence of bubbles or price crashes requires a relatively high proportion of case-based decision-makers in the market. It is not clear whether the analytical computation of the probability of the occurrence of such phenomena in this model is possible. Future work has, therefore, to deal with simulations of the model, which would enable the estimation of the frequency of such phenomena.

6.6 CRRA Utility

Up to now, the assumption of risk-neutrality has been made. Suppose, instead that the investors in this economy have a utility function with constant relative risk-aversion. The coefficient of relative risk-aversion is denoted by $(1-\beta)$ so that the utility functions can be parameterized in the following way:

$$\begin{aligned} u_\beta(x) &= x^\beta, \beta \in (0;1] \\ u_\beta(x) &= \ln x, \beta = 0. \end{aligned}$$

Diversification is still not allowed[142]. Parameterizing the utility function in this way has two

[142] As has already been stated above, the diversification constraint affects only the behavior of risk-averse expected utility maximizers. Since case-based decision makers are in general unwilling to diversify, it is of interest to study the model with diversification constraints first, hence to ask whether the risk attitude of the expected utility maximizers alone will allow them to drive the case-based decision makers to extinction. The assumption of no-diversification is abandoned in section 7.

effects: first, it changes the cut-off price at which the expected utility maximizers are indifferent between holding a and b; second, it alters the critical values of the aspiration level which determine the patterns of behavior of the case-based decision-makers. The second effect does not have much influence on the results derived in the previous section. Just replace $\bar{u} \gtreqless 1$, $\bar{u} \gtreqless (1+r)$ and $\bar{u} \gtreqless (1+\delta)$ by $\bar{u} \gtreqless u_\beta(1)$, $\bar{u} \gtreqless u_\beta(1+r)$ and $\bar{u} \gtreqless u_\beta(1+\delta)$, respectively to obtain the new critical values of $\bar{u}$ and determine which of the three cases discussed above is relevant.

Changing the cut-off price may, however, have a significant effect on the results. Since now the expected utility maximizers are risk-averse, they are ready to hold a only if its price is relatively low, i.e. only if its expected return exceeds the return of b so as to compensate for the higher risk. Hence, at the critical price $p^{eu}(\beta)$, the expected return of a is higher than those of b. It is a priori not clear whether this excessive return can compensate for the risk of a, when the population shares of investor types evolve according to the replicator dynamic, which benefits both high expected return and low risk.

First, I compute the cut-off price $p^{eu}(\beta)$at which the expected utility maximizers with coefficient of relative risk-aversion equal to $(1-\beta)$ are indifferent between a and b, given their belief that there are only expected utility maximizers in the market:

$$\begin{aligned}(1+r)^\beta &= q\left(\frac{p^{eu}(\beta)+\delta}{p^{eu}(\beta)}\right)^\beta + (1-q)\left(\frac{p^{eu}(\beta)}{p^{eu}(\beta)}\right)^\beta, \text{ for } \beta \in (0;1] \\ \ln(1+r) &= q\ln\left(\frac{p^{eu}(\beta)+\delta}{p^{eu}(\beta)}\right) + (1-q)\ln\left(\frac{p^{eu}(\beta)}{p^{eu}(\beta)}\right), \text{ for } \beta = 0.\end{aligned}$$

Solving for $p^{eu}(\beta)$, one obtains:

$$p^{eu}(\beta) = \left\{\begin{array}{ll} \frac{\delta}{\sqrt[\beta]{\frac{(1+r)^\beta-(1-q)}{q}}-1}, & \text{for } \beta \in (0;1] \\ \frac{\delta}{\sqrt[q]{(1+r)}-1}, & \text{for } \beta = 0. \end{array}\right\}. \tag{6.74}$$

Note that for $\beta = 1$, $p^{eu}(1) = p^{eu} = \frac{q\delta}{r}$ holds.

Consider the case in which the cut-off price exceeds 1 so that the expected utility maximizers hold the risky asset independently of its price in the market. According to (6.74), this is equivalent to:

$$\begin{aligned} q &\geq \frac{(1+r)^\beta - 1}{(1+\delta)^\beta - 1}, \text{ for } \beta \in (0;1] \\ q &\geq \frac{\ln(1+r)}{\ln(1+\delta)}, \text{ for } \beta = 0. \end{aligned}$$

The following two propositions generalize the results of propositions 6.5 and 6.8[143].

Proposition 6.11 *For a given β, let $p^{eu}(\beta) \geq 1$ and assume that the aspiration level satisfies*

$$\bar{u} \in (u_\beta(1+r); u_\beta(1+\delta)).$$

1. *If $\beta \in (0;1]$ and $q \in \left[\frac{(1+r)^\beta-1}{(1+\delta)^\beta-1}; \frac{(1+\delta)r}{(1+r)\delta}\right)$, then there exists an $\tilde{e}(\beta) \in (0;1)$ such that e_t^* is a submartingale below $\tilde{e}(\beta)$ and a supermartingale above $\tilde{e}(\beta)$.*
2. *If $\beta \in (0;1]$ and $q \in \left[\frac{(1+\delta)r}{(1+r)\delta}; 1\right]$, then e_t^* is a submartingale on $[0;1]$. The case-based decision-makers disappear with probability 1.*
3. *If $\beta = 0$ and $q \in \left[\frac{\ln(1+r)}{\ln(1+\delta)}; \frac{(1+\delta)r}{(1+r)\delta}\right)$, then there exists an $\tilde{e}(0) \in (0;1)$ such that e_t^* is a submartingale below $\tilde{e}(0)$ and a supermartingale above $\tilde{e}(0)$.*
4. *If $\beta = 0$ and $q \in \left[\frac{(1+\delta)r}{(1+r)\delta}; 1\right]$, then e_t^* is a submartingale on $[0;1]$. The case-based decision-makers disappear with probability 1.*

Proposition 6.12 *For a given β, let $p^{eu}(\beta) \geq 1$ and assume that the aspiration level satisfies*

$$\bar{u} \in (u_\beta(1); u_\beta(1+r)).$$

1. *If $\beta \in (0;1]$ and $q \in \left[\frac{(1+r)^\beta-1}{(1+\delta)^\beta-1}; \frac{(1+\delta)r}{(1+r)\delta}\right)$, then there exists an $\hat{e}(\beta) \in (0;1)$ such that e_t^* is a submartingale below $\hat{e}(\beta)$ and a supermartingale above $\hat{e}(\beta)$.*
2. *If $\beta \in (0;1]$ and $q \in \left[\frac{(1+\delta)r}{(1+r)\delta}; 1\right]$, then e_t^* is a submartingale on $[0;1]$. The case-based decision-makers disappear with probability 1.*
3. *If $\beta = 0$ and $q \in \left[\frac{\ln(1+r)}{\ln(1+\delta)}; \frac{(1+\delta)r}{(1+r)\delta}\right)$, then there exists an $\hat{e}(0) \in (0;1)$ such that e_t^* is a submartingale below $\hat{e}(0)$ and a supermartingale above $\hat{e}(0)$.*
4. *If $\beta = 0$ and $q \in \left[\frac{(1+\delta)r}{(1+r)\delta}; 1\right]$, then e_t^* is a submartingale on $[0;1]$. The case-based decision-makers disappear with probability 1.*

As in the case of a linear utility function, a relatively low probability of high dividends insures the survival of the case-based decision-makers. Moreover, the upper bound of q for which the case-based decision-makers survive does not depend on β. However, if q exceeds this bound,

[143] It might seem that the following two propositions for the cases of high and low aspiration level are identical. However, the cut-off points $\tilde{e}(\beta)$ and $\hat{e}(\beta)$ are different in the two cases, therefore the claims are stated seperately.

the case-based decision-makers cannot compensate for the higher returns of the expected utility maximizers by a lower risk and vanish with probability 1 in the limit. $\tilde{e}(\beta) > \hat{e}(\beta)$ holds in analogy with proposition 6.9.

The results derived in this section, however, severely rely on the assumption that diversification is not possible for risk-averse investors. They suggest that the excess return achieved by holding the risky asset at a lower price is still not sufficiently high to compensate for the riskiness of the strategy of the expected utility maximizers. Allowing for diversification, allows the expected utility maximizers not only to reduce the riskiness of their strategy, but to achieve even higher excess returns. Therefore, it can be expected that expected utility maximizers with relatively high coefficients of relative risk-aversion will be able to drive the case-based decision-makers out of the market even for values of $q < \frac{(1+r)^{\beta}-1}{(1+\delta)^{\beta}-1}$.

6.7 CRRA Utility with Diversification

Suppose first that the expected utility maximizers have a logarithmic utility function, i.e. $\beta = 0$ in the terms of the last section, and assume that diversification is possible, whereas short sales are still forbidden. Suppose that the expected utility maximizers expect the price in the next period to be p_{t+1} and know the true distribution of the dividends, as well as the riskless interest rate. Still, it is assumed that they act as if only expected utility maximizers were present in the market. Their decision problem, therefore, can be stated as:

$$\max_{\gamma_t^{eu} \in [0;1]} q \ln \left[\frac{p_{t+1} + \delta}{p_t} \gamma_t^{eu} + (1+r)\left(1 - \gamma_t^{eu}\right) \right] + (1-q) \ln \left[\frac{p_{t+1}}{p_t} \gamma_t^{eu} + (1+r)\left(1 - \gamma_t^{eu}\right) \right],$$

where γ_t^{eu} now denotes the proportion of the income *each* expected utility maximizer invests into the risky asset[144]. The first-order condition of this problem is easily seen to reduce to:

$$\gamma_t^{eu} = \frac{(1+r)\left(\frac{p_{t+1}+q\delta}{p_t} - (1+r)\right)}{\left(\frac{p_{t+1}+\delta}{p_t} - (1+r)\right)\left((1+r) - \frac{p_{t+1}}{p_t}\right)}.$$

Since short sales are forbidden, $\gamma_t^{eu} \in [0;1]$ must hold and the optimal portfolio is, therefore,

144 Since all expected utility maximizers are identical and since diversification is allowed, they will solve identical optimization problems in each period and will hold identical portfolios at each price p_t. Hence, γ_t^{eu} also denotes the proportion of the income the expected utility maximizers invest into the risky asset, as in the previous sections.

determined by:

$$\gamma_t^{eu}(p_t) = \left\{ \begin{array}{ll} 1, & \text{if } p_t < \frac{p_{t+1}(p_{t+1}+\delta)}{(1+r)(p_{t+1}+(1-q)\delta)} \\ \frac{(1+r)\left(\frac{p_{t+1}+q\delta}{p_t}-(1+r)\right)}{\left(\frac{p_{t+1}+\delta}{p_t}-(1+r)\right)\left((1+r)-\frac{p_{t+1}}{p_t}\right)}, & \text{if } p_t \in \left[\frac{p_{t+1}(p_{t+1}+\delta)}{(1+r)(p_{t+1}+(1-q)\delta)}; \frac{p_{t+1}+q\delta}{1+r}\right] \\ 0, & \text{if } p_t > \frac{p_{t+1}+q\delta}{1+r}. \end{array} \right\}.$$

Note that $p_t < \frac{p_{t+1}+q\delta}{1+r}$ is the condition that the expected return of the risky asset exceeds the return of the riskless one and that it implies that $\frac{p_{t+1}+\delta}{p_t} > (1+r)$. At the same time,

$$p_t < \frac{p_{t+1}\left(p_{t+1}+\delta\right)}{(1+r)\left(p_{t+1}+(1-q)\,\delta\right)}$$

implies that $\frac{p_{t+1}}{p_t} < (1+r)$. Hence, for $\gamma_t^{eu}(p_t) \in [0;1]$, the no-arbitrage restrictions are fulfilled. The short sale constraints insure that the demand for a is well-defined even if these restrictions are not satisfied.

Remark 6.1 *If $\frac{p_{t+1}(p_{t+1}+\delta)}{(1+r)(p_{t+1}+(1-q)\delta)} \geq 1$ holds, then the young expected utility maximizers invest their whole endowment into the risky asset a, independently of its current price p_t.*

It is a standard result that $\gamma_t^{eu}(p_t)$ is a decreasing function:

Lemma 6.2 $\frac{\partial \gamma_t^{eu}(p_t)}{\partial p_t} \leq 0.$

The following corollary obtains:

Corollary 6.2 *For each expected price p_{t+1} there exists a unique equilibrium price p_t^* in a market populated only by expected utility maximizers.*

In an equilibrium with rational expectations, the price of the risky asset must satisfy in each period:

$$\begin{aligned} p_{t+1} &= p_t \\ \gamma_t^{eu}(p_t) &= p_t. \end{aligned}$$

Proposition 6.13 *In an economy populated only by expected utility maximizers with a logarithmic utility function, the equilibrium price under rational expectations is given by $\min\left\{p_{\log}^{eu}; 1\right\}$ with*

$$p_{\log}^{eu} = \frac{1+r+\delta-\sqrt{(1+r+\delta)^2-4q\delta\,(1+r)}}{2r}.$$

$p_{\log}^{eu} \geq 1$ holds, iff $q \geq \frac{(1+\delta)r}{(1+r)\delta}$.

Note that the values of q for which the price under rational expectations exceeds 1 coincide with the values derived in proposition 6.11 and 6.12 which guarantee that case-based decision-makers disappear with probability 1 in the limit. Moreover, as the following corollary states, if q satisfies

$$q \geq \frac{(1+\delta)\,r}{(1+r)\,\delta},$$

then the expected utility maximizers are ready to invest their whole endowment in the risky asset a, independently of the price p_t:

Corollary 6.3

$$\frac{p_{\log}^{eu}\left(p_{\log}^{eu}+\delta\right)}{(1+r)\left(p_{\log}^{eu}+(1-q)\,\delta\right)} \geq 1 \tag{6.75}$$

holds if $q \geq \frac{(1+\delta)r}{(1+r)\delta}$.

The following proposition can now be stated in analogy to propositions 6.11 and 6.12:

Proposition 6.14 *Let* $p_{\log}^{eu} \geq 1$. *If the aspiration level of the case-based decision-makers satisfies*[145] $\bar{u} \in (0; \ln(1+\delta))$, *then the share of the expected utility maximizers converges to* 1 *with probability* 1.

The proposition shows that case-based decision-makers with an aspiration level exceeding the utility of 1 unit of consumption good disappear, when logarithmic expected utility maximizers are present in the market. The ability of the expected utility maximizers to diversify combined with a logarithmic utility function reduces their exposure to risk sufficiently so that they outperform the case-based decision-makers. This result is a special case of the more general one obtained by Blume and Easley (1992, 2001), Hens and Schenk-Hoppé (2001) and of Evstigneev, Hens and Schenk-Hoppé (2002, 2003), who show that expected utility maximizers with logarithmic utility function and correct beliefs follow the most successful strategy in a financial market and drive all other strategies to extinction. Note that since the logarithmic utility maximizers in this model act as in an equilibrium with rational expectations in an economy populated by expected utility maximizers identical to themselves, they indeed have correct beliefs in the limit for $e \to 1$[146]. Moreover, since their utility function lets them act as if they maximized the expected growth of their wealth, i.e. as if they maximized the expected value of the function given by the replicator dynamic, the result that they drive the case-based decision-makers out of the market remains true for all values of $p_{\log}^{eu}$.

[145] The result is stated for case-based decision makers with logarithmic utility function.

[146] This makes their strategy similar to the strategy called λ^* in the works of Hens and Schenk-Hoppé (2001) and of Evstigneev, Hens and Schenk-Hoppé (2002, 2003), which operates with rational beliefs only in the limit, when its wealth share converges to 1.

It is straightforward to extend the result of proposition 6.14 to the more general case of constant relative risk-aversion. Denote the price under rational expectations for an arbitrary coefficient of relative risk-aversion $(1-\beta)$ by p_{β}^{eu} and note that it increases with decreasing relative risk-aversion.

Lemma 6.3 $\frac{\partial p_{\beta}^{eu}}{\partial \beta} \geq 0$ *for all* $\beta \in (0;1]$.

It can be further shown that for each $\beta \in (0;1]$, the case-based decision-makers survive for a certain range of the parameters.

Proposition 6.15 *Suppose that diversification is allowed. Let* $\beta \in (0;1]$.
If $\bar{u} \in (u_{\beta}(1+r); u_{\beta}(1+\delta))$ *and*

1. $q \in \left[\frac{r}{r+(\delta-r)(1+\delta)^{\beta-1}}; \frac{(1+\delta)r}{(1+r)\delta}\right)$*, then there exists a cut-off point* $\tilde{e}_d(\beta) \in (0;1)$ *such that* e_t^* *is a supermartingale above* $\tilde{e}_d(\beta)$ *and a submartingale below* $\tilde{e}_d(\beta)$.

2. $q \in \left[\frac{(1+\delta)r}{(1+r)\delta}; 1\right]$*, then the share of the case-based decision-makers converges to* 0 *almost surely.*

If $\bar{u} \in (u_{\beta}(1); u_{\beta}(1+r))$ *and*

1. $q \in \left[\frac{r}{r+(\delta-r)(1+\delta)^{\beta-1}}; \frac{(1+\delta)r}{(1+r)\delta}\right)$*, then there exists a cut-off point* $\hat{e}_d(\beta) \in (0;1)$ *such that* e_t^* *is a supermartingale above* $\hat{e}_d(\beta)$ *and a submartingale below* $\hat{e}_d(\beta)$.

2. $q \in \left[\frac{(1+\delta)r}{(1+r)\delta}; 1\right]$*, then the share of the case-based decision-makers converges to* 0 *almost surely.*

The condition $q \geq \frac{r}{r+(\delta-r)(1+\delta)^{\beta-1}}$ implies that the expected utility maximizers wish to invest their whole initial endowment into the risky asset, independently of its price, whereas $q < \frac{(1+\delta)r}{(1+r)\delta}$ is necessary for the survival of the case-based decision-makers. When these two conditions are met simultaneously, the case-based decision-makers do not vanish in the limit, according to propositions 6.5 and 6.8.

The case of a logarithmic utility function $(\beta = 0)$ represents the limit case, in which

$$\frac{r}{r+(\delta-r)(1+\delta)^{\beta-1}} = \frac{(1+\delta)r}{(1+r)\delta}$$

holds. Since the cut-off price below which the expected utility maximizers invest all their initial endowment in the risky asset is decreasing in β, this means that lower values of q (than $\frac{(1+\delta)r}{(1+r)\delta}$) are needed to satisfy a condition analogous to (6.75) formulated for p_{β}^{eu}. Proposition 6.15 states that for each $\beta \in (0;1]$, there is an interval of values of q for which this condition is fulfilled and at the same time the survival condition $q < \frac{(1+\delta)r}{(1+r)\delta}$ is satisfied. For these values of q, the share of case-based decision-makers remains positive with probability 1 in the limit. Since these intervals become smaller and smaller, as the value of β decreases, it is easier for more risk-averse expected utility maximizers to drive the case-based decision-makers out of the market[147]. For each $\beta > 0$, however, the case-based decision-makers survive almost surely, at least for some values of the parameters.

6.8 Conclusion

The chapter presents a first attempt to analyze an asset market in which both expected utility maximizers and case-based decision-makers are present. It has been demonstrated that in a stationary state both types of investors can coexist, holding identical portfolios. The price of the risky asset is then equal to its fundamental value. However, in this case it is empirically impossible to distinguish between the two types of investors and the hypotheses of expected utility maximization and rational expectations cannot be rejected.

The analysis of the stationary state in which the proportion of expected utility maximizers is 1 shows that in general this stationary state need not be stable. Especially, if the fundamental value of the risky asset satisfies certain conditions, case-based decision-makers with relatively high aspiration level retain a strictly positive mass with probability one in the limit. Therefore, according to the definition of Blume and Easley (1992), the case-based decision-makers not only survive, but also dominate the market. In this sense they are also able to influence the price dynamics. By switching between the two assets they can cause predictability of price movements, excessive volatility and bubbles, which burst with probability one.

Alternatively, if the fundamental value of the risky asset is too high or too low, the case-based decision-makers vanish from the market with probability one.

For case-based decision-makers with relatively low aspiration levels the results are similar. They

147 In the sense that there is a greater range of parameters for which this happens.

vanish with probability one, if the fundamental value of the asset is too high, but survive and dominate the market with probability 1, else. Moreover, in certain periods they can even drive the expected utility maximizers out of the market, causing the price of the risky asset to fall to 0. This effect seems to be similar to the so called ”noise trader risk” discovered by De Long, Shleifer, Summers and Waldmann (1990), although the expected utility maximizers in this model have no rational expectations. However, the share of expected utility maximizers becomes positive again and in the limit both types of investors coexist in the market.

Some of the results obtained for the case of risk-neutral expected utility maximizers are generalized for expected utility maximizers with a utility function exhibiting constant relative risk-aversion. It turns out that even expected utility maximizers with logarithmic utility function do not always drive the case-based decision-makers to extinction. However, these results rely severely on the fact that diversification is not possible for risk-averse investors.

The ability to diversify allows the expected utility maximizers with logarithmic utility function to drive the case-based decision-makers out of the market. However, for lower degrees of risk-aversion, there are always values of the parameters for which the case-based decision-makers survive in a strictly positive proportion. This confirms the result established in the literature that logarithmic expected utility maximizers with correct beliefs perform best and accumulate the whole market wealth in the limit.

The analysis of the model, therefore, answers the two questions stated in the introduction by identifying conditions under which case-based decision-makers survive in the presence of expected utility maximizers and discussing their influence on prices. However, these results apply for the special case of one-period memory (on the side of the case-based decision-makers) and of constant expectations (on the side of the expected utility maximizers). Further research will have to allow for a longer memory and for Bayesian adaptation of expectations in order to analyze the issue of the efficiency of these learning rules in an evolutionary setting.

A final note has to be made on the issue of introducing expected utility maximizers with rational expectations. It is straightforward to see that with a linear utility function and non-binding short-sales-constraints, the expected returns of the two assets must be identical in each period of time. Therefore, the replicator dynamic selects for the less risky strategy in each period. Hence, the results about the instability of the stationary state $e = 1$ remain valid even if the expected utility maximizers have rational expectations with respect to the behavior of the case-based decision-makers and to the evolutionary dynamic of the economy.

Appendix

Proof of proposition 6.2:

In order to show the existence of an equilibrium, it is sufficient to demonstrate that the system of equations formulated in conditions (i), (ii), (iii) and (iv) of the definition has a solution:

$$e_t^* = \frac{\tilde{v}_t^{*eu}}{\tilde{v}_t^*} e_{t-1} \tag{6.76}$$

$$\gamma_t^{*eu} = \gamma_t^{eu}(p_t^*) \tag{6.77}$$

$$\gamma_t^{*cb} = \gamma_t^{cb}(p_t^*)$$

$$\gamma_t^{*eu} e_t^* + \gamma_t^{*cb}(1 - e_t^*) = p_t^* \tag{6.78}$$

First note that since

$$p_{t-1} = \gamma_{t-1}^{eu} e_{t-1} + \gamma_{t-1}^{cb}(1 - e_{t-1}),$$

(6.76) can be written as:

$$e_t^* = \frac{\left[\frac{p_t^* + \delta_t}{p_{t-1}} \gamma_{t-1}^{eu} + (1+r)\left(1 - \gamma_{t-1}^{eu}\right)\right] e_{t-1}}{p_t^* + \delta_t + (1+r)\left(1 - \gamma_{t-1}^{eu} e_{t-1} - \gamma_{t-1}^{cb}(1 - e_{t-1})\right)} =: f(p_t^*), \tag{6.79}$$

where p_t^* is determined from (6.77) and (6.78), taking e_t as given. It follows that $f(p_t^*)$ depends on e_t through p_t^*. Moreover, $f(p_t^*)$ is continuous in p_t^* and can only take values between 0 and 1 for all possible values of e_t and, thus, of p_t^* between 0 and 1. Therefore, it suffices to show that $f(p_t^*(e_t))$ has a fixed point in order to prove the existence of an equilibrium. Since for a given e_t, multiple equilibria can emerge, it will be shown that in each possible case, equilibria can be selected in such a way that a fixed point argument applies.

It is necessary to consider several cases depending on the values of the parameters p^{eu} and $\bar{u}$.

Let first $p^{eu} \geq 1$ and let $\gamma_{t-1}^{cb} = 1$. It follows that $p_{t-1}^* = 1$. Now, if $1 + \delta_t \geq \bar{u}$ holds, then $p_t^*(e_t) = 1$ for all e_t and is, thus, continuous in e_t, which guarantees the existence of an equilibrium according to the Brouwer's fixed point theorem, Mas-Collel, Whinston and Green (1995, p. 952).

If $1 + \delta_t < \bar{u}$, then $\gamma_t^{cb} = 0$ and $p_t^*(e_t) = e_t$, which is again a continuous function.

Now let γ_{t-1}^{cb} be arbitrary. The decision of the case-based decision-makers depends on the comparison

$$\frac{p_t^* + \delta_t}{p_{t-1}} = \frac{e_t + \gamma_t^{cb}(1 - e_t) + \delta_t}{p_{t-1}} \lesseqgtr \bar{u}.$$

Hence, for a given e_t, the share of case-based decision-makers holding a in a temporary equilibrium satisfies[148]:

$$\gamma_t^{cb} = \left\{ \begin{array}{ll} 1, & \text{if } \frac{1+\delta_t}{p_{t-1}} > \bar{u} > 1+r \\ \gamma_{t-1}^{cb}, & \text{if } \min\left\{\frac{e_t+\gamma_{t-1}^{cb}(1-e_t)+\delta_t}{p_{t-1}}; 1+r\right\} > \bar{u} \\ \left(1-\gamma_t^{cb}\right), & \text{if } \max\left\{\frac{e_t+\gamma_{t-1}^{cb}(1-e_t)+\delta_t}{p_{t-1}}; 1+r\right\} < \bar{u} \\ 0, & \text{if } \frac{e_t+\gamma_{t-1}^{cb}(1-e_t)+\delta_t}{p_{t-1}} < \bar{u} < 1+r \end{array} \right\}.$$

It follows that for $\bar{u} \leq 1+r$:

$$p_t^*(e_t) = \left\{ \begin{array}{ll} e_t + \gamma_{t-1}^{cb}(1-e_t), & \text{if } \frac{e_t+\gamma_{t-1}^{cb}(1-e_t)+\delta_t}{p_{t-1}} \geq \bar{u} \\ e_t, & \text{if } \frac{e_t+\gamma_{t-1}^{cb}(1-e_t)+\delta_t}{p_{t-1}} < \bar{u} \end{array} \right\}.$$

If $\bar{u}p_{t-1} - \delta_t > 1$, then $p_t^* = e_t$, which is a continuous function and therefore a fixed point exists.

If $\bar{u}p_{t-1} - \delta_t \leq 1$, both parts of $p_t^*(e_t)$ matter and $p_t^*(e_t)$ is obviously not continuous. Nevertheless, $p_t^*(e_t)$ is continuous but for upward jumps[149]. Moreover, this property is preserved if $p_t^*(e_t)$ is subjected to a monotone transformation, see Milgrom and Roberts (1994, p. 445). Now note that since for $p^{eu} \geq 1$ the choice of $\gamma_t^{eu} = 1$ is optimal in each period, it follows that in each period the expected utility maximizers invest a larger share of their income into the risky asset than the case-based decision-makers do. This implies that the share of the expected utility maximizers is increasing in the price of the risky asset p_t, $\frac{\partial f}{\partial p_t^*} \geq 0$. Hence, $f(\cdot)$ is a monotone transformation of p_t^*. Therefore, $f(\cdot)$ is also continuous but for upward jumps. Moreover, f transfers $[0;1]$ into $[0;1]$. Theorem 1 of Milgrom and Roberts (1994, p. 446) ascertains that such functions have a fixed point. Therefore an equilibrium share e_t^* of expected utility maximizers exists in this case.

For $\bar{u} > 1+r$:

$$p_t^*(e_t) = \left\{ \begin{array}{ll} 1, & \text{if } \frac{1+\delta_t}{p_{t-1}} \geq \bar{u} \\ e_t + \left(1-\gamma_{t-1}^{cb}\right)(1-e_t), & \text{if } \frac{1+\delta_t}{p_{t-1}} < \bar{u} \end{array} \right\}.$$

In both cases, $p_t^*(e_t)$ is continuous in e_t and, therefore, the Brouwer's fixed point theorem applies and an equilibrium exists.

Now suppose that $p^{eu} \in (0;1)$, $\frac{p^{eu}+\delta_t}{p_{t-1}} \geq \bar{u}$ and γ_{t-1}^{cb} is arbitrary. Then the equilibrium share of

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148 The points of indifference are omitted here for simplicity.

149 A function $g : [0;1] \to [0;1]$ is continuous but for upward jumps, if for all $x' \in [0;1]$

$$\lim\sup_{x \uparrow x'} \leq g(x') \leq \lim\inf_{x \downarrow x'}$$

holds, see Milgrom and Roberts (1994, p. 445).

case-based decision-makers willing to hold a in a temporary equilibrium is given by:

$$\gamma_t^{cb} = \left\{ \begin{array}{ll} \gamma_{t-1}^{cb}, & \text{if } \bar{u} \leq 1 + r \text{ and } \frac{p_t + \delta_t}{p_{t-1}} \geq \bar{u} \\ 0, & \text{if } \bar{u} \leq 1 + r \text{ and } \frac{p_t + \delta_t}{p_{t-1}} < \bar{u} \\ 1, & \text{if } \bar{u} > 1 + r \text{ ,} \end{array} \right\}.$$

where the last line follows from the fact that

$$p_t \geq p^{eu}, \text{ if } \gamma_t^{cb} = 1$$

must hold in equilibrium. For $\bar{u} \leq 1 + r$ the equilibrium price is therefore:

$$p_t^* \left(e_t \right) = \left\{ \begin{array}{ll} p^{eu}, & \text{if } e_t + \gamma_{t-1}^{cb} \left(1 - e_t \right) \geq p^{eu} \\ e_t + \gamma_{t-1}^{cb} \left(1 - e_t \right), & \text{if } e_t + \gamma_{t-1}^{cb} \left(1 - e_t \right) < p^{eu} \text{ and } \frac{e_t + \gamma_{t-1}^{cb} (1 - e_t) + \delta_t}{p_{t-1}} \geq \bar{u} \\ e_t, & \frac{e_t + \gamma_{t-1}^{cb} (1 - e_t) + \delta_t}{p_{t-1}} < \bar{u} \end{array} \right\},$$

if $p^{eu} > \gamma_{t-1}^{cb}$ and

$$p_t^* \left(e_t \right) = \left\{ \begin{array}{ll} \gamma_{t-1}^{cb} \left(1 - e_t \right), & \text{if } \gamma_{t-1}^{cb} \left(1 - e_t \right) \geq p^{eu} \\ p^{eu}, & \text{if } \gamma_{t-1}^{cb} \left(1 - e_t \right) < p^{eu} \end{array} \right\},$$

if $p^{eu} \leq \gamma_{t-1}^{cb}$.

Note that when $p^{eu} > \gamma_{t-1}^{cb}$ holds,

$$\gamma_{t-1}^{cb} \left(1 - e_{t-1} \right) < p^{eu}$$

must hold at $(t-1)$. Therefore, the mass of case-based decision-makers is not sufficient to drive the equilibrium price of the risky asset to p^{eu}. It follows that at least some of the expected utility maximizers must hold a at $(t-1)$. If

$$\gamma_{t-1}^{cb} \left(1 - e_{t-1} \right) + e_{t-1} \leq p^{eu},$$

then all of the expected utility maximizers will hold a in $(t-1)$, hence

$$\gamma_{t-1}^{eu} = 1 > p^{eu} > \gamma_{t-1}^{cb}$$

obtains. Else, it must be that the expected utility maximizers are indifferent between a and b, therefore:

$$\gamma_{t-1}^{cb} \left(1 - e_{t-1} \right) + \gamma_{t-1}^{eu} e_{t-1} = p^{eu}$$

must hold. But this can only be true, if

$$\gamma_{t-1}^{eu} > p^{eu} > \gamma_{t-1}^{cb}.$$

This argument demonstrates that the share of expected utility maximizers holding a at time $(t-1)$ exceeds those of case-based decision-makers holding a at $(t-1)$ and, therefore, $\frac{\partial f}{\partial p_t^*} \geq 0$ holds. Since $p_t^* \left(e_t \right)$ is continuous but for upward jumps, the fixed point argument of Milgrom and Roberts (1994, p. 446) applies to the function f and an equilibrium e_t^* exists.

For the case $p^{eu} \leq \gamma_{t-1}^{cb}$, $p_t^* \left(e_t \right)$ is a continuous function in e_t and, therefore, a fixed point argument applies for f and guarantees the existence of an equilibrium.

For the case $\bar{u} > 1 + r$,

$$p_t^* (e_t) = \left\{ \begin{array}{ll} 1 - e_t, & \text{if } 1 - e_t \geq p^{eu} \\ p^{eu}, & \text{if } 1 - e_t < p^{eu} \end{array} \right\}$$

holds. p_t^* is again continuous in e_t so that an equilibrium exists.

Now consider the case $p^{eu} \in (0;1)$, $\frac{p^{eu}+\delta_t}{p_{t-1}} < \bar{u}$ and let γ_{t-1}^{cb} be arbitrary. If $\bar{u} \leq 1 + r$, the equilibrium price is given by:

$$p_t^* (e_t) = \left\{ \begin{array}{ll} e_t, & \text{if } e_t \leq p^{eu} \\ p^{eu}, & \text{if } e_t > p^{eu} \end{array} \right\},$$

which is a continuous function of e_t and, therefore, an equilibrium exists. The same equation describes the price in the case, in which $\bar{u} > 1 + r$, but $\gamma_{t-1}^{cb} = 1$.

For $\bar{u} > 1 + r$ and $\gamma_{t-1}^{cb} < 1$ the equilibrium price depends on the parameters in the following way:

1. If $p^{eu} + \gamma_{t-1}^{cb} - 1 \geq 0$:

$$p_t^* (e_t) = \left\{ \begin{array}{ll} e_t + (1 - e_t) \left(1 - \gamma_{t-1}^{cb}\right), & \text{if } e_t + (1 - e_t) \left(1 - \gamma_{t-1}^{cb}\right) \leq p^{eu} \\ p^{eu}, & \text{if } e_t + \left(1 - \gamma_{t-1}^{cb}\right) (1 - e_t) > p^{eu} \end{array} \right\},$$

2. If $p^{eu} + \gamma_{t-1}^{cb} - 1 < 0$ and $\bar{u}p_{t-1} - \delta_t \geq 1 - \gamma_{t-1}^{cb}$:

$$p_t^* (e_t) = \left\{ \begin{array}{ll} p^{eu}, & \text{if } (1 - e_t) \left(1 - \gamma_{t-1}^{cb}\right) \leq p^{eu} \\ (1 - e_t) \left(1 - \gamma_{t-1}^{cb}\right), & \text{if } (1 - e_t) \left(1 - \gamma_{t-1}^{cb}\right) > p^{eu} \end{array} \right\},$$

3. If $p^{eu} + \gamma_{t-1}^{cb} - 1 < 0$ and $\bar{u}p_{t-1} - \delta_t < 1 - \gamma_{t-1}^{cb}$:

$$p_t^* (e_t) = \left\{ \begin{array}{ll} p^{eu}, & \text{if } (1 - e_t) \left(1 - \gamma_{t-1}^{cb}\right) \leq p^{eu} \\ (1 - e_t) \left(1 - \gamma_{t-1}^{cb}\right), & \text{if } (1 - e_t) \left(1 - \gamma_{t-1}^{cb}\right) > p^{eu} \text{ and } \frac{\left(1-\gamma_{t-1}^{cb}\right)(1-e_t)+\delta_t}{p_{t-1}} \leq \bar{u} \\ (1 - e_t), & \text{if } \frac{\left(1-\gamma_{t-1}^{cb}\right)(1-e_t)+\delta_t}{p_{t-1}} > \bar{u} \end{array} \right\}.$$

In cases 1. and 2, $p_t^* (e_t)$ is continuous. Hence, a fixed point of $f(\cdot)$ exists.

Consider, therefore, case 3. If

$$(1 - p^{eu}) > \gamma_{t-1}^{cb} > p^{eu},$$

then $\gamma_{t-1}^{cb} > \gamma_{t-1}^{eu}$. In this case define a function g such that:

$$g(\rho_t^* (c_t)) = \frac{\frac{\rho_t^*(c_t)+\delta_t}{p_{t-1}}\gamma_{t-1}^{cb} + (1 + r) \left(1 - \gamma_{t-1}^{cb}\right)}{\rho_t^* (c_t) + \delta_t + (1 + r) \left(1 - \gamma_{t-1}^{cb} c_t - \gamma_{t-1}^{eu} (1 - c_t)\right)},$$

where

$$\rho_t^* (c_t) = p_t^* (1 - c_t).$$

g is, thus, the equilibrium equation for the share of case-based decision-makers. It is easily verified that $g(\cdot)$ increases in ρ_t^*, as long as $\gamma_{t-1}^{cb} > \gamma_{t-1}^{eu}$ holds. Moreover, since $p_t^*(e_t)$ is continuous but for downward jumps, $\rho_t^*(c_t)$ is continuous but for upward jumps. Hence, an equilibrium share c_t^* exists and the equilibrium share e_t^* is obtained as $e_t^* = 1 - c_t^*$.

If $\gamma_{t-1}^{cb} < p^{eu}$, then $\gamma_{t-1}^{cb} < \gamma_{t-1}^{eu}$. In this case, it is useful to write $p_t^*(e_t)$ as a correspondence in the following way[150]:

$$p_t^*(e_t) = \left\{ \begin{array}{ll} p^{eu}, & \text{if } (1-e_t)\left(1-\gamma_{t-1}^{cb}\right) \leq p^{eu} \\ (1-e_t)\left(1-\gamma_{t-1}^{cb}\right), & \text{if } (1-e_t)\left(1-\gamma_{t-1}^{cb}\right) > p^{eu} \text{ and } \frac{(1-e_t)+\delta_t}{p_{t-1}} < \bar{u} \\ \left\{(1-e_t)\left(1-\gamma_{t-1}^{cb}\right); (1-e_t)\right\}, & \text{if } \frac{(1-e_t)+\delta_t}{p_{t-1}} \geq \bar{u} \text{ and } \frac{\left(1-\gamma_{t-1}^{cb}\right)(1-e_t)+\delta_t}{p_{t-1}} < \bar{u} \\ (1-e_t), & \text{if } \frac{\left(1-\gamma_{t-1}^{cb}\right)(1-e_t)+\delta_t}{p_{t-1}} \geq \bar{u} \end{array} \right\}.$$

This correspondence is illustrated in figure 13, where

$$\tilde{p}_t = \bar{u}p_{t-1} - \delta_t$$

denotes the price at which the case-based decision-makers who have observed a are indifferent between a and b.

The figure shows that although $p_t^*(e_t)$ does not satisfy the conditions of the fixed point theorem of Kakutani, see Mas-Collel, Whinston and Green (1995, p. 953), it still has a fixed point due to the fact that

$$(1-e_t) = \tilde{p}_t \text{ for } \frac{(1-e_t)+\delta_t}{p_{t-1}} = \bar{u}$$

and

$$(1-e_t)\left(1-\gamma_{t-1}^{cb}\right) = \tilde{p}_t \text{ for } \frac{\left(1-\gamma_{t-1}^{cb}\right)(1-e_t)+\delta_t}{p_{t-1}} = \bar{u}$$

Since $\gamma_{t-1}^{cb} < \gamma_{t-1}^{eu}$, $f(\cdot)$ is monotonically increasing in p_t. The monotone transformation of the correspondence, however, does not change the fixed point property and therefore $f(\cdot)$ also has a fixed point.■

Proof of proposition 6.3:

(i) Let first $e_{t-1} = 1$. It follows that $(1-e_{t-1}) = 0$. Since

$$e_t^* = \frac{\left[\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}}\gamma_{t-1}^{eu} + (1+r)\left(1-\gamma_{t-1}^{eu}\right)\right]e_{t-1}}{\frac{p_t^*(e_t^*)+\delta_t}{p_{t-1}}\gamma_{t-1}^{eu}e_{t-1} + (1+r)\left(1-\gamma_{t-1}^{eu}e_{t-1}\right)} = 1,$$

the claim of the proposition obtains.

[150] In some of the previously considered cases $p_t^*(e_t)$ is also a correspondence. Therefore, the functional form given selects one equilibrium in case of multiplicity. It turns out, however, that in this last case, the whole correspondence $p_t^*(e_t)$ is necessary to guarantee the existence of a fixed point.

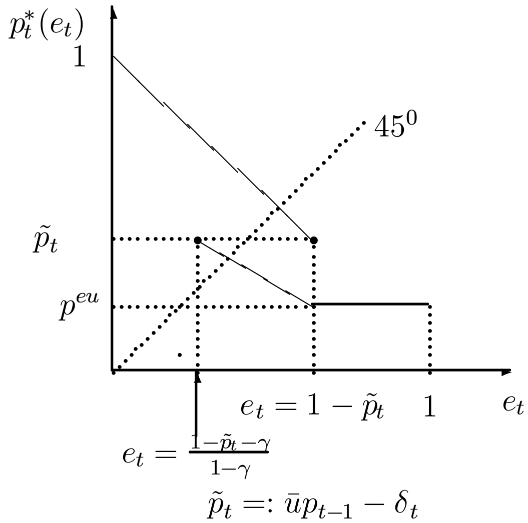

Figure 13

(ii) If $p_{t-1} > 0$ and $e_{t-1} = 0$, then $e_t^* = \frac{\tilde{v}_t^{eu} \cdot 0}{\tilde{v}_t} = 0$. Therefore, if it can be insured that the demand for a of the case-based decision-makers is strictly positive over time, the mass of the expected utility maximizers will remain 0 in every period of time. Hence, a condition is needed that insures that $\gamma_t^{*cb} > 0$ for each t. Assume, as in the proposition that $\gamma_0^{cb} > 0$. Let $(t-1)$ be some period in which $\gamma_{t-1}^{cb} > 0$ and note that those case-based decision-makers who have case $(a; v_t\,(a))$ in their memory either observe a return of 1 or a return of $1 + \frac{\delta}{p_{t-1}}$ if the price of the asset does not change between period $(t-1)$ and period t. In both cases, the return observed exceeds the aspiration level $\bar{u} < 1$, guaranteeing that a proportion of at least γ_{t-1}^{cb} of the case-based decision-makers holds a at t. The rest of the case-based decision-makers have observed the case $(b; (1+r))$ and since $(1+r) > 1 > \bar{u}$ is a satisfactory return, they invest in b. Therefore, the proportion of case-based decision-makers holding a, as well as the price of a remain constant over the time: $\gamma_t^{cb} = p_t^* = \gamma_0^{cb} > 0$ for each $t \geq 1$.■

Proof of proposition 6.4:

The assumption

$$\bar{u} < \frac{p^* + \min\{\delta_t\}}{p^*} = 1$$

guarantees that as long as the price of a remains constant over the time, none of the case-based decision-makers will ever switch away from the initially chosen portfolio. Indeed, if a case-based decision-maker remembers $(b; v_t(b))$,

$$v_t(b) = 1 + r > 1 > \bar{u}$$

holds and $\alpha_t^{cb} = b$. Alternatively, if a case-based decision-maker remembers $(a; v_t(a))$, then two possibilities occur: if $\delta_t = \delta$,

$$v_t(a) = \frac{p^* + \delta}{p^*} > 1 > \bar{u};$$

if $\delta_t = 0$, then

$$v_t(a) = \frac{p^* + 0}{p^*} = 1 > \bar{u}.$$

In both cases, the alternative chosen last is satisfactory and is chosen again. Hence, $\gamma_t^{cb} = \gamma_0^{cb}$ for each $t \geq 1$.

Let first $p^{eu} > 1$. Since short-sales are forbidden, the market price satisfies $p^* \leq 1 < p^{eu}$ in each period, therefore in each period $\gamma_t^{eu} = 1$ holds. At the same time, given that the case-based decision-makers start with $\gamma_0^{cb} = 1$, $\gamma_t^{cb} = 1$ will hold in each t, as the argument above demonstrated. This means that the equilibrium price $p_t^* = p^* = 1$ in each t is indeed constant over the time. Moreover, since the case-based decision-makers and the expected utility maximizers hold the same portfolio consisting only of risky assets, their returns are equal for each possible dividend realization in each period of time:

$$\tilde{v}_t^{eu} = \tilde{v}_t^{cb} = \frac{p^* + \delta_t}{p^*}.$$

Therefore $e_t^* = e_0 = e$ for all t. This result does not depend on the initial share e_0 of the expected utility maximizers.

Let now $p^{eu} \leq 1$. Now the short-sale constraint is not binding. Therefore, the expected utility maximizers will choose a if $p^* < p^{eu}$, b if $p^* > p^{eu}$ and will be indifferent at p^{eu}. If p^* exceeded p^{eu}, then the expected utility maximizers would hold b in each period, whereas at least some of the case-based decision-makers would have to hold a to guarantee that the price remains positive. Therefore, both types of investors would hold different portfolios and would therefore achieve different returns in general. Hence, e_t could not remain constant over time.

Alternatively, if $p^* < p^{eu}$, then all expected utility maximizers would hold a, whereas at least some of the case-based decision-makers would have to hold b, because else the price of a would jump to 1. Hence, in this case e_t would not remain constant over time, either. The only possible stationary state occurs therefore at $p^* = p^{eu}$. Note that this is the equilibrium price if the portfolios of both types of investors satisfy:

$$\gamma_t^{cb} = \gamma_t^{eu} = p^{eu}$$

for each t. Moreover, these portfolios are indeed optimal at the equilibrium price p^{eu}. But since now both types of investors hold identical portfolios, their returns are equal for each dividend realization in each period of time:

$$\tilde{v}_t^{eu} = \tilde{v}_t^{cb} = p^{eu}\frac{p^* + \delta_t}{p^*} + (1 - p^{eu})(1 + r).$$

Therefore, their shares in the population remain constant over time.■

Proof of proposition 6.5:

As was pointed out in the main text, the average returns of the case-based decision-makers and the expected utility maximizers are equal if both types hold a and only differ in periods in which the case-based decision-makers hold b, whereas the expected utility maximizers hold a. In such periods, the expected (since it depends on a random dividend payment) equilibrium share of case-based decision-makers is given by:

$$E\left[c_t^* \mid c_{t-1}^*\right] = q\frac{(1+r)\,c_{t-1}^*}{(1+r)\,c_{t-1}^* + \frac{(1+\delta)}{p_{t-1}^*}\left(1 - c_{t-1}^*\right)} + (1-q)\frac{(1+r)\,c_{t-1}^*}{(1+r)\,c_{t-1}^* + \frac{1}{p_{t-1}^*}\left(1 - c_{t-1}^*\right)}.$$

Now note that since in $(t-1)$ only the expected utility maximizers hold a,

$$p_{t-1}^* = e_{t-1}^* = 1 - c_{t-1}^*.$$

Therefore, the evolution of c_t^* depends on the comparison:

$$E\left[c_t^* \mid c_{t-1}^*\right] = q\frac{(1+r)\,c_{t-1}^*}{(1+r)\,c_{t-1}^* + (1+\delta)} + (1-q)\frac{(1+r)\,c_{t-1}^*}{(1+r)\,c_{t-1}^* + 1} \gtreqless c_{t-1}^*,$$

which is equivalent to:

$$r\,(1+\delta) + r\,(1+r)\,c_{t-1}^* - q\delta\,(1+r) \gtreqless (1+\delta)\,(1+r)\,c_{t-1}^* + (1+r)^2\,c_{t-1}^{*2}$$

$$r\left(1+\delta+(1+r)\,c_{t-1}^*\right) - (1+r)\,c_{t-1}^*\left(1+\delta+(1+r)\,c_{t-1}^*\right) - q\delta\,(1+r) \gtreqless 0$$

$$\left(1+\delta+(1+r)\,c^*_{t-1}\right)\left[r-(1+r)\,c^*_{t-1}\right]-q\delta\,(1+r)\gtreqless 0. \qquad (6.80)$$

It is clear that for $c^*_{t-1}\geq\frac{r}{1+r}$,

$$E\left[c^*_t\mid c^*_{t-1}\right]<c^*_{t-1},$$

therefore, c^*_t is a supermartingale and since $e^*_t+c^*_t=1$ in each period, it follows that e^*_t is a submartingale if $e^*_{t-1}\leq\frac{1}{1+r}$. For $c^*_{t-1}\to 0$, the l.h.s. of (6.80) becomes:

$$(1+\delta)\,r-q\delta\,(1+r)=(1+\delta-p^{eu}r)\,r-p^{eu}r>0 \text{ for } p^{eu}<\frac{(1+\delta)}{(1+r)},$$

since per assumption $p^{eu}=\frac{q\delta}{r}$ and $\delta>r$ hold. If $p^{eu}<\frac{(1+\delta)}{(1+r)}$, the continuity of the l.h.s. of (6.80) guarantees that

$$E\left[c^*_t\mid c^*_{t-1}\right]>c^*_{t-1}$$

holds in some surrounding of 0. Hence, c_t is a submartingale for c^*_{t-1} close to 0. It follows that for some $\tilde{c}\in\left(0;\frac{r}{1+r}\right)$

$$E\left[c^*_t\mid c^*_{t-1}=\tilde{c}\right]=\tilde{c}.$$

The assertion of the first part of the proposition now follows by defining $\tilde{e}=1-\tilde{c}$.

If $p^{eu}>\frac{(1+\delta)}{(1+r)}$, the l.h.s. of (6.80) is negative for all $c^*_{t-1}\in[0;1]$ and, therefore,

$$E\left[e^*_t\mid e^*_{t-1}\right]>e^*_{t-1} \text{ for all } e^*_{t-1}\in[0;1]$$

obtains. It follows that e^*_t is a submartingale on $[0;1]$. Hence, the convergence theorem for submartingales, see theorem 35.5 in Billingsley (1995, p. 468), applies, i.e. e^*_t converges almost surely. It follows that on almost each dividend path

$$\lim_{t\to\infty}\frac{e^*_t}{e^*_{t-1}}=\lim_{t\to\infty}\frac{(1+\delta_t)}{1+(1-e^*_t)\,r+\delta_t}=1$$

must hold, which is only possible, if $e^*_t\to 1$ with probability 1.■

Proof of proposition 6.6:

Assume that $e_t\in\left[\max\left\{p^{eu};1-p^{eu}+\frac{p^{eu}r^2}{1-p^{eu}}\right\};1\right)$.

The proposition will be proved separately for those periods in which the case-based decision-makers hold a and those periods in which they hold b. First note that if $\gamma^{cb}_t=1$ holds, then the case-based decision-makers continue to hold a at time $t+1$, iff $\delta_{t+1}=\delta$ so that in this case

$$p_{t+1}=p_t=p^{eu}.$$

By assumption, the share of case-based decision-makers satisfies

$$c^*_t=1-e^*_t\leq 1-p^{eu}<p^{eu},$$

since $p^{eu}<\frac{1}{2}$. If the price cannot rise higher than p^{eu}, the average return of the case-based decision-makers is, therefore:

$$\tilde{v}_{t+1}^{cb} = 1 + \frac{\delta}{p^{eu}},$$

whereas the average return of the population is given by

$$\tilde{v}_{t+1} = p^{eu} + \delta + (1+r)(1-p^{eu}) = 1 + r - rp^{eu} + \delta,$$

as long as $e_{t+1}^* < p^{eu}$ holds. Furthermore, since $1 + \frac{\delta}{p^{eu}} > 1 + \delta > \bar{u}$, the young case-based decision-makers invest in a as well so that $\gamma_{t+1}^{cb} = 1$.

Alternatively, if $\delta_{t+1} = 0$, then the highest return that the case-based decision-makers can achieve from a is $1 < \bar{u}$, therefore the young case-based decision-makers will choose b, achieving an average return of at most 1. Since this average return is smaller than the average return of the population, given by:

$$\tilde{v}_{t+1} = p^{eu} + (1+r)(1-p^{eu}) = 1 + r - rp^{eu},$$

the mass of the case-based decision-makers decreases, making it possible for the expected utility maximizers to sustain the price of a at p^{eu} at time $(t+1)$.

If $\delta_{t+1} = \delta$, then the returns and the behavior of the investors in $(t+2)$ is described exactly as in $(t+1)$, except in the case, in which the share of the case-based decision-makers has risen above p^{eu} and does not allow the expected utility maximizers to reduce the price of the risky asset to its fundamental value. This can happen, if the initial c_t^* is relatively high, so that:

$$c_{t+1}^* = \frac{\left(1 + \frac{\delta}{p^{eu}}\right)}{1 + r - rp^{eu} + \delta} c_t^* > p^{eu}. \tag{6.81}$$

It is, therefore, shown that if p^{eu} is sufficiently large, $c_{t+1}^* < p^{eu}$ holds for all values of $c_t^* \in (0; 1 - p^{eu})$. Indeed, rewrite (6.81) as

$$c_t^* > \frac{p^{eu}(1 + r - rp^{eu} + \delta)}{1 + \frac{\delta}{p^{eu}}}.$$

To exclude the case, in which the inequality in (6.81) holds, it is necessary that:

$$\frac{p^{eu}(1 + r - rp^{eu} + \delta)}{1 + \frac{\delta}{p^{eu}}} > 1 - p^{eu},$$

or that

$$-rp^{eu^3} + p^{eu^2}(2 + r + \delta) - (1 - \delta) p^{eu} - \delta > 0. \tag{6.82}$$

Note first that for $\delta = \frac{r}{2q}$ ($p^{eu} = \frac{1}{2}$), the l.h.s. is negative and that for $\delta = \frac{r}{q}$ ($p^{eu} = 1$), the l.h.s. is positive. Using now the fact that $p^{eu} = \frac{q\delta}{r}$, rewrite (6.82) as:

$$q^2\delta^2(1 - q) + \delta q(2q + qr + r) - r > 0$$

and since the l.h.s. of this expression is a convex quadratic function, there exists a $\hat{\delta}$, such that for every $\delta > \hat{\delta}$ (6.82) is satisfied.

The expected value of the share of the case-based decision-makers at time $(t+2)$, given their share at time t, can then be written as[151]:

$$\begin{aligned} E\left[c^*_{t+2} \mid c^*_t, \gamma^{cb}_t = 1\right] &= c^*_t q \frac{\left(1+\frac{\delta}{p^{eu}}\right)}{R+\delta}\left[q\frac{1+\frac{\delta}{p^{eu}}}{R+\delta} + (1-q)\frac{1}{R}\right] + \\ &\quad + c^*_t (1-q)\frac{1}{R}\left[q\frac{(1+r)}{R+\delta} + (1-q)\frac{(1+r)}{R}\right], \end{aligned}$$

where $R = 1 + r - rp^{eu}$. Using simple algebra and the fact that $p^{eu} = \frac{q\delta}{r}$ shows that

$$E\left[c^*_{t+2} \mid c^*_t, \gamma^{cb}_t = 1\right] > c^*_t,$$

if and only if

$$(q+r)\,R\,(R(1+r) + \delta(1-q)) + (1-q)(1+r)(R + \delta(1-q))(R+\delta) > (R+\delta)^2 R^2 \tag{6.83}$$

holds. If

$$p^{eu} = \frac{q\delta}{r} = 1,$$

meaning that $R = 1$ and $q\delta = r$, condition (6.73) simplifies to:

$$(1+\delta)\,r\,(\delta - r + qr) > 0,$$

which is always satisfied, since $\delta > r$ holds by assumption. On the other hand, for

$$p^{eu} = \frac{q\delta}{r} = \frac{1}{2}$$

and, hence, $R = 1 + \frac{r}{2}$ and $q\delta = \frac{r}{2}$, (6.83) is equivalent to

$$\frac{qr}{2} + \frac{qr^2}{4} + \frac{3r^3}{16} - \frac{1}{2} - r - \delta r - \delta^2 r - \delta r^2 > 0,$$

which is never satisfied, since

$$\begin{aligned} \frac{qr}{2} &< \frac{r}{2} < \frac{1}{2} \\ \frac{qr^2}{4} &< r^2 < r \\ \frac{3r^3}{16} &< r^3 < r^2 < \delta r \end{aligned}$$

hold according to the assumption that $\delta > r$, $r \in (0;1)$ and $q \in (0;1)$. Therefore,

$$E\left[c^*_{t+2} \mid c^*_t, \gamma^{cb}_t = 1\right] > c^*_t$$

[151] In fact, as above, it should be taken into account that the share of the case-based decision-makers might exceed p^{eu} in $(t+2)$ if the risky asset pays a high dividend. However, this will only increase the expected value of c^*_{t+2}. Since the argument relies on showing that the expected value of c^*_{t+2} exceeds c^*_t, neglecting this effect has no influence on the results.

holds for $\delta = \frac{r}{q}$ and since the expected value of c^*_{t+2} is continuous in δ, it follows that the process c^*_t, c^*_{t+2}, c^*_{t+4}... is a submartingale in some surrounding of $\delta = \frac{r}{q}$. At the same time,

$$E\left[c^*_{t+2} \mid c^*_t, \gamma^{cb}_t = 1\right] < c^*_t$$

holds for $\delta = \frac{r}{2q}$. By continuity of the expected value of c^*_{t+2}, there is, therefore, a value for δ, $\bar{\delta} \in \left(\frac{r}{2q}; \frac{r}{q}\right)$ such that the expected value of c^*_{t+2} exceeds c^*_t for $\delta > \bar{\delta}$.

Now suppose that $\gamma^{cb}_t = 0$. Similar arguments as those stated above allow to write the expected value of c^*_{t+2} as:

$$\begin{aligned} E\left[c^*_{t+2} \mid c^*_t, \gamma^{cb}_t = 0\right] &= c^*_t q \frac{(1+r)}{R+\delta}\left[q\frac{\left(1+\frac{\delta}{p^{eu}}\right)}{R+\delta} + (1-q)\frac{1}{R}\right] + \\ &\quad + c^*_t (1-q)\frac{(1+r)}{R}\left[q\frac{\left(1+\frac{\delta}{p^{eu}}\right)}{R+\delta} + (1-q)\frac{1}{R}\right]. \end{aligned}$$

Again, one should take into account that the mass of the case-based decision-makers could increase above p^{eu} in period $(t+1)$, when the dividend of the risky asset is low. However, this would require that:

$$\frac{(1+r)}{1+r-rp^{eu}} c^*_t > p^{eu},$$

or, equivalently

$$e^*_t < 1 - p^{eu} + \frac{r{p^{eu}}^2}{1+r},$$

which is excluded by the assumptions made.

Using simple algebra and the fact that $p^{eu} = \frac{q\delta}{r}$ shows that

$$E\left[c^*_{t+2} \mid c^*_t, \gamma^{cb}_t = 0\right] > c^*_t$$

holds if

$$(1+r)\left[R(1+r) + \delta(1-q)\right]\left[R + \delta(1-q)\right] > (R+\delta)^2 R^2 \tag{6.84}$$

is satisfied. Note that for $p^{eu} = 1$, condition (6.84) is equivalent to

$$(1+r)(1+\delta-r) > 0,$$

which is always satisfied. For $p^{eu} = \frac{1}{2}$, (6.84) becomes

$$\frac{3r^2}{8} + \frac{r}{4} + \frac{r^2\delta}{2} + \frac{r\delta}{2} + \frac{\delta}{2} > 0,$$

which is obviously satisfied for all positive values of r and δ. Since the expected value of c^*_{t+2} is continuous in δ, it follows that there is a $\check{\delta} \in \left[\frac{r}{2q}; \frac{r}{q}\right)$ such that

$$E\left[c_{t+2}^* \mid c_t^*, \gamma_t^{cb} = 0\right] > c_t^*$$

for all $\delta > \check{\delta}$. Now choose the maximal of the three values $\hat{\delta}$, $\check{\delta}$, $\bar{\delta}$ and denote it by $\tilde{\delta}$. Let $\tilde{p}^{eu} = \frac{q\tilde{\delta}}{r}$. It follows that $\tilde{p}^{eu} \in \left(\frac{1}{2}; 1\right)$ and that

$$E\left[c_{t+2}^* \mid c_t^*\right] > c_t^*,$$

for $p^{eu} > \tilde{p}^{eu}$ and

$$e_t \in \left[\max\left\{p^{eu}; 1 - p^{eu} + \frac{p^{eu} r^2}{1 - p^{eu}}\right\}; 1\right).$$

Since c_{t+2}^* and e_{t+2}^* sum to 1, it follows that

$$\begin{aligned} E\left[e_{t+2}^* \mid e_t^*, \gamma_t^{cb} = 0\right] &< e_t^* \\ E\left[e_{t+2} \mid e_t^*, \gamma_t^{cb} = 1\right] &< e_t^*, \end{aligned}$$

if $p^{eu} > \tilde{p}^{eu}$ and

$$e_t \in \left[\max\left\{p^{eu}; 1 - p^{eu} + \frac{p^{eu} r^2}{1 - p^{eu}}\right\}; 1\right)$$

are fulfilled simultaneously.■

Proof of proposition 6.7:

In lemma 19 in Sciubba (1999, p. 40), it is demonstrated that a supermartingale bounded between $[0; 1]$ and starting below 1 cannot converge to its upper boundary with probability 1. The following argument follows closely the proof of proposition 17 in Sciubba (1999, pp. 40-41). Suppose that e_t^* converges to 1 with strictly positive probability and denote the event on which this happens by Θ. Now consider e_t^* on the event Θ and suppose that on Θ $\Pr\{e_t^* \to 1\} = 1$. Denote by $\Theta_0 \subseteq \Theta_1 \subseteq ...\Theta_t \subseteq ...\Theta$ the natural filtration of Θ. Since $\Pr\{\Theta\} > 0$, and since the process of the dividends is i.i.d., the Law of Large Numbers applies and the distribution of dividends on Θ coincides with the distribution of the dividends on Φ, the set of all possible dividend paths. Especially, $\Pr\{\delta_t = \delta \mid \Theta_{t-1}\} = \Pr\{\delta_t = \delta\} = q$. Therefore, the process e_t^* on Θ can be described in exactly the same way, as the process e_t^* on Φ and, therefore, e_t^* is a supermartingale on Θ. But, according to Lemma 19 in Sciubba (1999, p. 40),

$$\Pr\{e_t^* \to 1 \mid \Theta\} \neq 1,$$

since e_t^* is a supermartingale bounded above by 1. Therefore, there is no event with positive probability on which $e_t^* \to 1$ occurs almost surely. Hence,

$$\Pr\{e_t^* \to 1\} = 0$$

and the case-based decision-makers survive with probability 1.■

Proof of proposition 6.8:

Since $p^{eu} \geq 1$ holds, the expected utility maximizers hold a in each period of time. The case-based decision-makers always choose b, since their aspiration level is between 1 and $(1+r)$. Therefore, the price of a is given by

$$p_t^* = e_t^* = 1 - c_t^*$$

for each t. The return of the case-based decision-makers is $(1+r)$ in each period, whereas the average return of the population is given by

$$\tilde{v}_t = e_t^* + \delta_t + (1 - e_t^*)(1+r) = 1 + \delta_t + c_t^* r.$$

Hence, $E\left[c_{t+1}^* \mid c_t^*\right]$ can be written as:

$$E\left[c_{t+1}^* \mid c_t^*\right] = \left[q \frac{(1+r)}{1 + c_t^* r + \delta} + (1-q)\frac{(1+r)}{1 + c_t^* r}\right] c_t^* \gtreqless c_t^*$$

This simplifies to:

$$(1 - c_t^*)\, r\, (1 + c_t^* r + \delta) - q\,(1+r)\,\delta \gtreqless 0. \tag{6.85}$$

For $c_t \to 0$ the left hand side becomes:

$$r\,(1+\delta) - q\,(1+r)\,\delta > 0 \text{ for } p^{eu} < \frac{1+\delta}{1+r}.$$

For $c_t^* = 1$, the l.h.s. of (6.85) is negative. Since the l.h.s. of (6.85) is a quadratic function with a negative coefficient in front of c_t^{*2}, it follows that for $p^{eu} < \frac{1+\delta}{1+r}$, there exists a unique $\hat{c} \in (0;1)$, for which the left hand side of (6.85) is 0. For $c_t^* > \hat{c}$, c_t^* is a supermartingale and vice versa. Now denote by $\hat{e} = 1 - \hat{c}$ the share of expected utility maximizers corresponding to the share $\hat{c}$ of case-based decision-makers. It follows that e_t^* is a supermartingale for $e_t^* > \hat{e}$ and a submartingale for $e_t^* < \hat{e}$.

If $p^{eu} \geq \frac{1+\delta}{1+r}$, then c_t^* is a supermartingale on the whole interval $[0;1]$. Therefore e_t^* is a submartingale on $[0;1]$. Hence, the convergence theorem for martingales applies, i.e. e_t^* converges almost surely. It follows that on almost each dividend path

$$\lim_{t\to\infty} \frac{e_t^*}{e_{t-1}^*} = \lim_{t\to\infty} \frac{\frac{(e_t^* + \delta_t)}{e_t^* + \delta_t + (1+r)\left(1 - e_{t-1}^*\right)}}{\frac{\left(e_{t-1}^* + \delta_{t-1}\right)}{e_{t-1}^* + \delta_{t-1} + (1+r)\left(1 - e_{t-2}^*\right)}} = 1.$$

must hold. For $e_t^* = e_{t-1}^* = e_{t-2}^*$, this implies:

$$\lim_{t\to\infty} \frac{(e_t^* + \delta_t)}{e_t^* + \delta_t + (1+r)(1 - e_t^*)} = \lim_{t\to\infty} \frac{(e_t^* + \delta_{t-1})}{e_t^* + \delta_{t-1} + (1+r)(1 - e_t^*)}.$$

Note, however that since δ_t is a stochastic process, this equality can only hold if the average return of the expected utility maximizers coincides with the average return of the society in each period of time, hence if $e_t^* \to 1$ with probability 1.■

Proof of proposition 6.9:

Rewrite conditions (6.80) and (6.85) as:

$$-\left(1+r\right)^2 c_t^{*^2} - \left(1+r\right) c_t^* \left(1+\delta-r\right) + \left(1+\delta\right) r - q\delta\left(1+r\right)$$

and

$$-r^2 c_t^{*^2} - \left(1+r\right) c_t^* \left(1+\delta-r\right) + \left(1+\delta\right) r - q\delta\left(1+r\right),$$

respectively. One easily sees that

$$\begin{aligned} & -\left(1+r\right)^2 c_t^{*^2} - \left(1+r\right) c_t^* \left(1+\delta-r\right) + \left(1+\delta\right) r - q\delta\left(1+r\right) \\ \leq \quad & -r^2 c_t^{*^2} - \left(1+r\right) c_t^* \left(1+\delta-r\right) + \left(1+\delta\right) r - q\delta\left(1+r\right) \end{aligned}$$

always holds. Hence, the sole positive root of (6.80) $\tilde{c}$ is always smaller than the sole positive root of (6.85), $\hat{c}$. Since $\tilde{e} = 1 - \tilde{c}$ and $\hat{e} = 1 - \hat{c}$, it follows that $\tilde{e} > \hat{e}$.■

Proof of proposition 6.10:

Since $e_t^* \geq \max\left\{p^{eu}; 1 - p^{eu} + \frac{rp^{eu^2}}{(1+r)}\right\}$, the mass of expected utility maximizers is sufficiently large to support the price of a at p^{eu} in periods $t+1$ and $t+2$. Since the aspiration level of the case-based decision-makers satisfies $\bar{u} \in (1; 1+r)$, they hold b in each period of time. Their average return is, therefore,

$$\tilde{v}_t^{cb} = \left(1+r\right),$$

whereas the average return of the population is

$$\tilde{v}_t = 1 + \delta_t + r - rp^{eu}.$$

Hence, $E\left[c_{t+1}^* \mid c_t^*\right]$ can be written as:

$$E\left[c_{t+1}^* \mid c_t^*\right] = \left[q\frac{\left(1+r\right)}{1+r-rp^{eu}+\delta} + \left(1-q\right)\frac{\left(1+r\right)}{1+r-rp^{eu}}\right] c_t^* \gtreqless c_t^*$$

This easily simplifies to:

$$-q\left(1+r\right)\delta \gtreqless -rp^{eu}\left(1+r-rp^{eu}+\delta\right)$$

and by using the fact that $p^{eu} = \frac{q\delta}{r}$, one obtains that c_t^* is a submartingale if

$$q\delta^2\left(1-q\right) > 0,$$

which is always satisfied for q and $\delta \in (0;1)$[152]. Since c_t^* is a submartingale and since $e_t^* = 1 - c_t^*$,

[152] For $q\delta = 0$ both the case-based decision makers and the expected utility maximizers hold only asset b and achieve identical returns in each period of time. The mass of the case-based decision makers, thus, remains constant.

it follows that e_t^* is a supermartingale on

$$\left[\max\left\{p^{eu}; 1-p^{eu}+\frac{rp^{eu^2}}{(1+r)}\right\}; 1\right].\blacksquare$$

Proof of lemma 6.1:

Rewrite (6.72) as:

$$c_{t+1}^{*2}+c_{t+1}^*\left(\delta+(1+r)\left(1-c_t^*\right)-1\right)-\delta=0. \tag{6.86}$$

For $c_{t+1}^*=1$ the l.h.s. becomes:

$$(1+r)\left(1-c_t^*\right)>0.$$

For $c_{t+1}^*=0$ the l.h.s. is

$$-\delta<0.$$

Since the l.h.s. of (6.86) is a convex quadratic function, it follows that it has exactly one root between 0 and 1.

To prove the second part of the assertion, compute the l.h.s. of (6.86) for $c_{t+1}^*=c_t^*$:

$$\left(c_t^*-1\right)\left(\delta-c_t^*r\right),$$

which is always negative, since $c_t^*<1$ and

$$\delta>r\geq c_t^*r$$

hold by assumption. Therefore, $c_t^*<c_{t+1}^*$ obtains.$\blacksquare$

Proof of proposition 6.11:

As was demonstrated in the proof of propositions 6.5, the condition for the existence of $\tilde{e}(\beta)$, as defined in the proposition, is that for $p^{eu}\geq 1$,

$$(1+\delta)r-q\delta(1+r)>0$$

holds. The condition $q\in\left[\frac{(1+r)^\beta-1}{(1+\delta)^\beta-1}; \frac{(1+\delta)r}{(1+r)\delta}\right)$ is, therefore, equivalent to

$$p^{eu}(\beta)\geq 1$$

and

$$(1+\delta)r-q\delta(1+r)>0$$

for $\beta\in(0;1]$, whereas the condition $q\in\left[\frac{\ln(1+r)}{\ln(1+\delta)}; \frac{(1+\delta)r}{(1+r)\delta}\right)$ is equivalent to

$$p^{eu}(0)\geq 1$$

and

$$(1+\delta)r-q\delta(1+r)>0.$$

Hence, these conditions indeed insure the existence of $\tilde{e}(\beta)$.

If, on the other hand, $q \geq \frac{(1+\delta)r}{(1+r)\delta}$, the argument in the proof of proposition 6.5 can be used to show that e_t^* is a submartingale on the whole interval $[0;1]$ and that the case-based decision-makers disappear with probability 1 in the limit.

It still remains to be shown that $\left[\frac{(1+r)^\beta-1}{(1+\delta)^\beta-1}; \frac{(1+\delta)r}{(1+r)\delta}\right)$ makes sense for each $\beta \in [0;1]$, i.e. that

$$\frac{(1+\delta)\,r}{(1+r)\,\delta} > \frac{(1+r)^\beta - 1}{(1+\delta)^\beta - 1}, \tag{6.87}$$

for $\beta \in (0;1]$ and

$$\frac{(1+\delta)\,r}{(1+r)\,\delta} > \frac{\ln(1+r)}{\ln(1+\delta)} \tag{6.88}$$

hold.

Consider first (6.87) and rewrite it as:

$$\frac{(1+\delta)\left[(1+\delta)^\beta - 1\right]}{\delta} > \frac{(1+r)\left[(1+r)^\beta - 1\right]}{r}.$$

Since $r < \delta$, (6.87) can be proved by showing that the function

$$h_\beta(x) = \frac{(1+x)\left[(1+x)^\beta - 1\right]}{x}$$

is increasing in x for all $\beta \in (0;1]$. Differentiate with respect to x to obtain:

$$h'_\beta(x) = \frac{1-(1-\beta x)(1+x)^\beta}{x^2}.$$

Note that $h'_1(x) = 1 > 0$ for all x and that

$$\lim_{\beta \to 0} h'_\beta(x) = 0 \text{ for all } x.$$

Differentiating $h'_\beta(x)$ with respect to β one obtains:

$$\frac{\partial h'_\beta(x)}{\partial \beta} = \frac{[x-(1-\beta x)\ln(1+x)](1+x)^\beta}{x^2}.$$

$\frac{\partial h'_\beta(x)}{\partial \beta} > 0$ is equivalent to

$$\beta > \frac{\ln(1+x) - x}{x\ln(1+x)}.$$

But $\frac{\ln(1+x)-x}{x\ln(1+x)} \leq 0$ for each $x \in (0;1)$ and hence, $\frac{\partial h'_\beta(x)}{\partial \beta} > 0$ holds for all $x \in (0;1)$ and all $\beta \in (0;1]$. Therefore, $h'_\beta(x) > 0$ for all $\beta \in (0;1]$, implying that $h_\beta(x)$ is strictly increasing in x for all $\beta \in (0;1]$.

For (6.88) it is enough to show that

$$\frac{(1+r)\ln(1+r)}{r} < \frac{(1+\delta)\ln(1+\delta)}{\delta}$$

holds, or that the function

$$h_0(x) = \frac{(1+x)\ln(1+x)}{x}$$

is increasing in x for $x \in [0;1]$. Since

$$h_0'(x) = \frac{x - \ln(1+x)}{x^2} \geq 0$$

is satisfied for all $x \in [0;1]$, $h_0(x)$ is indeed increasing in x and the claim holds.■

Proof of proposition 6.12:

As proposition 6.8 shows, the condition which is necessary for the survival of the case-based decision-makers is $p^{eu} \geq 1$ and

$$r(1+\delta) - q(1+r)\delta > 0,$$

or

$$q < \frac{(1+\delta)}{(1+r)\delta}. \tag{6.89}$$

As in the proof of proposition 6.11,

$$q \geq \frac{(1+r)^{\beta} - 1}{(1+\delta)^{\beta} - 1},$$

and

$$q \geq \frac{\ln(1+r)}{\ln(1+\delta)}$$

reflect the fact that the price $p^{eu}(\beta) \geq 1$. But in proposition 6.11, it has been shown that these conditions for q make sense. Hence, the result obtains.■

Proof of lemma 6.2:

Denote by $\rho_1 = p_{t+1} + \delta$ and by $\rho_2 = p_{t+1}$. Therefore, $\gamma_t^{eu}(p_t)$ can be written as

$$\gamma_t^{eu}(p_t) = \frac{(1+r)(q\rho_1 + (1-q)\rho_2 - p_t(1+r))p_t}{(\rho_1 - p_t(1+r))(p_t(1+r) - \rho_2)}.$$

Now differentiate with respect to p_t to obtain:

$$\begin{aligned}
&\frac{\partial\gamma_t^{eu}(p_t)}{\partial p_t} \\
=\ & (1+r)\frac{[q\rho_1 + (1-q)\rho_2 - 2p_t(1+r)](\rho_1 - p_t(1+r))(p_t(1+r) - \rho_2)}{(\rho_1 - p_t(1+r))^2(p_t(1+r) - \rho_2)^2} \\
& + (1+r)\frac{[2p_t(1+r) - \rho_1 - \rho_2](1+r)(q\rho_1 + (1-q)\rho_2 - p_t(1+r))p_t}{(\rho_1 - p_t(1+r))^2(p_t(1+r) - \rho_2)^2}.
\end{aligned}$$

Simple algebra shows that the sign of the derivative is determined by the sign of the polynomial:

$$-(1+r)^2 p_t^2((1-q)\rho_1 + q\rho_2) + 2p_t\rho_1\rho_2(1+r) - \rho_1\rho_2(q\rho_1 + (1-q)\rho_2), \tag{6.90}$$

which is quadratic in p_t. The discriminant of (6.90) can be computed to be:

$$\frac{D}{4} = -\left(1+r\right)^2 \rho_1 \rho_2 q \left(1-q\right) \left(\rho_1 - \rho_2\right)^2 < 0,$$

and since the coefficient in front of p_t^2 is less than 0, it follows that (6.90) is always negative. Hence, $\frac{\partial \gamma_t^{eu}(p_t)}{\partial p_t} < 0$ follows.■

Proof of corollary 6.2:

Since $\gamma_t^{eu}\left(p_t\right)$ is continuous and decreasing in p_t and since it maps the interval $[0;1]$ of possible prices p_t into $[0;1]$ it follows that it has exactly one fixed point. But the fixed point $\gamma_t^{eu}\left(p_t\right) = p_t$ is exactly the equilibrium condition for the market in which only expected utility maximizers are present.■

Proof of proposition 6.13:

The conditions for an equilibrium under rational expectations can be combined to obtain:

$$\frac{\left(1+r\right)\left(p_{\log}^{eu} + q\delta - p_{\log}^{eu}\left(1+r\right)\right) p_{\log}^{eu}}{\left(p_{\log}^{eu} + \delta - p_{\log}^{eu}\left(1+r\right)\right)\left(p_{\log}^{eu}\left(1+r\right) - p_{\log}^{eu}\right)} = p_{\log}^{eu},$$

where the l.h.s. represents the value of demand for the risky asset, $\gamma_t^{eu}\left(p_t\right)$, the r.h.s. represents the value of supply and $p_{\log}^{eu}$ denotes the price under rational expectations. Simplifying, one obtains the following quadratic equation for $p_{\log}^{eu}$:

$$-r^2 p_{\log}^{eu^2} + \left(1+r+\delta\right) r p_{\log}^{eu} - \left(1+r\right) q\delta = 0.$$

For $q > 0$, the discriminant of this equation is strictly positive:

$$\begin{aligned}
D &= \left(1+r+\delta\right)^2 r^2 - 4r^2\left(1+r\right) q\delta = \\
&= r^2\left(1+r^2+\delta^2+2r+2\delta+2\delta r - 4q\delta - 4rq\delta\right) > \\
&> r^2\left(1+r^2+\delta^2+2r+2\delta+2\delta r - 4\delta - 4r\delta\right) = \\
&= r^2\left(1+r-\delta\right)^2 > 0
\end{aligned}$$

and, therefore, the equation has two roots:

$$p_{\log,1}^{eu} = \frac{1+r+\delta+\sqrt{\left(1+r+\delta\right)^2 - 4q\delta\left(1+r\right)}}{2r}$$

and

$$p_{\log,2}^{eu} = \frac{1+r+\delta-\sqrt{\left(1+r+\delta\right)^2 - 4q\delta\left(1+r\right)}}{2r},$$

both of which are easily seen to be positive. Consider first $p_{\log,1}^{eu}$.

$$p_{\log,1}^{eu} > 1$$

is equivalent to

$$\begin{aligned} 1+r+\delta+\sqrt{(1+r+\delta)^2-4q\delta(1+r)} &> 2r, \text{ or} \\ 1-r+\delta+\sqrt{(1+r+\delta)^2-4q\delta(1+r)} &> 0, \end{aligned}$$

which is always fulfilled. Therefore $p^{eu}_{\log,1} > 1$ is satisfied for all values of the parameters. Note, however that an equilibrium price equal to 1 can only obtain, if

$$p^{eu}_{\log} < \frac{p^{eu}_{\log}\left(p^{eu}_{\log}+\delta\right)}{(1+r)\left(p^{eu}_{\log}+\delta(1-q)\right)} \tag{6.91}$$

is satisfied in order to insure that the unconstrained value of demand at $p^{eu}_{\log}$ exceeds 1. However, it is easy to see that $p^{eu}_{\log,1}$ does not satisfy this condition. Indeed (6.91) can be written as:

$$p^{eu}_{\log,1} < \frac{\delta(q-r)}{r},$$

which is equivalent to

$$1+r+\delta+\sqrt{(1+r+\delta)^2-4q\delta(1+r)}-2\delta(q-r) < 0.$$

Since however $1+\delta > 2\delta q$ holds, this cannot be fulfilled for any values of the parameters. Therefore, $p^{eu}_{\log,1}$ is not an equilibrium price of the economy without short sale constraints.

It turns out that $p^{eu}_{\log,2}$ satisfies condition (6.91). Indeed, for $p^{eu}_{\log,2}$ it becomes:

$$1+r+\delta-2\delta(q-r) < \sqrt{(1+r+\delta)^2-4q\delta(1+r)},$$

which simplifies to:

$$-\delta q(1-q)-r(1+r)-\delta r(1-r)-2\delta rq < 0,$$

which is obviously true for all values of δ, q and r.

On the other hand, $p^{eu}_{\log,2} \geq 1$ is equivalent to:

$$1+r+\delta-\sqrt{(1+r+\delta)^2-4q\delta(1+r)} \geq 2r, \text{ or}$$

$$1-r+\delta \geq \sqrt{(1+r+\delta)^2-4q\delta(1+r)},$$

which (since both sides are positive) is equivalent to:

$$\begin{aligned} (1-r+\delta)^2 &\geq (1+r+\delta)^2-4q\delta(1+r) \\ q &\geq \frac{r(1+\delta)}{\delta(1+r)}. \blacksquare \end{aligned}$$

Proof of corollary 6.3:

Since $q \geq \frac{r(1+\delta)}{\delta(1+r)}$, it follows that $p^{eu}_{\log,2} \geq 1$. Observe first that

$$\frac{p_{t+1}(p_{t+1}+\delta)}{(1+r)(p_{t+1}+(1-q)\delta)} \tag{6.92}$$

is increasing in p_{t+1} and in q. Since now the next period price expected by the expected utility maximizers satisfies $p_{t+1} \geq 1$, (6.92) becomes:

$$\begin{aligned}
\frac{p_{t+1}(p_{t+1}+\delta)}{(1+r)(p_{t+1}+(1-q)\delta)} &\geq \frac{(1+\delta)}{(1+r)(1+(1-q)\delta)} \geq \\
&\geq \frac{(1+\delta)}{(1+r)\left(1+\left(1-\frac{r(1+\delta)}{\delta(1+r)}\right)\delta\right)} \geq \\
&\geq 1. \blacksquare
\end{aligned}$$

Proof of proposition 6.14:

Since $p^{eu}_{\log} \geq 1$, it follows that $q \geq \frac{r(1+\delta)}{\delta(1+r)}$ and therefore, according to corollary 6.3, the critical price (6.92) exceeds 1. Therefore, the expected utility maximizers wish to invest their whole initial endowment in asset a in each period of time independently of its price p_t. Hence, the condition for the survival of the case-based decision-makers is identical to those in propositions 6.5 and 6.8, namely $q < \frac{r(1+\delta)}{\delta(1+r)}$. Since now $q \geq \frac{r(1+\delta)}{\delta(1+r)}$, only part 2 of each of propositions 6.5 and 6.8 applies and e^*_t is a submartingale on $[0;1]$. Therefore, the case-based decision-makers disappear with probability $1.\blacksquare$

Proof of lemma 6.3:

Let p^{eu}_β denote the price under rational expectations for $\beta \in (0;1]$. Since I am interested in comparative statics with respect to β, it is useful to write the maximization problem of an expected utility maximizer, who expects the price tomorrow to be $p_{t+1} = p$, if the price today is also equal to p:

$$\max_{\gamma^{eu}_t \in [0;1]} q\left[\left(1+\frac{\delta}{p}\right)\gamma^{eu}_t + (1+r)(1-\gamma^{eu}_t)\right]^\beta + (1-q)\left[\gamma^{eu}_t + (1+r)(1-\gamma_{t^{eu}})\right]^\beta$$

and observe that the first order condition simplifies to:

$$\gamma^{eu}_t(p) = \frac{(1+r)\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}} - 1\right]}{\frac{\delta}{p} + r\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}} - 1\right]}, \tag{6.93}$$

which describes the optimal γ^{eu}_t, as long as the term on the r.h.s. of (6.93) is between $[0;1]$. If the r.h.s. of (6.93) exceeds 1 or lies below 0, γ_t^{eu} takes the values 0 and 1, respectively[153]:

$$\gamma_t^{eu}(p) = \left\{ \begin{array}{ll} 0, & \text{if } \frac{(1+r)\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}}-1\right]}{\frac{\delta}{p}+r\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}}-1\right]} < 0 \\ \frac{(1+r)\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}}-1\right]}{\frac{\delta}{p}+r\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}}-1\right]}, & \text{if } \frac{(1+r)\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}}-1\right]}{\frac{\delta}{p}+r\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}}-1\right]} \in [0;1] \\ 1, & \text{if } \frac{(1+r)\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}}-1\right]}{\frac{\delta}{p}+r\left[\left(\frac{(1-q)r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}}-1\right]} > 0 \end{array} \right\}.$$

Note as well that at the interior solution, $\gamma_t^{eu} \in [0;1]$ implies

$$\left(\frac{(1-q)\,r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}} - 1 \geq 0,$$

or equivalently:

$$\left(\frac{(1-q)\,r}{q\left(\frac{\delta}{p}-r\right)}\right) \leq 1,$$

since $\frac{1}{\beta-1} < 0$. Since p_β^{eu} is the price under rational expectations, $\gamma_t^{eu}\left(p_\beta^{eu}\right) = p_\beta^{eu}$ must hold. The first derivative of the r.h.s. of (6.93) with respect to β is negative. To see this, denote

$$\left(\frac{(1-q)\,r}{q\left(\frac{\delta}{p}-r\right)}\right)^{\frac{1}{\beta-1}} =: z$$

and differentiate γ_t^{eu} w.r. to z:

$$\frac{\partial \gamma_t^{eu}}{\partial z} = \frac{(1+r)\,\frac{\delta}{p}}{\left(\frac{\delta}{p} + r\,(1-z)\right)^2} > 0.$$

But z itself is increasing in β, since

$$\frac{\partial z}{\partial \beta} = -\frac{1}{(\beta-1)^2}\left[\frac{r\,(1-q)}{q\left(\frac{\delta}{p}-r\right)}\right]^{\frac{1}{\beta-1}} \ln \frac{r\,(1-q)}{q\left(\frac{\delta}{p}-r\right)}$$

and

$$\frac{r\,(1-q)}{q\left(\frac{\delta}{p}-r\right)} \leq 1.$$

Hence, $\frac{\partial \gamma_t^{eu}}{\partial \beta} \leq 0$ holds. This means that for $\beta > \beta'$, $\gamma_{\beta,t}^{eu}(p) \leq \gamma_{\beta',t}^{eu}(p)$ for every p.

153 For the interior solutions, the no-arbitrage conditions are satisfied. For the corner solutions, the short sale constraints prevent arbitrage even if the no-arbitrage conditions fail.

Moreover, $\frac{\partial \gamma_t^{eu}}{\partial p} \leq 0$, since

$$\frac{\partial z}{\partial p} = \frac{1}{\beta - 1} \left(\frac{(1-q)\, r}{q \left(\frac{\delta}{p} - r \right)} \right)^{\frac{2-\beta}{\beta-1}} \frac{(1-q)\, r\delta}{qp^2 \left(\frac{\delta}{p} - r \right)^2} < 0$$

and $\frac{\partial \gamma_t^{eu}}{\partial z} > 0$. Combining these two results means that for each β γ_t^{eu} is a falling function in the price, whereas keeping the price constant, γ_t^{eu} decreases in β. Hence, $\frac{\partial p_\beta^{eu}}{\partial \beta} \leq 0$ must hold, as the figure 14 schematically illustrates.■

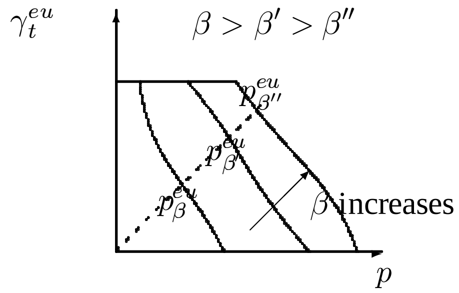

Figure 14

Proof of proposition 6.15:

Consider first the case of $p_\beta^{eu} = 1$ and note that the value of q, for which $\gamma_t^{eu} = p_\beta^{eu} = 1$ is still an interior solution of (6.93) is given by:

$$\frac{(1+r) \left[\left(\frac{(1-q)r}{q(\delta - r)} \right)^{\frac{1}{\beta-1}} - 1 \right]}{\delta + r \left[\left(\frac{(1-q)r}{q(\delta - r)} \right)^{\frac{1}{\beta-1}} - 1 \right]} = 1,$$

which simplifies to

$$q = \frac{r}{r + (\delta - r)(1+\delta)^{\beta - 1}}. \tag{6.94}$$

From $\frac{\partial \gamma_t^{eu}}{\partial z} > 0$ and

$$\frac{\partial z}{\partial q} = \frac{1}{\beta - 1} \left(\frac{(1-q) r}{q \left(\frac{\delta}{p} - r \right)} \right)^{\frac{2-\beta}{\beta-1}} \frac{-r \left(\frac{\delta}{p} - r \right)}{q^2 \left(\frac{\delta}{p} - r \right)^2} > 0,$$

it follows that $\frac{\partial \gamma_t^{eu}}{\partial q} > 0$ and since γ_t^{eu} falls in price, it follows that the equilibrium price is increasing in the probability of high dividend q. Therefore, for values of q higher than (6.94) the price under rational expectations (for a given β) is equal to 1, see figure 15.

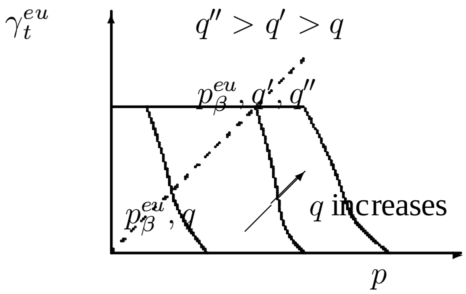

Figure 15

Hence, for all current values of $p_t \leq 1$, the expected utility maximizers, who believe that $p_{t+1} = 1$, invest their whole initial endowment into the risky asset. Therefore, the interval of values

$$q \in \left[\frac{r}{r + (\delta - r)(1+\delta)^{\beta-1}}; \frac{(1+\delta) r}{(1+r) \delta} \right)$$

corresponds to the case, in which the expected utility maximizers invest their whole initial endowment into a, independently of its price p_t and the case-based decision-makers survive, according to propositions 6.8 and 6.5. It remains to show that this conditions can be fulfilled simultaneously. Indeed,

$$\frac{r}{r + (\delta - r)(1+\delta)^{\beta-1}} < \frac{(1+\delta) r}{(1+r) \delta}$$

is equivalent to

$$(\delta - r)\left[(1+\delta)^{\beta} - 1\right] > 0,$$

which is always satisfied for $\beta > 0$. Note that the logarithmic utility function represents the limit case, $\beta = 0$, in which the equality holds.■

Chapter 7. Conclusion

In this chapter, I summarize the main findings of the thesis and give an outlook for future research.

7.1 Main Results

The thesis has analyzed the behavior of case-based investors in financial markets. The dynamics of portfolio holdings have been derived and discussed in the context of an individual portfolio choice problem, as well as in a market environment with endogenous prices. The price process in an economy populated by case-based decision-makers has been studied. Last, the fitness of case-based decisions has been addressed and conditions for survival of case-based investors in a market with expected utility maximizers provided.

7.1.1 Portfolio Choice with Case-Based Decisions

The analysis of the individual portfolio choice problem shows that a case-based investor with endogenous memory can learn to choose an optimal portfolio if he adapts his aspirations in the ”realistic-ambitious” manner, as proposed by Gilboa and Schmeidler (1996). However, in general, investors using case-based reasoning make suboptimal decisions in the limit. Low aspiration levels lead to satisficing behavior and imply that an investor may hold a suboptimal or an underdiversified portfolio, or fail to use arbitrage possibilities. Relatively high aspiration levels, on the other hand, cause frequent trades and constant switching among the available portfolio in the limit. Investors with high aspiration levels exhibit similar behavior to those of ”overconfident” traders described by Odean (1999), since they expect to achieve unrealistically high returns. These results are robust with respect to an adaptation rule, according to which the aspirations are set equal to the lowest or to the highest utility realization achieved, or to a linear combination of both.

Hypothetical reasoning helps the investor to choose the optimal portfolio in the limit, if he acquires information about the returns of all portfolios available in each period of time. Analogous results are obtained if the information is acquired for a finite number of periods, but the aspiration level of the decision-maker is relatively low. Investors with high aspiration levels fail to behave optimally even if the number of periods during which information is acquired is large, as long as it is finite.

The introduction of similarity considerations into a portfolio choice problem does not change these results qualitatively. If the similarity function is concave, it can be shown that investors with relatively high aspiration levels will not only trade too frequently, but also hold undiversified portfolios in the limit.

With similarity considerations, the ”realistic-ambitious” adaptation rule of Gilboa and Schmeidler (1996), however, does not lead to optimal choice in the limit in general. If the similarity function is concave, the decision-maker learns to choose the best undiversified portfolio in the limit. Introducing convexities into the similarity function, improves the quality of learning. By allowing the investor to better differentiate between acts, the limit choice can become arbitrary close to the expected utility maximizing act.

7.1.2 Asset Prices in an Economy with Case-Based Decision-Makers

In order to study the price dynamic, a notion of equilibrium for an overlapping generations model with case-based decision-makers is defined and studied. The existence of equilibrium in an economy populated with case-based investors is proved under quite general conditions. It is shown that degenerate equilibria with 0-asset prices are a typical feature of such markets. Conditions excluding such degenerate equilibria are identified, but it is questionable whether these conditions are economically meaningful. Therefore, conditions which guarantee the existence of at least one non-degenerate equilibrium are stated and consequently used in the analysis.

The equilibrium constructed is subsequently used to analyze the price dynamic in an economy populated only by case-based investors. In a model in which diversification is not allowed and the memory of the investors is short, it is found that asset prices and portfolio holdings remain constant over time if the aspiration levels in the economy are relatively low. Nevertheless, asset prices and portfolio holdings in a stationary state need not coincide with those predicted by expected utility maximization under rational expectations.

The presence of investors with high aspiration levels in the economy leads to cycles which can be stochastic or deterministic. These investors switch constantly among the available portfolios and cause excessive price volatility and predictability of returns. They buy at high prices and sell at low prices, thus lowering their expected returns. In contrast, investors with low aspirations do not change their portfolios over time, but might end up holding a suboptimal portfolio or even ignoring arbitrage possibilities present in the market. If their mass in the economy is relatively large, effects such as the equity premium puzzles might be observed even in an economy with risk-neutral investors.

Allowing the investors to remember all past cases experienced by his predecessors with the same aspiration level does not necessarily lead to optimal behavior. Only if the aspiration level is appropriately chosen, does an investor learn to choose the optimal portfolio (at the equilibrium price) in the limit. Excessively high aspirations again lead to frequent trading and excessive volatility.

In contrast to the individual portfolio choice problem, introducing hypothetical reasoning in a market environment does not lead to optimal decisions even if the returns of all available portfolios are observed in each period of time. Moreover, it is shown that the usage of hypothetical cases might even deteriorate limit choices compared to a situation in which the memory of the investors is completely endogenous.

Allowing for diversification and introducing a similarity function on the set of problem-act / price-portfolio pairs does not change the results significantly. In this case, investors with high aspiration levels can cause a bubble on an asset to emerge. The bubble may arise, even if the underlying asset has a 0 fundamental value. The bubble bursts in finite time with probability 1 and never reappears again. Moreover, as long as the similarity function is concave, the no-diversification result, derived in the context of individual portfolio choice also holds in a market environment. Especially, investors with high aspirations hold a diversified portfolio only for a finite number of periods. Afterwards, they either choose one of the corner portfolios forever or switch constantly among the undiversified portfolio available.

7.1.3 Fitness of Case-Based Decisions

The last chapter of the thesis analyzes the issue of survival of case-based investors in the presence of expected utility maximizers in the market. A replicator dynamic selecting for the type of investors with higher average returns is introduced. It is shown that case-based investors can coexist with expected utility maximizers by imitating them. The resulting equilibrium replicates the equilibrium under rational expectations.

Case-based investors can also survive in strictly positive proportion even if their strategy differs from the one of the expected utility maximizers. Especially, if the portfolio held by the case-based decision-makers is less risky than the one of the expected utility maximizers, then both types of investors coexist in the market in strictly positive proportions almost surely in the limit.

Moreover, if the share of case-based decision-makers is sufficiently large and their aspiration level relatively high, they can influence market prices by causing bubbles, excessive volatility and predictability of returns. Case-based investors with relatively low aspiration levels can drive the expected utility maximizers out of the market for a finite number of periods and cause an asset with positive fundamental value to be traded at a 0-price in the market.

The conditions for survival of the case-based investors, derived for a linear utility function, are further generalized for the case of constant relative risk-aversion. It is found that only expected utility maximizers with a logarithmic utility function are able to drive the case-based investors out of the market for all parameter values. For all coefficients of relative risk-aversion lower than 1, parameter values are identified for which the wealth share of the case-based decision-makers remains positive almost surely in the limit.

7.2 Outlook

This thesis is only a first attempt to examine the behavior of case-based decision-makers in a market environment and their influence on prices and asset returns. It shows that case-based reasoning is not always inferior to expected utility maximization and that it might help explain observed phenomena in real and experimental markets. Nevertheless, more research has to be done and further questions have to be answered, before the meaning of case-based reasoning in financial markets is fully understood.

In the course of the discussion, I have assumed that case-based investors act on their own account, deciding directly which portfolio to choose. However, given the small amount of information they possess, it seems much more natural to assume that case-based decision-makers rely on institutional investors to invest their wealth. Hence, instead of choosing an asset directly, they would choose a broker, whose investment rule is in general unknown. Thus, the situation is clearly one of structural ignorance. Hypothetical reasoning is impossible in this case and the application of case-based decision theory to evaluate the performance of a broker already chosen seems quite natural.

Such a model, however, might be very complex, since it has to capture the interaction of different brokers, i.e. different investment rules in an asset market, whereas the wealth available to a broker is determined by his past performance and the aspiration levels of the small investors. Nevertheless, it might provide interesting results about the investment rules which survive according to such "case-based selection", as well as about the dynamics of asset prices in a more realistic model.

Up to now, no interaction between case-based investors (or consumers) and the supply side has been modelled in the literature. Of course, this interaction can be neglected in a competitive market, such as the one discussed in this thesis. In an oligopolistic market, however, this interaction gains importance, especially, since it has been shown that case-based decision-makers might leave profit opportunities unused.

A monopolist who knows that his consumers behave according to the case-based decision theory might try to influence their choices, by creating cases in which the utility realizations exceed the aspiration level (e.g. through advertising) and, thus, might be able to sell at higher prices extracting a greater proportion of the consumer rent[154].

In a further step, the interaction between oligopolists in a market with case-based consumers should be analyzed. If the goods produced are similar, then each producer will have to take into account that a positive experience with his own product positively influences the evaluation of the rival product as well. Therefore, prices have to be strategically set and publicity planned, so as to maximize profits.

Although the proposed direction for further research can also find application in the context of financial markets, in which the market-maker has a monopolistic position or brokers engage

[154] There is still little literature discussing how a monopolist can increase his profits in the presence of boundedly rational consumers. Sarafidis (2004) provides an analysis for the case of time-incosistent consumers and shows that the monopolist does not necessarily profits from the bias of the consumers and might be even interested in eliminating it.

in an oligopolistic competition, the results would easily generalize to different economic contexts, such as competition in consumption good markets, in which publicity and perceptions of similarity play an important role.

Last, but not least, few words have to be said about the role that case-based reasoning could play in the economic theory. In the introduction, I have argued that the case-based reasoning models learning by induction. The economic literature has proposed few models which capture this type of learning, the most prominent of which is the evolutionary approach used in game theory. The advantage of this approach consists in the fact that the individual using it need not have any prior notion about the quality of the different alternatives. Instead, the past performance is used to determine those alternatives which survive in the selection process. Of course, both approaches use different selection methods, therefore a comparison of the results obtained in these two ways would help to understand the driving forces of evolution in economic environment.

The common feature of both approaches consists in the fact that they can be used for selecting among ”meta”-strategies and ”meta”-rules, without the necessity of constraining the possible outcomes in advance and formulating ex-ante priors about the probabilities of success[155]. Hence, these two approaches allow to construct models of acquiring nomologic knowledge. Although in this thesis this aspect of the case-based reasoning has been neglected, such an approach will surely lead to interesting and stimulating results. The work of Gilboa and Schmeidler (2003) on inductive inference show the possibilities which the usage of case-based reasoning opens in this respect.

To summarize, the case-based decision theory opens large perspectives for modelling economic problems in a different and more intuitive way. It allows to introduce and examine the influence of features not present in standard theoretical models. This thesis has taken one step to understand the consequences of case-based reasoning in financial markets. It has answered some questions, but much more research is necessary, before the role and the meaning of case-based decisions in economic problems is clarified and their predictive power tested.

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155 Gilboa, Postlewaite and Schmeidler (2004) argue that the formulation of a prior might be impossible in some situations due to lack of information.

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