CIV2202.9: AREA AND VOLUMES

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PREVIEW

Introduction

Areas and Volumes are often required in the context of design, eg. we might need the surface area of a lake, the area of crops, of a car park or a roof, the volume of a dam embankment, or of a road cutting.

Volumes are often calculated by integrating the area at regular intervals eg. along a road centreline, or by using regularly spaced contours. We simply use what you already know about numerical integration from numerical methods).

Objectives

After completing this topic you should be able to

calculate the areas of polygons and irregular figures and the volumes of irregular and curved solids

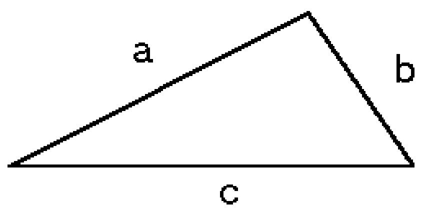
Readings

Read *Muskett,* Chapter 8

REQUIRED

AREAS

Triangles



if $s = (a + b + c) / 2$ then $\text{area} = \sqrt{S.(S-a)(S-b)(S-c)}$

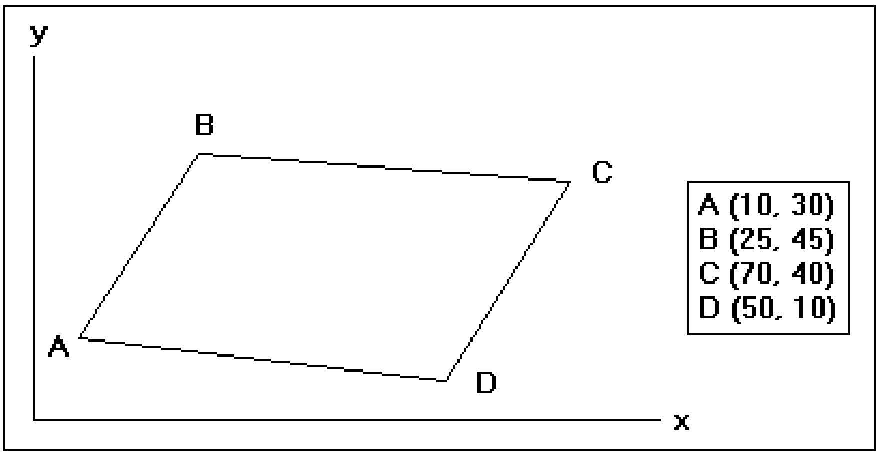
COORDINATES

Calculating area of a polygon from Coordinates:

If the coordinate points are numbered clockwise:

$$\text{area} = \frac{1}{2}\sum_{i=1}^{n}(N_i \,.\, E_{i+1} \;-\; E_i \,.\, N_{i+1})$$

This formula is not easy to remember, so let's look at a practical application:



Arrange the data in columns as shown below, repeating the last line at the top.

	-	+
D	50	10
A	10	30
B	25	45
C	70	40
D	50	10

Diagram to go in here

area = (1/2) x (10 x 10 + 25 x 30 + 45 x 70 + 40 x 50
– 50 x 30 - 10 x 45 - 25 x 40 - 70 x 10)
= 1175

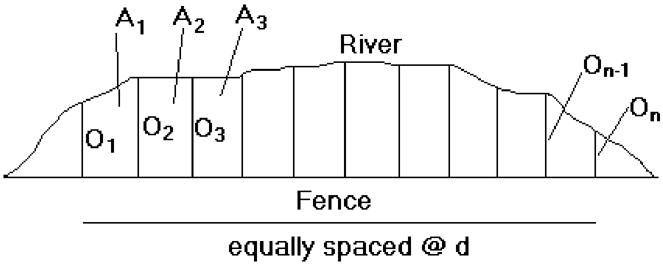
The result is –ve if lettered anti-clockwise, but the magnitude will be correct.

The formula can be used on a polygon with any number of sides. The points **MUST** be listed in sequential order.

TRAPEZOIDAL RULE

Calculating areas with the Trapezoidal Rule

(as used in integrating functions)



$A_1 = d \cdot (O_1 + O_2) / 2$
$A_2 = d \cdot (O_2 + O_3) / 2$
$A_3 = d \cdot (O_3 + O_4) / 2$

Hence, the total area is:

$$A = (d/2) \cdot [O_1 + 2.O_2 + 2.O_3 + \ldots + 2.O_{n-1} + O_n]$$

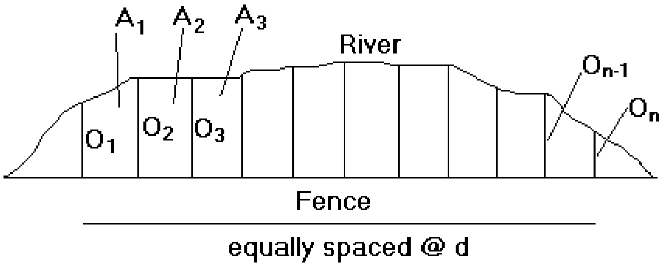
The Trapezoidal Rule assumes **straight line segments** on the boundary.

SIMPSON'S RULE

Doing better with Simpson's Rule

Simpson's Rule assumes a parabola fitted to 3 adjacent points, rather than the straight lines between adjacent points assumed by the Trapezoidal Rule.

This may be more accurate than the Trapezoidal Rule because boundaries are often curved.



It can be shown that:

$$A_1 + A_2 = (d/3).(O_1 + 4.O_2 + O_3)$$

the total area, A is:

$$A = (d/3).(O_1 + 4.O_2 + 2.O_3 + 4.O_4 + 2.O_5 + ... + 4.O_{n-1} + O_n)$$

Note that **an even number of segments is required**.

(If you have (2n+1) strips, you can always use Simpson's Rule for 2n strips, and the trapezoidal rule for the last).

Example, d = 20 m

offsets are 0 5.49 9.14 8.53 10.67 12.50 9.75 4.57 1.80 0

Simpson's Rule: $A_{SR} = 1254 \text{ m}^2$

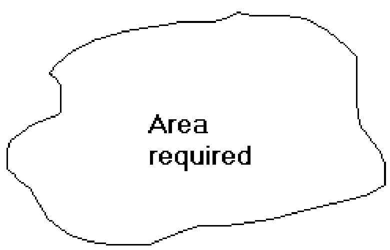
Trap. Rule: $A_{TR} = 1250 \text{ m}^2$

Difference is 0.3%

We'll see that when it comes to calculating volumes, Simpson's Rule is usually superior.

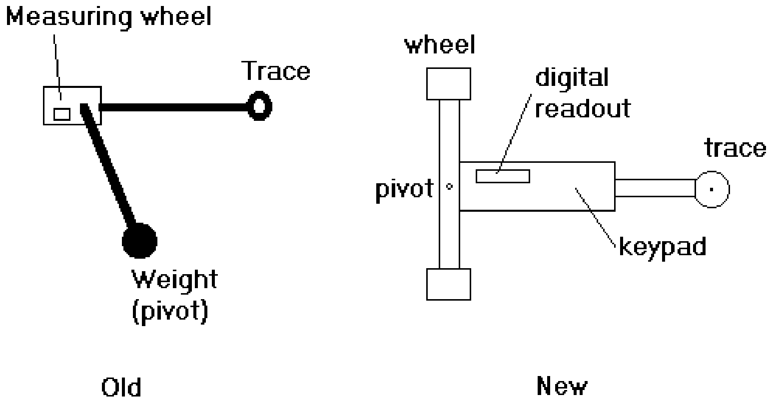
PLANIMETER

A planimeter is a mechanical device for measuring irregular areas:



Two types of instrument are common. The older one (on the left, below) uses a weight as a pivot. The more modern instrument (right, below), uses a rolling axle instead, so it can measure larger areas. Both now have a digital readout.

On each instrument, a measuring wheel rotates as the device moves, and registers the area. The most recent of the digital devices actually calculates coordinates of the cursor position, and calculates area by the coordinate method discussed earlier.



If the whole of the (old) instrument is within the boundary of the area to be found, you must also add the area of a whole circle.

It is usual to make **any measurement at least twice**. Planimeters are particularly well suited to calculating the areas of contours.

VOLUMES

Volumes can be calculated in a number of ways. It is common to calculate the area of each of several equally spaced slices (either vertical cross-sections, or horizontal contours), and integrate these using Simpson's Rule or similar.

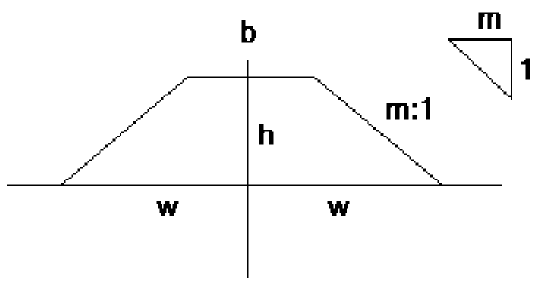
A second method is to use **spot levels**, and calculate the volume of a series of wedges or square cells.

CROSS-SECTIONS

Cross-sections are well suited for calculating volumes of roads, pipelines, channels, dam embankments, etc. Formulae are given below for the most common cross-section cases.

Horizontal ground

Man-made structures usually have constant side slopes : eg (simple case)

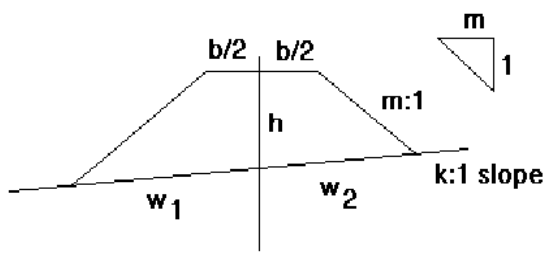


$w = b/2 + m.h$

\ Area $= h . [2.w + b] / 2 \quad = h.(b + m.h)$

Constant Cross-fall

In general, ground is not flat. If the cross-fall is constant,



$w_1 = (b/2 + m.h).[k / (k-m)]$

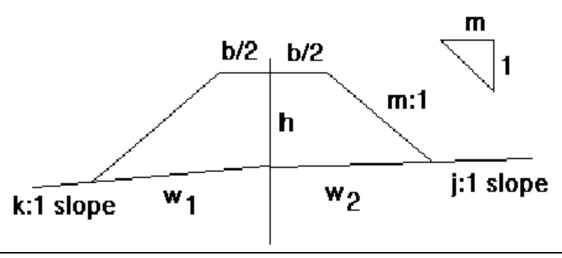
$w_2 = (b/2 + m.h).[k / (k+m)]$

$\text{Area} = [(b/2 + m.h).(w_1 + w_2) - b^2/2] / (2.m)$

This is a **Two-Level Section**, since two points define the cross fall.

Non-Constant Cross-fall

Three Level Section (eg. centreline plus one point on either side.



This time, w_2 = (b/2 + m.h).[j / (j+m)] (same as before, except j replaces k)

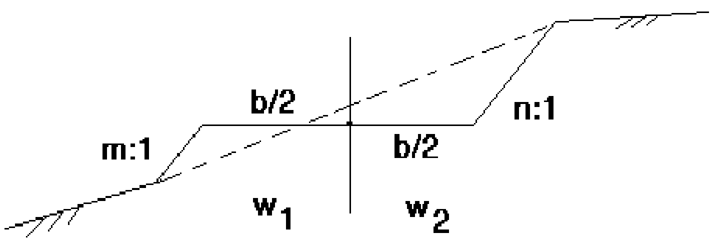
Area = [(b/2 + m.h).(w_1 + w_2) - b^2/2] / (2.m) (as before)

Many Level section

Other, more complex cases are possible, eg. n-Level Section. However, these are handled more conveniently using the **Coordinate Method** (described under "Areas"), and a computer.

Part Cut and Fill

Also, consider the problem when part cut and part fill at a cross-section:



We want the area of cut and area of fill separately. The intersection is rarely at the centreline.

Calculating Volumes

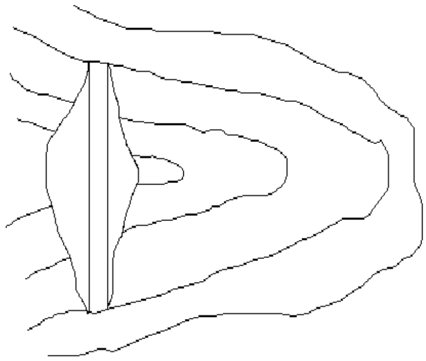
Having calculated the area of each cross-section, the volume is calculated by Simpson's Rule or a similar technique. (Effectively, you are integrating the area function with respect to distance).

CONTOURS

The Second major way to calculate volumes is to calculate areas of contours (which are like horizontal cross-sections) and again integrate that function using Simpson's Rule or similar.

To calculate the area of a contour, use a planimeter, counting squares, division into triangles etc.

Example



The reservoir is 23.3 m deep, with the following contour areas:

contour (m)	130	120	110	106.7
area (m^2)	610000	150000	1100	0

Simpson's Rule

Volume of reservoir:

$$V_{SR} = (h/3).(A_1 + 4.A_2 + A_3)$$
$$= (10 / 3).(610000 + 4 \times 150000 + 1100)$$
$$= 4.04 \times 10^6 \; m^3$$
$$= 4040 \text{ ML}$$

NB. There is also a small amount of volume at the bottom of the reservoir, which can be approximated by 1/3 times the area of the base times the height:

volume = 1/3 x area x height = 1100 x 3.3 / 3 = 1210 m^3 = 1.2 ML

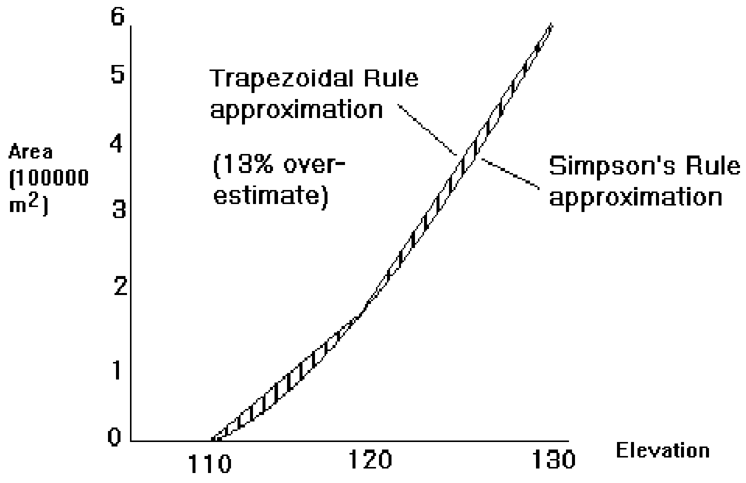
Trapezoidal Rule

Volume of Reservoir:

$$V_{TR} = (h / 2).(A_1 + 2.A_2 + A_3)$$
$$= (10 / 2).(610000 + 2 \times 150000 + 1100)$$
$$= 4.56 \times 10^6 \; m^3 = 4560 \text{ ML}$$

cf. Simpson's Rule result of 4040 ML - 13% different!

Why the Difference?



As you can see, Simpson's Rule is a better fit to the curve of the area vs elevation function.

Cross-Sections

You can use a similar approach to calculate volumes when using areas of cross-sections. Recall the simple formula for calculating the cross-sectional area for a One-Level Section:

Area $= h.(b + m.h) = b.h. + m.h^2$

ie. a second degree polynomial in h.

Hence we would not expect the Trapezoidal Rule to be a good fit, since the function to be integrated is quadratic.

Let's look at two cross-sections though:

$$A_1 = b.h_1 + m.h_1^2$$
$$A_2 = b.h_2 + m.h_2^2$$

What if $h_2 = h_1 + Dh$? where Dh is small compared to h_1 (say <10% of h_1).

$$A_2 = b.h_2 + m.h_2^2$$
$$= b.(h_1 + Dh) + m.(h_1 + Dh)^2$$

$\approx b.h_1 + b.Dh + m.h_1^2 + 2m.h_1.Dh$ (ignoring the $m.Dh^2$ term)

$= b.h_1 + m.h_1^2 + b.Dh + 2m.h_1.Dh$

$= A_1 + b.Dh + 2m.h_1.Dh$

which makes the difference between A_2 and A_1, a linear function of Dh.

For this case (ie. when Dh is small compared to h_1, which means that the cross-sectional areas are changing only slowly), the Trapezoidal Rule will work well.

In the reservoir example, the areas of the contours are very different (1100, 150000, 610000 - ie. **not** changing slowly), and hence Simpson's Rule is more appropriate, since it assumes a second degree polynomial.

Other (more sophisticated) methods of numerical integration are also available!

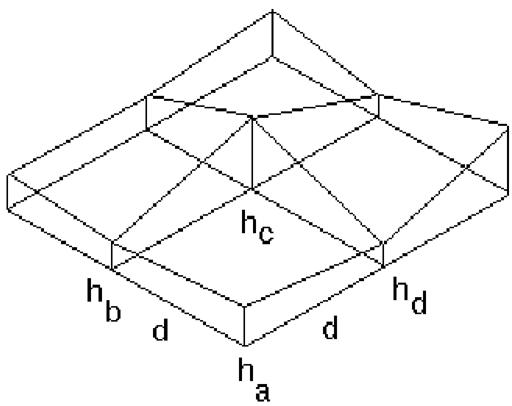
SPOT LEVELS

The following methods are particularly useful for large, open excavations such as tanks, borrow pits etc.

The area is divided into a grid, and levels obtained at the intersection points.

The spacing of the grid depends on the terrain, accuracy required, and resources available. Generally, one surface is horizontal (eg. base of excavation).

Regular grid of points



Volume of 1 cell $\approx$ $(d^2/4)(h_a + h_b + h_c + h_d)$

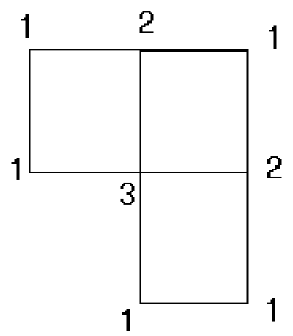
Summing over all cells, some heights will be counted 1, 2, 3, or 4 times.

\ $V = (d^2/4)\,(Sh_1 + 2.Sh_2 + 3.Sh_3 + 4.Sh_4)$

where h_i = heights counted i times

eg. h_b and h_d are counted twice, h_c is counted four times, h_a is counted once

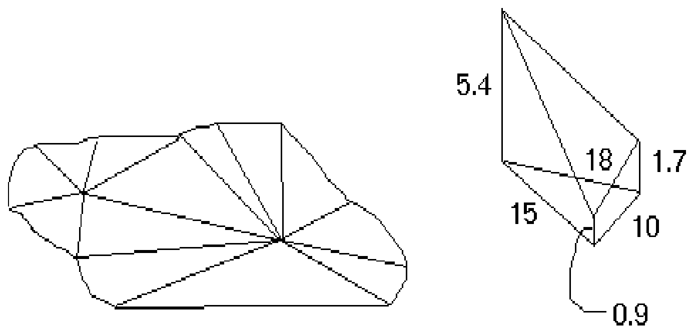
3 times occurs as follows :



You usually finish up with some extra volume around the edge made up of triangles and trapezia. Calculate this volume by methods described earlier.

For grids, a better match to the surface may be obtained if each rectangle is divided into 2 triangles. The formula is similar to the earlier one except that some points may now occur 5 or 6 times. **See *Muskett* Section 8.6**. If there is a transition between cut and fill smaller cels should be used..

Irregular spoil heap



Spot levels located by tacheometry or similar. Volume calculated by multiplying areas of triangle by the average of the corner heights.

For the example wedge, area of the base is:

A $= \sqrt{S.(S - a)(S - b)(S - c)}$
$= \sqrt{21.5(21.5 - 10)(21.5 - 15)(21.5 - 18)}$
$= 75\ m^2$

V $= 75 \times (0.9 + 1.7 + 5.4) / 3 \quad = 200\ m^3$

THE PRISMOIDAL FORMULA

A prismoid is defined as a solid whose end faces lie in parallel planes and consist of any two polygons, not necessarily of the same number of sides, the longitudinal faces being plane surfaces extended between the end planes. Lines joining the end phases are straight lines as shown on the above diagram.

The longitudinal faces may take the form of triangles, parallelograms, or trapeziums.

Let d = length of the prismoid measured perpendicular to the two end parallel planes
A1 = area of cross-section of one end plane
A2 = area of cross-section of the other end plane
Am = the mid area = the area of the plane midway between the end planes and parallel to them
V = volume of prismoid

$$V = \frac{d}{6}(A_1 + A_2 + 4A_m)$$

PAPUS THEOREM

If a plane figure is revolved about a line in the same plane, the volume of the solid generated is equal to the product of area and the distance travelled by its centroid. (Centroid is NOT the centre of mass, but centre of the plane).

Volume of revolution = (Area of plane)(Distance centroid moved) of a plane.

AB be the horizontal line through A

Effect of Curvature

The previous discussion has assumed straight (parallel) sections. Commonly road and dam embankments are curved. Sections are not parallel.

Read *Musket* 8.7

REQUIRED

MASS HAUL DIAGRAMS

Read *Muskett,* 8.8

REQUIRED

For Linear works, eg. roads, railways, channels, it is normal to tabulate cut and fill along the route. These values are displayed on the longitudinal section. **The aim is to balance cut and fill where this is economical**. Sometimes it is cheaper to waste (dump) material in one place, and borrow it at another.

Summing cut/fill with distance gives a **Mass Haul Diagram**. Consideration for shrinkage after compaction is required:

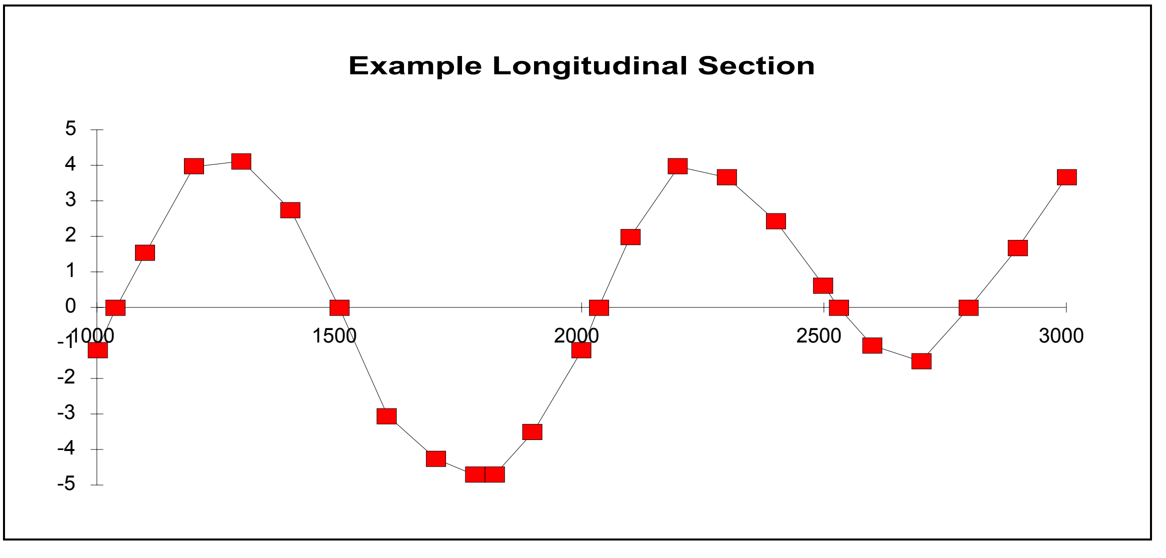
eg.	rock	1.4	ie. rock typically **expands** by 40%
	clay	0.9	ie. clat typically **shrinks** by 10%

These are typical figures only. Soil and/or rock compaction tests are required to determine real values.

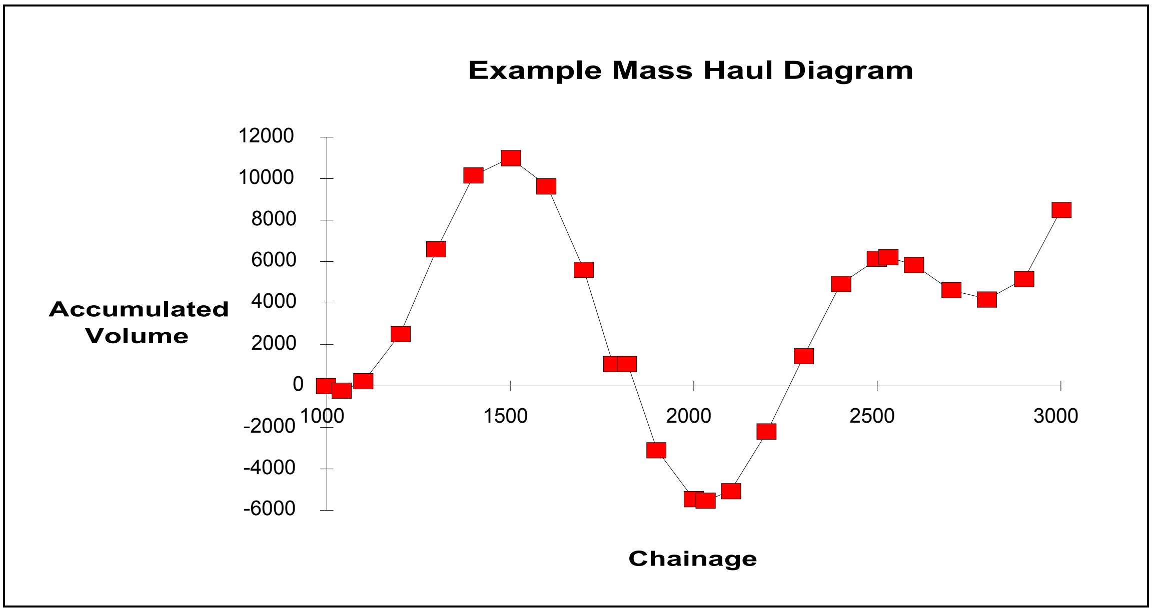
EXAMPLE

Chainage (m)	Centre Height	Cut	Fill	Shrinkage Factor	Corrected Volume	Accum. Volume
1000	F1.22					0
1040	0		230		-230	-230
1100	C1.52	480		0.90	+430	+200
1200	C3.96	2560		0.90	+2300	+2500
1300	C4.12	4560		0.90	+4100	+6600
1400	C2.74	3940		0.90	+3550	+10150
1500	0	950		0.90	+850	+11000
1600	F3.05		1350		-1350	+9650
1700	F4.27		4010		-4010	+5640
1780	F4.72		4600		-4600	+1040
1820	F4.72		Bridge			+1040
1900	F3.51		4130		-4130	-3090
2000	F1.22		2370		-2370	-5460
2035	0		60		-60	-5520
2100	C1.98	510		0.90	+460	-5060
2200	C3.96	3180		0.90	+2860	-2200
2300	C3.66	4055		0.90	+3650	+1450
2400	C2.44	3860		0.90	+3470	+4920
2500	C0.61	1320		0.90	+1190	+6110
2530	0	100		0.90	+90	+6200
2600	F1.07		350		-350	+5850
2700	F1.52		1230		-1230	+4620
2800	0		420		-420	+4200
2900	C1.68	1080		0.89	+960	+5160
3000	C3.66	3730		0.89	+3320	+8480

Example Longitudinal Section



Example Mass Haul Diagram



Notes on the Example

Tabulate the cut and fill - cut is positive.

The **shrinkage constant** modifies cut volumes because soil is usually denser when compacted into an embankment.

Natural surface (NS) and Finished Surface (FS) are plotted.
Accumulated volume (Mass Haul Diagram) is plotted.

Movement of Material

1. if the curve is above the base line, then material moves to the right

2. if the curve is below the base line, material moves to the left

MHD rising: region of cut and maximum occurs at end of cut.

MHD falling: region of fill and minimum occurs at end of fill.

Balance Lines

Any horizontal line drawn across the MHD represents a balance line - cut and fill are balanced within that distance (ie. cut = fill). The Base line is also a balance line (but not the only one).

Should only balance cut and fill while haul cost is less than cost of winning and cost of dumping.

Free Haul

Normally a Bill of Quantities includes a price for excavation which includes a certain haul distance - the free haul. For greater distances, an extra charge is made - the over haul.

Economical balance lines need to consider free haul, over haul, and costs of winning and dumping.

Other Notes

It is preferable to haul downhill.

Dumping and borrowing may cause undesirable environmental effects - need to rehabilitate stockpiles and borrow pits (particularly).

There may also be a shortage of suitable material. Some soil is unsuitable for use in embankments.