The Smallest Parts Partition Function

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Tuesday, March 13, 2012

ABSTRACT

Let $\mathrm{spt}(n)$ denote the number of smallest parts in the partitions of n. In 2008, Andrews found surprising congruences for the spt-function mod 5, 7 and 13. We discuss recent work on analytic, arithmetic and combinatorial properties of the spt-function. Some of this work is joint with George Andrews (PSU) and Jie Liang (UCF).

THE PARTITION FUNCTION

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$p(n)$

A *partition* of n is a nonincreasing sequence of positive integers whose sum is n.

$n = 1$

$$1$$

$n = 2$

$$2, \quad 1+1$$

$n = 3$

$$3, \quad 2+1, \quad 1+1+1$$

$n = 4$

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1$$

$n = 5$

$$5, \quad 4{+}1, \quad 3{+}2, \quad 3{+}1{+}1, \quad 2{+}2{+}1, \quad 2{+}1{+}1{+}1, \quad 1{+}1{+}1{+}1{+}1$$

Let $p(n)$ denote the number of partitions of n

n	$p(n)$
1	1
2	2
3	3
4	5
5	7
⋮	
10	42
⋮	
100	190569292
⋮	
1000	24061467864032622473692149727991
⋮	
10000	36167251325636293988820471890953695495016030339315650 422081868605887952568754066420592310556052906916435144

- HARDY - RAMANUJAN (1918)

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}} \qquad \text{(Hardy-Ramanujan 1918)}$$

- EULER:

$$\sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}$$

$$= \frac{1}{1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \cdots}$$

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DEDEKIND:

- $$\sum_{n=0}^{\infty} p(n) q^{n-\frac{1}{24}} = \frac{1}{\eta(\tau)},$$

  where

  $$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n),$$

  $q = \exp(2\pi i \tau)$.

- $$\eta(\tau+1) = \exp(\pi i/12)\,\eta(\tau)$$
  $$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$$
  $$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \nu_\eta(A)\,(c\tau+d)^{1/2}\,\eta(\tau)$$

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HECKE OPERATORS

- For $\ell \geq 5$ prime

$$Z_\ell(z) := \sum_{n=-s_\ell}^{\infty} \left(\ell^3 \, p(\ell^2 n - s_\ell) + \ell \chi_{12}(\ell) \left(\frac{1-24n}{\ell} \right) p(n) + p\left(\frac{n+s_\ell}{\ell^2} \right) \right) q^{n-\frac{1}{24}},$$

where $s_\ell = (\ell^2 - 1)/24$.

- ATKIN (1968): Then the function $Z_\ell(z)\,\eta(z)$ is a modular function on the full modular group $\Gamma(1)$.
- EXAMPLE:

$$Z_5(z)\eta(z) = j(z) - 750.$$

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Partition Congruences

- RAMANUJAN (1915)

$$p(5n+4) \equiv 0 \pmod 5$$
$$p(7n+5) \equiv 0 \pmod 7$$
$$p(11n+6) \equiv 0 \pmod{11}$$

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5 \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6}$$

- ATKIN (1965)

$$p(11^3 \cdot 13 \cdot n + 237) \equiv 0 \pmod{13}$$

$$\vdots$$

$$p(5^3 \cdot 7^4 \cdot 13^3 \cdot 17^3 \cdot 19^4 \cdot 37^4 \cdot 113 \cdot 337^3 \cdot 661^3 \cdot 1049^3 n + \\ + 12788270520615768872783247697214202\,99) \\ \equiv 0 \pmod{113}$$

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- ONO (2000)
  Let $\ell \geq 5$ be prime and $m \geq 1$. Then there are ∞ly many (A, B):
  $$p(An + B) \equiv 0 \pmod{\ell^m}$$
- KOLBERG (1962)
  There ∞ly many n for which $p(n)$ is even and ∞ly many n for which $p(n)$ is odd.
- CONJECTURE
  For each fixed r there are ∞ly many n such that
  $$p(n) \equiv r \pmod 3$$

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The Rank and Crank

- DYSON's RANK (1944) The *rank* of a partition is the largest part minus the number of parts. Then the residue of the rank mod 5 (resp. mod 7) divides the partitions of $5n+4$ (resp. $7n+5$) into 5 (resp. 7) equal classes.
- ANDREWS-G. CRANK (1988) The *crank* of a partition is the largest part if the partition has no ones, otherwise it is difference between the number of parts larger than the number of ones and the number of ones. Then the residue of the crank mod 5 (resp. mod 7, resp. mod 11) divides the partitions of $5n+4$ (resp. $7n+5$, resp. $11n+6$) into 5 (resp. 7, resp. 11) equal classes.

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Partition	Rank	Rank mod 11	Crank	Crank mod 11
6	5	5	6	6
5+1	3	3	0	0
4+2	2	2	4	4
4+1+1	1	1	-1	10
3+3	1	1	3	3
3+2+1	0	0	1	1
3+1+1+1	-1	10	-3	8
2+2+2	-1	10	2	2
2+2+1+1	-2	9	-2	9
2+1+1+1+1	-3	8	-4	7
1+1+1+1+1+1	-5	6	-6	5

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FOKKINK, FOKKINK AND WANG (1995)

- $$\sum_{\substack{\pi\in\mathscr{D}\\|\pi|=n}} (-1)^{\#(\pi)-1}\, s(\pi) = d(n),$$

  where is the $\mathscr{D}$ is set of partitions into distinct parts, and $s(\pi)$ is the smallest part of π.

- EXAMPLE $(n = 6)$

π	$(-1)^{\#(\pi)-1}s(\pi)$
6	6
$5+1$	-1
$4+2$	-2
$3+2+1$	1
	$---$
	$4 = d(6)$

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spt-function

- Andrews (2008) defined the function $\mathrm{spt}(n)$ as the total number of appearances of the smallest parts in the partitions of n. For example,

$$\dot{4}, \quad 3+\dot{1}, \quad \dot{2}+\dot{2}, \quad 2+\dot{1}+\dot{1}, \quad \dot{1}+\dot{1}+\dot{1}+\dot{1}.$$

Hence, $\mathrm{spt}(4) = 10$.

n	$\mathrm{spt}(n)$
1	1
2	3
3	5
4	10
5	14
6	26
$\vdots$	
10	119
$\vdots$	
100	1545832615
$\vdots$	
1000	600656570957882248155746472836274
$\vdots$	

- BRINGMANN-MAHLBURG (2009)

$$\mathrm{spt}(n) \sim \frac{2\sqrt{6n}}{\pi}\, p(n) \sim \frac{1}{\sqrt{8n}}\, e^{\pi\sqrt{\frac{2n}{3}}}$$

- ANDREWS (2008)

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n \\
&= \sum_{n=1}^{\infty} (q^n + 2q^{2n} + 3q^{3n} + \cdots) \cdot \frac{1}{(1-q^{n+1})(1-q^{n+2})\cdots} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2 (q^{n+1};q)_\infty} \\
&= q + 3q^2 + 5q^3 + 10q^4 + 14q^5 + 26q^6 + 35q^7 + \cdots,
\end{aligned}$$

where $(a)_\infty = (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$.

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NOTATION

$$(a)_0 := (a;q)_0 := 1,$$
$$(a)_n := (a;q)_n := (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$$

when n is a nonnegative integer.

$$(a)_\infty := (a;q)_\infty := \prod_{m=1}^{\infty}(1-aq^{m-1})$$

if $|q| < 1$.

SPT and Rank-Crank Moments

- ANDREWS (2008)

$$\sum_{n=1}^{\infty} \mathrm{spt}(n) q^n$$

$$= \frac{1}{(q)_\infty} \left(\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-q^n)^2} \right)$$

- $$\mathrm{spt}(n) = \frac{1}{2}\left(M_2(n) - N_2(n)\right)$$

  where

$$M_k(n) := \sum_m m^k M(m,n) \qquad (k\text{th crank moment})$$

$$N_k(n) := \sum_m m^k N(m,n) \qquad (k\text{th rank moment})$$

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SPT and Maass Forms

BRINGMANN (2008)

- $$\mathcal{M}(z) := \sum_{n=0}^{\infty} (12\mathrm{spt}(n) + (24n-1)p(n))\, q^{n-1/24} - \frac{3\sqrt{3}i}{\pi} \int_{-\overline{z}}^{i\infty} \frac{\eta(\tau)\, d\tau}{(-i(z+\tau))^{3/2}}$$

  Then
  $$\mathcal{M}\left(\frac{az+b}{cz+d}\right) = \frac{(cz+d)^{3/2}}{\nu_\eta(A)} \mathcal{M}(z).$$

- $\mathcal{M}(24z)$ is a weight $\frac{3}{2}$ weak Maass form $\mathcal{M}(z)$ on $\Gamma_0(576)$ with Nebentypus χ_{12}.

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SPT and HECKE OPERATORS

- Define

$$\mathcal{A}(z) := \sum_{n=0}^{\infty} a(n) q^{n-\frac{1}{24}},$$

where $a(n) = 12\mathrm{spt}(n) + (24n-1)p(n)$, and

$$\mathcal{M}_\ell(z) = T(\ell^2)\mathcal{A}(z) := \sum_{n=-s_\ell}^{\infty} \left(a(\ell^2 n - s_\ell) + \chi_{12}(\ell)\left(\frac{1-24n}{\ell}\right) a(n) + \ell\, a\left(\frac{n+s_\ell}{\ell^2}\right)\right) q^{n-\frac{1}{24}},$$

for $\ell \geq 5$ prime.

- ONO (2011): Then $\left(\mathcal{M}_\ell(z) - \chi_{12}(\ell)(1+\ell)\mathcal{A}(z)\right)\eta^{\ell^2}(z)$ is an entire modular form of weight $\frac{1}{2}(\ell^2+3)$.

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SPT and ATKINs U-operator

- $$\begin{aligned}\alpha_\ell(z) &= U(\ell)\mathcal{A}(z) - \ell\chi_{12}(\ell)\mathcal{A}(\ell z)\\ &:= \sum_{n=0}^{\infty}\left(a(\ell n - s_\ell) - \ell\chi_{12}(\ell)\, a\left(\frac{n}{\ell}\right)\right) q^{n-\frac{\ell}{24}},\end{aligned}$$

  for $\ell \geq 5$ prime.
- G. (preprint): Then $\alpha_\ell(z)\frac{\eta^{2\ell}(z)}{\eta(\ell z)}$ is an entire modular form of weight $\ell + 1$ on $\Gamma_0(\ell)$.

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spt-Congruences

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$$spt(13n+6) \equiv 0 \pmod{13}. \quad (3)$$

- G. (unpublished)

$$\sum_{n=1}^{\infty} \mathrm{spt}(5n-1)q^n + 5\sum_{n=1}^{\infty} \mathrm{spt}(n)q^{5n}$$
$$= \frac{5}{2}\sum_{n=1}^{\infty}(\sigma(5n)-\sigma(n))q^n \times \prod_{n=1}^{\infty}\frac{1}{(1-q^{5n})}$$
$$+ \frac{25q}{2}\left(1+\sum_{n=1}^{\infty}(\sigma(n)-5\sigma(5n))q^n\right) \times \prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}$$

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- G. (TAMS, to appear): For a, b, $c \geq 3$,

$$\mathrm{spt}(5^a n + \delta_a) + 5\,\mathrm{spt}(5^{a-2} n + \delta_{a-2}) \equiv 0 \pmod{5^{2a-3}},$$
$$\mathrm{spt}(7^b n + \lambda_b) + 7\,\mathrm{spt}(7^{b-2} n + \lambda_{b-2}) \equiv 0 \pmod{7^{\lfloor \frac{1}{2}(3b-2) \rfloor}},$$
$$\mathrm{spt}(13^c n + \gamma_c) - 13\,\mathrm{spt}(13^{c-2} n + \gamma_{c-2}) \equiv 0 \pmod{13^{c-1}},$$

where δ_a, λ_b and γ_c are the least nonnegative residues of the reciprocals of 24 mod 5^a, 7^b and 13^c respectively.

- G. (2008); ONO (2011): If $\left(\frac{1-24n}{\ell}\right) = 1$ then

$$\mathrm{spt}(\ell^2 n - \tfrac{1}{24}(\ell^2 - 1)) \equiv 0 \pmod{\ell},$$

for any prime $\ell \geq 5$. This follows from $(\mathcal{M}_\ell(z) - \chi_{12}(\ell)(1+\ell)\mathcal{A}(z) \equiv 0 \pmod{\ell})$.

- AHLGREN, BRINGMANN and LOVEJOY (2011) If $\left(\frac{-23-24n}{\ell}\right) = 1$ then

$$\mathrm{spt}(\ell^{2m} n + d_{\ell,2m}) \equiv 0 \pmod{\ell^m},$$

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  for any prime $\ell \geq 5$.

SPT mod 2 and 3

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- FOLSOM and ONO (2008) If $\ell\geq 5$ is prime then

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The SPT-Crank - ANDREWS, G. and LIANG (preprint)

- For a partition π, define $s(\pi)$ as the smallest part in the partition with $s(\ ) = \infty$ for the empty partition. We define the following subset of vector partitions

$$S := \{\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in V \ : \ 1 \leq s(\pi_1) < \infty$$

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We call the elements of S, S-**partitions**.

- Let $\omega_1(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$. Then the number of S-partitions of n with crank m counted according to the weight ω_1 is defined to be

$$N_S(m,n) = \sum_{\substack{\vec{\pi} \in S, |\vec{\pi}|=n \\ \text{crank}(\vec{\pi})=m}} \omega_1(\vec{\pi}).$$

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The number of S-partitions of n with crank congruent to m modulo t counted according to the weight ω_1 is denoted by $N_S(m,t,n)$, so that

$$N_S(m,t,n) = \sum_{k=-\infty}^{\infty} N_S(kt+m,n) = \sum_{\substack{\vec{\pi}\in S, |\vec{\pi}|=n \\ \text{crank}(\vec{\pi})\equiv m \pmod{t}}} \omega_1(\vec{\pi}). \tag{5}$$

By considering the transformation that interchanges π_2 and π_3 we have

$$N_S(m,n) = N_S(-m,n), \tag{6}$$

so that

$$N_S(m,t,n) = N_S(t-m,t,n). \tag{7}$$

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Theorem

$$N_S(k,5,5n+4) = \frac{spt(5n+4)}{5} \qquad \text{for } 0 \le k \le 4, \qquad (8)$$

$$N_S(k,7,7n+5) = \frac{spt(7n+5)}{7} \qquad \text{for } 0 \le k \le 6. \qquad (9)$$

EXAMPLE

$\vec{\pi}_1 = (1, 1+1+1, -)$	$+1$	3
$\vec{\pi}_2 = (1, -, 1+1+1)$	$+1$	-3
$\vec{\pi}_3 = (1, 1+1, 1)$	$+1$	1
$\vec{\pi}_4 = (1, 1, 1+1)$	$+1$	-1
$\vec{\pi}_5 = (1, 1+2, -)$	$+1$	2
$\vec{\pi}_6 = (1, -, 1+2)$	$+1$	-2
$\vec{\pi}_7 = (1, 2, 1)$	$+1$	0
$\vec{\pi}_8 = (1, 1, 2)$	$+1$	0
$\vec{\pi}_9 = (1, 3, -)$	$+1$	1
$\vec{\pi}_{10} = (1, -, 3)$	$+1$	-1
$\vec{\pi}_{11} = (1+2, 1, -)$	-1	1
$\vec{\pi}_{12} = (1+2, -, 1)$	-1	-1
$\vec{\pi}_{13} = (1+3, -, -)$	-1	0
$\vec{\pi}_{14} = (2, 2, -)$	$+1$	1
$\vec{\pi}_{15} = (2, -, 2)$	$+1$	-1
$\vec{\pi}_{16} = (4, -, -)$	$+1$	0

spt-Crank EXAMPLE (continued)

$$\sum_{\vec{\pi}\in S, |\vec{\pi}|=4} \omega_1(\pi) = \sum_m N_S(m,4) = 13 - 3 = 10 = \mathrm{spt}(4). \qquad (10)$$

$$\begin{aligned} N_S(0,5,4) &= \omega_1(\vec{\pi}_7) + \omega_1(\vec{\pi}_8) + \omega_1(\vec{\pi}_{13}) + \omega_1(\vec{\pi}_{16}) \\ &= 1 + 1 - 1 + 1 = 2. \end{aligned}$$

Similarly

$$\begin{aligned} N_S(0,5,4) = N_S(1,5,4) = N_S(2,5,4) = N_S(3,5,4) = N_S(4,5,4) = 2 \\ = \frac{\mathrm{spt}(4)}{5}. \end{aligned}$$

Nonnegativity Theorem

Table of spt-crank coefficients $N_S(m, n)$

n \ m	0	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0	0
4	2	2	1	1	0	0	0	0	0	0	0
5	2	2	2	1	1	0	0	0	0	0	0
6	4	4	3	2	1	1	0	0	0	0	0
7	5	4	4	3	2	1	1	0	0	0	0
8	7	7	6	5	3	2	1	1	0	0	0
9	10	9	8	6	5	3	2	1	1	0	0
10	13	13	11	10	7	5	3	2	1	1	0
11	17	16	15	12	10	7	5	3	2	1	1
12	24	24	21	18	14	11	7	5	3	2	1

ANDREWS, G. and LIANG (preprint); DYSON (preprint)

Theorem

$$N_S(m,n) \geq 0, \tag{11}$$

for all (m,n).

$$\begin{aligned}
&\sum_{n=1}^{\infty} N_S(m,n) q^n \\
&= \sum_{j=0}^{\infty} \frac{q^{j^2+mj+2j+m+1}}{(q;q)_j (q;q)_{m+j}} (q;q)_j \sum_{h=0}^{j} \begin{bmatrix} j \\ h \end{bmatrix} \frac{q^{h^2+h}}{(q;q)_h (1-q^{m+j+h+1})} \\
&= \sum_{j=0}^{\infty} \sum_{h=0}^{\infty} \frac{q^{j^2+mj+2hj+2j+m+hm+2h^2+3h+1}}{(q^{j+h+1};q)_m (q;q)_h^2 (q;q)_j (1-q^{m+j+2h+1})}.
\end{aligned} \tag{12}$$

Problems

1. Find a statistic on partitions that explains (11) combinatorially. More precisely, find a statistic s-rank : $\mathscr{P} \longrightarrow \mathbb{Z}$ and a weight function $\varphi : \mathscr{P} \longrightarrow \mathbb{N}$ such that

$$\sum_{\pi \in \mathscr{P},\, |\pi| = n} \varphi(\pi) = \mathrm{spt}(n), \quad \text{and} \tag{13}$$

$$\sum_{\substack{\pi \in \mathscr{P},\, |\pi| = n \\ \text{s-rank}(\pi) = m}} \varphi(\pi) = N_S(m, n), \tag{14}$$

   for $m \in \mathbb{Z}$ and $n \geq 1$.
2. Find a crank-type result that explains the congruence $\mathrm{spt}(13n + 6) \equiv 0 \pmod{13}$.

It is straightforward to interpret the generating function in (12) in terms of Durfee squares and rectangles for fixed i. The problem is to interpret the result so that something like (13) and (14) hold.

SPT PARITY AND SELF-CONJUGATE S-PARTITIONS

- The map $\iota : S \longrightarrow S$ given by

$$\iota(\vec{\pi}) = \iota(\pi_1, \pi_2, \pi_3) = (\pi_1, \pi_3, \pi_2),$$

  is a natural involution.
- An S-partition $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ is called **self-conjugate** if $\pi_2 = \pi_3$. These are the fixed points of ι.
- EXAMPLE: Self-conjugate S-partitions of 5.

	weight
$\vec{\pi}_1 = (1, 1+1, 1+1)$	$+1$
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$$\sigma(q) = \sum_{n=0}^{\infty} S(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q;q)_n},$$
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NOT QUITE CORRECT CLAIM BY FOLSOM and ONO

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$$\begin{aligned}\mathrm{spt}(507) &= 6047032737556285225064\\ &= 2^3\cdot 3\cdot 251\cdot 236699\cdot 1703123\cdot 24900893\end{aligned}$$

In fact, if $p\equiv 23 \pmod{24}$ is prime and

$$n=\frac{1}{24}(p^3\cdot m^2+1),$$

where $(m,6p)=1$, then $24n-1=p^3\cdot m^2$ and

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How about $n=268181$? $24n-1=23^5$ and

$$\mathrm{spt}(268181)=17367\cdots 2073 \qquad \text{(574 decimal digits)}$$

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- $\mathrm{spt}(n)$ is odd and if and only if $24n - 1 = pm^2$ for some prime $p \equiv 23 \pmod{24}$ and some integer m.
- COUNTEREXAMPLE: $n = 507$, $24n - 1 = 23^3 = 23 \cdot 23^2$:

$$\begin{aligned}\mathrm{spt}(507) &= 6047032737556285225064 \\ &= 2^3 \cdot 3 \cdot 251 \cdot 236699 \cdot 1703123 \cdot 24900893\end{aligned}$$

In fact, if $p \equiv 23 \pmod{24}$ is prime and

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where $(m, 6p) = 1$, then $24n - 1 = p^3 \cdot m^2$ and

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FOLSOM and ONOs PROOF

- $$\mathcal{L}(z) = \frac{1}{\theta(z)}\left(\sum_n \frac{(12n-1)q^{6n^2-1/24}}{1-q^{12n-1}} - \sum_n \frac{(12n-5)q^{6n^2-25/24}}{1-q^{12n-5}}\right)$$
- Then
$$\mathcal{L}(24z) - \mathcal{A}(24z)$$
is a weakly holomorphic weight $\frac{3}{2}$ modular form on $\Gamma_0(576)$ with Nebentypus χ_{12}, where
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- $$\mathcal{L}(z) \equiv \sum_{n\geq 1}\sum_{m\geq 0}\left(q^{(12n-1)(12n+24m+1)} + q^{(12n-5)(12n+24m+5)}\right)$$
$$+\sum_{n\geq 1}\sum_{m\geq 0}\left(q^{(12n+1)(12n+24m-1)} + q^{(12n+5)(12n+24m-5)}\right)$$

- $$q^{-1}\mathcal{S}(24z) = \sum_{n\geq 1} \mathrm{spt}(n) q^{24n-1} \equiv \mathcal{L}(24z) \pmod 2,$$
so that
$$\mathrm{spt}(n) \equiv \sum_{\substack{1\leq d_1 < d_2 \\ d_1 d_2 = 24n-1}} 1 \pmod 2,$$
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RESULTS OF ANDREWS, DYSON and HICKERSON (1988)

- For $m \equiv 1 \pmod{24}$ let $T(m)$ denote the number of inequivalent solutions of

$$u^2 - 6v^2 = m,$$

with $u + 3v \equiv \pm 1 \pmod{12}$ minus the number with $u + 3v \equiv \pm 5 \pmod{12}$.

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$$m = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r},$$

be the prime factorisation where each $p_i \equiv 1 \pmod{6}$ or p_i is the negative of a prime $\equiv 5 \pmod{6}$.
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$$T(m) = T\left(p_1^{e_1}\right) T(\left(p_2^{e_2}\right) \cdots T\left(p_r^{e_r}\right),$$

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$$T\left(p^e\right) = \begin{cases} 0 & \text{if } p \not\equiv 1 \pmod{24} \text{ and } e \text{ is odd}, \\ 1 & \text{if } p \equiv 13, 19 \pmod{24} \text{ and } e \text{ is even}, \\ (-1)^{e/2} & \text{if } p \equiv 7 \pmod{24} \text{ and } e \text{ is even}, \\ e+1 & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = 2, \\ (-1)^e(e+1) & \text{if } p \equiv 1 \pmod{24} \text{ and } T(p) = -2 \end{cases}$$

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PROOF OF SPT PARITY - PART 1

$$24n - 1 = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} q_1^{b_1} q_2^{b_2} \cdots q_s^{b_s},$$

where each $p_j \equiv 5 \pmod 6$ and $q_j \equiv 1 \pmod 6$ so that

$$1 - 24n = (-p_1)^{a_1} (-p_2)^{a_2} \cdots (-p_r)^{a_r} q_1^{b_1} q_2^{b_2} \cdots q_s^{b_s},$$

and

$$a_1 + a_2 + \cdots + a_r \equiv 1 \pmod 2.$$

We have

$$T(1-24n) = T\left((-p_1)^{a_1}\right) T\left((-p_2)^{a_2}\right) \cdots T\left((-p_r)^{a_r}\right) T\left(q_1^{b_1}\right) T\left(q_2^{b_2}\right) \cdots$$

Now suppose $\mathrm{spt}(n)$ is odd so that $T(1-24n) \neq 0$. At least one of the a_j is odd, say a_1. Since $T(1-24n) \neq 0$ we deduce that $p_1 \equiv 23 \pmod{24}$, and the factor $T((-p_1)^{a_1}) = \pm(a_1+1)$ is even. If $j \neq 1$, a_j is odd and $p_j \equiv 23 \pmod{24}$ then the factor $T((-p_j)^{a_j})$ would also be even and we would get that $\mathrm{spt}(n)$ is even, which is a contradiction. Therefore each a_j is even for $j \neq 1$.

Similarly each b_j is even. Hence each exponent in the factorisation is even except a_1. So

$$\frac{1}{2}T((-p_1)^{a_1}) = \pm\frac{1}{2}(a_1+1) \equiv 1 \pmod{2},$$

$a_1 \equiv 1 \pmod{4}$ and

$$24n - 1 = p^{4a+1}m^2,$$

where $p \equiv 23 \pmod{24}$ is prime and $(m,p) = 1$. Conversely, if

$$24n - 1 = p^{4a+1}m^2,$$

where $p \equiv 23 \pmod{24}$ is prime and $(m,p) = 1$ then it easily follows that $\frac{1}{2}T(1-24n)$ is odd, and $\mathrm{spt}(n)$ is odd. This completes the proof of the parity result.

PROOF OF SPT PARITY - PART 2

It remains to show that

$$\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(1-q^n)}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n}.$$

First we show that

$$\frac{1}{(-q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(1-q^n)}=\sum_{n=0}^{\infty}\frac{1}{(q^2;q^2)_n}\left((q)_{2n}-(q)_\infty\right)$$

We need

$$\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}=\frac{1}{(t;q)_\infty}\qquad\text{(EULER)}$$

$$\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(1-q^n)}=\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q^2;q^2)_{n-1}}{(q)_n}$$
$$=\sum_{n=1}^{\infty}\frac{q^n}{(q)_n(q^{2n};q^2)_\infty}=\sum_{n=1}^{\infty}\frac{q^n}{(q)_n}\sum_{k=0}^{\infty}\frac{q^{2nk}}{(q^2;q^2)_k}$$

(by EULER)

$$=\sum_{k=0}^{\infty}\frac{1}{(q^2;q^2)_k}\sum_{n=1}^{\infty}\frac{q^{n(2k+1)}}{(q)_n}$$
$$=\sum_{k=0}^{\infty}\frac{1}{(q^2;q^2)_k}\left(\frac{1}{(q^{2k+1};q)_\infty}-1\right),$$

again by EULER.
By multiplying by $(q)_\infty$ we have

$$\frac{(q)_\infty}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^n(-q;q)_{n-1}}{(1-q^n)}=\sum_{k=0}^{\infty}\frac{1}{(q^2;q^2)_k}\left((q)_{2k}-(q)_\infty\right),$$

Finally we show that

$$\sum_{n=0}^{\infty} \frac{1}{(q^2;q^2)_n}\left((q)_{2n}-(q)_\infty\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}q^{n^2}}{(q;q^2)_n}.$$

We need

$$\sum_{n=0}^{\infty}\left(\frac{(t;q)_\infty}{(a;q)_\infty} - \frac{(t;q)_n}{(a;q)_n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{(q^{-1}q;q)_n(at^{-1})^n}{(qt^{-1};q)_n} + \frac{(t;q)_\infty}{(a;q)_\infty}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\right.$$

$$\left. + \sum_{n=1}^{\infty}\frac{t^{-1}q^n}{1-t^{-1}q^n} - \sum_{n=1}^{\infty}\frac{tq^n}{1-tq^n} - \sum_{n=1}^{\infty}\frac{at^{-1}q^n}{1-at^{-1}q^n}\right)$$

and

$$\sum_{n=0}^{\infty} \frac{(a;q)_\infty (b;q)_\infty}{(q;q)_\infty (c;q)_\infty} - \frac{(a;q)_n (b;q)_n}{(q;q)_\infty (c;q)_\infty}$$

$$= \frac{(a;q)_\infty (b;q)_\infty}{(q;q)_\infty (c;q)_\infty} \left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} - \sum_{n=1}^{\infty} \frac{aq^n}{1-aq^n} \right.$$

$$\left. \sum_{n=1}^{\infty} \frac{(cb^{-1};q)_n b^n}{(a;q)_n (1-q^n)} \right)$$

Thus

$$\sum_{n=0}^{\infty}\left(\frac{1}{(q^2;q^2)_\infty}-\frac{1}{(q^2;q^2)_n}\right)=\frac{1}{(q^2;q^2)_\infty}\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}$$

and

$$\sum_{n=0}^{\infty}\left((q^2;q^2)_\infty-(q^2;q^2)_n\right)=\sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2}}{(q;q^2)_n}+(q;q^2)_\infty\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{(q^2;q^2)_n} \left((q;q)_{2n} - (q;q)_\infty\right)$$

$$= \sum_{n=0}^{\infty} \left((q;q^2)_n - (q;q^2)_\infty + (q;q^2)_\infty - \frac{(q;q)_\infty}{(q^2;q^2)_n} \right)$$

$$= \sum_{n=0}^{\infty} \left((q;q^2)_n - (q;q^2)_\infty\right) + (q;q)_\infty \sum_{n=0}^{\infty} \left(\frac{1}{(q^2;q^2)_\infty} - \frac{1}{(q^2;q^2)_n} \right)$$

$$= -\sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2}}{(q;q^2)_n} - (q;q^2)_\infty \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} + \frac{(q;q)_\infty}{(q^2;q^2)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{n^2}}{(q;q^2)_n}.$$

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