Homeomorphic Meshes in $\Re^3$

Nimish R. Shah
Mentor Graphics, 1001 Ridder Park Drive, San Jose, CA 95131, USA.
email: nimish_shah@mentorg.com

Abstract

For employing the finite element method for solving partial differential equations, a preliminary step is to decompose the given geometric domain into a mesh. One of the requirements of automatic mesh generation is that the generated mesh should be homeomorphic (that is, topologically equivalent) to the given geometry. We present sufficient conditions that guarantee such a homeomorphism.

Our approach is based on the well-studied octree decomposition [9] of the given domain. The root of the octree is a cube enclosing the domain. A node of the octree is either a leaf cube or has eight equal sized cubes as children. For a leaf cube C and some geometry X, let $C_X = C \cap X$ denote the restricted cube *of C with respect to X. The sufficient conditions require that the restricted cubes of all leaves of the octree with respect to the domain and with respect to the boundary of the domain be topological closed balls. We describe how to construct appropriate edges, triangles and tetrahedra of a mesh so that the mesh is homeomorphic to the domain. The vertex set of the mesh constructed by our method is the set of centroids of all non-empty restricted cubes.*

Keywords. Homeomorphism, mesh-generation, octrees, simplicial complexes, triangulations.

1 Introduction

One approach to solve partial differential equations numerically is to use the finite element method. As a preliminary step, this method involves the generation of discretizations of the geometric domain over which the partial differential equation is being solved, see e.g. [13]. The process of discretizing the domain is called *mesh generation*. Surveys of mesh generation methods can be found in [5, 12]. In this paper, we study octree based meshes and describe conditions which guarantee a homeomorphism between the generated mesh and the given geometric domain in $\Re^3$. Below, we introduce some definitions from topology [6, 8].

Basic definitions. A *simplex* σ_T is the convex hull of an affinely independent point set T. Its *dimension* is $k = \dim \sigma_T = \operatorname{card} T - 1$, where $\operatorname{card} T$ denotes the cardinality of T. σ_T is also referred to as a *k-simplex*. For $d = 0, 1, 2, 3$, a d-simplex is respectively called a *vertex*, an *edge*, a *triangle* and a *tetrahedron*. If $U \subseteq T$, then σ_U is called a *face* of σ_T. A *simplicial complex* $\mathcal{K}$ is a finite collection of simplices such that the following properties hold.

(i) If $\sigma_U \in \mathcal{K}$ and $V \subseteq U$, then $\sigma_V \in \mathcal{K}$.

(ii) If $\sigma_U, \sigma_V \in \mathcal{K}$, then $\sigma_U \cap \sigma_V \in \mathcal{K}$.

The first property says that all faces of $\sigma_U \in \mathcal{K}$, including the empty set, are in $\mathcal{K}$ and the second property says that two simplices in $\mathcal{K}$ intersect in a common face. The *vertex set* of $\mathcal{K}$ is $\operatorname{vert} \mathcal{K} = \bigcup_{\sigma_T \in \mathcal{K}} T$, the *dimension* of $\mathcal{K}$ is $\dim \mathcal{K} = \max_{\sigma \in \mathcal{K}} \dim \sigma$, and the *underlying space* of $\mathcal{K}$ is $|\mathcal{K}| = \bigcup_{\sigma \in \mathcal{K}} \sigma$. A *subcomplex* of $\mathcal{K}$ is a simplicial complex $\mathcal{L} \subseteq \mathcal{K}$.

A *geometric triangulation* of a point set $S \subseteq \Re^d$ is a simplicial complex $\mathcal{K}$ with $\operatorname{vert} \mathcal{K} \subseteq S$ and $|\mathcal{K}| = \operatorname{conv} S$.[1] The popular Delaunay triangulation [2, 3, 7] of a point set is an example of a geometric triangulation. In this paper we shall be interested in a topological triangulation of a topological space in $\Re^3$. A *topological triangulation* of a topological space X is a simplicial complex homeomorphic to X. A bijection $f : X \to Y$ is a *homeomorphism* between X and Y if f and f^{-1} are continuous. If such an f exists, then X and Y are *homeomorphic* or are *homeomorphs*, writ-

[1] $\operatorname{conv} S$ denotes the convex hull of S.

ten $X \approx Y$. A simplicial complex $\mathcal{K}$ is *homeomorphic* to X if the underlying space of $\mathcal{K}$ is homeomorphic to X.

Background. The meshes that we consider in this paper are simplicial complexes. Schroeder and Shephard [9] present a method for generating meshes automatically. Though one of the issues addressed by their paper is topological equivalence between the mesh and the domain, the paper does not guarantee a topoloical equivalence. Chew [1] describes a method to construct a triangulation of a surface imbedded in $\Re^3$. Extending the idea in [1], Edelsbrunner and Shah [4] present sufficient conditions which guarantee topological equivalence between a given geometric domain X and a simplicial complex meshing X. The conditions are formulated in terms of restricted Voronoi cells, which are intersections of Voronoi cells [14] and X, see [4, 10] for details. The simplicial complex obtained by their method is a subcomplex of the Delaunay triangulation of a carefully chosen point set (vertex set). It is not clear how to choose this vertex set. In this paper we extend the ideas in [4] to octree based triangulations. This approach implicitly addresses the vertex selection issue.

Outline. In section 2 we define octree triangulations $\mathcal{O}_X$ restricted by X. In sections 3 and 4 we present conditions that guarantee a homeomorphism between $|\mathcal{O}_X|$ and X. We conclude the paper with some remarks in section 5.

Remark. The intent of this paper is to present a possibly useful technique for homeomorphic mesh generation without going into tedious technical details. Some proofs are either omitted or only sketched briefly. Complete proofs can be found in [11] along with extensions to 3-manifolds in $\Re^d$.

2 Restricted Octree Triangulations

An *octree* [9] is a geometric division of $\Re^3$ into a finite tree of 3-cubes[2]. Each 3-cube is either a leaf of the tree or an internal node. If a 3-cube is an internal node of the tree, then it is *split* into 8 equal volume children. Two 3-cubes are *neighbors* if one contains a face[3] of another. An edge e of a 3-cube C is *split* if any neighbor C' of C incident to e is split. A *corner* of a 3-cube is one of its 8 vertices. An octree is *balanced* if an edge of any unsplit cube contains at most one corner in its interior. If a k-face, $k > 0$, of one 3-cube of a balanced octree is contained in another, then the two cubes have equal edge length, or the ratio of the larger edge length to the smaller is $2:1$. If $k = 0$, then the ratio can also possibly be $4:1$. A balanced octree is *strictly balanced* if the $4:1$ ratio is disallowed. Throughout this paper we shall be concerned only with strictly balanced octrees. We shall further assume that the 3-cubes are rectilinear, that is each facet[4] of a 3-cube is orthogonal to some unit vector $\mathbf{e}_i$, $1 \leq i \leq 3$.

Let p be the centroid of some 3-cube C_p. For a given octree τ, let $S = S_\tau$ denote the set consisting of all centroids p such that C_p is a leaf cube of τ. Let $X \subset \Re^3$ be a topological space of interest. For $p \in S$, let $C_{p,X} = C_p \cap X$ be the cube C_p *restricted by* X. For $T \subseteq S$, let

$$C_T = \bigcap_{p \in T} C_p,$$

and

$$C_{T,X} = \bigcap_{p \in T} C_{p,X} = C_T \cap X.$$

For $T \subseteq S$, define μ_T to be the maximum subset T' of S such that $C_T = C_{T'}$. Observe that C_T is a (possibly empty) cube. Define ℓ_T so that C_T is a $(4-\ell_T)$-cube. Note that $1 \leq \ell_T \leq 4$. Before we define the octree triangulation $\mathcal{O}_X$ restricted by X, we need the following lemma.

LEMMA 2.1 Let τ be a strictly balanced octree and let $S = S_\tau$. Let $T \subseteq S$ be such that $C_T \neq \emptyset$. Then,

(i) conv $\mu_T \subseteq \bigcup_{p \in \mu_T} C_p$, and

(ii) if $(\mu_T)_\perp$ denotes the projection of μ_T on the $(\ell_T - 1)$-flat orthogonal to aff C_T,[5] then the points in $(\mu_T)_\perp$ are in convex position.

PROOF. (i) A facet f of a 3-cube C_p, $p \in \mu_T$, is *free* if it is contained in the boundary of $\bigcup_{p \in \mu_T} C_p$. See figure 2.1 for an illustration in $\Re^2$. Let f_p be a

[2] Later, we will use 2-cube to denote a square, 1-cube to denote an edge and a 0-cube to denote a vertex.

[3] A *face* of a 3-cube is any of its vertex, edge or bounding square.

[4] A *facet* of a d-cube is any of its $(d-1)$-faces.

[5] aff Y denotes the affine hull of Y.

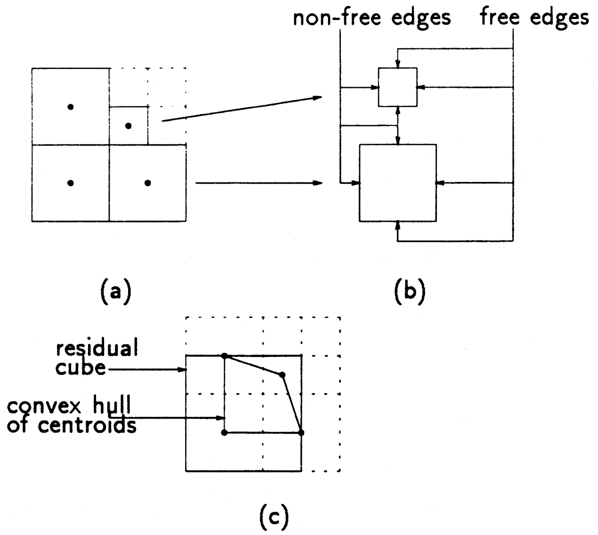

Figure 2.1: Residual 2-cube (square) in $\Re^2$. (a) An ensemble of four squares of interest. (b) Free and non-free facets (edges). (c) The residual cube contains the convex hull of the centroids of the four squares.

free facet for some $p \in \mu_T$. The closed half-space of aff f_p containing p also contains all points in μ_T because τ is strictly balanced. The intersection of all such half-spaces for all free facets is the ***residual cube*** P, a convex polytope contained in $\bigcup_{p\in\mu_T} C_p$. Since $\mu_T \subseteq P$, it follows that conv $\mu_T \subseteq P \subseteq \bigcup_{p\in\mu_T} C_p$.

(ii) Let q be any point in $(\mu_T)_\perp$. Without loss of generality assume that (a) the point $h \cap \text{aff}\, C_T$ is the origin, (b) h is identified with $\Re^k$, $k \leq 3$, and (c) q is the point $(1, 1, \ldots, 1) \in \Re^k$. It can be verified by enumerating all possible configurations of μ_T under the restriction that τ be strictly balanced that the halfspace $H : \langle q, x\rangle < \langle q, q\rangle \subset h$ contains $(\mu_T)_\perp - \{q\}$. We omit the details. □

Assume that the points in $S = S_\tau$ are indexed so that if $p \in S$ has an index smaller that $q \in S$, then the volume of C_p is no greater than that of C_q. Now we are ready to construct $\mathcal{O}_X = \mathcal{O}_{S,X}$. For every subset $T \subseteq S$ such that $C_{T,X} \neq \emptyset$, we construct a simplicial complex $\mathcal{O}(T)$ by induction on ℓ_T. For $\ell_T = 1$, define $\mathcal{O}(T) = \{\sigma_T\}$. For $\ell_T = 2$, define $\mathcal{O}(T) = \{\sigma_T\} \cup T$. For $\ell_T = 3$, card $\mu_T = 3$ or card $\mu_T = 4$. If card $\mu_T = 3$, define $\mathcal{O}(T)$ to be the simplicial complex consisting of the triangle formed by the three points along with its edges and vertices. Otherwise, let $\mu_T = \{a, b, c, d\}$ such that $\ell_{\{a,c\}} = \ell_{\{b,c\}} = \ell_{\{b,d\}} = \ell_{\{d,a\}} = 2$. Let a be the point with the smallest index among the points in μ_T. Define

$$\mathcal{O}(T) = \{\sigma_{\{a,b,c\}}, \sigma_{\{a,b,d\}}\} \cup \bigcup_{T' \subseteq \mu_T, \ell_{T'}=2} \mathcal{O}(T').$$

By Lemma 2.1(ii), the points in the projection of T into a plane orthogonal to aff C_T are in convex position and so it follows that $\mathcal{O}(T)$ must be a simplicial complex. Now let $\ell_T = 4$. Let p be the point in μ_T with the smallest index. Let $\Delta_T = \bigcup_{T' \subseteq \mu_T, \ell_{T'}=3} \mathcal{O}(T')$. Define $\mathcal{O}(T) = \Delta_T \cup \{\text{conv}\,(\sigma \cup \{p\}) \mid \sigma \in \Delta_T \text{ does not contain } p\}$. It is shown in [11] that $\mathcal{O}(T)$ is a simplicial complex. Finally define

$$\mathcal{O}_X = \mathcal{O}_{S,X} = \bigcup_{C_{T,X} \neq \emptyset} \mathcal{O}(T).$$

Lemma 2.1(i) guarantees that $\mathcal{O}_X$ is a simplicial complex. $\mathcal{O}_X$ is the *octree simplicial complex of S restricted by X*. Note that $\mathcal{O}_X$ is uniquely defined for S once we assume a specific indexing of its points.

Intuitively, we construct $\mathcal{O}(T)$ by inductively obtaining its triangulated "boundary" and then triangulating the "interior". For example, if $\ell_T = 4$ and card $\mu_T = 8$, then a possible collection of points for μ_T is $\{(1,1,1), (2,-2,2), (-2,-2,2), (-2,2,2)\} \cup \{(1,1,-1), (2,-2,-2), (-2,-2,-2), (-2,2,-2)\}$ with $\{(0,0,0)\}$ being the intersection of all 8 cubes. Let T' be the set of points with positive 3rd coordinate ($\ell_{T'} = 3$). See figure 2.2. If the smallest indexed point is, say, $p = (1,1,-1)$, then $\mathcal{O}(T)$ contains simplices conv $\{p, a, b, c\}$ and conv $\{p, a, b, d\}$, among others.

See figure 2.3 for an illustration of restricted quad tree triangulations. (A quad tree is a 2-dimensional counterpart of an octree.) Observe that $\mathcal{O}_{X_1}$ and $\mathcal{O}_{X_2}$ are homeomorphic to X_1 and X_2 respectively, while $\mathcal{O}_{X_3}$ is not homeomorphic to X_3.

3 Homeomorphic Triangulation for Compact Manifolds

In this section we shall state the conditions under which the restricted octree triangulation is a topological triangulation of X. These conditions apply only

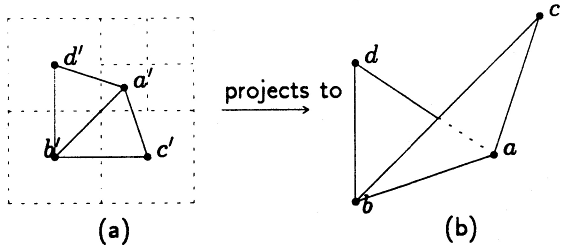

Figure 2.2: Constructing $\mathcal{O}_X$. $T' = \{a, b, c, d\}$, where $a = (1, 1, 1), b = (-2, -2, 2), c = (2, -2, 2)$ and $d = (-2, 2, 2)$. $\ell_{T'} = 2$. The index of a is smaller than that of b, c and d. (a) A triangulation of points a', b', c', d' where x' denotes the projection of $x \in T'$ on the 2-flat orthogonal to $C_{T'}$. (b) The triangulation $\mathcal{O}(T')$ obtained by projecting the triangulation in (a) into $\Re^3$.

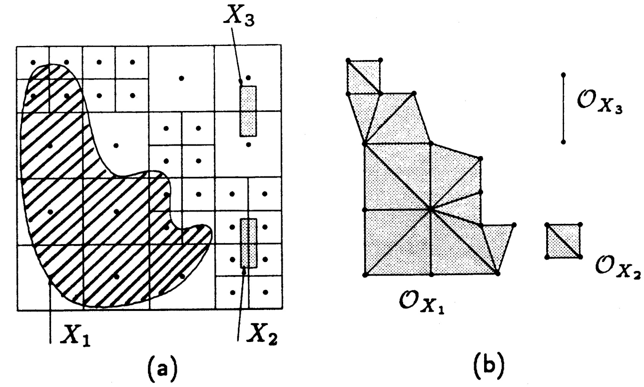

Figure 2.3: (a) Quad tree based decomposition of three geometries X_1, X_2 and X_3. (b) Restricted quad ree triangulations. $\mathcal{O}_{X_1}$ and $\mathcal{O}_{X_2}$ are homeomorphic to X_1 and X_2 but $\mathcal{O}_{X_3}$ is not homeomorphic to X_3.

when X is a compact m-manifold, $0 \leq m < 4$. First, we introduce some terminology.

Topological balls and manifolds. Homeomorphs of open, half-open, and closed balls of various dimensions play an important role in the forthcoming discussion. For $k \geq 0$, let $\mathbf{o}$ be the origin of $\Re^k$ and define

$$\begin{aligned} H^k &= \{x = (\xi_1, \ldots, \xi_k) \in \Re^k \mid \xi_k \geq 0\}, \\ B^k &= \{x \in \Re^k \mid |x\mathbf{o}| \leq 1\}, \text{ and} \\ S^{k-1} &= \{x \in \Re^k \mid |x\mathbf{o}| = 1\}. \end{aligned}$$

For convenience, we define $\Re^k = H^k = B^k = S^k = \emptyset$ if $k < 0$. An *open k-ball* is a homeomorph of $\Re^k$, a *half-open k-ball* is a homeomorph of H^k, a *closed k-ball* is a homeomorph of B^k, and a $(k-1)$-*sphere* is a homeomorph of S^{k-1}. For $k \geq 1$ these are disjoint classes of spaces, that is, open balls, half-open balls, closed balls, and spheres are pairwise non-homeomorphic. This is not true for $k = 0$: open, half-open, and closed 0-balls are points, and a 0-sphere is a pair of points.

$X \subseteq \Re^3$ is a *k-manifold without boundary* if each $x \in X$ has an open k-ball as a neighborhood in X. $X \subseteq \Re^3$ is a *k-manifold with boundary* if each $x \in X$ has an open or half-open k-ball as a neighborhood in X. The set of points without open k-ball neighborhood forms the *boundary*, $\mathrm{bd}\, X$, of X. Note that the boundary of a half-open k-ball is an open $(k-1)$-ball, which is therefore without boundary. The *interior* of a manifold X is $\mathrm{int}\, X = X - \mathrm{bd}\, X$; it is the set of points with open k-ball neighborhoods. A manifold $X \subseteq \Re^3$ is *compact* if it is closed and bounded. A manifold $Y \subseteq X$ is a *submanifold* of X.

Generic intersection and closed ball properties. Let $P \subseteq \Re^3$ be a convex polyhedron of dimension ℓ and let $X \subseteq \Re^3$ be an m-manifold without boundary. We say that P *intersects X generically* if $X \cap P = \emptyset$ or $X \cap P$ is an $(m + \ell - 3)$-manifold and $X \cap \mathrm{int}\, P = \mathrm{int}\, X \cap P$. If X has a non-empty boundary, then P *intersects X generically* if P intersects X and $\mathrm{bd}\, X$ generically.

Let $X \subseteq \Re^3$ be a compact m-manifold, with or without boundary. Let τ be a strictly balanced tree decomposing X and let $S = S_\tau$. S has the *generic intersection property* for X if for every subset $T \subseteq S$, C_T intersects X generically. S has the *closed ball property* for X if for every $T \subseteq S$ with $\ell = m + 1 - \ell_T$, the following two conditions hold.

(i) $C_{T,X}$ is either empty or a closed ℓ-ball, and

(ii) $C_{T,\mathrm{bd}\, X}$ is either empty or a closed $(\ell-1)$-ball.

Remark. The generic intersection and closed ball properties were first defined for restricted Voronoi cells in [4]. See [4] for a related discussion on non-degeneracy.

THEOREM 3.1 Let $X \subseteq \Re^3$ be a compact m-manifold. Let τ be a strictly balanced octree such that $S = S_\tau$

has the generic intersection property and the closed ball property for X. Then $X \approx |\mathcal{O}_{S,X}|$.

To prove the theorem, we shall need the following lemma.

LEMMA 3.2 Let X, m, τ and S be as in Theorem 3.1. Define $\mathsf{C}_i = \{C_{T,X} \mid C_{T,X} \text{ is an } i\text{-ball}\}$, $0 \leq i \leq m$. There exist simplicial complexes $\mathcal{K}_i$ and homeomorphisms $h_i : \bigcup \mathsf{C}_i \to \bigcup \mathcal{K}_i = |\mathcal{K}_i|$, so that $\mathcal{K}_{i-1} \subseteq \mathcal{K}_i$ and h_i agrees with η_{i-1} on $\bigcup \mathsf{C}_{i-1}$. Furthermore there exists a bijection $\beta : \text{vert}\, \mathcal{K}_m \to \bigcup_{0\leq i \leq m} \mathsf{C}_i$ so that $\sigma_T \in \mathcal{K}_m$ iff the elements of $\beta(T)$ can be arranged to form a linear chain under the inclusion relation. □

Lemma 3.2 can be proved by first proving Lemma 4.2 in [4] and then following the proof of Theorem 4.3 in [4]. We omit the details. Lemma 3.2 implies that the underlying space of $\mathcal{K} = \mathcal{K}_m$ is homeomorphic to $X = \bigcup \mathsf{C}_m$. To complete the proof of Theorem 3.1 we will show that $|\mathcal{K}| \approx |\mathcal{O}_X|$.

We need some terminology. For $T \subseteq S$ such that $C_{T,X} \neq \emptyset$, let h_T denote the $(\ell_T - 1)$-flat orthogonal to aff C_T and containing the origin, let s_T denote the point in aff $C_T \cap |\mathcal{O}(T)|$, let $h_T(q)$ denote the orthogonal projection of $q \in \Re^3$ into h_T and let $h_T(Q) = \{h_T(q) \mid q \in Q\}$ for $Q \subseteq \Re^3$. We note that $h_T(|\mathcal{O}(T)|)$ is a star polytope[6]. Furthermore, the point $h_T(s_T)$ lies in its kernel (proof omitted). An *ℓ-chain* of $T \subseteq S$ is a chain C of subsets of μ_T under the inclusion relation such that if $T_1, T_2 \in \mathsf{C}$ and $T_1 \subseteq T_2$, then $\ell_{T_1} < \ell_{T_2}$. Let C_T denote the set of all ℓ-chains of T. We define the ***star subdivision*** of $h_T(|\mathcal{O}(T)|)$, sd T, as follows. If $\ell_T = 1$, sd $T = \{h_T(s_T)\}$. For $\ell_T > 1$, sd $T = \{\text{conv}\,\{h_T(s_{T'}) \mid T' \in \mathsf{C}\} \mid \mathsf{C} \in \mathsf{C}_T\}$.

sd T induces a subdivision sd$'\,T$ of $|\mathcal{O}(T)|$ (thru the inverse of the orthogonal projection into h_T). It can be verified that $\sigma_V \in \text{sd}'\,T$ iff $\sigma_{h_T(V)} \in \text{sd}\,T$.

Define the ***star subdivision*** of $\mathcal{O}_X = \mathcal{O}_{S,X}$ to be

$$\text{sd}'\,\mathcal{O}_X = \bigcup_{C_{T,X}\neq\emptyset} \text{sd}'\,T.$$

Since $|\mathcal{O}_X| = |\text{sd}'\,\mathcal{O}_X|$, we have $|\mathcal{O}_X| \approx |\text{sd}'\,\mathcal{O}_X|$. Thus to complete the proof of Theorem 3.1, we only need to show that $|\mathcal{K}| \approx |\text{sd}'\,\mathcal{O}_X|$.

The function $f : \text{vert}\,\text{sd}'\,\mathcal{O}_X \to \text{vert}\,\mathcal{K}$ defined by $f(s_T) = \beta^{-1}(C_{T,X})$, where β is the bijection in Lemma 3.2, is a bijection. By construction of $\mathcal{O}_X$ and $\mathcal{K}$, $\sigma_V \in \mathcal{K}$ iff $\sigma_{f^{-1}(V)} \in \text{sd}'\,\mathcal{O}_X$. It follows that $|\mathcal{K}| \approx |\text{sd}'\,\mathcal{O}_X|$. □

Remark. A generalization of the above theorem to compact 3-manifolds in $\Re^d$ can be found in [11].

4 Extension to General Topological Spaces

The previous section described sufficient conditons for $\mathcal{O}_X$ to be homeomorphic to X when X was a manifold. In engineering practice, one encounters geometries which are not necessarily manifolds. In this section we extend the sufficient conditions to cover more general geometries. To present the conditions succinctly, we use finite regular CW complexes. Intuitively, the gist of the conditions is that X should be decomposed into balls appropriately. See also [4, 10].

A closed ball is called a *cell*, or a *k-cell* if its dimension is k. A finite collection of non-empty cells, $\mathcal{C}$, is a *regular CW complex* if the cells have pairwise disjoint interiors, and the boundary of each cell is the union of other cells in $\mathcal{C}$. Let $X \subseteq \Re^3$ be a topological space, let τ be a strictly balanced octree decomposing X and let $S = S_\tau$. S has the *extended closed ball property* for X if there is a regular CW complex $\mathcal{C}$, with $X = \bigcup \mathcal{C}$, that satisfies the following properties for every $T \subseteq S$ with $C_{T,X} \neq \emptyset$:

(i) there is a regular CW complex $\mathcal{C}_T \subseteq \mathcal{C}$ so that $C_{T,X} = \bigcup \mathcal{C}_T$,

(ii) the set $\mathcal{C}_T^\circ = \{\gamma \in \mathcal{C} \mid \text{int}\,\gamma \subseteq \text{int}\,C_T\}$ contains a unique cell, η_T, so that $\eta_T \subseteq \gamma$ for every $\gamma \in \mathcal{C}_T^\circ$,

(iii) if η_T is a j-cell then $\eta_T \cap \text{bd}\,C_T$ is a $(j-1)$-sphere, and

(iv) for each integer k and each k-cell $\gamma \in \mathcal{C}_T^\circ - \{\eta_T\}$, $\gamma \cap \text{bd}\,C_T$ is a closed $(k-1)$-ball.

Furthermore, S has the *extended generic intersection property* for X if for every $T \subseteq S$ and every $\gamma \in \mathcal{C}_T - \mathcal{C}_T^\circ$

[6] A ***polytope*** is the underlying space of a simplicial complex. A ***star*** polytope is a polytope P for which there exists a point $p \in P$ such that for every point $q \in P$, the edge pq is contained in P. The set of all such points p is called the ***kernel*** of P.

there is a $\delta \in C^{o}_{T}$ so that $\gamma \subseteq \delta$.

For sake of brevity, we omit a discussion of the above conditions and state the following theorem.

THEOREM 4.1 Let $X \subseteq \Re^3$ be a topological space. Let τ be a strictly balanced octree such that $S = S_\tau$ has the extended generic intersection property and the extended closed ball property for X. Then $X \approx |\mathcal{O}_{S,X}|$.

5 Concluding Remarks

We have discussed octree triangulations restricted by a topological space X and presented sufficient conditions that guarantee a homeomorphism between the triangulation and the space. Our method is based on decomposing X by cubes, requiring the intersections of the cubes and X to be balls and then appropriately connecting the centroids of the cubes to obtain the restricted octree triangulation. Homeomorphic triangulations were studied in [4]. The octree decomposition of the space X studied in this paper gives an automatic way of generating the vertex set of the triangulation which was a sore point of the method in [4]. It would be interesting to see how this approach performs (the number of simplices generated, the quality of simiplices, etc.) on geometries generated by solid modelers. This would require verification of the conditions in sections 3 and 4, which in turn might put constraints on the functional description of the geometry of the domain.

References

[1] L. P. Chew. Guaranteed-quality mesh generation for curved surfaces. *In* "Proc. 9th Ann. Sympos. Comput. Geom.", 1993, 274–280.

[2] B. N. Delaunay. Sur la sphère vide. *Izv. Akad. Nauk SSSR, Otdelenie Matematicheskii i Estestvennyka Nauk* 7 (1934), 793–800.

[3] H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag, Heidelberg, Germany, 1987.

[4] H. Edelsbrunner and N. R. Shah. Triangulating topological spaces. *In* "Proc. 10th Ann. Sympos. Comput. Geom.", 1994, 285–292.

[5] K. Ho-Le. Finite element mesh generation methods: a review and classification. *Comput. Aided Design* **20** (1988), 27–38.

[6] J. R. Munkres. *Elements of Algebraic Topology.* Addison-Wesley, Redwood City, California, 1984.

[7] F. P. Preparata and M. I. Shamos. *Computational Geometry – an Introduction.* Springer-Verlag, New York, 1985.

[8] J. J. Rotman. *An Introduction to Algebraic Topology.* Springer-Verlag, New York, 1988.

[9] W. J. Schroeder and M. S. Shephard. A combined octree/Delaunay method for fully automatic 3-d mesh generation. *Int. J. Numr. Methods Eng.* **29** (1990), 37–55.

[10] N. R. Shah. *Topological Modeling with Simplicial Complexes.* PhD thesis, University of Illinois at Urbana-Champaign, Urbana, Illinois, 1994.

[11] N. R. Shah. 2^d tree based topological triangulations. Manuscript, 1994.

[12] M. S. Shephard. Approaches to the automatic generation and control of finite element meshes. *Appl. Mech. Rev.* **41** (1988), 169–185.

[13] G. Strang and G. J. Fix. *An Analysis of the Finite Element Method.* Prentice Hall, Englewood Cliffs, New Jersey, 1973.

[14] G. F. Voronoi. Nouvelles applications des paramètres continus à la théorie des formes quadratiques. *J. Reine Angew. Math.* **133** (1907), 97–178.