On the Number of Simplicial Complexes in $\Re^d$

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Abstract

Using a simplex-crossing counting technique we prove: if the number of non-improperly intersecting simplices with vertices in a set S of n labelled points in $\Re^d$ is $O(n^{\lceil d/2 \rceil})$, then there are $2^{\Theta(n^{\lceil d/2 \rceil})}$ different geometric simplicial complexes with vertices in S.

1 Introduction

In this paper we consider the problem of counting the number of combinatorially different geometric simplicial complexes with vertices in a fixed set of n labelled points in $\Re^d$, the d-dimensional real space. Geometric simplicial complexes consist of geometric simplices rather than topological simplices. Precise definitions are given in Section 2.

A related problem of counting the number of combinatorially different triangulations with vertices in a fixed labelled point set is considered in [3, 8]. Let $t_d(n)$ and $s_d(n)$ denote the maximum number of different topological and geometric triangulations respectively of S^d, the d-dimensional sphere, with n being the number of vertices. Kalai [8] showed that $c_1 n^{\lfloor d/2 \rfloor} \leq \log t_d(n) \leq c_2 n^{\lceil d/2 \rceil} \log n$ for some constants c_1, c_2. In [3], Dey showed that $\log s_d(n) = O(n^{\lceil d/2 \rceil})$ if at most $O(n^{\lceil d/2 \rceil})$ $\lceil d/2 \rceil$-simplices can be embedded in $\Re^d$ without any crossing. Actually, this upper bound also holds for $\log r_d(n)$, where $r_d(n)$ is the maximum number of geometric triangulations possible with n points in $\Re^d$. By a geometric triangulation of a point set in $\Re^d$ we mean a triangulation of the convex hull of the point set with geometric simplices. The only known lower bound for $\log r_d(n)$ is $\Omega(n)$.

Let $\kappa_d(S)$ denote the number of different geometric simplicial complexes with vertices in a set S of labelled points in $\Re^d$, and let

$$\kappa_d(n) = \max_{S \subset \Re^d, |S|=n} \kappa_d(S).$$

In contrast to geometric triangulations, it is easy to establish an $\Omega(n^{\lceil d/2 \rceil})$ lower bound on the the logarithm of $\kappa_d(n)$. However, the upper bound on the number of geometric triangulations does not provide an upper bound on the number of geometric simplicial complexes. This is because, for $d > 2$, not all simplicial complexes in $\Re^d$ are extendible to a triangulation of the underlying point set. For example, the boundary complex of the Schönardt polytope [11] is not extendible to a triangulation of the corresponding vertex set.

Previous results on the number of simplicial complexes dealt with all possible simplicial complexes on n vertices in all dimensions. Let *simp*(n) denote this number. It follows from the results of [9, 10] that $\log simp(n) = \Theta(\binom{n}{\lfloor n/2 \rfloor})$. This paper concentrates on counting the number of simplicial complexes in a fixed dimension $\Re^d$. Specifically, we show that $\log \kappa_d(n) = O(n^{\lceil d/2 \rceil})$ matching the lower bound if $O(n^{\lceil d/2 \rceil})$ simplices can be embedded in $\Re^d$ without crossing. In light of the result of Goodman and Pollack [7], this bound for a fixed point set can be extended to cover all point sets of some fixed cardinality. More specifically, they show that there are at most $2^{O(n \log n)}$ combinatorially different configurations of n points in $\Re^d$. This result combined with ours shows that there are at most $2^{O(n^{\lceil d/2 \rceil} + n \log n)}$ combinatorially different geometric simplicial complexes with n points in $\Re^d$ provided at most $O(n^{\lceil d/2 \rceil})$ simplices are embeddable in $\Re^d$ without crossing.

The rest of the paper is organised as follows. In the next section, we introduce some terminology and

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present the statement of our main result. In section 3, we prove a crossing result. Our method is an extension of the method in [4], where it was used to prove a bound on the number of crossings of triangles in $\Re^3$. Section 4 generalises the argument for counting triangulations in [3] to establish the main result. In section 5 we state some open problems.

2 Definitions and Preliminaries

A *d-simplex* σ_T is the convex hull of an affinely independent point set T of size $d+1$. σ_V is a *face* of σ_T if $V \subseteq T$. A *(geometric) simplicial complex* $\mathcal{K}$ is a finite collection of simplices satisfying the following properties.

(a) If $\sigma_T \in \mathcal{K}$ and $V \subseteq T$, then $\sigma_V \in \mathcal{K}$, and

(b) If $\sigma_V, \sigma_U \in \mathcal{K}$, then $\sigma_V \cap \sigma_U = \sigma_{V \cap U}$.

$\mathcal{K}$ is a *k-complex* if the largest dimension of a simplex in $\mathcal{K}$ is k. For any collection $\mathcal{L}$ of simplices (not necessarily a simplicial complex), we define

$$\mathcal{L}^{(j)} = \{\sigma \in \mathcal{L} \mid \sigma \text{ is a } j\text{-simplex}\}.$$

$\mathcal{L}^{(0)}$ is the set of *vertices* of $\mathcal{L}$.

Two simplicial complexes $\mathcal{K}_1, \mathcal{K}_2$ with vertices in a labelled point set are *combinatorially different* if and only if there exists a simplex σ_V such that $\sigma_V \in \mathcal{K}_1$ and $\sigma_V \notin \mathcal{K}_2$ or $\sigma_V \in \mathcal{K}_1$ and $\sigma_V \notin \mathcal{K}_2$. Let $\kappa_d(S)$ denote the number of different geometric simplicial complexes with vertices in a labelled fixed point set $S \subseteq \Re^d$, and let

$$\kappa_d(n) = \max_{S \subseteq \Re^d, |S|=n} \kappa_d(S).$$

We prove the following.

Theorem 1. $\log \kappa_d(n) = \Theta(n^{\lceil \frac{d}{2} \rceil})$, *if at most* $O(n^{\lceil d/2 \rceil})$ *simplices are embeddable in* $\Re^d$ *without crossing.*

It is easy to see that $\log \kappa_d(n) = \Omega(n^{\lceil \frac{d}{2} \rceil})$. Let $p(t) = (t, t^2, \ldots, t^d) \in \Re^d$ be a point on the moment curve [5]. Let $S = \{p(i) \mid 1 \leq i \leq n\}$ and let $\tau = \lceil \frac{d}{2} \rceil$. Let $\mathcal{K}$ denote the collection of all simplices σ_T, $T \subseteq S$, $|T| \leq \tau$. Then for any two simplices, $\sigma_U, \sigma_V \in \mathcal{K}$, $|U| + |V| \leq d+1$. Since S is in general position, it follows that $\mathcal{K}$ is a simplicial complex. Let $\mathcal{L}$ denote the collection of $(\tau-1)$-simplices in $\mathcal{K}$. Clearly, the cardinality of $\mathcal{L}$ is $\Theta(n^\tau)$. For every $\mathcal{L}' \subseteq \mathcal{L}$, $(\mathcal{K} - \mathcal{L}) \cup \mathcal{L}'$ is a simplicial complex. This proves the lower bound.

The combinatorial bounds proved in this paper are based on the following proposition.

Conjecture 2. *If* $\mathcal{K}$ *is a simplicial complex embedded in* $\Re^d$, *then the total number of simplices in* $\mathcal{K}$ *is* $O(n^{\lceil \frac{d}{2} \rceil})$, *where* n *is the number of vertices of* $\mathcal{K}$.

If $\mathcal{K}$ is a d-complex, the conjecture is true by a result of [6]. It is widely believed that the conjecture is true in general. Two simplices σ_U and σ_V have an *improper intersection* if they intersect but the intersection is not $\sigma_{U \cap V}$ (that is, the intersection is not a common face). Conjecture 2 says that the size of a collection of simplices, with vertices from amongst n fixed points in $\Re^d$, such that no two simplices in the collection have an improper intersection is $O(n^\tau)$, where $\tau = \lceil \frac{d}{2} \rceil$. Two simplices σ_U and σ_V *cross* if they have an improper intersection and $U \cap V = \emptyset$. A collection of simplices is *crossing-free* if no two simplices in the collection cross. An improper intersection is a *non-crossing intersection* if it is not a crossing. To prove Theorem 1, we will need a bound on the size of a collection of crossing-free simplices.

Remark. Since the total number of simplices with vertices in a fixed point set $S \subseteq \Re^d$ of size n is $O(n^{d+1})$, it follows that $\kappa_d(S) = \binom{O(n^{d+1})}{O(n^\tau)} = 2^{O(n^\tau \cdot \log n)}$ if conjecture 2 is true.

3 A Lower Bound on the Number of Crossings

Let $\mathcal{L}$ be some collection of simplices with vertices from a labelled fixed point set $S \subseteq \Re^d$ of cardinality n. Further suppose that if $\sigma_T \in \mathcal{L}$ and $V \subseteq T$, then $\sigma_V \in \mathcal{L}$. Let t_k denote the cardinality of $\mathcal{L}^{(k)}$, $0 \leq k \leq d$. As before, we let $\tau = \lceil \frac{d}{2} \rceil$. Let $x^{(d)}(n, j, t_j)$ denote the number of distinct crossings of j-simplices in $\mathcal{L}$. Below, we shall prove a lower bound on $x^{(d)}(n, j, t_j)$ when $\bigcup_{0 \leq k < j} \mathcal{L}^{(k)}$ is a simplicial complex. Note that this requirement and Conjecture 2 imply that $t_k < c_k \cdot n^\tau$ for some constants c_k, $0 \leq k < j$. We shall need the following lemma which can be found in [3, 4].

Lemma 3. *For $k_1 + k_2 \geq d$, let $\Delta_1 \subseteq \Re^d$ be a k_1-simplex that improperly intersects a k_2-simplex $\Delta_2 \subseteq \Re^d$. Then there exists an ℓ_1-face σ_1 of Δ_1 and an ℓ_2-face σ_2 of Δ_2 such that $\ell_1 + \ell_2 = d$ and σ_1 crosses σ_2.*

Using Lemma 3 and Conjecture 2, we give below a bound on the number of j-simplices in $\mathcal{L}$ if no two j-simplices of $\mathcal{L}$ cross.

Lemma 4. *If Conjecture 2 is true then the following holds. If $t_k < c_k \cdot n^\tau$ for some constants c_k, $0 \leq k < j$, then there exists a constant c so that if $t_j > c \cdot n^\tau$, then there exists a pair of crossing j-simplices in $\mathcal{L}$.*

PROOF. Conjecture 2 guarantees a pair of improperly intersecting j-simplices if $t_j > b_1 \cdot n^\tau$ for some constant b_1. Suppose that there is no crossing pair amongst the t_j j-simplices in $\mathcal{L}$. The outline of the proof is the following. We shall remove from $\mathcal{L}$ one of the two j-simplices involved in a non-crossing intersection. We show that we remove at most $b_2 \cdot n^\tau$ j-simplices by this process, for some constant b_2. At the end, we are left with at least $(c - b_2) \cdot n^\tau$ j-simplices such that no two of them have a non-crossing intersection. If $c - b_2 > b_1$, then Conjecture 2 contradicts the supposition that there is no crossing pair of j-simplices.

We remove j-simplices involved in non-crossing intersections according to the following procedure. We let $\mathcal{L}'$ denote the current set of j-simplices; initially, $\mathcal{L}'$ is the same as $\mathcal{L}^{(j)}$, but it changes as we remove j-simplices. For a j-simplex $\sigma_V \in \mathcal{L}'$, let

$$\Sigma_V = \{\sigma \in \mathcal{L}' \mid \sigma, \sigma_V \text{ have a non-crossing intersection}\}.$$

We remove all simplices in Σ_V from $\mathcal{L}'$. In order to keep track of the number of simplices removed, we use the following charging scheme. Let $\sigma_U \in \Sigma_V$, and let $I = U \cap V$. Then, we charge σ_{U-I} one unit for the removal of σ_U. By Lemma 3, σ_{U-I} has an improper intersection with σ_V. We claim that σ_{U-I} gets charged at most $\binom{j+1}{|I|}$ units. First, we show that σ_{U-I} is never charged at a later step. Suppose it is charged at a later step for the removal of some simplex $\sigma_{U'}$ from $\Sigma_{V'}$. However, since σ_{U-I} is a face of $\sigma_{U'}$, it follows that $\sigma_{U'}$ has an improper intersection with σ_V. Since we supposed that there are no crossings, this intersection must be non-crossing. But then, $\sigma_{U'}$ would be removed from $\mathcal{L}$ when Σ_V was processed. Thus, it cannot be present in $\Sigma_{V'}$ at a later step. This also means that σ_{U-I} was not charged by an earlier step. Finally, it is clear that the step of removing the simplices in Σ_V can charge σ_{U-I} at most $\binom{j+1}{|I|}$ units. Let $b_3 = \Sigma_{0 \leq k < j}\, c_k$. Since the size of $\bigcup_{0 \leq k < j} \mathcal{L}^{(k)}$ is at most $b_3 \cdot n^\tau$, it follows that the total number of j-simplices removed is at most $b_2 \cdot n^\tau$, where $b_2 = 2^{j+1} \cdot b_3$. □

Below, we prove a lower bound on $x^{(d)}(n, j, t_j)$.

Lemma 5. *Let $j \geq \tau$. If for some constants $c, c_{j-1}, t_{j-1} < c_{j-1} \cdot n^\tau$ and there exists a pair of crossing j-simplices whenever $t_j > c \cdot n^\tau$, then there exist constants c', h so that*

$$x^{(d)}(n, j, t_j) \geq c' \cdot \binom{n}{2j+2} \cdot \left(t_j \Big/ \binom{n}{j+1} \right)^{1+\gamma_j},$$

when $t_j > h \cdot n^\tau$. Here, $\gamma_j = \frac{j+1}{j+1-\tau} > 1$.

PROOF. Since we are interested in a lower bound, we can assume that $\mathcal{L}^{(j)}$ realises the lower bound for $x^{(d)}(n, j, t_j)$. Let bound denote the term $c' \cdot \binom{n}{2j+2} \cdot \left(t_j / \binom{n}{j+1} \right)^{1+\gamma_j}$. We shall proceed by induction on $t = t_j$. We choose $d = c+1$. We have at least $t - c \cdot n^\tau \geq n^\tau$ crossings since there is a crossing for every j-simplex above $c \cdot n^\tau$.

First, we dispense away with the case where n is no greater than the constant $n_0 = 2j+2$. In this case, t is also a constant, and we can make bound ≤ 1 by simply choosing a sufficiently small c'. Thus, the lower bound holds in this case since as we saw above, we have at least n^τ crossings. For the rest of the induction step, we will assume that $n > n_0$. We have two cases.

Case 1 (Base Case). $h \cdot n^\tau \leq t \leq (h + c_{j-1}) \cdot n^\tau$.

Since we have at least n^τ crossings, it suffices to show that bound $\leq n^\tau$. Since $n > 2j+2$, $\binom{n}{j+1} \geq b_1 \cdot n^{j+1}$ for some constant b_1. Since $t \leq (h + c_{j-1}) \cdot n^\tau$, we have

$$\text{bound} \leq c' \cdot \frac{n^{2j+2} \cdot (h + c_{j-1})^{1+\gamma_j} \cdot n^{\tau \cdot (\gamma_j + 1)}}{b_1^{(1+\gamma_j)} \cdot n^{(j+1)\cdot(\gamma_j+1)}} = b_2 \cdot n^\tau,$$

where $b_2 = \frac{c' \cdot (h + c_{j-1})^{1+\gamma_j}}{b_1^{(1+\gamma_j)}}$ is a constant. $b_2 < 1$ if c' is small enough.

Case 2 (Induction Step). $t > (h + c_{j-1}) \cdot n^\tau$.

Let $T(w)$ denote the set of j-simplices in $\mathcal{L}^{(j)}$ that are not incident to the vertex $w \in \mathcal{L}^{(0)}$ and let

$t(w) = |T(w)|$. Now $t(w) \geq t - t_{j-1} > h \cdot n^\tau$. For every pair of crossing j-simplices Δ_1 and Δ_2, we count all vertices except those incident to Δ_1 and Δ_2. Alternatively, this count can be obtained by summing up all crossings between j-simplices in $T(w)$ for each vertex w. Thus, we have $(n-2j-2) \cdot x^{(d)}(n,j,t) = \sum_{w\in\mathcal{L}^{(0)}} x^{(d)}(n-1,j,t(w)) \geq c' \cdot \binom{n-1}{2j+2} \cdot \left(\sum_{w\in\mathcal{L}^{(0)}} t(w)^{1+\gamma_j} / \binom{n-1}{j+1}^{1+\gamma_j}\right)$ by induction, since $t(w) > h \cdot n^\tau$. Now $\sum_{w\in\mathcal{L}^{(0)}} t(w) = (n-j-1) \cdot t$. Thus $\sum_{w\in\mathcal{L}^{(0)}} t(w)^{1+\gamma_j} \geq n \cdot \left(\frac{(n-j-1)\cdot t}{n}\right)^{1+\gamma_j}$. This implies that $x^{(d)}(n,j,t) \geq c' \cdot \frac{n}{n-2j-2} \cdot \binom{n-1}{2j+2} \cdot \left(\frac{(n-j-1)\cdot t}{n \cdot \binom{n-1}{j+1}}\right)^{1+\gamma_j}$
$\geq c' \cdot \binom{n}{2j+2} \cdot \left(\frac{t}{\binom{n}{j+1}}\right)^{1+\gamma_j}$. □

By the pigeon-hole principle, it follows that there is at least one j-simplex in $\mathcal{L}^{(j)}$ that crosses many other j-simplices. Specifically, we have the following lemma.

Lemma 6. *Let Conjecture 2 hold and let $j \geq \tau$. Then there exists a j-simplex in $\mathcal{L}$ that crosses at least $\frac{h_j \cdot t^{\gamma_j}}{n^{(\gamma_j-1)\cdot(j+1)}}$ other j-simplices of $\mathcal{L}$ for some constant $h_j > 0$, when $t = t_j > h \cdot n^\tau$.*

4 Counting the Number of Simplicial Complexes

Let $S \subseteq \Re^d$ be a labelled fixed point set of cardinality n. Let $\mathcal{F}(j)$ denote the set of all j-simplicial complexes with vertices in S. Let $\Delta(j)$ denote the set of all j-simplices with vertices in S. For a simplicial complex $\mathcal{K} \in \mathcal{F}(j-1)$ and a collection of j-simplices $T \subseteq \Delta(j)$, define $\mathcal{L}(j,T,\mathcal{K}) = \{\mathcal{K} \cup T' \mid T' \subseteq T,\ \mathcal{K} \cup T' \text{ is a simplicial complex}\}$. Thus $\mathcal{L}(j,T,\mathcal{K})$ is the collection of j-simplicial complexes $\mathcal{K}'$ such that j-simplices of $\mathcal{K}'$ come from T and the k-dimensional simplices of $\mathcal{K}'$ are the same as those in $\mathcal{K}$, $0 \leq k < j$. Define

$$F(j,t,\mathcal{K}) = \max_{T\subseteq\Delta(j), |T|=t} |\mathcal{L}(j,T,\mathcal{K})|,$$

and

$$F(j,t) = \sum_{\mathcal{K}\in\mathcal{F}(j-1)} F(j,t,\mathcal{K}).$$

Observe that $F\left(j, \binom{n}{j+1}\right) = |\mathcal{F}(j)|$. We shall show that $F(j,t) = 2^{O(n^\tau)}$ for $0 \leq j \leq d$ if Conjecture 2 holds. Since $\kappa_d(S) = \sum_{0\leq j\leq d} F\left(j, \binom{n}{j+1}\right)$, it follows that $\kappa_d(S) = 2^{O(n^\tau)}$. Thus to establish Theorem 1, we only need to prove the following lemma.

Lemma 7. $F(j,t) = 2^{O(n^\tau)}$ *if Conjecture 2 holds.*

Proof. We shall use induction, both on j and t. We shall inductively assume that $|\mathcal{F}(j-1)| = 2^{O(n^\tau)}$ and show that for every $\mathcal{K} \in \mathcal{F}(j-1)$, $F(j,t,\mathcal{K}) = 2^{O(n^\tau)}$, whence it follows that $F(j,t) = \sum_{\mathcal{K}\in\mathcal{F}(j-1)} 2^{O(n^\tau)} = 2^{O(n^\tau)}$, and so $|\mathcal{F}(j)| = 2^{O(n^\tau)}$.

For $j < \tau$, the number of j-simplices with vertices in S is bounded by $n^{j+1} \leq n^\tau$. Thus $|\Delta(j)|$ is bounded by $O(n^\tau)$, and so the size of the power set of $\Delta(j)$ is at most $2^{O(n^\tau)}$. This implies that $F(j,t,\mathcal{K}) = 2^{O(n^\tau)}$ for any complex $\mathcal{K} \in \mathcal{F}(j-1)$. Because of the above argument and since $|\mathcal{F}(0)| \leq 2^n$, it follows that the inductive hypothesis holds for $j < \tau$. In the following, we consider the case when $j \geq \tau$, and induct on t.

First, we dispense away with the case $j = d$. Let $\mathcal{K} \in \mathcal{F}(d-1)$. Let $\Gamma \subseteq \Delta(d)$ be a collection of d-simplices so that for every $\sigma \in \Gamma$, $\mathcal{K} \cup \{\sigma\}$ is a d-complex. We claim that a $(d-1)$-simplex σ_U of $\mathcal{K}$ can be incident to at most 2 d-simplices in Γ. Suppose not and let $\sigma_{U\cup\{p_1\}}$, $\sigma_{U\cup\{p_2\}}$, and $\sigma_{U\cup\{p_3\}}$ be three d-simplices of Γ incident to σ_U. At least two points from p_1, p_2 and p_3, say p_1 and p_2, lie on the same side of the hyperplane aff(U). But then $\sigma_{U\cup\{p_2\}}$ improperly intersects some $(d-1)$-face σ' of $\sigma_{U\cup\{p_1\}}$. Since $\sigma' \in \mathcal{K}$, it follows that $\mathcal{K} \cup \{\sigma_{U\cup\{p_2\}}\}$ is not a simplicial complex, contradicting the assumption that $\sigma_{U\cup\{p_2\}} \in \Gamma$. Thus at most 2 d-simplices of Γ can be incident to a $(d-1)$-simplex of $\mathcal{K}$. Since Conjecture 2 implies that the size of $\mathcal{K}^{(d-1)}$ is $O(n^\tau)$, it follows that the size of Γ is $O(n^\tau)$. For any t, $F(j,t,\mathcal{K}) \leq |\mathcal{L}(j,\Gamma,\mathcal{K})|$ and so we have $F(j,t,\mathcal{K}) = 2^{O(n^\tau)}$. In the following, we only consider the case when $\tau \leq j < d$.

Let λ_j be a large enough constant (to be determined later) so that $\lambda_j > h$ and $h_j > \frac{\gamma_j}{\lambda_j}$, where h is the constant in Lemma 5 and h_j is the constant in Lemma 6. Fix a complex $\mathcal{K} \in \mathcal{F}(j-1)$, and consider a set $T \subseteq \Delta(j)$ of size t that realizes the maximum $F(j,t,\mathcal{K})$. When $t \leq \lambda_j \cdot n^\tau$, the number of subsets T' of T is bounded by $2^{O(n^\tau)}$. The bound on $F(j,t,\mathcal{K})$ follows.

Let $t > \lambda_j \cdot n^\tau$. We show that $F(j,t,\mathcal{K}) \leq C^{n^\tau} \cdot f(j,t)$, where $C = (2\lambda)^{\lambda + \frac{1}{\lambda^{\gamma_j - 2}}}$ and $f(j,t) = \left(\frac{t}{n^\tau}\right)^{-\lambda_j \cdot \frac{n^{\tau\cdot\gamma_j}}{t^{\gamma_j-1}}}$. Certain useful properties of $f(j,t)$ are discussed in appendix. In particular, property (P.1) implies that $f(j,t) \leq 1$ for $\lambda_j \cdot n^\tau \leq t \leq \binom{n}{j+1}$, implying that $F(j,t,\mathcal{K}) = 2^{O(n^\tau)}$. We divide the proof into two cases.

Case 1 (Base Case). $\lambda_j \cdot n^\tau < t \leq 2\lambda_j \cdot n^\tau$.

Assume $\lambda_j > 2$. Since the number of subsets of T is at most $2^{2\lambda_j \cdot n^\tau}$, we have

$$\begin{aligned} F(j,t,\mathcal{K}) &\leq 2^{2\lambda_j \cdot n^\tau} \\ &\leq (2\lambda_j)^{\lambda_j \cdot n^\tau} \cdot (t/n^\tau)^{\lambda_j \cdot \frac{n^{\tau\cdot\gamma_j}}{t^{\gamma_j - 1}}} \cdot f(j,t) \\ &\leq (2\lambda_j)^{\lambda_j \cdot n^\tau} \cdot (2\lambda_j)^{\frac{\lambda_j \cdot n^{\tau\cdot\gamma_j}}{\lambda_j^{\gamma_j - 1} \cdot n^{\tau\cdot(\gamma_j - 1)}}} \cdot f(j,t) \\ &= C^{n^\tau} \cdot f(j,t). \end{aligned}$$

Case 2 (Induction Step). $t > 2\lambda_j \cdot n^\tau$.

Since $\mathcal{K}$ is a simplicial complex, by Conjecture 2 the number of k-simplices in $\mathcal{K}$ is $O(n^\tau)$ for $0 \leq k < j$. So Lemma 7 applies with $\mathcal{L} = \mathcal{K} \cup T$. Let σ be the j-simplex in T that crosses at least $h_j \cdot \frac{t^{\gamma_j}}{n^{(\gamma_j - 1)\cdot(j+1)}} \geq \frac{\gamma_j \cdot t^{\gamma_j}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}}$ other j-simplices of T. We get the following recurrence.

$$F(j,t,\mathcal{K}) \leq F(j,t-1,\mathcal{K}) + F(j, t - \frac{\gamma_j \cdot t^{\gamma_j}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}}, \mathcal{K}).$$

Let $\rho = t/n^\tau$. Then $2\lambda_j < \rho < n^{j+1-\tau}$.

$$\begin{aligned} t - \frac{\gamma_j \cdot t^{\gamma_j}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}} &= \rho \cdot n^\tau - \frac{\gamma_j \cdot \rho^{\gamma_j} \cdot n^{\tau\cdot\gamma_j}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}} \\ &= \rho \cdot n^\tau \left(1 - (\gamma_j/\lambda_j) \cdot \rho^{\frac{\tau}{j+1-\tau}} / n^\tau\right) \\ &> \rho \cdot n^\tau (1 - \gamma_j/\lambda_j) \\ &> \lambda_j \cdot n^\tau \quad \text{if } \lambda_j > 2\gamma_j. \end{aligned}$$

So we can apply the inductive assumption and get

$$\begin{aligned} F(j,t,\mathcal{K}) &\leq F(j,t-1,\mathcal{K}) + F(j, t - \frac{\gamma_j \cdot t^{\gamma_j}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}}, \mathcal{K}) \\ &< C^{n^\tau} \cdot f(j,t-1) + C^{n^\tau} \cdot f(j, t - \frac{\gamma_j \cdot t^{\gamma_j}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}}) \\ &< C^{n^\tau} \cdot f(j,t) \quad \text{by property (P.5) of } f(j,t). \end{aligned}$$

We note that property (P.5) applies only when ρ is larger than the maximum of $(2\lambda'_j)^{1/\gamma_j}$ and $e^{\frac{1}{\gamma_j - 1}} + \frac{1}{n^\tau}$. ρ can be coerced to be always larger than this maximum by choosing a sufficently large λ_j since $\rho > 2\lambda_j$.

To wrap up the proof, we simply choose λ_j large enough so that it satisfies the requirements of the above proof and so that the properties of $f(j,t)$ discussed in the appendix hold. □

5 Concluding Remarks

We have derived a tight upper bound on $\log \kappa_d(n)$ based on a conjecture that any simplicial complex has at most $O(n^{\lceil d/2 \rceil})$ simplices embeddable in $\Re^d$ without crossing. A natural question that arises is whether these bounds extend to topological simplicial complexes. Let Δ_{n-1} denote a geometric $(n-1)$-simplex and let $\mathcal{L}$ be the collection of all j-faces of Δ_{n-1}, $0 \leq j < d+1$. Let $\mathcal{L}'$ be a subcomplex of $\mathcal{L}$ and let $g : \left(\bigcup_{\sigma \in \mathcal{L}'} \sigma\right) \rightarrow \Re^d$ be an embedding. Then $\mathcal{K} = \{g(\sigma) \mid \sigma \in \mathcal{L}'\}$ is a *topological simplicial complex* in $\Re^d$. The vertex set of $\mathcal{K}$ is $g(\mathcal{L}'^{(0)})$. Although we assumed a linear embedding of simplices, our result is valid for any fixed map $g' : \left(\bigcup_{\sigma \in \mathcal{L}} \sigma\right) \rightarrow \Re^d$ such that g' restricted to each $\sigma \in \mathcal{L}$ is an embedding. However, our counting method fails when several such maps are considered. Hence, the result does not immediately extend to topological simplicial complexes since it is possible to embed the simplices of $\mathcal{L}$ in $\Re^d$ in more than one way.

Related to determining the number of geometric simplicial complexes is the question of determining the number of geometric triangulations, $r_d(n)$, on n vertices in $\Re^d$. Clearly, the upper bound on $\log \kappa_d(n)$ holds for $\log r_d(n)$, see also [3]. As mentioned in section 1, the lower bound on $\log r_d(n)$ is $\Omega(n)$. Reducing the huge gap between the upper and lower bounds on $\log r_d(n)$ remains a challenge till date.

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Appendix

Properties of $f(j,t) = \left(\frac{t}{n^\tau}\right)^{-\lambda_j \cdot \frac{n^{\tau\cdot\gamma_j}}{t^{\gamma_j - 1}}}$. (We shall assume that $t \leq n^{j+1}$.)

P.1. $f(j,t) \leq 1$ for $n^\tau \leq t \leq \binom{n}{j+1}$.

This is easy to see since $\frac{t}{n^\tau} \geq 1$, and the exponent is negative.

P.2. $f'(j,t) = \frac{\partial}{\partial t} f(j,t) > \lambda_j \cdot \frac{n^{\tau\cdot\gamma_j}}{t^{\gamma_j}} \cdot f(j,t)$ if $t > e^{\frac{1}{\gamma_j - 1}} \cdot n^\tau$.

Again, this is straightforward since $f'(j,t) = f(j,t) \cdot \lambda_j \cdot \frac{n^{\tau\cdot\gamma_j}}{t^{\gamma_j}} \cdot \left((\gamma_j - 1) \cdot \ln\left(\frac{t}{n^\tau}\right) - 1\right)$. This implies that $f(j,t)$ is a monotonically increasing function of t when $t > e^{\frac{1}{\gamma_j-1}} \cdot n^\tau$.

P.3. $f(j,t-1) < f(j,t) \cdot \frac{t^{\gamma_j}}{t^{\gamma_j} + \lambda_j \cdot n^{\tau\cdot\gamma_j}}$ if $t > e^{\frac{1}{\gamma_j-1}} \cdot n^\tau + 1$.

By the mean value theorem, $f(j,t) - f(j,t-1) = f'(j,t')$ for some $t - 1 \leq t' \leq t$. By property (P.2), $f(j,t) - f(j,t-1) > \lambda_j \cdot \frac{n^{\tau\cdot\gamma_j}}{t^{\gamma_j}} \cdot f(j,t-1)$. (P.3) follows.

P.4. $f\left(j, t - \frac{\gamma_j \cdot t^{\gamma_j}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}}\right) \leq \lambda'_j \cdot \frac{n^{\tau\cdot\gamma_j}}{t^{\gamma_j}} \cdot f(j,t)$ where $\lambda'_j = (4^{\gamma_j})^{2^\tau}$ is a constant, provided $\lambda_j > 2\gamma_j$ and $j < d$.

Let $\mu = \frac{\gamma_j \cdot t^{\gamma_j - 1}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}}$. Since $\frac{t^{\gamma_j-1}}{n^{\tau\cdot\gamma_j}} = \left(\frac{t}{n^{j+1}}\right)^{\frac{\tau}{j+1-\tau}} \leq 1$ and $\lambda_j > 2\gamma_j$, we have $0 < \mu < \frac{1}{2}$. As a result, $1 + \mu \leq \frac{1}{(1-\mu)^{\gamma_j - 1}}$. This is easy to see when $\gamma_j \geq 2$ since we have $\frac{1}{(1-\mu)^{\gamma_j-2}} \geq 1 \geq 1 - \mu^2$. The only case when $\gamma_j < 2$ is when the dimension d is even and $j = d$. But this is precluded since $j < d$.

Observe that $f(j,t) = \left(\frac{t}{n^\tau}\right)^{\frac{-\gamma_j}{\mu}}$. Also, $f(j,t(1-\mu)) = f\left(j, t - \frac{\gamma_j \cdot t^{\gamma_j}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}}\right) = a(j,t,\mu) \cdot b(j,t,\mu)$, where $a(j,t,\mu) = \left(\frac{t}{n^\tau}\right)^{\frac{-\gamma_j}{\mu\cdot(1-\mu)^{\gamma_j-1}}}$, and $b(j,t,\mu) = \left((1-\mu)^{\frac{-1}{\mu}}\right)^{\frac{\gamma_j}{(1-\mu)^{\gamma_j-1}}}$. Now,

$$\begin{aligned} a(j,t,\mu) &= (t/n^\tau)^{\frac{-\gamma_j}{\mu\cdot(1-\mu)^{\gamma_j-1}}} == f(j,t)^{\frac{1}{(1-\mu)^{\gamma_j-1}}} \\ &\leq f(j,t)^{1+\mu} \quad \text{since } 1+\mu \leq \frac{1}{(1-\mu)^{\gamma_j-1}} \\ &\qquad \text{and } f(j,t) \leq 1 \\ &= f(j,t) \cdot n^{\tau\cdot\gamma_j}/t^{\gamma_j}. \end{aligned}$$

We claim that $g(\mu) = (1-\mu)^{\frac{-1}{\mu}} \leq 4$ for $0 < \mu < \frac{1}{2}$. To see this, note that by Taylor series expansion $\ln g(\mu) = \sum_{i\geq 0} \frac{\mu^i}{i+1}$, which is an increasing function of $\mu > 0$. Since e^x is an increasing function of x, it follows that $g(\mu) \leq g(\frac{1}{2}) = 4$ for $0 < \mu < \frac{1}{2}$. Now,

$$\begin{aligned} b(j,t,\mu) &= \left((1-\mu)^{\frac{-1}{\mu}}\right)^{\frac{\gamma_j}{(1-\mu)^{\gamma_j-1}}} \\ &\leq 4^{\frac{\gamma_j}{(1-\mu)^{\gamma_j-1}}} \quad \text{since } 4 \geq (1-\mu)^{\frac{-1}{\mu}} \\ &< (4^{\gamma_j})^{2^\tau} \quad \text{since } 0 < \mu < 1/2 \text{ and } \gamma_j - 1 \leq \tau. \end{aligned}$$

P.5. $f(j,t-1) + f\left(j, t - \frac{\gamma_j \cdot t^{\gamma_j}}{\lambda_j \cdot n^{\tau\cdot\gamma_j}}\right) < f(j,t)$ when $j < d$, $t > \max\{e^{\frac{1}{\gamma_j-1}} \cdot n^\tau + 1, (2\lambda'_j)^{1/\gamma_j} \cdot n^\tau\}$ and $\lambda_j > 2\lambda'_j$, where λ'_j is the constant in (P.4).

Because of properties (P.3) and (P.4), it suffices to show that

$$\frac{t^{\gamma_j}}{t^{\gamma_j} + \lambda_j \cdot n^{\tau\cdot\gamma_j}} + \lambda'_j \cdot \frac{n^{\tau\cdot\gamma_j}}{t^{\gamma_j}} \leq 1,$$

which is equivalent to showing that

$$\left(\frac{t}{n^\tau}\right) \geq \left(\frac{\lambda_j \cdot \lambda'_j}{\lambda_j - \lambda'_j}\right)^{1/\gamma_j}.$$

The inequality holds because the term on the right is bounded by $(2\lambda'_j)^{1/\gamma_j}$.