Lower bounds for the number of crossing-free subgraphs of K_n

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Abstract

We obtain lower bounds for the maximum number of triangulations, perfect matchings, spanning trees and polygonizations of a set of n points in the plane. Our results improve previously known values. It is also proved that the number of perfect matchings and spanning trees is minimum when the points are in convex position.

1 Introduction and preliminaries

Depending on how the complete graph K_n is drawn in the plane by means of points and straight line segments connecting them, some of its subgraphs will have crossings, i.e. at least one pair of open line segments will intersect, or else will be crossing-free. The problem of bounding the number of crossing-free subgraphs of K_n has been studied in the last years, particularly the case of crossing-free Hamiltonian cycles, which correspond to simple polygonizations of a set of n points. Newborn and Moser [12] introduced the problem and found a configuration with $\Omega(10^{n/3})$ simple polygonizations. By introducing other configurations or analyzing more thoroughly existing ones, this lower bound has been successively improved over the last years [2, 8, 5].

On the other hand it seems considerably difficult to obtain sharp upper bounds for this problem. In [1] a fundamental result was proven, namely that the number of crossing-free subgraphs of any plane drawing of K_n (even if one allows non rectilinear edges) never exceeds a fixed exponential in n. An upper bound of $173\,000^n$ on the number of triangulations can be found in [14].

In this paper we introduce a particular configuration of n points giving $\Omega(4.642^n)$ different polygonizations, thus improving previous results. The analysis is based on generating functions and an application of Darboux's lemma. The same configuration provides a large number of crossing-free subgraphs of several kinds. In particular, we prove that the number of triangulations is $\Omega(8^n n^{O(1)})$, the number of Euclidean perfect matchings is $\Omega(3^n n^{O(1)})$, and the number of Euclidean spanning trees is $\Omega(9.35^n)$. These results improve those appearing in [14].

The basic configuration C_n we will analyze is depicted in Figure 1a. It consists on $2n$ points, $p_1, \ldots, p_n$ on the upper chain L_1 and $q_1, \ldots, q_n$ on the lower chain L_2. Both chains are convex with opposed concavity; for every i and j the line connecting p_i and p_j leaves all points of L_2 below, and the line connecting q_i and q_j leaves all points of L_1 above. The numbering of the

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points in both chains is from left to right. When it comes to compute bounds for sets of n points we will just use $C_{n/2}$.

For a configuration to be useful in this setting, the number of crossings has be to be relatively small, it is $2\binom{n/2}{4} + \binom{n/2}{2} \sim n^4/48$ for $C_{n/2}$, and it is desirable to have many symmetries in order to simplify the analysis.

We need as prerequisites several results from enumerative combinatorics. First, the classical result that the number of triangulations of a convex polygon with $n+2$ vertices is the Catalan number $C_n = \binom{2n}{n}/(n+1) = \Theta(n^{-3/2}4^n)$. Also that the number of Euclidean perfect matchings of $2n$ points in convex position (classically referred to as non-crossing configurations of chords on a circle) is again the Catalan number C_n. And finally that the number of Euclidean spanning trees of $n+1$ points in convex position is equal to $\binom{3n}{n}/(2n+1) = \Theta(n^{-3/2}(27/4)^n)$, a generalized Catalan number. The first result goes back to Euler; for the second one see [11, 4]; the third one can be found in [4] (see also [13]).

From now on a *matching* will be an Euclidean perfect matching, a *tree* will be an Euclidean spanning tree, and polygonizations will always be simple. Points will always be assumed to be in general position, in our case no three of them collinear.

2 Triangulations, matchings and trees

In [14] one can find a set of n points admitting in the order of 6.75^n triangulations (this is a correction, using the results in [9], of the claimed value of 9.08^n). Also in [14] sets of n points are shown with 2.618^n matchings and 7.10^n trees. The configuration C_n allows us to increase these values.

Theorem 2.1 *There are sets of n points with 1) $\Omega(8^n n^{O(1)})$ triangulations; 2) $\Omega(3^n n^{O(1)})$ matchings; and 3) $\Omega(9.35^n)$ trees.*

Proof. In all three cases we will compute either the exact value or an asymptotic lower bound for the respective number of triangulations, matchings and trees in C_n. Substituting n by $n/2$ when necessary will yield the desired bounds.

1) Any triangulation of C_n has to use necessarily the diagonals $p_1p_2, p_2p_3, \ldots, p_np_1, q_1q_2, q_2q_3, \ldots, q_nq_1$ and p_1q_1, p_nq_n. Hence we have a decomposition of the convex hull into two convex n-gons and one non-convex $2n$-gon P. For triangulating P we start at edge p_1q_1; it can joined to either p_2, the next point in L_1, or to q_2, the next point in L_2, and once the choice has been made we are confronted with a new choice between L_1 and L_2. In the end there have been $n-1$ selections from L_1 and $n-1$ from L_2. Thus the number of triangulations in C_n is equal to $C_{n-2}^2\binom{2n-2}{n-1} = \Theta(64^n n^{-7/2})$.

2) In any matching of C_n there will be k points of L_1 matched with k points of L_2, with $n-k$ even and $0 \le k \le n$. The unmatched points will form two convex sets of size $n-k$. Thus the number of matchings is

$$\sum_{\substack{k=0 \\ n-k \text{ even}}}^{n} \binom{n}{k}^2 C^2_{(n-k)/2}.$$

This is a sum of positive unimodal terms which could be accurately estimated by standard methods [3]. However it is enough for our purposes to note that the larger term occurs when $k = n/3$ and that Stirling's estimate gives $\binom{n}{n/3}^2 C^2_{n/3} = \Theta(9^n n^{-4})$.

3) Consider the configuration C_{um}, where $n = 2um$ and u is to be determined. Take any tree on L_1 and any matching of size m between L_1 and L_2. Now take any tree on the $um - m$

unmatched points in L_2. Finally add an extra edge in L_2 to produce a tree in the whole configuration. In this way we get

$$t_{um}t_{um-m}\binom{um}{m}^2 \sim \left[(27/4)^{2u-1}\left(\frac{u^u}{(u-1)^{u-1}}\right)^2\right]^{n/2u}$$

different trees, where $t_{n+1} = \binom{3n}{n}/(2n+1)$ is the number of trees for n points in convex position as mentioned above. Setting $g(u) = (27/4)^{(2u-1)/2u}u(u-1)^{(1-u)/u}$, we aim at maximizing $g(u)$. Elementary calculus shows that the maximum is achieved at $u_0 = 1+3\sqrt{3}/2$. The desired value is then $g(u_0) = 9.35$. □

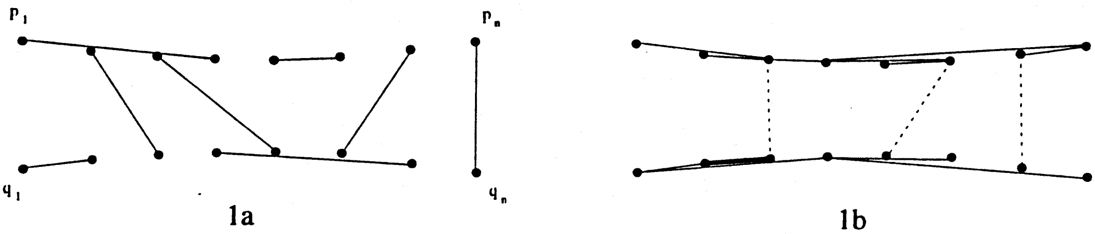

Figure 1: Matchings and trees in C_n.

We close this Section by proving an absolute lower bound on the number of matchings and trees of any configuration (note that the result is trivial for the number of polygonizations).

Theorem 2.2 *The number of matchings of a set n points in the plane is minimum when the points are in convex position. The same result holds for the number of trees.*

Proof. We need the well-known fact that the number C_n of matchings of $2n$ points in convex position, a Catalan number, satisfies the recurrence $C_n = C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0$. Now let P be a set of $2n$ points, a_1 a point in the convex hull of P and $a_2, \ldots, a_{2n}$ an ordering of the remaining points in polar order with respect to a_1. If in a matching of P point a_1 is joined to a_{2i} then, by induction on n, the remaining $2n-2$ points can be matched in at least $C_{i-1}C_{n-i}$ ways. Hence the number of matchings is at least $C_0C_{n-1} + \ldots + C_{n-1}C_0 = C_n$. The proof for the number of trees is analogous and uses the recurrence $t_n = \sum_{i+j+k=n+1} t_it_jt_k$ for the number t_n of trees of n points in convex position (see [13]). □

3 Polygonizations

Let C be a simple curve in C_n starting at p_1 and ending at q_n, and let k be the number of jumps on C from L_1 to L_2 and from L_2 to L_1. Let also $\{p_{i_1}, \ldots, p_{i_k}\}$ and $\{q_{j_1}, \ldots, q_{j_k}\}$ be the extremes of the jumps, with $i_1 \leq i_2 \leq \ldots \leq i_k$ and $j_1 \leq j_2 \leq \ldots \leq j_k$. Among all curves starting at p_1 and ending at q_n, we consider those in which the points $\{p_{i_1}, \ldots, p_{i_k}\}$ ($\{q_{j_1}, \ldots, q_{j_k}\}$) are visited in exactly this order in the curve (see Figure 2a). We will denote by S the family of such curves.

Obviously, one can close these curves by adding an adequate extra point on L_2. Therefore, asymptotically the number of polygonizations of the $2n+1$ points will be no less than the number of curves in S. If we take $C \in S$ and consider how C visits the n points on L_1 and L_2, we have the curves shown in Figure 2b. Notice that none of the jumps in the figure can be enveloped by another jump because $\{p_{i_1}, \ldots, p_{i_k}\}$ $(\{q_{j_1}, \ldots, q_{j_k}\})$ are visited in exactly this order. Now, let us consider the four types of curves shown in Figure 3.

Type 1: the curve starts and ends at two points not in L_2.
Type 2: the curve starts at a point not in L_2 and ends at q_1.
Type 3: the curve starts at a point not in L_2 and ends at q_n.
Type 4: as type 1, but q_1 and q_n must be directly joined, a point q_i is visited before q_1, and a point q_j is visited after q_n.

Notice that, for example, two curves of type 2 can visit the points $\{q_1, \ldots, q_n\}$ in the same order but they are considered different if they have different jumps. We are ready for the following result.

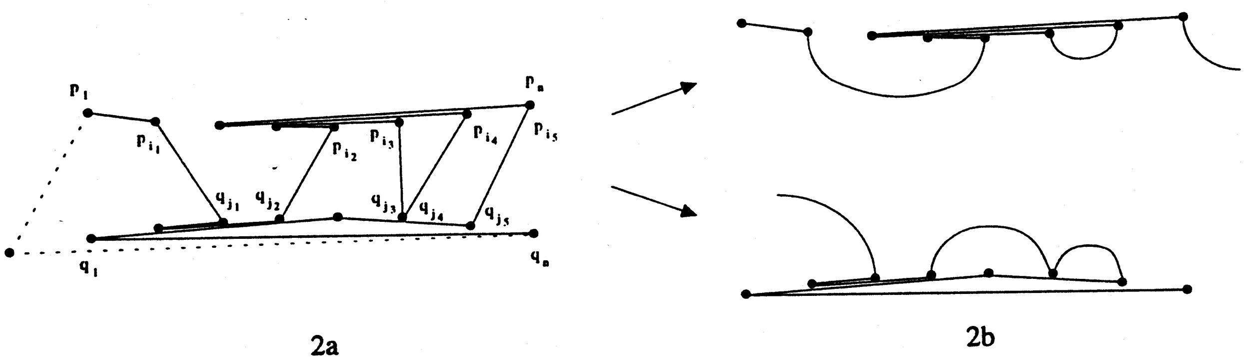

Figure 2: Polygonizations in C_n.

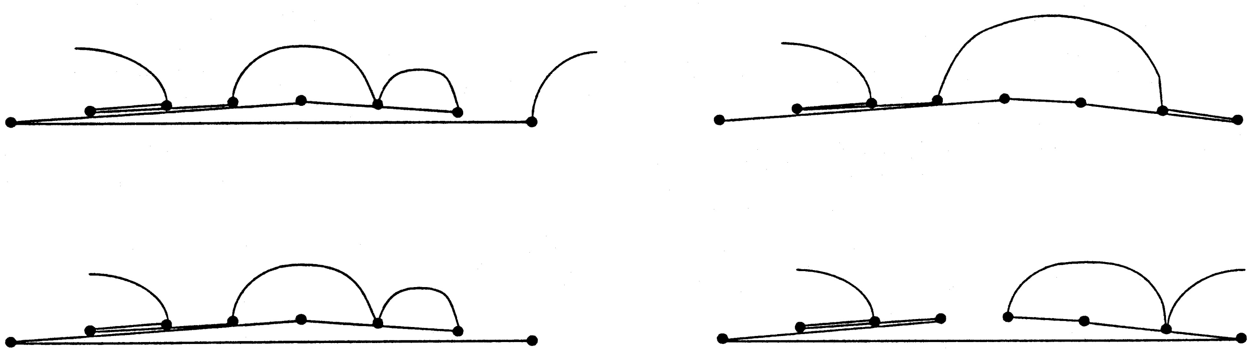

Figure 3: Four types of curves.

Theorem 3.1 *There are sets of n points with $\Omega(4.64^n)$ polygonizations.*

Proof. We will denote by $g_1(n)$, $g_2(n), g_3(n), g_4(n)$ the number of curves of type 1, 2, 3, 4, respectively. Then, recurrence formulas for $g_i(n)$ (see [6, 5]) can be obtained because every curve of any of the four types is formed by shorter curves of the same types. They are

$$\begin{aligned}
g_1(n) &= g_3(n) + g_4(n) + \textstyle\sum_{i=2}^{n-1} g_2(i)g_3(n-i) + \sum_{i=4}^{n-1} g_4(i)g_1(n-i), \quad n \geq 1;\\
g_2(n) &= g_2(n-1) + g_3(n-1) + \textstyle\sum_{i=1}^{n-3} g_1(i)g_2(n-i-1), \quad n \geq 3;\\
g_3(n) &= g_1(n-1) + g_2(n-1) + g_3(n-1) + \textstyle\sum_{i=1}^{n-3} g_1(i)g_2(n-i-1), \quad n \geq 3;\\
g_4(n) &= \textstyle\sum_{i=2}^{n-2} g_2(i)g_2(n-i), \quad n \geq 4,
\end{aligned}$$

with suitable initial conditions.

Let $G_i(z) = \sum_{n \geq 0} g_i(n)z^n$ be the generating function of $g_i(n)$, for $1 \leq i \leq 4$. From the equations above we obtain:

$$\begin{aligned}
G_1(z)G_2(z) &= (1/z - 2)G_2(z) + z - 1;\\
G_4(z) &= (G_2(z) - z)^2;\\
G_1(z)G_4(z) &= (1 - z + z^2)G_1(z) + (z-2)G_2(z) - 2G_4(z) + z,
\end{aligned}$$

and from these

$$G_2(z)^3 - zG_2(z)^2 + (-1 + 3z - z^2)G_2(z) + z(z-1)^2 = 0.$$

The singularities of the solutions to this equation must be zeros of the discriminant $\Delta(z) = \frac{1}{108}\left(32z^5 - 117z^4 + 168z^3 - 112z^2 + 36z - 4\right)$ (see [10]). Solving $\Delta(z) = 0$ we obtain the real root $z_1 = 0.2154185247$ as the closest to zero. Therefore, by Darboux's theorem [7]

$$g_2(n) = c_2 n^{-(\nu+1)/\nu}(1/z_1)^n + o\left((1/z_1)^n n^{-(\nu+1)/\nu}\right),$$

where ν is equal to 2 or 3, and similarly for $g_3(n)$.

On the other hand, if $g_3(n,k)$ denotes the number of curves of type 3 with k jumps then $g_3(n) = \sum_k g_3(n,k)$. If $g(n)$ is the number of curves in S with $n/2$ points on L_1 and $n/2$ points on L_2, we have $g(n) = \sum_k g_3(n/2,k)^2$. Using the inequalities

$$g_3(n)^2 > g(2n) > \frac{g_3(n)^2}{n}$$

then $\lim_{n\to\infty} g(n)^{1/n} = 1/z_1 = 4.642126305.$ □

4 Concluding remarks

We have analyzed a particular configuration of points in the plane with a number of crossing-free subgraphs of several kinds which improves previous results. Whereas for triangulations and matchings our formulas are tight, a deeper analysis might eventually show the existence of a larger number of trees and polygonizations. We can prove however an upper bound of $O(5.61^n)$ on the number of polygonizations of $\mathcal{C}_{n/2}$.

If we consider the related problem of maximizing the number of (Euclidean) bipartite matchings among a set of $n/2$ red points and $n/2$ blue points, one can prove a lower bound of $\Omega(5^{n/2}n^{O(1)})$ again using the configuration $\mathcal{C}_n$ with alternating colours in both chains.

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