Watchman Routes in the Presence of a Pair of Convex Polygons
(EXTENDED ABSTRACT)

Laxmi P Gewali
Department of Computer Science
University of Nevada, Las Vegas
Las Vegas, NEVADA 89154

Simeon Ntafos
Computer Science Program
University of Texas at Dallas
Richardson, TEXAS 75080

ABSTRACT: Given a set of polygonal obstacles in the plane, the shortest watchman route problem asks for a route from which each point in the exterior of the polygons is visible to some point along the route. We present an $O(n^2)$ time algorithm for computing such a route for a pair of convex polygons, where n is the total number of edges in the polygons.

1. Introduction

Computing collision-free paths in the presence of obstacles is a well known problem in computational geometry having applications in robotics and computer graphics. Visibility properties and shortest path properties are related to each other. Visibility graphs have been used to compute shortest paths in the presence of obstacles [LW79], and all pair shortest paths inside a simple polygon have been used to compute visibility graphs. Collision-free paths having visibility properties have also been considered [CN91, CE*91, MW91]. The watchman route problem asks for a route from a point s to itself such that each point in the given space is visible to some point along the route [CN91]. An optimum watchman route is a watchman route of minimum length. The optimum watchman route problem is known to be NP-hard for polygons with holes and can be solved in linear time for simple orthogonal polygons [CN88]. An $O(n^4)$ algorithm for computing the optimum watchman route for simple polygons is reported in [CN91]. The watchman route problem in the exterior of a single polygon has been explored. When the polygon is monotone, convex, or rectilinear a linear time algorithm for computing the optimum external watchman route is given in [NG90] which also contains an $O(n^4)$ algorithm for the case of simple polygon. Recently, an approximation algorithm for computing a watchman route inside an orthogonal polygon with holes is reported in [MM95]; this algorithm computes a watchman route whose length is within $O(\log n)$ of the minimum possible length and the time complexity of the algorithm is $O(n^9)$.

In this paper we consider the problem of computing the shortest watchman route in the presence of a pair of convex polygons. This problem was first considered in [GL93] which contains an $O(n^3)$ time algorithm to obtain the solution. We present an improved algorithm requiring $O(n^2)$ time.

2. Preliminaries

Consider convex polygons P and Q whose vertices in clockwise order are $p_1, p_2, ..., p_k$ and $q_1, q_2, ..., q_r$, respectively. Let $n = k + r$. A **watchman route** for P and Q is a closed path lying outside of P and Q and such that each point in the exterior of P and Q is visible to some point along the route. The optimum watchman route is a watchman route of shortest length. A **commpon supporting segment** of P and Q is a segment that is tangent to both of them. There are exactly four common supporting segments for a pair of convex polygons, two of which are bridge segments and the other two are cross segments. Consider three consecutive vertices q_{i-1}, q_i, and q_{i+1} of Q. The half-lines obtained by extending edges $\overline{q_i, q_{i-1}}$ and $\overline{q_i, q_{i+1}}$ (with end points

q_{i+1} and q_{i-1}) are called the **legs** of q_i. The vertices of P and Q can be distinguished into four categories: A vertex q_i of Q is a **type 1** (respectively, **type 2**) vertex if polygon P is completely inside (respectively, outside) the cone formed by the legs of q_i. If only one leg (respectively, both legs) intersect P then it is a **type 3** (respectively, **type 4**) vertex (Figure 1).

We distinguish four kinds of watchman routes (Figure 2): Watchman routes that wrap polygon P (respectively, Q) are **P-wrapping watchman routes** (respectively, **Q-wrapping watchman routes**). Similarly, watchman routes that wrap both P and Q (respectively, neither P nor Q) are **P-Q-Wrapping watchman routes** (respectively, **Non-wrapping watchman routes**). In [NG91] it is shown that the optimum watchman route for a single polygon must touch its boundary. This property does not hold for a pair of polygons. The shortest P-Q-wrapping route is given by the convex hull of $P \cup Q$. Given P and Q, the convex hull of P and Q can be computed in $O(n)$ time by using the merge hull technique[PS85]. Due to space limitation proofs and other details are omitted from this extended abstract.

Lemma 1 *The shortest P-Q-wrapping watchman route can be computed in $O(n)$ time.*

In the rest of the paper we show that the shortest P-wrapping and the shortest non-wrapping watchman routes can be computed in $O(n)$ and $O(n^2)$ time, respectively. This results in the following theorem.

Theorem 1 *The shortest watchman route for a pair of convex polygons can be computed in $O(n^2)$ time.*

3. P-Wrapping Watchman Routes

Lemma 2 *The shortest P-Wrapping watchman route must touch the boundary of P at its right most vertex p_r [GL93].*

Consider the wedge formed by the legs of vertex q_i. The area obtained by removing the polygon Q from the wedge is the **trimmed wedge** of q_i (Figure 3). A route that connects the two legs of a trimmed wedge internally is said to **span** that wedge.

Observation 1: Any route that encircles P and spans the trimmed wedge of a vertex of Q is a watchman route. If such a route spans the trimmed wedge of q_i then it is a **P-wrapping route with respect to q_i**.

A P-wrapping watchman route with respect to q_i can span the trimmed wedge of q_i in four distinct ways as shown in Figure 4. This results in three types of P-wrapping routes: (i) a **0-leg** route (with respect to q_i) extends to the outside of the trimmed wedge of q_i from both legs, (ii) a **1-leg** route extends to the outside from only one leg, and (iii) a **2-leg** route does not extend to the outside.

Observation 2: The shortest 0-leg P-wrapping watchman route is trivially given by the boundary of P.

3.1 1-Leg P-Wrapping Watchman Routes

Observe that the shortest P-wrapping watchman route with respect to a type 2 or a type 3 vertex must be a 1-leg route. Intuitively, a 1-leg P-wrapping watchman route can be obtained by stretching the boundary route of P to meet the leg/legs of q_i not intersected by P. The shortest such route with respect to a given vertex q_i can be found by constructing the images of P and Q and computing the shortest path from a carefully selected object point to its image point.

Lemma 3 *The shortest 1-leg P-wrapping watchman route with respect to a given type 2 or type 3 vertex can be computed in $O(n)$ time.*

To find the shortest 1-leg P-wrapping route, we could apply Lemma 3 repeatedly to compute 1-leg watchman routes with respect to each type 1 and type 2 vertices of Q and report the one having the least length. This clearly takes $O(n^2)$ time. The following lemma establishes that the time complexity can be reduced to $O(n)$.

Lemma 4 *The shortest 1-leg P-wrapping watchman route can be computed in $O(n)$ time.*

3.2 2-Leg P-Wrapping Watchman Routes

The shortest P-wrapping watchman route with respect to a type 1 vertex of Q must be a 2-leg route and hence it reflects from both legs. Such a route can be determined by constructing images of P and Q by treating both legs as mirrors and finding the shortest path between the rightmost vertex of P and its second image. In general, the shortest 2-leg P-wrapping watchman route with respect to a type 1 vertex reflects from the legs by performing at most four take-offs and at most four landings on P and Q (shown by dark dots in Figure 5). We call these take-off and landing points the **contact points** of the route with the polygons.

Lemma 5 *The shortest 2-leg P-wrapping watchman route with respect to a type 1 vertex can be computed in $O(n)$ time.*

To compute the shortest 2-leg P-wrapping watchman route naively, we could apply Lemma 5 repeatedly to all type 1 vertices of Q and select the one having the minimum length, and this would require $O(n^2)$ time. By a careful analysis of the change of contact points due to the change of leg vertices, shortest 2-leg P-wrapping watchman routes can be computed in optimum time. A 2-leg P-wrapping watchman route with respect to a given type 1 vertex can be determined by knowing the contact points of the route with P and Q. If we advance the leg vertex clockwise from q_i to q_{i+1} then the contact points may advance. *The key observation here is that if the leg vertex moves in the clockwise direction (from q_i to q_{i+1}) and results in the movement of the contact points then the contact points also move in the clockwise direction.* We need to efficiently compute new contact points as legs are advanced. Note that after a few advances some contact points may actually disappear. The detail is omited and we state the result in the following lemma.

Lemma 6 *The shortest 2-leg P-wrapping watchman route can be computed in $O(n)$ time.*

4. Non-Wrapping Watchman Routes

A route that spans the trimmed wedge of p_i and q_j without wrapping P or Q is a non-wrapping watchman route. The shortest such route that spans the trimmed wedge of p_i and q_j is denoted by $U(p_i, q_j)$. The legs through which $U(p_i, q_j)$ reflects depends on the relative orientations of their trimmed wedges. The relative orientations of trimmed wedges can be classified into eighteen distinct cases.

Lemma 7 *$U(p_i, q_j)$ can be computed in $O(n)$ time.*

To determine the shortest non-wrapping watchman route we could compute $U(p,q)$'s for all combinations of p and q and select the shortest one, and this would require $O(n^3)$ time. By using the idea of advancing supporting segments (Lemma 6) the total computation time can be reduced to $O(n^2)$.

Lemma 8 *The shortest non-wrapping watchman route can be computed in $O(n^2)$ time.*

5. Conclusion

Computing the shortest watchman routes in the presence of polygonal obstacles is intractable [CN88]]. Recently, an approximation algorithm of time complexity $O(n^9)$ for computing the shortest watchman routes when the polygons are restricted to be orthogonal has been reported [MM95]. This clearly highlights the need for faster exact/approximation algorithms. We presented an $O(n^2)$ time algorithm to compute the shortest watchman route in the presence of a pair of convex polygons. The bottle-neck of the algorithm is the computation of non-wrapping routes and faster ways to compute them will also reduce the overall complexity of the algorithm. It would be interesting to extend our approach to compute the shortest watchman route in the presence of K convex polygons, where K is a fixed constant.

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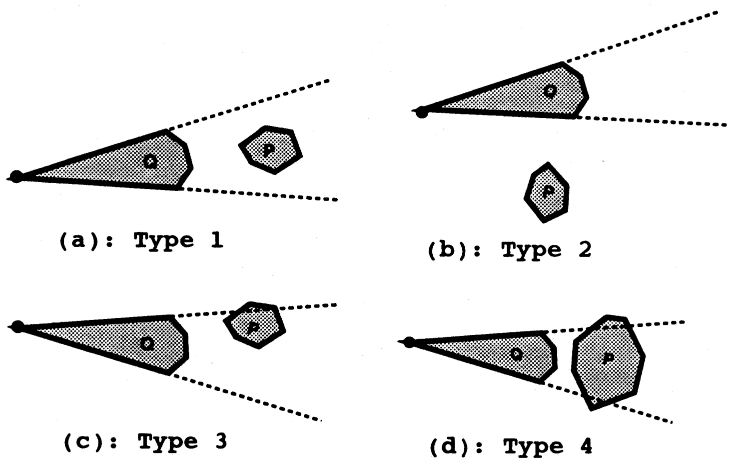

Figure 1: Four types of vertices

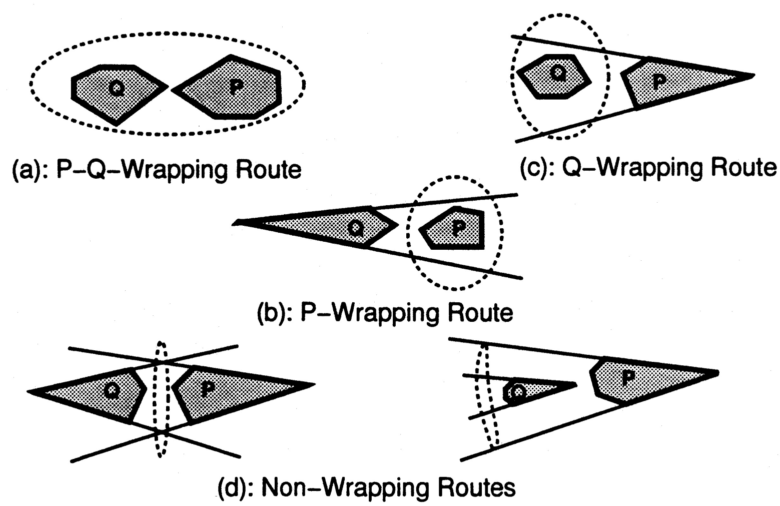

Figure 2: Four Kinds of Watchman Routes

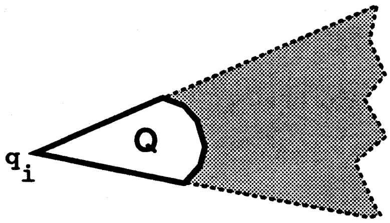

Figure 3: Illustrating a Trimmed Wedge

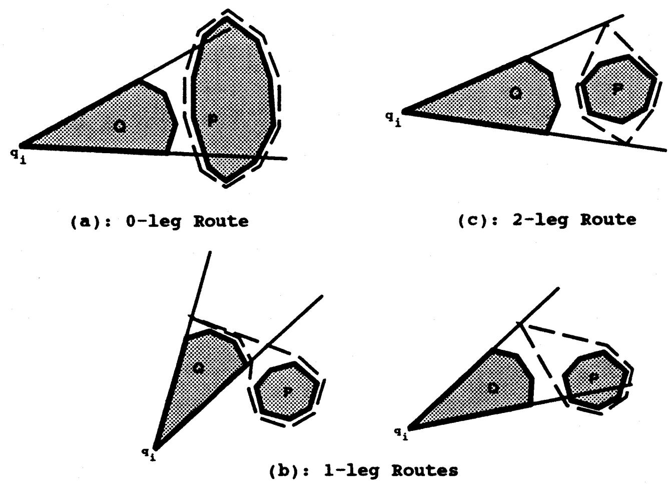

Figure 4: P–Wrapping Watchman Routes

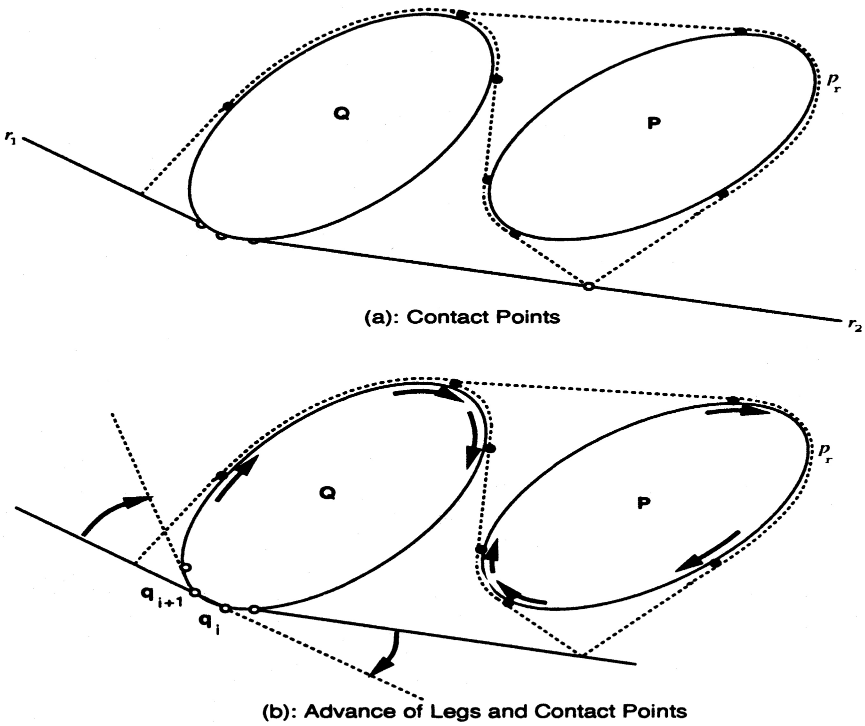

Figure 5