Visibility Graphs of Polygonal Rings*

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Abstract

In this paper we show that the visibility graph of a *polygonal ring* (i.e., a convex polygon with one convex hole inside it) is a circular-arc graph by giving an algorithm that explicitly constructs a circular-arc model of the graph. The running time of the algorithm is proportional to the number of vertices in the polygonal ring. Therefore the result implies that, given a polygonal ring as input, several optimization problems, such as CLIQUE, DOMINATING SET, and INDEPENDENT SET, which are NP-hard for general visibility graphs, are linear-time solvable on the visibility graphs of polygonal rings.

Keywords: circular-arc graph, linear algorithm, polygonal ring, visibility graph

1 Introduction

The *visibility graph* of a polygon with holes is a graph with a vertex for every polygon and hole vertex and an edge between two vertices in the graph if their corresponding polygon or hole vertices can see each other, that is, if the line segment connecting them does not intersect the exterior of the polygon or the interior of any hole. Note that this definition allows two vertices to see each other even when the line segment connecting them touches the boundaries of the polygon or holes. Little is known about the combinatorial structure of visibility graphs (with or without holes), and neither a polynomial time algorithm nor an NP-hardness proof is known for the recognition of visibility graphs. Nevertheless, polynomial time recognition algorithms do exist for some special classes of polygons; in particular, the visibility graphs of spiral polygons can be recognized in linear time [1, 2] and the visibility graphs of orthogonal staircases can be recognized in cubic time (see [6]). We refer the reader to a survey by O'Rourke [6] for more information about visibility graphs.

In this paper we study the visibility graph of a *polygonal ring* — a convex polygon with one convex hole inside it. We show that the visibility graph of a polygonal ring is a circular-arc graph, that is, it can be represented as the intersection graph of arcs on a circle. Our approach is to present an algorithm that explicitly constructs a circular-arc model of the visibility graph. The algorithm runs in time proportional to the number of vertices in the polygonal ring, and thus implies linear algorithms on such visibility graphs for several optimization problems, such as CLIQUE, DOMINATING SET, and INDEPENDENT SET, which are NP-hard for general visibility graphs.

2 Preliminaries

Throughout the paper, we assume that a polygon P on m vertices is represented by a *boundary chain* $[p_0, p_1, \ldots, p_{m-1}, p_0]$, where each p_i is a vertex of P and the interior of P lies to the right as the boundary chain is traversed in order. We use $P[p_i, p_j]$, subscript arithmetic modulo m, to denote $[p_i, p_{i+1}, \ldots, p_j]$ and call it a *P-chain*. For

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any P-chain $P' = P[p_i, p_j]$, we refer to p_i and p_j as the *head*, denoted $h(P')$, and *tail*, denoted $t(P')$, of P' respectively. A Q-chain is defined in a similar manner. We denote a polygonal ring consisting of a convex polygon $P = [p_0, p_1, \ldots, p_{m-1}, p_0]$ and a convex hole $Q = [q_0, q_1, \ldots, q_{n-1}, q_0]$ by $P \backslash Q$. Note that P and Q do not share any vertex and that the interior of $P \backslash Q$ is the intersection of the interior of P and the exterior of Q (see Figure 1).

It is easy to see that for any vertex p_i of P, p_i together with all the vertices on P that p_i sees form a P-chain which will be denoted by P_i. Similarly, all the vertices on Q that p_i sees form a Q-chain which will be denoted by Q_i. If p_i sees all vertices of P, then we assume that the head $h(P_i)$ and tail $t(P_i)$ of P_i are the first and last vertices, respectively, in the segment of the boundary of P that p_i sees when the segment is traversed clockwisely. Similarly, if p_i sees all vertices of Q, we assume that the head $h(Q_i)$ and tail $t(Q_i)$ of Q_i are the first and last vertices, respectively, in the segment of the boundary of Q that p_i sees when the segment is traversed clockwisely. We denote the vertex set of P by V_P, the vertex set of Q by V_Q, and the vertex set of $P \backslash Q = V_P \cup V_Q$ by $V_{P \backslash Q}$. Given $V' \subseteq V$ we denote by $G[V']$ the subgraph of G induced by V'.

A graph is a *circular-arc graph* if there is a one-to-one correspondence between its vertex set and a family $\mathcal{F}$ of arcs on a circle such that two vertices are adjacent if and only if their corresponding arcs have a nonempty intersection. The family $\mathcal{F}$ is called the *circular-arc model* of the graph. In this paper, we assume that the circumference of the circle has length n. An open arc clockwisely from point x to point y is denoted by (x, y), and its corresponding closed arc is denoted by $[x, y]$. If $\mathcal{F}$ is a circular-arc model for a graph $G = (V, E)$ then $Arc(v)$ denotes the arc in $\mathcal{F}$ corresponding to $v \in V$. If $Arc(v)$ starts at x and ends at y, then $Arc(v).head = x$ and $Arc(v).tail = y$.

The following facts about a polygonal ring $P \backslash Q$ can be easily established.

Fact 2.1 *Any vertex of P sees a P-chain and a Q-chain each with at least two vertices.*

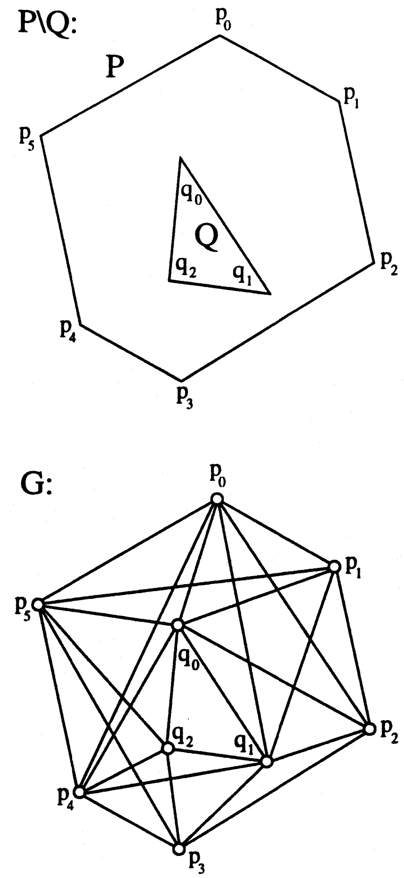

Figure 1: A polygonal ring $P \backslash Q$ and its visibility graph G

Fact 2.2 *Any vertex of Q sees a P-chain consisting possibly of one vertex and a Q-chain with at least three vertices.*

Fact 2.3 *If two vertices of P see each other then there is a vertex of Q which sees both of these two vertices.*

Fact 2.4 *Let p_i and p_j be two vertices of P that do not see each other. Then for the Q-chains Q_i and Q_j seen by p_i and p_j respectively, either they do not intersect, or only intersect at one end, or only intersect at the two ends.*

3 The algorithm

Now we give an algorithm for constructing a circular-arc model of the visibility graph $G = (V, E)$ of $P\backslash Q$. For the moment we assume that Q contains no three collinear vertices; later we will show how to handle this case. The algorithm consists of the following three major steps: (1) first construct a circular-arc model for $G[V_Q]$, (2) then add arcs for vertices of P to the model so that the adjacencies between V_P and V_Q are satisfied, and (3) finally adjust the arcs corresponding to vertices of P to obtain a circular-arc model of G.

Algorithm: Polygonal-Ring

Input: A polygonal ring $P\backslash Q$ with $P = [p_0, p_1, \ldots, p_{m-1}, p_0]$ and $Q = [q_0, q_1, \ldots, q_{n-1}, q_0]$.

Output: A circular-arc model of the visibility graph $G = (V, E)$ of $P\backslash Q$.

Comments: All arithmetic is taken modulo n, where n is the number of vertices in Q.

1. {*Preprocessing.*}

   For each vertex p_i of P, compute the heads and tails of the P-chain P_i and Q-chain Q_i that p_i sees.

2. {*Construct a circular-arc model for* $G[V_Q]$.}

   For each vertex q_j, $0 \leq j \leq n-1$, of Q, set $Arc(q_j) = [j, j+1]$;

3. {*Add arcs for vertices of P to the model to obtain a circular-arc model for* $G - E(G[V_P])$. }

   For each vertex p_i, $0 \leq i \leq m-1$, of P, set $Arc(p_i) = (Arc(h(Q_i)).head + 1/2, Arc(t(Q_i)).head + 1/2)$;

4. {*Adjust the arcs corresponding to vertices of P to obtain a circular-arc model of G. This is done by adjusting the heads and tails of these arcs to meet the adjacency relation amongst vertices of P while maintaining the adjacencies between* V_P *and* V_Q.}

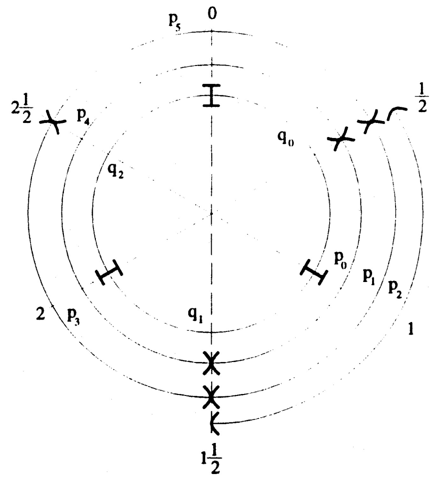

Figure 2: The model after Step 3

4.1 For each j, $0 \leq j \leq n-1$, let H_j be the P-chain formed by

$$\{p : p \in V_P \text{ and } Arc(p).head = j + 1/2\}$$

and let T_j be the P-chain formed by

$$\{p : p \in V_P \text{ and } Arc(p).tail = j + 1/2\}$$

4.2 {*Adjust the starting points of the arcs corresponding to vertices in* V_P.}
For each H_j $(0 \leq j \leq n-1)$ and each vertex p_l in H_j set $Arc(p_l) = [Arc(p_l).head + \epsilon * (\pi_j(p_l) - 1), Arc(p_l).tail)$ where $\pi_j(p_l)$ is the position of p_l in H_j $(1 \leq \pi_j(p_l) \leq |H_j|)$ and ϵ is a small number, say $1/n^2$;

4.3 {*Adjust the end points of the arcs corresponding to vertices in* V_P.}
For each T_j $(0 \leq j \leq n-1)$ and each vertex p_k of T_j, if $t(P_k)$ is in H_j then set $Arc(p_k) = [Arc(p_k).head, Arc(t(H_j)).head]$.

The construction of a circular-arc model for the polygonal ring $P\backslash Q$ in Figure 1 is shown in Figure 2, Figure 3 and Figure 4. The correctness of the algorithm can be proved by using the facts outlined in the previous section, and such a correctness proof will be given in the full paper.

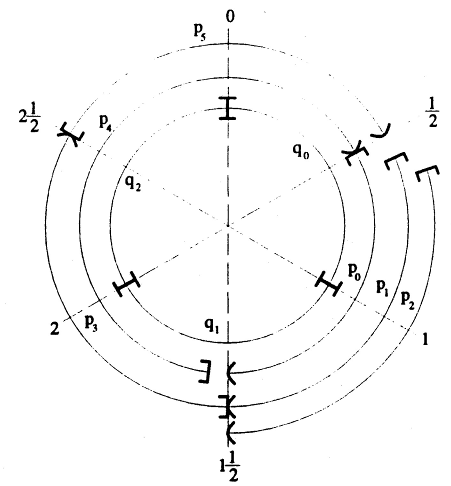

Figure 3: The model after Step 4.2

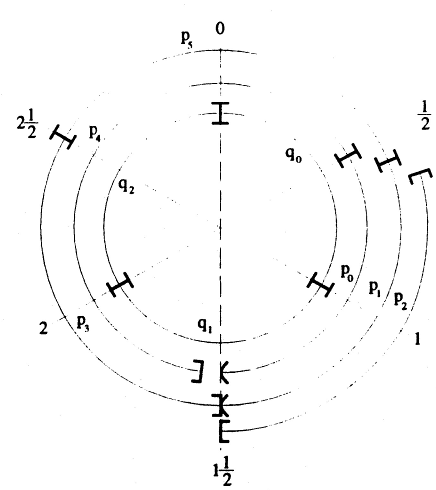

Figure 4: The final model

To handle the case when Q contains sets of three or more collinear points, let $V_Q^* = \{q_0^*, \ldots, q_{n^*-1}^*\} \subseteq V_Q$ such that no three vertices of V_Q^* are collinear and every vertex $q \in V_Q$ lies on a line segment $q_i^* q_{i+1}^*$ for some $0 \leq i \leq n^* - 1$ (subscript arithmetic modulo n^*). Then we modify the algorithm in the following way. Construct arcs for vertices in V_Q^* according to the preceding algorithm. Now, for each $q \in V_Q - V_Q^*$ lying on a segment $q_i^* q_{i+1}^*$ for some $0 \leq i \leq n^* - 1$, add the arc $Arc(q) = [i+1, i+1]$.

It is easy to see that Steps 2-4 of the algorithm can be implemented in linear time. In fact, Step 1 can also be carried out in linear time by scanning the boundary chains carefully. Therefore we can state our main result:

Theorem 3.1 *The visibility graph of a polygonal ring is a circular-arc graph whose circular-arc model can be constructed in time proportional to the number of vertices on the polygonal ring.*

Because certain NP-hard problems on circular-arc graphs, with their corresponding circular-arc models as input, can be solved in time proportional to the number of arcs in the circular-arc models [3], the above theorem enables us to solve these problems for the visibility graphs of polygonal rings very efficiently.

Corollary 3.2 *Given a polygonal ring $P \backslash Q$ as input, the following problems can be solved in linear time for the visibility graph of $P \backslash Q$:* CLIQUE, DOMINATING SET *and* INDEPENDENT SET

These results are interesting in view of the fact that CLIQUE [1], DOMINATING SET, and INDEPENDENT SET [5] are known to be NP-hard for the visibility graphs of polygons with holes.

4 Conclusion

In this paper we gave a constructive proof that the visibility graph of a polygonal ring is a circular-arc graph, and presented a linear-time algorithm to construct a circular-arc model. However, the problem of determining whether a given graph is the visibility graph of a polygonal ring remains open.

We have listed several problems for which efficient algorithms for circular-arc graphs yield efficient algorithms for the visibility graphs of polygonal rings. On the other hand, the complexity status of several interesting problems on the visibility graphs of polygonal rings is still unknown. A particularly interesting problem is the vertex colouring problem, which is NP-complete for circular-arc problems but polynomial-time solvable for proper circular-arc graphs [4]. Notice that the visibility graph of a polygonal ring is generally not a proper circular-arc graph, but the set of vertices in P induces a proper circular-arc graph. Will the restriction of the vertex colouring problem to the visibility graphs of polygonal rings make the problem tractable?

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