On the Difficulty of Greedy Tetrahedralization Of Points In 3-D[1]

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Abstract

We show that a set of points in E^3 is not always greedy tetrahedralizable if the definition of greedy tetrahedralization is a straight-forward extension of the E^2 counterpart. Sorted lists in ascending order of edge length and triangular area, are considered. If the definition of greedy tetrahedralization is modified such that a set of points in E^3 is always tetrahedralizable under the greedy approach, we can show that the problem is NP-complete.

1 Introduction

Triangulation of a set of points is a fundamental problem in computational geometry and in many applications, such as surface interpolation, finite-element computation, etc. This problem has many variations when constraints are imposed on the triangulation, for example, Delaunay triangulation [6], greedy triangulation [2,3,7,8,9], minimum-weight triangulation [2,5], MinMax-angle triangulation [1], just to name a few.

While the above problem in E^2 have been extensively studied, few results are known in E^3 [12]. This is because in general, the problems in E^3 are more complex than those in E^2. In E^2, the greedy triangulation method adds one edge or a set of edges of the triangulation at a time until the point set has been totally triangulated according to the ordering of some parameters. If this edge does not intersect any of the previously selected edges, it is added to the triangulation, otherwise, it is discarded. It can be shown that triangulation in E^2 is always possible for the greedy approach even when other selection criteria are applied.

[1]This work is supported by NSERC grant OPG0041629.

We shall present in this paper a result related to greedy triangulation in E^3. When the greedy approach is applied to point sets in E^3, it is a common belief that this approach, even though might not give an optimal tetrahedralization (triangulation in E^3), can at least tetrahedralize the point set. In this paper, we show that tetrahedralization is not always possible when edges are added to the tetrahedralization as long as they do not intersect the previously selected elements (which may be edges, triangular facets, or tetrahedra). Edges and their induced triangles, which do not intersect each other, can 'interlock' each other such that tetrahedralization including these edges is impossible (Section 2).

As a 'greedy' method never undoes what it did earlier, one way to solve this problem is not to add the selected edge or set of edges to the tetrahedralization as soon as it creates 'interlocks', even though there is no intersection with the previously selected elements. However, checking interlock is not easy. In Section 3, we show that the checking interlock is NP-complete when the greedy elements are selected in ascending order according to their edge lengths and triangular areas.

2 Greedy Tetrahedralizability of Point Set in 3-D

Let G be the sorted list of edges or triangles according to some parameters. **Greedy tetrahedralization** of S upon G is a tetrahedralization of S obtained by repeatedly selecting the front element from the remainder of list G as long as the selected element itself and its induced component do not intersect with any greedy element. Two elements (that is, line segments, triangular facets, and tetrahedra) are said to **intersect** each other if they share at least one of their interior points. A **greedy element** is any previously selected element or any component induced by the previously selected elements (e.g., a greedy triangle is a component induced by its three greedy edges and a greedy tetrahedron is one induced by its four greedy triangular facets).

As a consequence, all greedy elements should not properly intersect each other, and the selected element from the remainder of G should not intersect with the greedy elements. A natural question is to ask whether or not the nonintersecting greedy elements, which are selected in this manner, can eventually tetrahedralize S. Unfortunately, it can be shown that nontetrahedralization is possible.

Given a point set S, a set of elements are said to **interlock** each other iff there does not exist any tetrahedralization of S which contains this set of elements as a subset of the tetrahedralization. It is easy to see that the set of interlocked elements contains at least three elements and the convex hull of the vertices of these interlocked elements is not tetrahedralizable.

Readers can easily verify the following theorem by examining Figure 1 and Figure 2, and we omit the proof here. In the theorem, G_{LA} denotes the list of the complete connecting edges of point set S sorted by their lengths in ascending order and G_{AA}, that of triangles sorted by their areas in ascending order.

Theorem 1 *There exists a set of points that cannot be greedy tetrahedralized upon each of G_{AA} and G_{LA}.*

3 Modified Greedy Tetrahedralization of S

We shall give a modified definition of greedy tetrahedralization and present an algorithm to compute the tetrahedralization upon G. Basically, the new element will not be selected if it leads to an interlock situation.

Modified Greedy tetrahedralization of S upon G, denoted by $GT(S)$, is a tetrahedralization of S obtained by repeatedly selecting the first element from the remainder of list G into the tetrahedralization as long as the element and its induced components do not intersect any greedy element nor interlock each other.

The greedy tetrahedralization of a set of points under the modified definition always exists because the introduction of the non-interlock condition ensures the tetrahedralizability of convex hull of the point set all the time.

3.1 Checking Tetrahedralizability

Given a set of greedy elements upon a sorted list, testing whether this set of greedy elements "interlock" each other is related to its complement question of deciding whether the given point set S has a tetrahedralization which includes the set of greedy elements, i.e., the **tetrahedralizability problem**. We shall show that the tetrahedralizability problem for the sorted lists G_{LA} and G_{AA} is NP-complete. The proof for NP-completeness follows the idea used in [12] for the non-convex polyhedron by transforming the satisfiability problem to the tetrahedralizability problem. In [12], *it has been shown that, for any Boolean formula in conjunctive normal form, one can construct a 3-dimensional polyhedron that is tetrahedralizable iff the Boolean formula is satisfiable.* The main tool in the construction is a **niche**, which, when attached to a 'basis' polyhedron Q, restricts the possible tetrahedralizations. In particular, niches can force certain tetrahedra to appear (clause satisfaction) and they can prevent certain pair of tetrahedra from appearing simultaneously (truth-setting).

The niche is a 'twisted prism' with edges $\overline{q_1p_2}$, $\overline{q_2p_3}$, and $\overline{q_3p_1}$, called **cut-in edges**, concave inward as shown in Figure 3. Because of these cut-in edges, the triangular face $\Delta q_1q_2q_3$ cannot be fully 'seen' by p_1, p_2, and p_3, this 'twisted prism' cannot be tetrahedralized. Let $\Delta p'_1p'_2p'_3$ be the triangle obtained by the intersection of the three extended planes containing $\Delta q_1q_2p_2$, $\Delta q_2q_3p_3$, $\Delta q_3q_1p_1$ with $\Delta p_1p_2p_3$. Since points in $\Delta p'_1p'_2p'_3$ (shaded area) might be able to see the whole triangular face $\Delta q_1q_2q_3$ and this niche can be tetrahedralized iff extra point(s) is (are) introduced in $\Delta p'_1p'_2p'_3$. In fact, any points within the cone or prism above $\Delta p'_1p'_2p'_3$ and bounded by the three extended planes of $\Delta q_1q_2p_2$, $\Delta q_2q_3p_3$, and $\Delta q_3q_1p_1$ can see the whole triangular face $\Delta q_1q_2q_3$ and can be used to tetrahedralize this niche. This cone or prism of visibility above $\Delta p'_1p'_2p'_3$, is called **illuminant**. It is proved that it is always possible to construct a niche whose illuminant is exactly equal to any given triangular cone or prism with base $\Delta p'_1p'_2p'_3$ **[Illuminant Lemma]**. If $\Delta q_1q_2q_3$ of a niche has to be realized in the tetrahedralization, then there must exist a point in the illuminant to be the fourth vertex of the tetrahedron containing q_1, q_2 and q_3. Note that truth setting of a variable can be achieved if exactly two points, t and f denoting 'true' and 'false', are inserted in the illuminant of a niche as exactly one of the tetrahedra, $T_t = \Delta tq_1q_2q_3$ and $T_f = \Delta fq_1q_2q_3$, can and must exist in the tetrahedralization. The niche for clause satisfaction can be constructed in a similar fashion.

In the NP-completeness proof [12], considerable efforts are spent to the construction of P, the 'basis' polyhedron Q with the attached niches corresponding to a given boolean formula E, such that E is satisfiable iff P is tetrahedralizable.

Our task is to construct a niche $p_1p_2p_3q_1q_2q_3$ with the greedy elements of G_{AA} or G_{LA}, such that its illuminant is exactly the triangular cone or the prism with base $\Delta p'_1p'_2p'_3$. To do so, we shall first show in the next lemma that such a niche is sufficiently represented by its three cut-in edges in the illuminant formation.

Lemma 1 *Given a set of points S and a set of three cut-in edges $\{\overline{q_3p_1}, \overline{q_1p_2}, \overline{q_2p_3}\}$ (Figure 3), assume that no points of S lie inside the niche defined by the three cut-in edges, $S \cup \{\overline{q_3p_1}, \overline{q_1p_2}, \overline{q_2p_3}\}$ is tetrahedralizable iff there exists a point $p \in S$ that lies inside the illuminant of the niche.*

Proof (Sketch): "If" case is easy. "Only if" case (Figure 3) is proved by contradiction. Assume $S \cup \{\overline{q_3p_1}, \overline{q_1p_2}, \overline{q_2p_3}\}$ is tetrahedralizable, and all points in S lie outside the niche and its illuminant. Then there must exist three points $s_1, s_2, s_3 \in S$, each associated with a cut-in edge, such that $\Delta q_3p_1s_1$, $\Delta q_1p_2s_2$ and $\Delta q_2p_3s_3$ are in the tetrahedralization and intersect with the niche. Without loss of generality, let us consider $\Delta q_2p_3s_3$, point s_3 is outside the plane(s) containing $\Delta q_3p_1q_1$ or $\Delta q_1p_2q_2$ (or both) of the niche. If s_3 lies outside the plane of $\Delta q_3p_1q_1$, then s_3 must lie above the cut-in edge $\overline{q_3p_1}$, otherwise $\Delta p_3q_2s_3$ would be intersected. Similarly s_3 must lie below $\overline{q_1p_2}$ if it lies outside the plane containing $\Delta q_1p_2q_2$. It can be shown that either $\overline{q_2s_3}$ or $\overline{p_3s_3}$ would replace $\overline{q_2p_3}$ to interlock with $\overline{q_3p_1}$ and $\overline{q_1p_2}$ and results a smaller illuminant.

With the same arguments, when $\Delta q_3p_1s_1$ and $\Delta q_1p_2s_2$ are considered, the cut-in edge $\overline{q_3p_1}$ would be replaced by $\overline{p_1s_1}$ or $\overline{q_3s_1}$ and the cut-in edge $\overline{q_1p_2}$ by $\overline{p_2s_2}$ or $\overline{q_1s_2}$ respectively. These three new cut-in edges would interlock each other and define a smaller illuminant. This process can be carried out recursively until it leads to a non-tetrahedralizable situation. □

We shall describe how to construct these three cut-in edges of a niche for a given illuminant according to G_{AA} and G_{LA}. Sections 3.2 and 3.3 prove the Illuminant Lemma for G_{AA} and G_{LA} respectively and the NP-completeness proof will be completed in Section 3.4.

3.2 Illuminant lemma for G_{AA}

Given an illuminant, we can construct its corresponding niche $p_1p_2p_3q_1q_2q_3$ as described in [12]. Extend the line segment $\overline{p'_1p'_2}$ to $\overline{p'_1p_2}$ such that $\overline{p'_1p_2}$ is slightly longer than $\overline{p'_1p'_2}$, and obtain $\overline{p'_2p_3}$ and $\overline{p'_3p_1}$ in a similar fashion.

Based on Lemma 1, the niche can be represented by three cut-in edges $\{\overline{q_1p_2}, \overline{q_2p_3}, \overline{q_3p_1}\}$. According to G_{AA}, elements are inserted according to the sizes of the triangles in ascending order, a niche is formed if the inserted greedy elements, i.e., triangles, include the three cut-in edges.

The main idea is to insert an extra point very close to each cut-in edge such that the triangle formed by each cut-in edge and this extra point is of very small area and thus is chosen as a greedy element. For example, let A be the current smallest area of triangle and let $A = \mid \overline{q_3p_1} \mid * \epsilon$ for some small positive real ϵ. Let $c(\overline{q_3p_1})$ be the cylinder with $\overline{q_3p_1}$ as axis and ϵ as radius. A similar cylinder is constructed for each line segment of the niche. Choose a point p''_1 such that it lies inside cylinder $c(\overline{p_1q_3})$, outside the niche, outside the cylinders of other line segments, and close to p_1. Then, $\Delta p_1p''_1q_3$ is the desired triangle corresponding to the cut-in edge $\overline{q_3p_1}$. Similarly, obtain the other two triangles, namely $\Delta p_2p''_2q_1$ and $\Delta p_3p''_3q_2$ for the cut-in edges, $\overline{q_1p_2}$ and $\overline{q_2p_3}$.

By the above construction, we ensure that the areas of triangles $\Delta p_1p''_1q_3$, $\Delta p_2p''_2q_1$ and $\Delta p_3p''_3q_2$ are currently the smallest. By Lemma 1, the three triangles containing the cut-in edges function as a niche. We conclude this by the following lemma.

Theorem 2 *There always exists a niche which satisfies the greedy criterion on G_{AA} and whose illuminant covers a given triangular cone/prism, and the niche can be constructed in polynomial time.* □

3.3 Illuminant Lemma for G_{LA}

Given an illuminant, we shall construct the corresponding niche which satisfies the greedy criterion on G_{LA}. The main idea of the construction is to ensure that the cut-in edges can be selected as greedy elements according to their edge lengths (Figure 5). Without loss of generality, let the lengths of $\overline{p'_3p'_2}$, $\overline{p'_3p'_1}$, and $\overline{p'_2p'_1}$ be in descending order. Points q_1, q_2, and q_3 are chosen such that they are very close to p'_1, p'_2, and p'_3, respectively. Properly choose p_1 such that the cut-in edge $\overline{q_1p_2}$ is slightly shorter than $\overline{p_1q_2}$ and the length of $\overline{q_1p_2}$ is almost equal to that of $\overline{p'_1p'_2}$, and also properly choose p_3 such that cut-in edge $\overline{q_3p_1}$ is slightly shorter than $\overline{p_3q_1}$. Thus, it is always possible to construct a niche such that two of the three cut-in edges can be included into the tetrehedralization as greedy elements. However, the third cut-in edge $\overline{q_2p_3}$

would be longer than $\overline{p_2q_3}$, and in order to prevent $\overline{p_2q_3}$ from being selected as a greedy element before $\overline{q_2p_3}$ is considered (if so, $\overline{q_2p_3}$ cannot be a greedy element), an extra edge $\overline{q_2p_4}$, slightly shorter than $\overline{p_2q_3}$, crossing the facet $\Delta p_3p'_2p_2$, nearly parallel to $\overline{p_2q_2}$, and lying outside the triangular cone, is inserted to the niche (Figure 4). Since $\overline{q_2p_4}$ lies outside the cone, by Lemma 1, p_4 cannot tetrahedralize the niche and these vertices of the niche themselves cannot form a tetrahedralization.

As p_2 is chosen close to p'_2, $\overline{p_2q_3}$ will be longer than both $\overline{q_1q_3}$ and $\overline{q_1p_2}$ because q_1 and q_3 are close to p'_1 and p'_3 respectively, and $\overline{p'_3p'_2}$ is longer than $\overline{p'_1p'_3}$ and $\overline{p'_1p'_2}$. Since $\overline{q_2p_4}$ crosses the triangular facet $\Delta q_3p_2q_1$, $\overline{p_2q_3}$ cannot be selected as a greedy element. By the above construction, $\overline{q_2p_2}$, $\overline{q_1p_1}$, $\overline{q_1q_2}$, $\overline{q_3p_3}$, $\overline{q_1p_2}$, $\overline{q_2p_1}$, $\overline{p_2p_1}$, $\overline{q_1q_3}$, and $\overline{q_2p_4}$, would be the first group of the shortest edges. This group of edges can all be selected since they and their induced components do not intersect. The next shortest edges will be $\overline{q_3p_2}$, but since it will induce a triangle $\Delta p_2q_1q_3$ intersecting edge $\overline{q_2p_4}$, $\overline{q_3p_2}$ cannot be selected. As we can show by case analysis that $\overline{p_4q_3}$ is longer than $\overline{q_2p_3}$ and $\overline{q_3p_1}$, $\overline{q_3p_1}$ and $\overline{q_2p_3}$ will be selected as greedy edges satisfying the greedy criterion on G_{LA}. By Lemma 1, the three cut-in edges, $\overline{q_1p_2}$, $\overline{q_2p_3}$ and $\overline{q_3p_1}$, will be chosen as greedy elements and define the niche. Thus, we have the following lemma.

Theorem 3 *There always exists a niche which satisfies the greedy criterion on G_{LA} and whose illuminant covers a given triangular cone/prism, and the niche can be constructed in polynomial time.*

3.4 Basis Polyhedron

The 'basis' polyhedron Q in [12] consists of a number of distorted wedges (one for each variable), each with a convex roof and sticking together. Each distorted wedge is of the shape as shown in Figure 6, where the base of the wedge consists of $2m+1$ vertices c_1, c_2, ..., c_{2m+1} lying on a parabola, where m is the number of clauses in the Boolean formula. These vertices bound m triangular faces $\Delta c_1c_2c_3$, $\Delta c_3c_4c_5$, ..., $\Delta c_{2m-1}c_{2m}c_{2m+1}$. A clause niche will be attached to each of these triangles. The top of the wedge is a 'roof', convex in shape and containing the variable's literal vertices. A variable niche will be attached to one side of roof $\Delta y_1y_2y_3$. In the construction of the polyhedron, there is no restriction about the sizes of these niches. In order to simplify our proof, we assume that the niches are relatively small in size when compared with the triangular faces to which these niches are attached. Besides, the polyhedron has to satisfy a number of constraints in order to fulfil the NP-completeness proof for the reduction from a given Boolean formula E. The detailed shape of the polyhedron is described in [12] and the understanding of the above is sufficient for proving our claims.

The same construction for P is adopted in our proof, but instead of tetrahedralizing P, we shall show in the following lemmas that E is satisfiable iff the vertex set of P with the set of cut-in edges of the niches is tetrahedralizable. Before we proceed to prove the main result, let us define $CH(P)$ as the convex hull of the vertices of polyhedron P. We then extend the results by Goodman and Pach [4] on tetrahedralizability that if C_1 and C_2 are two nonintersecting convex polyhedra, then $CH(C_1 \cup C_2) - C_1 \cup C_2$ is tetrehedralizable.

Lemma 2 *Let $\{C_1, C_2, ..., C_n\}$ be a set of nonintersecting convex polyhedra (sharing at most one common edge) such that $CH(\cup_{i=1}^{k} C_i) \cap C_{k+1} = \phi$ for $1 \leq k < n$. Then $CH(\cup_{i=1}^{n} C_i) - \cup_{i=1}^{n} C_i$ is tetrahedralizable.*

Lemma 3 *$CH(P) - P$ is tetrahedralizable.*

Proof (Sketch): The difference of $CH(P)$ and P consists of three parts: (1) the part associated with the roof, (2) the part associated with niches (at the root and the base triangles) and (3) the convex polyhedron formed by c_1, c_{2m+1}, z_f, y_1 (Figure 6). Since part (3) is convex and hence tetrahedralizable. The tetrahedralizability of the first two parts can be proved with Lemma 2 by considering the volume of the convex hull of these roofs minus all the roofs and the volume of the convex hull formed by the niche and its basis facet minus the niches itself. □

Theorem 4 *There exists a tetrahedralization which includes all the niches of P for $CH(P)$ iff P is tetrahedralizable iff Boolean formula E is satisfiable.*

Theorem 5 *The tetrahedralizability problem for G_{AA} and G_{LA} is NP-complete.*

Proof: This proof is based on the construction given in [12]. For any Boolean formula E, a nonconvex polyhedron P is constructed with a number of niches corresponding to the variables and clauses in E. Let us consider the vertices of P and apply the modified greedy tetrahedralization to this vertex set. We want to show that the modified greedy tetrahedralization of P according to G_{AA} or G_{LA} will eventually include all the niches, i.e., the three cut-in edges of each niche (Section 3.1). The tetrahedralization problem at that stage is equivalent to the determination whether or not the Boolean formula E is satisfiable, and thus is NP-complete.

Depending on the sorted list for G_{AA} or G_{LA}, different types of niches (as described in Sections 3.2 and 3.3) are constructed. In both cases, and niche with its triangular face $\Delta q_1 q_2 q_3$ as part of $CH(P)$ can be made relatively very small when compared with the facet on which the niche is located. Thus, the other vertices of P will be far away from the niche and would not affect the tetrahedralization of the niche. As for G_{AA}, the extra points should be inserted to each cut-in edges of the niches in such a way that they would not form any small area triangles with other vertices in P other than with their associated cut-in edges. This is achievable as these niches for variables/clauses all lie on a plane. □

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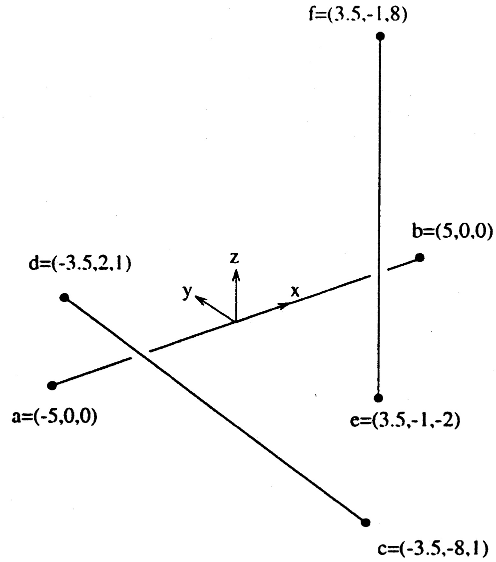


Figure 1: Nontetrahedralized Example for G_{LA}

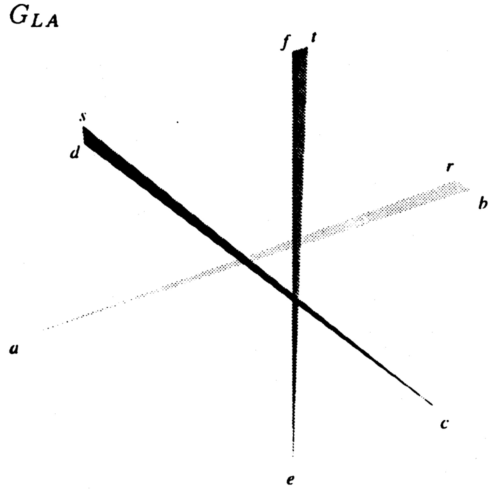


Figure 2: Example for G_{AA}

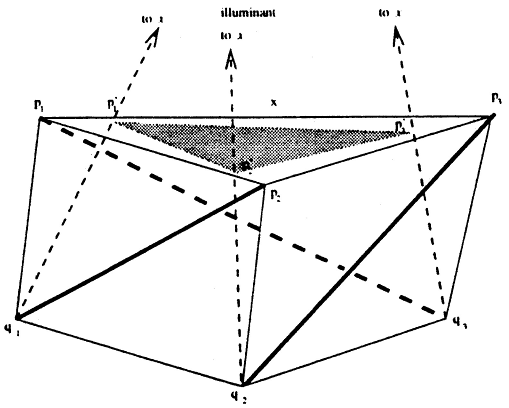


Figure 3: A Niche and Its Illuminant

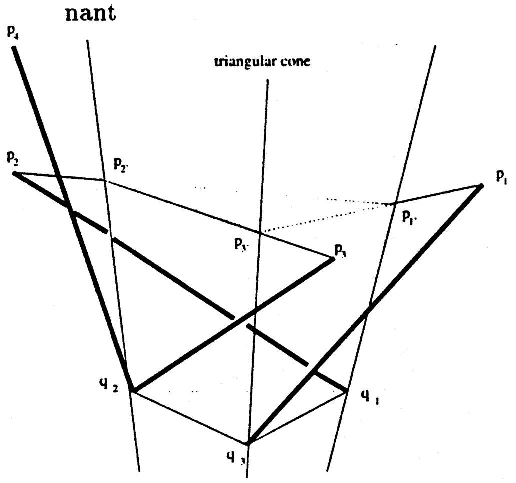
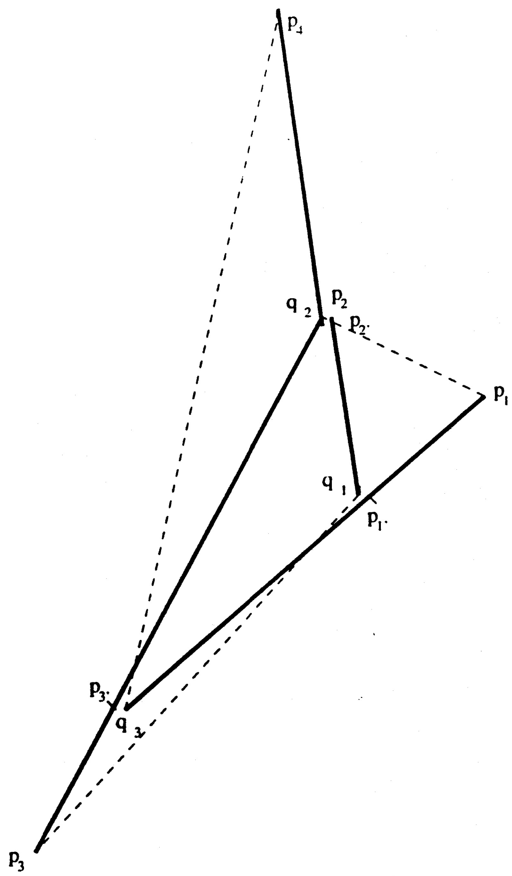


Figure 4: A Special Niche Tetrahedralizable by x



$\overline{p_3'p_2'} > \overline{p_3'p_1'} > \overline{p_2'p_1'}$
$\overline{q}_1p_3$ is slightly shorter than $\overline{p}_1q_2$
$\overline{q}_2p_1$ is slightly shorter than $\overline{p}_3q_1$
$\overline{q}_3p_4$ is slightly shorter than $\overline{p}_2q_3$

Figure 5: An Example for Constructing Length Niche

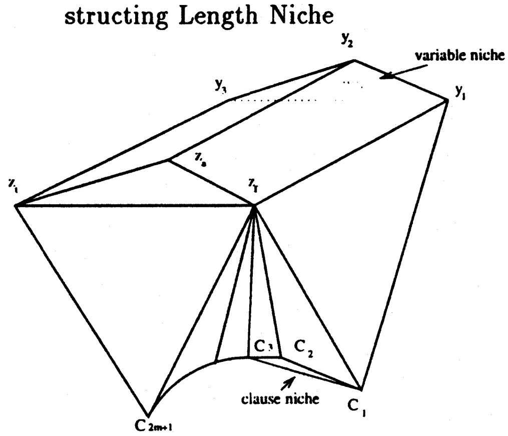


Figure 6: A "Distorted" Wedge