# ISOMORPHIC TRIANGULATIONS WITH SMALL NUMBER OF STEINER POINTS
(Extended Abstract)

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## Abstract

We present two algorithms for constructing isomorphic (i.e. adjacency preserving) triangulations of two simple $n$ vertex polygons $P$ and $Q$ with $k$ and $l$ reflex vertices, respectively. The first algorithm computes isomorphic triangulations of $P$ and $Q$ by introducing at most $O((k+l)^2)$ Steiner points and has running time $O(n+(k+l)^2)$. The second algorithm computes isomorphic triangulations of $P$ and $Q$ by introducing at most $O(kl)$ Steiner points and has running time $O(n + kl \log n)$. The number of Steiner points introduced by the second algorithm is also worst-case optimal. Unlike the $O(n^2)$ algorithm of Aronov, Seidel and Souvaine [1] our algorithms are sensitive to the number of reflex vertices of the polygons. In particular, our algoruthms have linear running time when $k + l \leq \sqrt{n}$ for the first algorithm, and $kl \leq n/\log n$ for the second algorithm.

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## 1 Introduction

For many problems in image analysis it is necessary to transform one picture into another [6]. This is often accomplished by introducing Steiner points in order to achieve isomorphic triangulations. Let $P, Q$ be two $n$ vertex polygons with vertices $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$, respectively. The (triangulated) polygons are called isomorphic if there exists a one-to-one correspondence $\phi$ of $\{1, \ldots, n\}$ into itself such that $\{v_i, v_j\}$ is an edge of $P$ if and only if $\{u_{\phi(i)}, u_{\phi(j)}\}$ is an edge of $Q$. In general, such an isomorphism may not exist (see [1]). To resolve this problem it is necessary to introduce new points at approriate locations within the polygons. Like in the case of Steiner trees [5] we call these points Steiner points. This raises the following question: "Give an efficient algorithm that constructs isomorphic triangulations of the polygons $P, Q$ by introducing the minimum possible number of Steiner points." A related question was first proposed by Goodman and Pollack in 1989 [1].

In this paper we consider the problem for the special case when the isomorphism between the two polygons is already given. This restriction arises naturally from the application of our problem to image analysis, i.e. the outline of the images to be processed predefine a fixed mapping between them. In [1] Aronov, Seidel and Souvaine [1] give two simple polygons on $n$ vertices requiring at least $\Omega(n^2)$ Steiner points. They give an algorithm for finding isomorphic triangulations of two simple polygons by introducing $O(n^2)$ Steiner points. The running time of their algorithm is $O(n^2)$.

We present two algorithms for constructing isomorhic triangulations between two simple polygons $P, Q$ with the same number $n$ of vertices. Let $k, l$ be the number of reflex vertices of $P, Q$, respectively. The first algorithm computes an isomorphism by introducing at most $O((k+l)^2)$ Steiner points and has running time $O(n+(k+l)^2)$. Steiner points are introduced via the technique of "optimal paths". The second algorithm computes an isomorphism by introducing at most $O(kl)$ Steiner points and has running time $O(n + kl \log n)$. In this case, Steiner points are introduced via the technique of "ray-shooting". Our results improve on those of [1] since our algorithms are sensitive to the number of Steiner points of the two polygons. Notice that when $k + l \leq \sqrt{n}$ the first algorithm is linear, and when $kl \leq n/\log n$ the second algorithm is also linear. Also when $k, l \leq \sqrt{n}$ both algorithms introduce a linear number of Steiner points.

In the next two sections we outline the algorithms and proofs of correctness; details will appear in the full paper.

# 2 Isomorphism via Optimal Paths

To make things easier, let us assume that we have two $n$ vertex polygons $P$ and $Q$ with vertices $v_1, \ldots, v_n$ and $u_1, \ldots u_n$, respectively, such that for all $i \leq n$,

$$(v_i, v_{i+1}), (u_i, u_{i+1}),$$

are edges of $P, Q$, respectively. We want to find isomorphic triangulations of $P$ and $Q$ in which $v_i$ is mapped to $u_i$, for $i = 1, \ldots, n$.

## 2.1 Algorithm

**Theorem 2.1** *Given two polygons $P$ and $Q$ with $k$ and $l$ reflex vertices respectively, we can find isomorphic triangulations for them in $O(n + (k+l)^2)$ time by introducing $O((k+l)^2)$ Steiner points.*

PROOF (OUTLINE) Assume that $P$ has $k$ reflex vertices $v_{s_1}, \ldots, v_{s_k}$ and $Q$ has $l$ reflex vertices $u_{t_1}, \ldots, u_{t_l}$. Our objective is to show that $P$ and $Q$ have isomorphic triangulations after introducing $O((k+l)^2)$ Steiner points.

Define the set $Z = \{s_1, \ldots, s_k\} \cup \{t_1, \ldots, t_l\}$ and suppose that the elements of $Z$ are $z_1, \ldots, z_m$ in sorted order where $m$ is at most $k + l$. $Z$ contains the indicess of the reflex vertices of $P$ and $Q$ (see Figure 1).

We now define two simple polygons $P'$ and $Q'$ as follows:

1. Polygon $P$:
   (a) For each vertex $v_{z_i}$ of $P$, compute the shortest polygonal path $P(i)$ path from $v_{z_i}$ to $v_{z_{i+1}}$, for $i = 1, \ldots, m$, where $m+1 = 1$. Let $P''$ (not $P'$) be the concatenation of all $P(i)$, for $i = 1, \ldots, n$. (see Figure 2).
   (b) We next observe that $P''$ has $m$ vertices (although some of them appear twice on $P''$). This follows from the observation that by the definition of $P''$ the vertices of $P''$ are of the form $v_{z_i}$ or reflex vertices of $P$ appearing on a shortest path from some $v_{z_i}$ to $v_{z_{i+1}}$, which in any case, are already accounted for in $Z$.
      i. Defining P':
      If it so happens that $P''$ is simple then take $P' = P''$, otherwise:
      ii. If $P''$ is not simple, notice that by splitting some of its vertices into two (these vertices are reflex vertices of $P$), we can obtain a simple polygon $P'$ totally contained in $P$. Clearly the number of vertices of $P'$ is at most $m + k$ (see Figure 3).
2. Polygon $Q$:
   (a) We define $Q'$ in a similar way but, using $Q$ in place of $P$.

Map $v_{z_i}$ on $P'$ to $u_{z_i}$ on $Q'$ and extend this mapping in the natural way to all duplicate vertices introduced in case (b.ii) introducing extra vertices as necessary on the edges of $P'$ and $Q'$ so as to ensure that every duplicate vertex of $P'$ (or $Q'$) maps to a different vertex of $Q'$ ($P'$). It is not hard to see that in this last step, the number of vertices of $P'$ and $Q'$ increases to at most $m + k + l \leq 2(k + l)$.

It now follows that $P'$ and $Q'$ have isomorphic triangulations with $O((k + l)^2)$ Steiner points.

Now $P \setminus P'$ and $Q \setminus Q'$ can be decomposed into $m$ simple (spiral) polygons $P_1, \ldots, P_m$ whose boundaries can be decomposed into a convex and a reflex chain. The convex chain is that joining $v_{z_i}$ to $v_{z_{i+1}}$ on $P$ (resp. $u_{z_i}$ to $u_{z_{i+1}}$ on $Q$ ) and the reflex chain is the shortest path from $v_{z_i}$ to $v_{z_{i+1}}$ in $P$ (resp. $u_{z_i}$ to $u_{z_{i+1}}$ in $Q$ ).

Now we can prove the following lemma.

**Lemma 2.1** *Let $P$ and $Q$ be spiral polygons with vertices $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$ such that $v_1, \ldots, v_k$ and $u_1, \ldots, u_k$ are all the reflex vertices of $P$ and $Q$, respectively. Then there are isomorphic triangulations of $P$ and $Q$ in which $v_i$ is mapped to $u_i$, $i = 1, \ldots, n$ containig at most $O(k^2)$ Steiner points.* ∎

Let $r_i$ be the number of reflex vertices of $P_i$, then by the previous paragraph, there are isomorphic triangulations between $P_i$ and $Q_i$ using $O(r_i^2)$ Steiner points. However since the sum $r_i + \cdots + r_m$ is $O(k + l)$ it follows easily that

$$r_i^2 + \cdots + r_m^2 = O((k + l)^2)$$

and our result now follows.

## 2.2 Complexity

The only part that has to be analyzed here is the construction of $P''$ and $Q''$. Next we will show how to do this in linear time.

First obtain a triangulation $H$ of $P$ using Chazelle's linear time algorithm [2], and consider the duat tree $T$ of $H$. Observe that $P''$ intersects each edge in $H$ at most twice. It is well known that the shortest path between any two points of $P$ corresponds to a path in $T$. Moreover by the previous observation each edge of $T$ is used at most twice. Now we can prove the following estimate on the time complexity.

**Lemma 2.2** *Calculating $P''$ (and $Q''$) can be done in linear time.*

Proof (Outline) Mark on $T$ those vertices corresponding to triangles containing vertices $v_{z_i}$. Delete from $T$ by standard leaf prunning techniques all those triangles not intersected by $P''$. We can now calculate in linear time the paths on $T$ from $v_{z_i}$ to $v_{z_{i+1}}$ and thus the shortest paths from $v_{z_i}$ to $v_{z_{i+1}}$ in total linear amortized time. ∎

This completes the outline of the proof of the theorem. ∎

# 3 Isomorphism via Ray-Shooting

The previous algorithm needs to introduce $O((k + l)^2)$ Steiner points. Thus the number of Steiner points is proportional to the square of the largest number of reflex vertices among the given polygons. In the sequel we give an algorithm that uses only $O(kl)$ Steiner points but at the cost of a $\log n$ time-overhead.

## 3.1 Algorithm

**Theorem 3.1** *Given two polygons $P$ and $Q$ with $k$ and $l$ reflex vertices respectively, we can find isomorphic triangulations for them in $O(n + kl \log n)$ time by introducing $O(kl)$ Steiner points.*

Proof (Outline) The idea is to establish the desired triangulation of the polygon $P$ on the vertices of a convex polygon (say, a circle). The introduction of Steiner points corresponds to a planar partition of the circle. Now we "overlay" the planar partition of $P$ with the planar partition of $Q$ keeping track of the Steiner points introduced. From this we can deduce easily the isomorphic triangulations of the polygons. Details are as follows.

1. Polygon $P$:
   (a) Map the $n$ vertices of $P$ on $n$ equidistant verices all lying on a circle.

(b) Move clockwise along the vertices and recursively "eliminate" all reflex vertices by forming a convex decomposition of the polygon $P$. To eliminate a reflex vertex, extend the two edges of the reflex vertex inside the polygon and shoot a ray within the region delimited by these two extended edges (see Figure 4) until it intersects an edge of the polygon, say the edge determined by the vertices $v_i, v_{i+1}$. Introduce this point of intersection between $v_i$ and $v_{i+1}$ as a Steiner point (see Figure 5). Now map this Steiner point between the images of these two vertices on the circle. In addition, form a planar subdivision that updates the polygon by incorporating the new edge between the reflex vertex and the Steiner point introduced.

(c) This introduction of Steiner points decomposes the polygon into $k+1$ convex regions. On the circle this corresponds to a planar subdivision into $k+1$ corresponding convex regions. In view of the equivalent numbering it also follows that any triangulation of a region in the polygon has an isomorphic triangulation on the corresponding region on the circle.

2. Polygon $Q$: (see Figure 5)

(a) In essence we execute the same algorithm as above with $Q$ replacing $P$ but we use the following "overlaying" procedure. We keep track of new Steiner points introduced via a standard straight-line planar subdivion data-structure [5]. We traverse the reflex vertices of $Q$ one at a time moving clockwise. For each such vertex we introduce a Steiner point on an edge of the polygon $Q$ (as in the case of the polygon $P$); in addition we introduce as Steiner points the points of intersection of the ray and the edges of the planar subdivision corresponding to $P$. After introducing these Steiner points we update the data structure and proceed to the next reflex vertex of $Q$.

## 3.2 Complexity

We observe the following Lemma.

**Lemma 3.1** *The above ray-shooting divides the polygon into subpolygons such that the boundary of each subpolygon can be broken into two chains one convex and the other not necessarily convex but totally contained in the boundary of the original polygon.* ∎

Using this and standard ray-shooting algorithms [3] it is easy to see that the complexity of introducing a Steiner point is $O(\log n)$ per Steiner point. Hence the total cost is $O(kl \log n)$. The overlaying procedure above can be performed in time $O(n+K)$, where $K$ is the number of Steiner points introduced, as in Guibas and Seidel [4].

An important feature of the algorithm is that it requires a date structure for maintaining and introducing Steiner points. Such a data structure is the planar subdivision [5]. The main cost is in the "overlaying" procedure of the two circular planar graphs. Planarity implies that each internal edge of the $Q$ intersects at most $k$ internal edges of $P$. This determines a total of at most $O(kl)$ points of intersection. These new points are introduced as the new Steiner points.

The resulting regions can now be triangulated thus resulting in equivalent triangulations for the given simple polygons. This completes the proof of the theorem. ∎

# 4 Comparison of the two algorithms

It easy to see that the first algorithm is superior to the second only when the polygons have the same number of reflex verices up to an $O(\log n)$ factor. More precisely this is true exactly when

$$\frac{k}{l} + \frac{k}{l} = O(\log n).$$

## 5 Lower Bounds

The number of Steiner points introduced by the second algorithm is worst-case optimal (up to a constant factor) with respect to the number of reflex vertices of the polygons. This follows from the following result.

**Theorem 5.1** *There exist two simple polygons $P, Q$ any isomorphic triangulation of which requires at least $\Omega(kl)$ Steiner points.*

Proof (Outline) A slightly modified version of the example given in [1] is sufficient to prove the lower bound. ■

## 6 Conclusion

We have given two algorithms for computing isomorphic triangulations of any two simple polygons with the same number $n$ of vertices. We related the number of Steiner points introduced with the number of reflex vertices, say $k, l$ of the two polygons, respectively. We proved that the number of Steiner points required to obtain isomorphic triangulations of the polygons is bounded by $kl$. The second algorithm is worst-case optimal in the number of Steiner points intorduced, while the first algorithm has superior running time when $\frac{k}{l} + \frac{k}{l} = O(\log n)$.

Our complexity bounds are valid under the assumption that the isomorphism between the two polygons is already given. If this is not the case our bounds (on the number of Steiner points introduced) still remain valid, however determing the "best" mapping still remains open.


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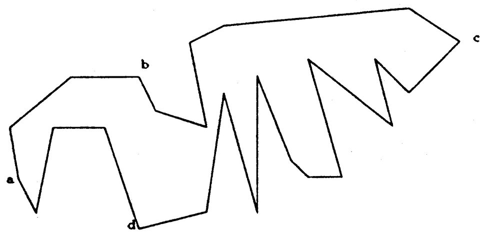


Figure 1: Polygon $P$; $a, b, c, d$ are the reflex vertices of $Q$.

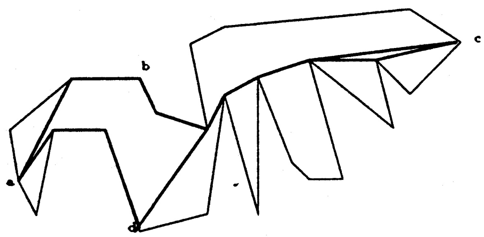


Figure 2: Computing shortest paths.

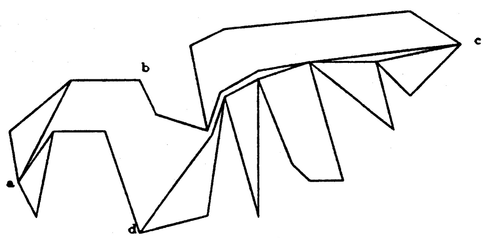


Figure 3: Splitting some vertices of the shortest path.

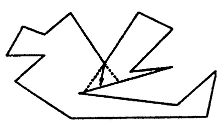

Figure 4: Introducing Steiner points via ray-shooting

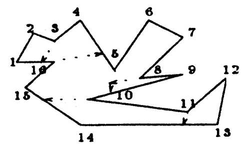


Figure 5: Polygon $P$: Introducing Steiner points via ray-shooting

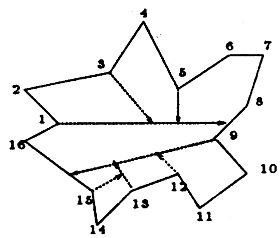


Figure 6: Polygon $Q$: Introducing Steiner points via ray-shooting

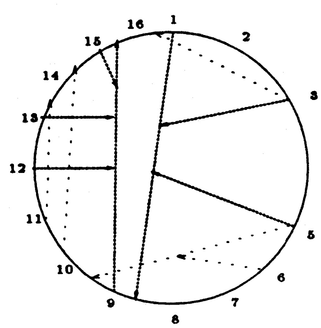


Figure 7: Mapping $P$ and $Q$ onto a convex polygon