List coloring and Euclidean Ramsey Theory (Abstract)

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Abstract

It is well known that one can color the plane by 7 colors with no monochromatic configuration consisting of the two endpoints of a unit segment, and it is not known if a smaller number of colors suffices. Many similar problems are the subject of Euclidean Ramsey Theory, introduced by Erdős et. al. in the 70s.

In sharp contrast we show that for any finite set of points K in the plane, and for any finite integer s, one can assign a list of s distinct colors to each point of the plane, so that any coloring of the plane that colors each point by a color from its list contains a monochromatic isometric copy of K. The proof follows from a general new theorem about coloring uniform simple hypergraphs with large minimum degrees from prescribed lists.

1 Euclidean Ramsey Theory

A well known problem of Hadwiger and Nelson is that of determining the minimum number of colors required to color the points of the Euclidean plane so that no two points at distance 1 have the same color. Hadwiger showed already in 1945 that 7 colors suffice, and Nelson as well as L. Moser and W. Moser noted that 3 colors do not suffice. These bounds have not been improved, despite a considerable amount of effort by various researchers.

A more general problem has been considered by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus [4, 5, 6] under the name Euclidean Ramsey Theory. The main question is the investigation of finite point sets K in the Euclidean space for which any coloring of an Euclidean space of a sufficiently high dimension $d \geq d_0(K, r)$ by r colors must contain a monochromatic copy of K. The conjecture is that this holds for a set K if and only if it can be embedded in a sphere. Another conjecture considered by these authors asserts that for any set K of 3 points in the plane, there is a coloring of the plane by 3 colors with no monochromatic copy of K.

Intriguing variants of these questions arise when one places some restrictions on the set of colors available in each point. This is related to the notion of list coloring introduced by Vizing [8] and by Erdős, Rubin and Taylor [7].

2 List coloring

The *list chromatic number* (or *choice number*) $\chi_\ell(G)$ of a graph $G = (V, E)$ is the minimum integer s such that for every assignment of a list L_v of s colors to each vertex v of G, there is a proper vertex coloring of G in which the color of each vertex is in its list. This notion was introduced independently by Vizing [8] and by Erdős, Rubin and Taylor [7]. In both papers the authors realized that this is a variant of usual coloring that exhibits several new properties, and that in general $\chi_\ell(G)$, which is always at least as large as the chromatic number of G, may be arbitrarily large even for graphs G of chromatic number 2.

It is natural to extend the notion of list coloring to hypergraphs. The list chromatic number $\chi_\ell(H)$ of a hypergraph H is the minimum integer s such that for every assignment of a list of s colors to each vertex of H, there is a vertex coloring of H assigning to each vertex a color from its list, with no monochromatic edges.

An interesting property of list coloring of graphs, which is not shared by ordinary vertex coloring, is the result that the list chromatic number of any (simple) graph with a large average degree is large. Indeed, it is shown in [1] that the list chromatic number of any graph with average degree d is at least $(\frac{1}{2} - o(1)) \log_2 d$, where the $o(1)$-term tends to zero as d tends to infinity. Our main combinatorial result is an extension of this fact to simple uniform hypergraphs.

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3 The new results

A hypergraph is called *simple* if every two of its distinct edges share at most one vertex. It is an r-graph if each of its edges contains exactly r vertices. We prove that the result of [1] can be extended to simple r-graphs. This is stated in the following theorem.

Theorem 1 *For every fixed $r \geq 2$ and $s \geq 2$, there is $d = d(r, s)$, such that the list chromatic number of any simple r-graph with n vertices and nd edges is greater than s.*

It is worth noting that the theorem provides a linear time algorithm for computing, for a given input simple r-graph, a number s such that its list chromatic number is at least s and at most $f(s)$ for some explicit function f. There is no such known result for ordinary coloring, and it is known that there cannot be one under some plausible hardness assumptions in Complexity Theory, as shown in [3].

The above result implies the following.

Theorem 2 *For any finite set X in the Euclidean plane and for any positive integer s, there is an assignment of a list of size s to every point of the plane, such that whenever we color the points of the plane from their lists, there is a monochromatic isometric copy of X.*

The proofs of both theorems can be found in [2].

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