# Splitting Ratios: Metric Details of Topological Line-Line Relations

**Konstantinos A. Nedas** and **Max J. Egenhofer**

National Center for Geographic Information and Analysis and Department of Spatial Information Science and Engineering
University of Maine, Orono, ME 04469-5711, USA
{kostas,max}@spatial.maine.edu

**Abstract**

Within the geographic domain, an important class of problems relies on geometric abstractions in the form of lines where, for instance, transportation networks and trajectories of movements are typically perceived or modeled at such a generalized geometric level. To support querying and computational comparisons, oftentimes multi-resolution models are needed to guide users from coarser to finer details. Within such a setting topological properties are coarse spatial information, whereas metric refinements offer finer details. The 9-intersection distinguishes 33 topological relations between two lines. This paper develops a model that captures *metric details* for line-line relations through *splitting ratios,* which are normalized values of lengths and areas of intersections. These ratios apply to the 9-intersection's non-empty values, thereby providing refinements of topological properties. Three such splitting ratios comprehensively refine 30 of the 33 topological relations: one for the lengths of common paths, one for the partitioning of lines through intersections, and another one for the areas enclosed by two lines with two or more common components. For the remaining three relations—disjoint, meet, and equal—no further metric refinements based on common parts are possible. The splitting ratios are integrated into a compact representation of detailed topological relations, thereby addressing topological and metric properties of arbitrarily complex line-line relations.

## Introduction

Modern GISs still rely heavily on quantitative descriptions of spatial objects and phenomena, both for storage and querying. There is significant evidence, however, that people think of space and communicate about spatial concepts using qualitative rather than quantitative terms (Lynch 1960; Hernández 1994; Regier 1995). An example is the approximate way in which people communicate directions to one another (i.e., the church is inside the square, which is a couple blocks down and to the left). The persistence on the classic quantitative paradigm renders GIS packages usable only by professionals or sophisticated users who often receive extensive training so that they become proficient in the formalizations of underlying spatial data models and their terminology. Non-expert users typically feel alienated, since they lack the necessary background and the technical jargon needed to comprehend and employ these tools, even for relatively simple tasks such as way-finding or spatial querying in order to find objects of interest around them.

Recent studies addressed the lack of commonsense formalizations of geographic knowledge in computers, by proposing formal and sound theories that allow reasoning about spatial relations, primarily in a qualitative manner (Egenhofer and Franzosa 1991; Randell *et al.* 1992). One such developed theory is the 9-intersection model (Egenhofer and Herring 1990), which focuses on binary topological relations between two regions, two lines, and a region and a line. The 9-intersection can be seen as one of the seminal efforts to incorporate *Naive Geography* concepts and reasoning into GISs (Egenhofer and Mark 1995). The internal representations of spatial relations and the mathematical operations that take place within this model are transparent to users, who are able to formulate queries by employing spatial predicates that correspond to natural-language terms such as *inside* or *overlap*, and also receive answers in a similar fashion.

The prominence of topology in the 9-interesection as the most critical aspect that people refer to when assessing spatial relationships in geographic space, has been confirmed by experiments in psychology and cartography (Lynch 1960; Stevens and Coupe 1978; Mark 1992). A critical factor that reinforces this view is that errors about spatial relations in human cognition are typically of metric, rather than topological nature (Tversky 1981; Talmy 1983). Despite its importance, however, topology *per se* is often insufficient in addressing people's needs. Metric details—though considered to be of lesser importance—are still required to capture the essence of spatial relations. Such circumstances arise when topology-based results to queries—even though exact—are underdetermined (i.e., do not provide enough detail so as to help accomplish the task at hand). Typical situations of the usefulness of metric enhancements are exemplified by people's tendencies to occasionally complement qualitative with quantitative information in order to resolve ambiguities in the description of spatial scenes. To reflect better human behavior, geographic information systems that rely on models such as the 9-intersection, need to incorporate mechanisms that will allow metric, in addition to the topological inferences among spatial entities. We follow the premise that *topology matters, while metric refines*

(Egenhofer and Mark 1995); hence, the metric enhancements, should be viewed only as extensions and supplements to the theory and not as the core of a qualitative geographic information system.

This paper focuses on binary relations between linear objects. The intent is to develop a comprehensive model for capturing metric details about such relations. Examples of entities that people often conceptualize as lines include road networks, sewer systems, rivers and streams, irrigation networks, aerial navigation routes, and satellite orbits. The critical components for line-line relations are the interiors and boundaries of the lines (Egenhofer 1994). When the interior or boundary of one line interacts with either the interior or boundary of the other line, certain metric properties can be captured about this interaction. For instance, a line may cross the interior of another, thus separating it into two distinct segments, the length of which could be measured. Purely quantitative measures, however, are undesirable because they do not take into consideration the relation to the objects for which they were derived. To describe details about topological relations, we consider the metric concept of *splitting*, which determines how a line's interior and exterior are partitioned by the other line's interior or boundary. *Splitting ratios* are normalized (i.e., scale-independent) values with respect to metric properties of line relations, such as the lengths of common parts or the area enclosed by two lines. These splitting ratios of line-line relations complement the metric refinements identified for region-region relations and line-region relations (Egenhofer and Shariff 1998).

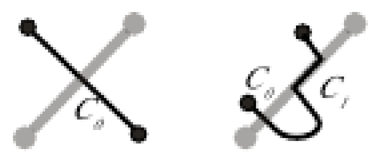
The remainder of this paper presents in detail the topological and metric models used to specify the geometry of spatial relations. Section 2 summarizes briefly the main concepts of the 9-intersection model, such as intersections and components, focusing on topological relations between linear objects as well as topological properties that characterize such relations. Section 3 introduces the rationale for splitting ratios and defines three types of line-splitting ratios: line alongness, interior splitting, and exterior splitting. Section 4 integrates such metric information into the same tabular representation that was used for the 9-intersection-based detailed topological relations (Clementini and di Felice 1998), yielding a *metrically enhanced classifying invariant*. Section 5 discusses conclusions.

## Topological Measures for Line-Line Relations

The 9-intersection model (Egenhofer and Herring 1990) provides a comprehensive framework for the description of topological relations between objects of type area, line, and point. The topological relation between two point sets, $A$ and $B$, is characterized by the binary value (empty, non-empty) of the set intersections of $A$'s interior ($A^\circ$), boundary ($\partial A$), and exterior ($A^-$), with the interior, boundary, and exterior of $B$ (Equation 1).

$$I(A,B) = \begin{pmatrix} A^\circ \cap B^\circ & A^\circ \cap \partial B & A^\circ \cap B^- \\ \partial A \cap B^\circ & \partial A \cap \partial B & \partial A \cap B^- \\ A^- \cap B^\circ & A^- \cap \partial B & A^- \cap B^- \end{pmatrix} \quad (1)$$

The content of the set intersections is a topological invariant (i.e., a topological property that is preserved under topological transformations such as rotation, scaling, and skewing). With nine set intersections and two possible values for each, the model distinguishes 512 possible topological relations, some of which cannot be realized depending on the dimension of the objects and the dimension of the embedding space. Those that cannot be realized are eliminated through a set of consistency constraints (Egenhofer and Franzosa 1991; Egenhofer 1994). One that applies to line-line relations, for example, is that the intersection of the exteriors of two lines in $R^2$ can never be empty. Eliminating impossible relations through constraints results in a set of 33 relations that can be realized between simple linear objects in $R^2$ (i.e., lines with exactly two boundary nodes and without any self-intersections). These relations are the focus of this work. The content invariant, although attractive due to its simplicity, is a coarse measure, incapable of differentiating situations that people often do. For example, the two spatial configurations in Figure 1 are distinct, while they are represented by the same 9-intersection matrix; therefore, in order to capture such finer details one has to consider additional invariants. Early work for invariants of line-line relations suggested using the type of interior intersections (touching or crossing) as an invariant (Herring 1991). Egenhofer and Franzosa (1995) developed a set of invariants that help establish topological equivalence between a model representation and a spatial configuration for region-region relations. Based on this model, Clementini and di Felice (1998) derived a complete set of invariants for line-line relations.



(a) (b)

Figure 1: Two configurations with different numbers of components.

An important invariant is the *number of components*. A component of a set $Y$ is the largest connected (non-empty) subset of $Y$ (Egenhofer and Franzosa 1995). Whenever any of the nine set intersections is separated into disconnected subsets, these subsets are the *components* of this set intersection. Hence, any non-empty intersection may have several distinct components, each of which may be characterized by its own topological properties. The number of components of an intersection is denoted by $\#(A \cap B)$. For example, for the relation of Figure 1a, $\#(L_1^\circ \cap L_2^\circ) = 1$, whereas for Figure 1b, $\#(L_1^\circ \cap L_2^\circ) = 2$. In addition, for Figure 1b, $C_o$ is a 0-dimensional component, whereas $C_1$ is a 1-dimensional component. An obvious dependency between the content and the component invariants is that any empty intersection has zero components, and every non-empty intersection has at least one component.

## Splitting Measures

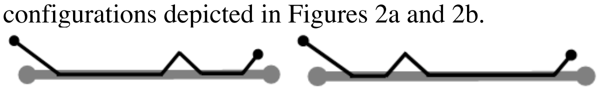
*Splitting* determines how a line's interior is divided by another line's interior or boundary. A special case of splitting pertains to the separation of the common exterior of the lines into bounded and unbounded components. To describe the degree of splitting, the metric concepts of the *length of a line* and the *area of a bounded exterior* are used. Among the entries of the 9-intersection for two simple lines, there are five intersections—between two boundaries, between boundary and interior, and between boundary and exterior—that cannot be evaluated with a length or area measure, because these intersections are 0-dimensional (Table 1). The intersection of the two interiors can be evaluated with a length measure only when it is 1-dimensional. The two intersections of one line's interior with the other line's exterior are always 1-dimensional when not empty. The intersection of the exteriors of the lines is always 2-dimensional.

Table 1: Area and Length Measures applied to the nine intersections of two lines.

| | $L_2^\circ$ | $\partial L_2$ | $L_2^-$ |
|---|---|---|---|
| $L_1^\circ$ | $length(L_1^\circ \cap L_2^\circ)$ | — | $length(L_1^\circ \cap L_2^-)$ |
| $\partial L_1$ | — | — | — |
| $L_1^-$ | $length(L_1^- \cap L_2^\circ)$ | — | $area(L_1^- \cap L_2^-)$ |

To normalize the length of the common interior we compare it with the length of $L_1$ (or the length of $L_2$). The length of the intersection between $L_1$'s interior and $L_2$'s exterior is normalized by the length of $L_1$. Similarly, the length of the intersection between $L_2$'s interior and $L_1$'s exterior is normalized by the length of $L_2$. The area of a bounded exterior is normalized by the area of a circle whose perimeter is equal to the sum of the lengths of the two lines. Such a circle encloses the largest bounded exterior area that two lines can form.

Two simple lines may form a topological configuration of arbitrary complexity with multiple components of the same or different intersection types; therefore, the metric refinements in the form of the splitting measures operate at the component level so as to help us describe adequately the different metric properties of each component. For instance, for the configuration of Figure 2a we calculate the metric properties separately for each intersection between the interiors of the lines. A global measure that would rely on the sum of all common interior segments would not help distinguish between the two topologically equivalent configurations depicted in Figures 2a and 2b.



(a) (b)

Figure 2: A global metric measure instead of one based on components would fail to add any refinement between the two topologically equivalent configurations.

### Line Alongness

In order to consider *line alongness*, the intersection of the interiors of two lines must be non-empty ( $A^\circ \cap B^\circ = \neg\varnothing$ ) and 1-dimensional. The interior of one line interacts with the interior of the other such that each line is separated into two sets of line parts: line segments that are in the common interior (i.e., common interior components) and line segments that are in the exterior of the other line. This separation makes a 1-dimensional object split another 1-dimensional object into two or more 1-dimensional parts (Figure 3).

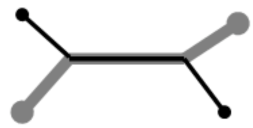


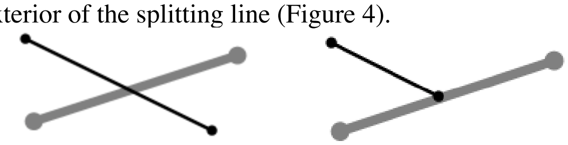
Figure 3. Line Alongness: the common interior separates each line into parts of inner and outer segments (more complex configurations may have multiple components in the intersection of the line interiors).

As the measure for the separation we employ the notion of the *line alongness ratio* ($LA$) as the ratio between the length of the common interior and the length of a line. There are two possible ratios: one with respect to the length of $L_1$ and another with respect to the length of $L_2$ (Equation 2). The range of the line alongness ratio is $0 \le LA \le 1$. When the common interior segment degenerates to a point, $LA$ reaches 0. If $L_1$ is entirely contained within the interior of $L_2$, then $LA_1$ becomes 1, and the same occurs for $LA_2$, when $L_2$ is entirely contained within the interior of $L_1$. If both $LA_1$ and $LA_2$ are 1, then the lines are equal. For arbitrarily complex configurations with multiple interior-interior intersections, a separate measure of line alongness is derived for each component.

$$LA = \frac{length(L_i^\circ \cap L_j^\circ)}{length(L_i)} \quad with \quad i,j \in \{1,2\}, i \neq j \qquad (2)$$

### Interior Splitting

If the interior or boundary of one line interacts with the interior of the other line, it separates the interior into left and right line segments according to some predetermined orientation. This involves a 1-dimensional object (i.e., common interior segment) or a 0-dimensional object (i.e., interior or boundary point) splitting a 1-dimensional object into two 1-dimensional parts, both of which intersect with the exterior of the splitting line (Figure 4).



(a) (b)

Figure 4. Interior Splitting: (a) one line's interior separates the other line's interior into two parts (the common interior could also be 1-dimensional) and (b) one line's boundary separates the other line's interior into two parts.

In order to consider *interior splitting*, the intersection of a line's closure with the interior of another line must be non-empty (i.e., $A^\circ \cap B^\circ = \neg\varnothing$ or $\partial A \cap B^\circ = \neg\varnothing$ or $A^- \cap B^\circ = \neg\varnothing$). A normalized measure for the interior splitting is the *interior splitting ratio* (*IS*) between the line segment of the split line located in the exterior of the splitting line and the length of the split line (Equation 3). This measure is evaluated separately for each applicable component intersection. For example, in a typical *cross*-like configuration (Figure 4a) there are four components.

$$IS = \frac{length(component(L_i^\circ \cap L_j^-))}{length(L_i)} \quad (3)$$

$$with \quad i, j \in \{1,2\}, i \neq j$$

The range of the interior splitting ratio is $0 \leq IS \leq 1$. It would be 0 if one line was entirely contained within another, or if the lines were equal, which means that either $A^\circ \cap B^-$ or $A^- \cap B^\circ$ or both would be empty. It reaches 1 for the components of one line when the interior-interior intersection becomes empty, for instance, when the case of Figure 4a degenerates to that of Figure 4b.

### Exterior Splitting

*Exterior splitting* occurs if parts of the two lines (interiors or boundaries or both) interact in such a way so as to form one or more closed regions (Figure 5). Hence, exterior splitting involves two 1-dimensional objects splitting a 2-dimensional object into two or more parts. Specifically, this type of splitting implies a partitioning of the common exterior of the two lines into two or more components: an unbounded exterior component and one or more *bounded* exterior components. The term *bounded* refers to the exterior-exterior intersections that are completely surrounded by the interiors of the two lines.

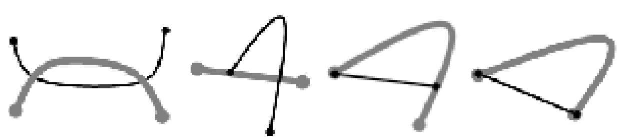



Figure 5: Exterior Splitting: a bounded exterior formed by (a) two interior-interior intersections; (b) one boundary-interior and one interior-interior intersection; (c) one boundary-boundary and one boundary-interior intersection; and (d) two boundary-boundary intersections.

A normalized measure for this property is the *exterior splitting ratio* (*ES*) as the ratio between the area of the bounded exterior that is formed by the two lines and the maximum bounded exterior that could possibly be formed by the same lines (Equation 4).

$$ES = \frac{4\pi(area(boundedComponent(L_1^- \cap L_2^-))}{(length(L_1) + length(L_2))^2} \quad (4)$$

The area of the maximum possible bounded exterior is equal to the area of a circle with perimeter equal to the sum of the lengths of the two lines. The range of the exterior splitting ratio is $0 < ES \leq 1$. It would reach zero if the bounded area were nonexistent. It becomes 1 if the two lines form only one bounded area, and there are two non-empty boundary-boundary ($\partial A \cap \partial B$) intersections (Figure 5d).

## Representations for Arbitrarily Complex Line-Line Relations

For a complex configuration, with many intersections of the same or different type between two simple lines, all of the measures developed may apply one or multiple times, depending on the number of existing components. In such a case one needs to develop a complete and efficient representation for all metric details that apply to the spatial scene such that it allows a smooth transition from the representation of simple to arbitrarily complex line-line relations. Completeness requires that all applicable measures be encoded. Efficiency requires that the form of representation be organized such that it can be easily understood. In the context of efficiency, it is also desirable to combine the topological and metric properties for a scene into a single form of representation. We base our representation technique on the concept of the *classifying invariant* (Clementini and di Felice 1998). The classifying invariant captures in a matrix the values of the topological properties needed to describe a scene involving two simple lines. In this section we extend this matrix to include the quantitative values of the metric details in addition to the qualitative values of the topological invariants. We call the resulting matrix a *metrically enhanced classifying invariant.*

The general structure of the classifying invariant for two simple lines, denoted as $Cl(L_1, L_2)$, is a matrix of four columns and *m* rows (Table 2). Each row defines an interior-interior, interior-boundary, or boundary-boundary intersection between the two lines. These are the most essential intersections since they determine how the two lines interact. The four columns give the qualitative values of several topological properties which are the *intersection sequence* $S(L_2)$, the *collinearity sense* $CS$, the *intersection type T*, and the *link orientation* $LO_{L_2}$. The generic entry $k_i$ represents the label of the intersection component. This set of topological invariants has been proven sufficient and necessary in order to establish topological equivalence with any configuration for a pair of simple lines.

Table 2: Representation of the classifying invariant in tabular form.

| $S(L_2)$ | $CS$ | $T$ | $LO_{L_2}$ |
|---|---|---|---|
| $k_0$ | $CS(k_0)$ | $T(k_0)$ | $LO_{L_2}(k_0, k_1)$ |
| $k_1$ | $CS(k_1)$ | $T(k_1)$ | $LO_{L_2}(k_1, k_2)$ |
| … | … | … | … |
| … | … | … | $LO_{L_2}(k_{m-2}, k_{m-1})$ |
| $k_{m-1}$ | $CS(k_{m-1})$ | $T(k_{m-1})$ | - |

The intersection sequence describes the order in which the various components occur. One first follows line $L_1$ from its first point and assigns numeric labels to the intersections until the last point is reached. The *intersection sequence* is then the sequence of numbers established by traversing line $L_2$ and recording the labels that were previously assigned to $L_1$. For example, the intersection sequence in Figure 6 is [0,1,3,2].

First establishing a clockwise orientation and then recording at the intersection node the sequence of incoming and outgoing arcs, starting from the boundary of one line, defines the *intersection type*. For instance, for Intersection 0 in Figure 6 the sequence is $< i_1, i_2, o_1, o_2 >$, assuming that we record the arcs starting from the incoming arc of the black line. The number of arcs in the sequence can be less than four. For example, for Intersection 2 in Figure 6 the sequence is $< i_1, o_1, i_2 >$. Although the choice of the first arc to start the sequence is arbitrary, their order must be preserved as it implicitly stores information about whether the intersections are *crossing* or *touching* (Herring 1991).

For 1-dimensional intersections, the collinearity sense distinguishes whether the segments that make these components are traversed following the same or the reverse orientation in the two lines. If the former is true the value of the collinearity sense is 1, if the latter holds it is -1, whereas for 0-dimensional intersections it takes the value of 0. For instance, since the 1-dimensional Intersection 2 (Figure 6) is traversed in reserve orientation, its collinearity sense is –1; Intersections 0, 1, and 3, however, are 0-dimensional, therefore, their collinearity sense is 0.

The link orientation depends on the notion of a link, which is the part of line $L_2$ located between two consecutive intersections $(h,k)$. If the cycle obtained by traversing the link $L_2(h,k)$ and coming back to $h$ traversing the line $L_1$ has a clockwise orientation, the link orientation value becomes *r* (i.e., right), otherwise *l* (i.e., left). Because the link orientation invariant depends on two consecutive intersections, its value is undefined for the last row of the classifying invariant matrix. Figure 6 demonstrates how these concepts apply for a complex configuration of two simple lines and the construction of its classifying invariant (Table 3).

For the last three invariants, there appears to be a one-to-one correspondence with the three metric ratios. The first correspondence is between the collinearity sense and the line alongness measure. Instead of using 1 and –1 to denote whether the segments along the common interior have the same or reverse orientation, we use a positive or negative value between 0 and 1, equal to the line alongness ratio. We arbitrarily choose the ratio with respect to $L_2$. For 0-dimensional intersections, the value of the collinearity

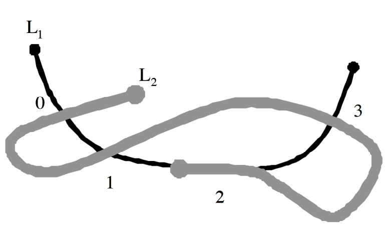



Figure 6: A complex configuration with four interior-interior intersection components formed by two simple lines.

Table 3: Tabular representation of the classifying invariant.

| $S(L_2)$ | $CS$ | $T$ | $LO_{L_2}$ |
|---|---|---|---|
| 0 | 0 | $(i_1, i_2, o_1, o_2)$ | *l* |
| 1 | 0 | $(i_1, o_2, o_1, i_2)$ | *r* |
| 3 | 0 | $(i_1, i_2, o_1, o_2)$ | *r* |
| 2 | -1 | $(i_1, o_1, i_2)$ | - |

sense remains 0 and represents the extreme case of the line alongness measure, where the common segment degenerates to a single point.

The second correspondence is between the intersection type and the line splitting ratio. The encoding sequence of the arcs can be extended with numeric information that relates each arc to a line splitting ratio measure between 0 and 1. The line splitting ratio for each arc is derived by dividing the length of the arc through the length of the line that contains it. The length of each arc is taken equal to the length of the line between the intersection which is being recorded and the immediate previous intersection or starting boundary of the line for inputting arcs, or the immediate next intersection or finishing boundary of the line for outputting arcs. The labels of the arcs (i.e., $i_1$, $i_2$, $o_1$, $o_2$) must also be recorded, because they may occur at different orders depending on the intersection type. Such information must be maintained so as to be able to distinguish different topological relations.

The third correspondence between a topological invariant and a splitting measure is between the link orientation and the exterior splitting ratio. The link orientation describes the orientation of the circular section between two consecutive intersections. This circular section, however, forms always a bounded exterior; therefore, one could combine the link orientation and the exterior splitting measure by recording only the value of the exterior splitting ratio for each bounded exterior component. The value is preceded by a *plus* sign if the link orientation is clockwise and by a *minus* sign if it is counter-clockwise. The topological configuration between two simple lines (Figure 6) is annotated with metric details (Figure 7). Table 4 displays the matrix for this scene's metrically enhanced classifying invariant.

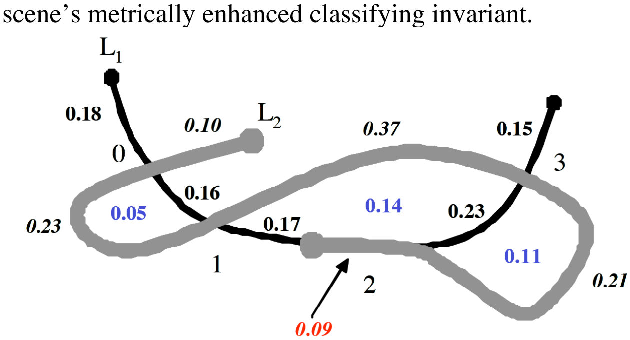



Figure 7: A complex configuration with four intersection components enhanced with metric details. Numbers in black represent the interior splitting ratio for each segment (**bold** for segments of $L_1$ and *italics* for segments of $L_2$). Numbers in red represent the line alongness measure. Numbers in blue represent the exterior splitting ratio for each bounded exterior component.

Table 4: Metrically enhanced classifying invariant matrix for the configuration of Figure 7.

| $S(L_2)$ | $CS$ | $T$ | $LO_{L_2}$ |
|---|---|---|---|
| 0 | 0 | $(i_1, i_2, o_1, o_2)$ (0.18, 0.10, 0.16, 0.23) | -0.05 |
| 1 | 0 | $(i_1, o_2, o_1, i_2)$ (0.16, 0.37, 0.17, 0.23) | 0.14 |
| 3 | 0 | $(i_1, i_2, o_1, o_2)$ (0.23, 0.37, 0.15, 0.21) | 0.11 |
| 2 | -0.09 | $(i_1, o_1, i_2)$ (0.17, 0.23, 0.21) | - |

## Conclusions

This paper introduced a computational model that extends topological information about binary relations between simple lines, based on the 9-intersection, with metric information in terms of *splitting ratios*. Three splitting ratios were derived: *line alongness*, which applies for 19 of the 33 relations; *interior splitting,* which applies for 30 relations; and *exterior splitting*, which can be applied to 23 relations. They all take values between 0 and 1 and grow linearly with the size of the intersection component that they measure. To encode splitting ratios we converted Clementini's and di Felice's (1998) matrix, which stores values of topological properties for detailed topological relations between lines, into the *metrically enhanced classifying invariant*. Such metric details of line-line relations may complement both coarse and detailed relations in spatial similarity retrieval in order to sort query results; they may help correct overshoots in sketched queries, restoring the proper topology for a query; and may guide the selection of appropriate terminology for spatial relations (Shariff *et al.* 1998).

## Acknowledgments

Konstantinos Nedas is supported by a University of Maine Graduate Research Assistantship. Max Egenhofer's work is supported by the National Science Foundation under grant numbers EPS-9983432 and IIS-9970123, the National Geospatial-Intelligence Agency under grant numbers NMA201-00-1-2009, NMA201-01-1-2003, and NMA401-02-1-2009, and the National Institute of Environmental Health Sciences, NIH, under grant number 1 R 01 ES09816-01.

## References

Clementini, E. and P. di Felice (1998) Topological Invariants for Lines. *IEEE Transactions on Knowledge and Data Engineering* **10**(1): 38-54.

Egenhofer, M. (1994) Definitions of Line-Line Relations for Geographic Databases. *Data Engineering* **16**(11): 479-481.

Egenhofer, M. and R. Franzosa (1991) Point-Set Topological Spatial Relations. *International Journal of Geographical Information Systems* **5**(2): 161-174.

Egenhofer, M. and R. Franzosa (1995) On the Equivalence of Topological Relations. *International Journal of Geographical Information Systems* **9**(2): 133-152.

Egenhofer, M. and J. Herring (1990) *Categorizing Binary Topological Relations Between Regions, Lines, and Points in Geographic Databases*, Department of Surveying Engineering, University of Maine, http://www.spatial.maine.edu/~max/ 9intReport.pdf.

Egenhofer, M. and D. Mark (1995) Naive Geography. *COSIT '95*, Lecture Notes in Computer Science **988:** 1-15.

Egenhofer, M. and R. Shariff (1998) Metric Details for Natural-Language Spatial Relations. *ACM Transactions on Information Systems* **16**(4): 295-321.

Hernández, D. (1994) *Qualitative Representation of Spatial Knowledge*. Lecture Notes in Computer Science **804**, New York, Springer-Verlag.

Herring, J. (1991) The Mathematical Modeling of Spatial and Non-Spatial Information in Geographic Information Systems. *Cognitive and Linguistic Aspects of Geographic Space*. D. Mark and A. Frank (eds.). Dordrecht, Kluwer**:** 313-350.

Lynch, K. (1960). *The Image of a City*. Cambridge, MA, MIT Press.

Mark, D. (1992) Counter-Intuitive Geographic Facts: Clues for Spatial Reasoning at Geographic Scales. *Theories and Methods of Spatio-Temporal Reasoning in Geographic Space*. A. Frank, I. Campari, and U. Formentini (eds.). Lecture Notes in Computer Science **639:** 305-317.

Randell, D., Z. Cui, and A. Cohn (1992) A Spatial Logic based on Regions and Connection. *3rd International Conference on Knowledge Representation and Reasoning*, Morgan Kaufmann: 165-176.

Regier, T. (1995) A Model of the Human Capacity for Categorizing Spatial Relations. *Cognitive Linguistics* **6**(1): 63-88.

Shariff, R., M Egenhofer, and D. Mark (1998) Natural-Language Spatial Relations between Linear and Areal Objects: The Topology and Metric of English-Language Terms. *International Journal of Geographical Information Systems* **12**(3): 215-246.

Stevens, A. and P. Coupe (1978) Distortions in Judged Spatial Relations. *Cognitive Psychology* **10**: 422-437.

Talmy, L. (1983) How Language Structures Space. *Spatial Orientation: Theory, Research, and Application.* H. Pick and L. Acredolo. New York, Plenum Press**:** 225-282.

Tversky, B. (1981) Distortions in Memory for Maps. *Cognitive Psychology* **13**: 407-433.