Product-based Causal Networks and Quantitative Possibilistic Bases

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Abstract

In possibility theory, there are two kinds of possibilistic causal networks depending if possibilistic conditioning is based on the minimum or on the product operator. Similarly there are also two kinds of possibilistic logic: standard (min-based) possibilistic logic and quantitative (product-based) possibilistic logic. Recently, several equivalent transformations between standard possibilistic logic and min-based causal networks have been proposed. This paper goes one step further and shows that product-based causal networks can be encoded in product-based knowledge bases. The converse transformation is also provided.

Introduction

Generally, uncertain pieces of information or flexible constraints can be represented in different equivalent formats. In possibility theory, possible formats can be:

- graphical-based representations, viewed as counterparts of probabilistic Bayesian networks [11,12], and
- logical-based representations which are simple extensions of classical logic.

In graphical representations [1,9,10], uncertain information is encoded by means of possibilistic causal networks which are composed of Directed Acyclic Graph (DAG) and conditional possibility distributions.

In logical representations [7], uncertain information is encoded by means of possibilistic knowledge bases which are sets of weighted formulas having the form (ϕ_i, α_i) where ϕ_i is a propositional formula and α_i is a positive real number belonging to the unit interval [0,1].

Each possibilistic causal network (resp. each possibilistic knowledge base) induces a ranking between possible interpretations of a language, called a possibility distribution.

The possibility degree associated with an interpretation is obtained by combining the satisfaction degrees of this interpretation with respect to each weighted formula of the knowledge base, or with respect to each conditional possibility degree of the causal network.

Two combination operators have been used [7]: minimum operator and product operator. Therefore, there are two kinds of causal networks: min-based possibilistic networks and product-based possibilistic networks.

Similarly, two kinds of possibilistic knowledge bases are defined: min-based possibilistic logic (standard possibilistic logic) and product-based possibilistic logic called also quantitative possibilistic logic.

In the rest of this paper, we only focus on product-based possibilistic causal networks and on quantitative possibilistic logic.

Even if graphical or logical representation can encode same pieces of uncertain information, they in general use different inference tools. For instance, some inference tools in possibilistic causal networks are simple adaptations of probabilistic propagation algorithms [9,10]. In possibilistic logic, the inference tools are based on SAT provers (satisfiability test of propositional formulas). Hence, it is very important to have equivalent transformations from one representation format to another in order to take advantage of these different inference tools.

Another need of these transformations is when we fuse uncertain information given in different formats provided by different sources. Indeed, existing fusion modes assume that all information is represented in a same format, which is not always the case in practice. Having transformations algorithms between different representations allow the use of existing fusion modes even if the information is represented in different formats.

In [2,5] equivalent transformations have been provided between min-based possibilistic knowledge bases and min-based causal networks. This paper goes one step further. It provides an encoding of product-based possibilistic causal networks into quantitative possibilistic knowledge bases, and conversely.

The rest of this paper is organised as follows. Next section gives a background on possibilistic logic and posibilistic causal networks. Section 3 studies the transformations between product-based graphs and quantitative possibilistic knowledge bases. Section 4 gives the converse transformation. Section 5 concludes the paper.

Backgrounds

This section only gives a very brief recalling on possibilistic logic and possibilistic causal networks. See [7] for more details on possibilistic logic, and [1,9,10] for more details on possibilistic causal networks.

Possibilistic logic

Let $\mathcal{L}$ be a finite propositional language and Ω be the set of all propositional interpretations. Let $\phi, \psi, \ldots$ be propositional formulas. $\omega \models \phi$ means that ω is a model of ϕ. A possibility distribution [7] π is a mapping from a set of interpretations Ω into a linearly-ordered scale, usually the unit interval [0,1]. $\pi(\omega)$ represents the degree of compatibility of the interpretation ω with available pieces of information.
By convention, $\pi(\omega) = 0$ means that ω is impossible to be the real world, $\pi(\omega) = 1$ means that ω is totally possible to be the real world, and $\pi(\omega) > \pi(\omega')$ means that ω is a preferred candidate to ω' for being the real world.
A possibility distribution is said to be normalized if there exists ω such that: $\pi(\omega) = 1$. In this paper, only normalized distributions are considered.
Given a possibility distribution π, two dual measures are defined:

- The possibility measure of a formula ϕ:
$$\Pi(\phi) = max\{\pi(\omega) : \omega \models \phi\},$$
which evaluates the extent to which ϕ is consistent with the available beliefs expressed by π.
- The necessity measure of a formula ϕ:
$$N(\phi) = 1 - \Pi(\neg\phi),$$
which evaluates the extent to which ϕ is entailed by the available beliefs.

A possibilistic knowledge base Σ is a set of weighted formulas:
$$\Sigma = \{(\phi_i, \alpha_i) : i = 1, ..., n\},$$
where ϕ_i is a propositional formula and $\alpha_i \in [0, 1]$ which represents the certainty level of ϕ_i.
Each piece of information (ϕ_i, α_i) of a possibilistic knowledge base can be viewed as a constraint which restricts a set of possible interpretations. If an interpretation ω satisfies ϕ_i then its possibility degree is equal to 1 (ω is completely compatible with the belief ϕ_i), otherwise it is equal to $1-\alpha_i$. (the more ϕ_i is certain, the less ω is possible). In particular, if $\alpha_i = 1$, then any interpretation falsifying ϕ_i is equal to 0, namely is impossible.
More formally, the possibility distribution associated with a weighted formula (ϕ_i, α_i) is [7] :
$$\pi_{(\phi_i,\alpha_i)(\omega)} = \begin{cases} 1 - \alpha_i & if\ \omega \not\models \phi_i \\ 1 & \text{otherwise} \end{cases} \tag{1}$$
More generally, the possibility distribution associated with Σ is the result of combining possibility distributions associated with each weighted formula (ϕ_i, α_i) of Σ, namely:
$$\pi_\Sigma(\omega) = \oplus\{\pi_{(\phi_i,\alpha_i)}(\omega) : (\phi_i, \alpha_i) \in \Sigma\}. \tag{2}$$
where $\oplus$ is either equal to the minimum operator (in standard possibilistic logic), or the product operator (*) (in product-based possibilistic logic).

In the rest of the paper, we only focus on the case where $\oplus = *$, Σ is then called product-based possibilistic knowledge base. Equation (2) can then be written as:
$$\pi_\Sigma(\omega) = \begin{cases} 1 & \text{if } \omega \text{ satisfies } \Sigma \\ *_{(\phi_i,\alpha_i)\in\Sigma,\omega\not\models\phi_i}(1-\alpha_i) & \text{otherwise} \end{cases} \tag{3}$$

Possibilistic causal networks

A possibilistic causal network [1, 9,10] is a graphical way to represent uncertain information. Let $V = \{A_1, A_2, ..A_n\}$ be a set of variables. We denote by D_i the domain associated with the variable A_i. The set of all interpretations is the cartesian product of all domains of the variables in V. When each variable is binary, we simply write $D_i = \{a_i, \neg a_i\}$. In this paper, for sake of simplicity, only binary variables are considered. A possibilistic graph, denoted by Π_G is a Directed Acyclic Graph (DAG), where nodes represent variables (for example the temperature of a patient,..) and edges encode the causal links between these variables.
When a link exists from the node A_i to the node A_j, A_i is called a parent of A_j. The set of the parents of a node A_j is denoted by $Par(A_j)$. Uncertainty is represented on each node by means of conditional possibility distributions which express the strength of the links between variables. Conditional possibility distributions are associated with the DAG in the following way:

- For root nodes A_i, we specify the prior possibility distributions $\Pi(a_i), \Pi(\neg a_i)$ with $max(\Pi(a_i), \Pi(a_i)) = 1$ (the normalisation condition) .
- For other nodes A_j, we specify conditional possibility distributions $\Pi(a_j \mid u_j)$ with:
$$max(\Pi(a_j \mid u_j), \Pi(\neg a_j \mid u_j)) = 1,$$
where a_j is an instance of A_j and u_j is an instance of $Par(A_j)$.

In possibilistic theory, two kinds of possibilistic conditioning are defined depending on whether the setting is qualitative or quantitative (for a detailed discussion on possibilistic conditioning see [5]):

- In ordinal setting, a min-based conditioning is defined as:
$$\Pi(\omega \mid \phi) = \begin{cases} 1 & if\ \pi(\omega) = \Pi(\phi) \text{ and } \omega \models \phi \\ \pi(\omega) & \pi(\omega) < \Pi(\phi) \text{ and } \omega \models \phi \\ 0 & \text{otherwise} \end{cases}$$
- In a numerical setting, a product-based conditioning is defined as:
$$\Pi(\omega \mid \phi) = \begin{cases} \frac{\pi(\phi)}{\Pi(\phi)} & \omega \models \phi \\ 0 & \text{otherwise} \end{cases}$$

In this paper, we only focus on product-based conditioning. Each product-based possibilistic graph (DAG and local conditional possibility distributions) induces a unique joint conditional possibility distributions using a so-called chain rule

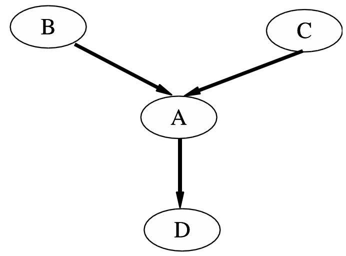

Figure 1: Example of a DAG

similar to the one used in probabilistic Bayesians networks. Let $\omega = a_1, a_2, ...a_n$, be an interpretation. We have:

$$\pi_{dag}(\omega) = *\{\Pi(a_i \mid u_i) : \omega \models a_i \wedge u_i, i = 1, .., n\} \quad (4)$$

where a_i is an instance of A_i and u_i is an instance of the parents of A_i.

Example 1 *Let us consider the product-based possibilistic causal network presented by the DAG of Figure 1.*
The local conditional possibility distributions are given in Tables 1 and 2.

Table 1: Initial conditional possibility distributions of $\Pi(A \mid B \wedge C)$

ABC	$\Pi(A \mid B \wedge C)$
abc	1
$ab\neg c$	.6
$a\neg bc$	1
$a\neg b\neg c$	.2
$\neg abc$	.2
$\neg ab\neg c$	1
$\neg a\neg bc$	.1
$\neg a\neg b\neg c$	1

Table 2: Initial conditional possibility distributions of $\Pi(B)$, $\Pi(C)$ and $\Pi(D \mid A)$

B	$\Pi(B)$	C	$\Pi(C)$	AD	$\Pi(D \mid A)$
b	1	c	1	ad	1
$\neg b$	.3	$\neg c$	.7	$a\neg d$	0
				$\neg ad$	.2
				$\neg a\neg d$	1

Using the chain rule defined in (4), we obtain a joint possibility distribution given in Table 3. For example:
$\pi(ab\neg cd) = \Pi(d \mid a) * \Pi(a \mid b\neg c) * \Pi(b) * \Pi(\neg c) = .42$

From Product-based graphs to a quantitative possibilistic base

In order to make easy the transformation from a product-based graph into a quantitative possibilistic base, a possibilistic causal network will be represented by a set of triples:

$$\Pi_G = \{(a_i, u_i, \alpha_i) : \alpha_i = \Pi(a_i \mid u_i) \neq 1\},$$

Table 3: Joint distribution using product-based chain rule

ABCD	$\pi(ABCD)$	ABCD	$\pi(ABCD)$
$abcd$	1	$\neg abcd$	.04
$abc\neg d$	0	$\neg abc\neg d$	.2
$ab\neg cd$	.42	$\neg ab\neg cd$	.14
$ab\neg c\neg d$	0	$\neg ab\neg c\neg d$	.7
$a\neg bcd$	.3	$\neg a\neg bcd$	.006
$a\neg bc\neg d$	0	$\neg a\neg bc\neg d$	.03
$a\neg b\neg cd$	.042	$\neg a\neg b\neg cd$	.042
$a\neg b\neg c\neg d$	0	$\neg a\neg b\neg c\neg d$	.21

where a_i is an instance of the variable A_i and u_i is an element of the cartesian product of the domain D_j of the variables $A_j \in Par(A_i)$.

Example 2 *Let us consider again the DAG presented in Figure 1 and Table 1-2. The codification is represented by:*
$\Pi_G = \{(\neg b, \emptyset, .3), (\neg c, \emptyset, .7), (a, b\neg c, .6), (a, \neg b\neg c, .2),$
$(\neg a, bc, .2), (\neg a, \neg bc, .1), (d, \neg a, .2), (\neg d, a, 0)\}$.

The construction of a possibilistic knowledge base Σ_{DAG} associated with a DAG is obtained immediately. It simply consists in replacing each triple (a, u, α) of the directed possibilistic graph Π_G by a possibilistic formula $(\neg a \vee \neg u, 1 - \alpha)$. Intuitively, this transformation is obtained by recalling first that (a, u, α) means that $\Pi(a \mid u) = \alpha$. Then, in the possibilistic base, the necessity measure is associated with conditional formulas where conditioning is equivalent to a material implication. Therefore, by definition $\Pi(a \mid u) = \alpha$ is equivalent to $N(\neg a \mid u) = 1 - \alpha$. By replacing the conditionning by the material implication, we obtain: $N(\neg a \vee \neg u) = 1 - \alpha$.
Formally, a possibilistic base Σ_{DAG} associated with the DAG is defined as follow:

$$\Sigma_{DAG} = \{(\neg a_i \vee u_i, 1 - \alpha_i) : (a_i, u_i, \alpha_i) \in \Pi_G\}. \quad (5)$$

Example 3 *The possibilistic knowledge base associated with Π_G of the example 2 is:*
$\Sigma_{DAG} = \{(b, .7), (c, .3), (\neg a \vee \neg b \vee c, .4), (\neg a \vee b \vee c, .8),$
$(a \vee \neg b \vee \neg c, .8), (a \vee b \vee \neg c, .9), (\neg d \vee a, .8), (d \vee \neg a, 1)\}$.

Proposition 1 *Let Π_G be a product-based possibilistic causal network and let Σ be a possibilistic base associated with Π_G using equation (5). We have:*

$$\forall \omega \in \Omega, \pi_\Sigma(\omega) = \pi_{dag}(\omega)$$

Where π_Σ is obtained using equation (3) and π_{dag} is obtained from equation (4).

Example 4 *We can check that the joint possibility distribution generated by the DAG of example 1 (Table 3) is the same as the one generated by the possibilistic knowledge base of example 3.*
For instance, let $\omega_1 = a \wedge b \wedge \neg c \wedge d$.
From table 3, we have: $\pi_{dag}(\omega_1) = .42$.
Let us consider the possibilistic base Σ defined in example 3. Using equation 3, we have:
$\pi_\Sigma(\omega_1) = (1 - .3) * (1 - .4) = .7 * .6 = .42$.
So, $\pi_{dag}(\omega_1) = \pi_\Sigma(\omega_1)$.

From quantitative possibilistic base to product-based graph

The converse transformation from a quantitative possibilistic base Σ into a product-based graph Π_G is less obvious. Indeed, we first need to establish some lemmas.
First, we need to define the notion of equivalent possibilistic knowledge bases:

Definition 1 *Two quantitative possibilistic knowledge bases Σ and Σ' are said to be equivalent if they induce the same possibility distributions, namely:*

$$\forall \omega, \pi_{\Sigma}(\omega) = \pi_{\Sigma'}(\omega).$$

The first lemma indicates that tautologies can be removed from quantitative possibilistic bases without changing possibility distributions.

Lemma 1 *If $(\top, \alpha_i) \in \Sigma$ then Σ and $\Sigma' = \Sigma - \{(\top, \alpha_i)\}$ are equivalent.*

The proof is immediate since only formulas which are falsified by a given interpretation are taken into account during the computation of possibility distributions. Removing tautologies is important since it avoids fictitious dependence relations between variables. For instance, the tautological formula $(a \vee \neg a \vee b, 1)$ might induce a link between B and A.
Next lemma concerns the reduction of a possibilistic base.

Lemma 2 (reduction) *Let Σ be a possibilistic base. Let $(x \vee p, \alpha)$ and $(x \vee \neg p, \alpha)$ be two formulas from Σ. Let $\Sigma' = \Sigma - \{(x \vee p, \alpha), (x \vee \neg p, \alpha)\} \cup \{(x, \alpha)\}$. Then, Σ' and Σ are equivalent.*

Next lemma shows that replacing (x, α) by $(x \vee p, \alpha), (x \vee \neg p, \alpha)$ does not change the induced possibilistic distribution.

Lemma 3 (Extension) *Let Σ be a possibilistic base. Let (x, α) be a formula in Σ. Let Σ' be defined as follows:*
$\Sigma' = \Sigma - \{(x, \alpha)\} \cup \{(x \vee p, \alpha), (x \vee \neg p, \alpha)\}$.
Then, Σ' and Σ are equivalent.

Next lemma shows how to handle redundancies in a quantitative possibilistic base.

Lemma 4 (Redundancies) *If $(x, \alpha) \in \Sigma$ and $(x, \beta) \in \Sigma$, then Σ and $\Sigma - \{(x, \alpha), (x, \beta)\} \cup \{(x, \alpha + \beta - \alpha * \beta)\}$ are equivalent.*

With the help of the four previous lemmas, the construction of a product-based causal network from a quantitative possibilistic knowledge base can be obtained from the following three steps:

- The first step consists in modifying the possibilistic base by removing tautologies and using simplification given by lemma 2.
- The second step constructs the graph by identifying the parents of each variable.
- The last step computes the conditional possibilities associated with the graph.

We will illustrate these steps by using the following quantitative possibilistic knowledge base:

Example 5 $\Sigma = \{(a \vee b \vee c, .7), (a \vee b \vee \neg c, .7),$
$(\neg a \vee c \vee \neg d, .7), (a \vee c \vee d, .9), (b \vee c, .8),$
$(\neg b \vee e, .2), (\neg d \vee f, .5), (a \vee b \vee \neg a, 1)\}$.

Step 1 consists in applying Lemma 1.

Example 6 *After applying step 1, the base of example 5 becomes:*
$\Sigma = \{(a \vee b, .7), (\neg a \vee c \vee \neg d, .7), (a \vee c \vee d, .9),$
$(b \vee c, .8), (\neg b \vee e, .2), (\neg d \vee f, .5)\}$.
It is obtained after removing the tautology $(a \vee b \vee \neg a, 1)$ and replacing the two formulas $(a \vee b \vee c, .7), (a \vee b \vee \neg c, .7)$ by $(a \vee b, .7)$.

Step 2 consists in constructing the graph. We start with an arbitrarily ordering of the variables $X_1, X_2, ...X_n$. Intuitively, we consider that parents of a variable X_i should be among $X_{i+1}...X_n$ (however, it can be empty). Then we decompose successively Σ into $\Sigma_{X_1} \cup \Sigma_{X_2} .... \cup \Sigma_{X_n}$ such as:

- Σ_{X_1} contains all the formulas of Σ which include an instance of X_1
- Σ_{X_2} contains all the formulas of $\Sigma - \Sigma_{X_1}$ which include an instance of X_2, and more generally,
- Σ_{X_i} contains all the formulas of $\Sigma - (\Sigma_{X_1} \cup .. \cup \Sigma_{X_{i-1}})$ which include an instance of X_i, for $i = 2, ..., n$.

The associated graph is such that nodes are the variables X_i of Σ, and parents of a variable X_i are the variables which are in Σ_{X_i}. If $\Sigma_{X_i} = \emptyset$ then the variable X_i is a root in the constructed graph.

Example 7 *Let us consider again the quantitative possibilistic knowledge base of example 6, namely*
$\Sigma = \{(a \vee b, .7), (\neg a \vee c \vee \neg d, .7), (a \vee c \vee d, .9),$
$(b \vee c, .8), (\neg b \vee e, .2), (\neg d \vee f, .5)\}$.
Σ contains six variables arbitrarily ordered in the following way: $X_1 = A, X_2 = B, X_3 = C, X_4 = D, X_5 = E, X_6 = F$. Then,we have

- $\Sigma_A = \{(a \vee b, .7), (\neg a \vee c \vee \neg d, .7), (a \vee c \vee d, .9)\}$, *Par(A)=*$\{$*B,C,D*$\}$.
- $\Sigma_B = \{(b \vee c, .8), (\neg b \vee e, .2)\}$ *; Par(B)=*$\{$*C,E*$\}$.
- $\Sigma_C = \emptyset$ *; Par(C)=* $\emptyset$.
- $\Sigma_D = \{(\neg d \vee f, .5)\}$ *; Par(D) =*$\{F\}$.
- $\Sigma_E = \emptyset$ *; Par(E)=* $\emptyset$.
- $\Sigma_F = \emptyset$ *; Par(F)=* $\emptyset$.

So we obtain the graph given in Figure 2.

Proposition 2 *Let G the graph obtained in the previous step. Then G is a DAG.*

Step 3 consists in computing conditional possibility distribution: $\Pi(X_i \mid Par(X_i))$ for each variable. The computation of conditional possibility distributions from Σ_{X_i} is obtained in three tasks:

a. Application of lemma 3 which consists in extending every formula $(x, \alpha) \in \Sigma_{X_i}$ to all instances of $X_j \in Par(X_i)$.

b. Application of lemma 4 which consists in replacing redundancies.

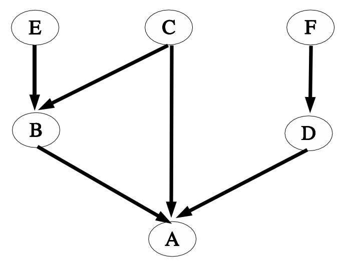

Figure 2: DAG associated with the knowledge base of example 5

c. Computation of conditional possibility degree for each instance of X_i and for each instance of parents of X_i.

The example below, illustrates the tasks (a) and (b). Example 9 illustrates task (c).

Example 8 *Let us apply tasks (a)-(b) to the possibilistic base Σ of example 6.*

Treatment of node A:

For the node A, we have Par(A)=$\{B, C, D\}$. The formulas $(a \vee b, .7), (\neg a \vee c \vee \neg d, .7)$ and $(a \vee c \vee d, .9)$ need to be extended for different instances of the parents of the variable A.

- *The extension of the formula $(a \vee b, .7)$ results in* $\{(a \vee b \vee c \vee d, .7), (a \vee b \vee \neg c \vee d, .7), (a \vee b \vee c \vee \neg d, .7), (a \vee b \vee \neg c \vee \neg d, .7)\}$.
- *The extension of the formula $(\neg a \vee c \vee \neg d, .7)$ gives:* $\{(\neg a \vee b \vee c \vee \neg d, .7), (\neg a \vee \neg b \vee c \vee \neg d, .7)\}$.
- *The extension of the formula $(a \vee c \vee d, .9)$ gives:* $\{(a \vee b \vee c \vee d, .9), (a \vee \neg b \vee c \vee d, .9)\}$.

After applying task 1 to Σ_A, we get:
$\Sigma_A = \{(a \vee b \vee c \vee d, .7), (a \vee b \vee \neg c \vee d, .7), (a \vee b \vee c \vee \neg d, .7), (a \vee b \vee \neg c \vee \neg d, .7), (\neg a \vee b \vee c \vee \neg d, .7), (\neg a \vee \neg b \vee c \vee \neg d, .7), (a \vee b \vee c \vee d, .9), (a \vee \neg b \vee c \vee d, .9)\}$.
Now, we need to apply lemma 4 in order to remove redundancies, and we obtain the final base Σ_A :
$\Sigma_A = \{(a \vee b \vee c \vee d, .97), (a \vee b \vee \neg c \vee d, .7), (a \vee b \vee c \vee \neg d, .7), (a \vee b \vee \neg c \vee \neg d, .7), (\neg a \vee b \vee c \vee \neg d, .7), (\neg a \vee \neg b \vee c \vee \neg d, .7), (a \vee \neg b \vee c \vee d, .9)\}$.

Treatment of node B:

For the node B, after applying Lemma 3 to the formulas $(b \vee c, .8), (\neg b \vee e, .2)$, we get Σ_B gives:
$\Sigma_B = \{(b \vee c \vee e, .8), (b \vee c \vee \neg e, .8), (\neg b \vee c \vee e, .2), (\neg b \vee \neg c \vee e, .2)\}$.

Treatment of node D:

$\Sigma_D = \{(\neg d \vee f, .5)\}$.

The other nodes are roots so, their respective bases are empties, $\Sigma_C = \Sigma_E = \Sigma_F = \emptyset$.

After applying Lemma 3 (extension) and Lemma 4 (redunduncies), the computation of conditional possibility degrees, for each instance of X_i and each instance of parents of X_i, becomes immediate. More precisely, let $\Sigma = \Sigma_{X_1} \cup \Sigma_{X_2} \cup ....\Sigma_{X_n}$ be the possibilistic knowledge base obtained from tasks (a)-(b). Let X_i be a variable and $Par(X_i) = \{Y_1, Y_2, ...Y_n\}$ be the set of its parents. Let x an instance of the variable X_i and $u = y_1 \wedge y_2 \wedge ... \wedge y_n$ be an instance of $Par(X_i)$. The local conditional distributions are defined as follow:

$$\Pi(x \mid u) = \begin{cases} 1 - \alpha_i & if\ (\neg x \vee \neg u, \alpha_i) \in \Sigma_{X_i} \\ 1 & \text{otherwise} \end{cases} \quad (6)$$

Example 9 *Let us continue example 8, the different bases Σ_{X_i} obtained are:*

$\Sigma_A = \{(a \vee b \vee c \vee d, .63), (a \vee b \vee \neg c \vee d, .7), (a \vee b \vee c \vee \neg d, .7), (a \vee b \vee \neg c \vee \neg d, .7), (\neg a \vee b \vee c \vee \neg d, .7), (\neg a \vee \neg b \vee c \vee \neg d, .7), (a \vee \neg b \vee c \vee d, .9)\}$.

$\Sigma_B = \{(b \vee c \vee e, .8), (b \vee c \vee \neg e, .8), (\neg b \vee c \vee e, .2), (\neg b \vee \neg c \vee e, .2)\}$.

$\Sigma_D = \{(\neg d \vee f, .5)\}$.

$\Sigma_C = \Sigma_E = \Sigma_F = \emptyset$.

The application of equation (5) gives us the local conditional possibility distributions, summarized in Tables 4-7:

Table 4: Final conditional possibility distributions of $\Pi(A \mid BCD)$

$A \mid BCD$	bcd	$bc\neg d$	$b\neg cd$	$b\neg c\neg d$
a	1	1	.3	1
$\neg a$	1	1	1	.1
$A \mid BCD$	$\neg bcd$	$\neg bc\neg d$	$\neg b\neg cd$	$\neg b\neg c\neg d$
a	1	1	1	1
$\neg a$	.3	.3	1	.03

Table 5: Final conditional possibility distributions of $\Pi(B \mid CE)$

$B \mid CE$	ce	$c\neg e$	$\neg ce$	$\neg c\neg e$
b	1	.8	1	.8
$\neg b$	1	1	.2	.2

Table 6: Final conditional possibility distributions of $\Pi(D \mid F)$

$D \mid F$	f	$\neg f$
d	1	.5
$\neg d$	1	1

Proposition 3 *Let Σ be a quantitative possibilistic base. Let Π_G be the DAG obtained from steps 1 to 3. Then,*

$$\forall \omega,\ \pi_\Sigma(\omega) = \pi_{dag}(\omega).$$

Table 7: Final conditional possibility distributions of $\Pi(C)$, $\Pi(E)$ and $\Pi(F)$

C	$\Pi(C)$		E	$\Pi(E)$		F	$\Pi(F)$
c	1		e	1		f	1
$\neg c$	1		$\neg e$	1		$\neg f$	1

Conclusion

This paper has proposed an equivalent transformation from a product-based causal networks to product-based possibilistic logic. The computational complexity of the transformation from a product-based causal network to a quantitative base is linear.

This result is similar, in term of complexity, to the transformation from a min-based causal network to a standard possibilistic base [4]. The converse transformation from a quantitative possibilistic base to a product-based causal network depends of the maximum number of parents of each variable. Thus, the cardinality of $\text{Par}(X_i)$ can be used as a criterion to rank order the variables. At each step, one can select the variable which has the least number of the parents. The transformation from a quantitative knowledge base to a graph is more interesting, in term of computation, than the transformation from a standard possibilistic base to a min-based causal network given in [4]. Indeed, the transformation given in [4] requires additional steps in order to provide the coherence of the min-based causal network. These different transformations presented in this paper will also allow bridging the gap between product-based causal network and penalty logic [8]. This link is possible given the narrow relations which exist between quantitative possibilistic logic and penalty logic[6,7].

References

1 N. Ben Amor, S. Benferhat, and K. Mellouli. Anytime propagation algorithm for min-based possibilistic graphs. Soft Computing, A fusion of foundations, methodologies and applications 8(2):150-161, 2003.

2 S.Benferhat, D.Dubois, S.Kaci and H.Prade. Graphical readings of possibilistic logic bases. UAI,2001.

3 S.Benferhat, D.Dubois, S.Kaci and H.Prade. Bridging logical, comparative and graphical possibilistic representation frameworks, ESQARU,2001.

4 S.Benferhat, D.Dubois, L.Garcia and H.Prade. On the transformation between possibilistic logic bases and possibilistic causal networks, IJAR,2002.

5 D.Dubois, H.Prade, the logical view of conditioning and its application to possibility and evidence theories, Int.J.Approx.Reason.4 (1), 23-46,1990.

6 D. Dubois and H. Prade, Epistemic entrenchment and possibilistic logic, Artificial Intelligence 50(1991), 223-239.

7 D. Dubois, J. Lang, and H. Prade, possibilistic logic, In D.MGabbay, C.J.Hogger and J.A.Robinson ,editors, Handbook of logic in Artificial Intelligence and Logic Programming, Volume 3, pages 439-513. Clarendon Press- Oxford 1994.

8 F. Dupin de Saint Cyr, J.Lang and T.Schiex. Penalty logic and its link with Dempster-Shafer theory. In proc of the 10th conf. on uncertainty in Artificial intelligence, pages 204-211 Morgan Kaufmann, 1994.

9 Fonck, P . 1992. Propagating uncertainty in a directed acyclic graph. In Proceedings of International Conference on Information Processing of Uncertainty in Knowledge Based Systems (IPMU'92), 17-20, 1992.

10 J.Gebhardt and R.Kruse, Background and perspectives of possibilistic graphical models, in proceeding of European Conference of Symbolic and Quantitative Approaches to Reasoning and Uncertainty, pages 108-121, Bad Honnef Germany, 1997.

11 F.V.Jensen, An introduction to Bayesian networks, UCL Press, University College London, 1996.

12 J.Pearl, Probabilistic Reasoning in Intelligent Systems: Networks of plausible inference, Morgan Kaufmann, Los Altos, CA, 1988.