Multi-Attribute Decision Tree Evaluation in Imprecise and Uncertain Domains

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Abstract

We present a decision tree evaluation method integrated with a common framework for analyzing multi-attribute decisions under risk, where information is numerically imprecise. The approach extends the use of additive and multiplicative utility functions for supporting evaluation of imprecise statements, relaxing requirements for precise estimates of decision parameters. Information is modeled in convex sets of utility and probability measures restricted by closed intervals. Evaluation is done relative to a set of rules, generalizing the concept of admissibility, computationally handled through optimization of aggregated utility functions. Pros and cons of two approaches, and tradeoffs in selecting a utility function, are discussed.

Introduction

In classic decision theory the different alternatives are merely objects of choice, and it is assumed that a decision maker can assign precise numerical values corresponding to the true value of each consequence, as well as precise numerical probabilities when uncertainty prevails. This is, however, seldom the case when dealing with decisions involving a set of different stakeholders with conflicting interests. Thus, the ordering of alternatives is a delicate matter and an equitable mathematical representation is crucial.

A variety of approaches for aggregating utility functions have been suggested for evaluations of decision problems involving multiple objectives. A number of techniques used in multi-attribute utility theory (MAUT) have been implemented in software such as SMART (Edwards 1977) and EXPERT CHOICE, the latter being based on the AHP method (Saaty 1980). However, most of these approaches require numerically precise data when analyzing and evaluating decision problems, a requirement that is often considered unrealistic in real-life situations. PRIME, described in (Salo and Hämäläinen 1995, 2001), is another approach to model and evaluate decision situations involving multiple attributes, supporting impreciseness of the input parameters. PRIME features a useful elicitation tour, where the decision maker makes interval-valued ratio estimates for value differences (Gustafsson et al. 2001). In the discrimination of alternatives, PRIME calculates value investigations of the problem when the value intervals are overlapping, although it is still possible to employ conventional decision rules such as *maximax* (Hurwicz 1951), *maximin* (Wald 1950), and *minimax-regret* (Savage 1951). Thus, PRIME in its current form is more concerned with the elicitation process of the input parameters, and to a less extent on evaluation techniques of imprecise data and comparisons between different courses of action. The system ARIADNE (Sage and White 1984) allows for the usage of imprecise input parameters, but does not discriminate between alternatives when these are evaluated into overlapping intervals.

The purpose of this paper is to present a method, integrated into a common framework, for multi-attribute evaluation under risk generalized to support the use of vague and numerically imprecise data. The work herein originates from earlier work on evaluating probabilistic decision situations involving a finite number of alternatives and consequences. Impreciseness is modeled in the form of interval utilities, probabilities, and weights, as well as comparisons, derived from sets of utility and probability measures. By doing so, the work conforms to classical statistical decision theory, avoiding problems with set membership functions emerging with the use of, e.g., fuzzy sets. We focus on extending the use of the simple additive utility function, often referred to as the weighted sum, and on the multiplicative utility function defined in MAUT.

Analyses with Multiple Attributes

Standard utility theory as well as multi-attribute utility theory has two important fundamentals, the first being that the preference relation $\geq_p$ ("preferred to") is transitive, and the second is that there cannot be any incomparability between two different courses of action, i.e., $\geq_p$ must be a weak order. It is important to be aware of that indifference is not the same as incomparability. When $a \sim_p b$ is concluded, a comparison must have been made resulting in that *a* and *b* belong to the same equivalence class in terms of preference.

Additive Weighting

A number of approaches to aggregate utility functions under a variety of attributes have been suggested, such as (Keeney and Raiffa 1976), (Saaty 1980), and (von Winterfeldt and Edwards 1986), where the simplest method is the *additive utility function*, sometimes referred to as the *weighted sum*.

An aggregated utility function must obey the assumption of preferential independence, i.e., when a subset of alternatives differ only on a subset $G_i \subset G$ of the set of attributes, then preferences between the alternatives must not depend on the common performance levels $G \setminus G_i$. When employing the additive utility function, the condition of *additive independence* must hold, meaning that changes in lotteries in one attribute will not affect preferences for lotteries in other attributes.

To express the relative importance of the attributes, weights are used as input parameters restricted by a normalization constraint $\sum w_j = 1$, where w_j denotes the weight of attribute G_j. A global utility function U using the additive utility function is then expressed as

$$U(x) = \sum_{i=1}^{n} w_i u_i(x) \qquad (1)$$

where w_i is the weight representing the relative importance of attribute G_i, and $u_i: X_i \rightarrow [0,1]$ is the increasing individual utility function for attribute G_i. X_i is the state space for attribute G_i. It is assumed that the u_i map to zero for the worst possible state regarding the attribute *i*, and map to one for the best. There are several different techniques for assessing the weights, e.g., pricing out (Keeney and Raiffa 1976), swing weighting (von Winterfeldt and Edwards 1986), and the reference lottery approach (Keeney and Raiffa 1976). In the reference lottery, the decision maker has two fictitious alternatives to choose from. The first alternative being a lottery, with one outcome having the best value for all attributes, and the other outcome having the worst value for all attributes. The probability p_i of ending up with the best value for all attributes, where the decision maker is indifferent between the lottery and the alternative of having the best value for attribute G_i and the worst value for all other attributes for sure, then equals w_i. Thus

$$w_i = U(u_1^-, u_2^-, ..., u_i^+, u_{i+1}^-, ..., u_n^-) \qquad (2)$$

where u_i^+ is the best utility for attribute G_i and u_j^- the worst utility for attribute G_j. If the weights are assessed in this manner but the decision maker cannot agree with assigning the weights consistent with $\sum w_j = 1$, the multiplicative model found in MAUT is more appropriate. As will be shown later, though, this has computational implications.

Multi-Attribute Utility Theory (MAUT)

The simple additive utility function ignores an important characteristic of decision problems involving multiple objectives, viz. the fact that two attributes may to some extent be substitutes or complements for one another. Therefore, the multiplicative utility function is introduced in (Keeney and Raiffa 1976). The independence axiom is introduced, meaning that preferences between two alternatives should not depend on the introduction of a third alternative. Further, every attribute must be *mutually utility independent* of all other attributes, which means that changes in sure levels of one attribute do not affect preferences for lotteries in the other attributes. In contrast to additive independence, utility independence allows the decision maker to consider two attributes to be substitutes or complements of each other. (1) is extended into a multiplicative form and the global utility function is usually expressed as

$$1 + KU(x_i) = \prod_{i=1}^{n} \left[Kk_i u_i(x_i) + 1 \right] \qquad (3)$$

where u_i: $X_i \rightarrow [0,1]$ is the increasing individual utility function for attribute G_i, and X_i is the state space for attribute G_i. As for the additive function, u_i map to zero for the worst possible state regarding attribute *i*, and map to one for the best. The constant K is the nonzero solution to

$$1 + K = \prod_{i=1}^{n} \left(1 + Kk_i\right) \qquad (4)$$

where the k_i represent scaling constants, similar in their meaning to weights, but without the normalization requirement. Further, k_i is regarded as the utility of a global outcome having the best impact on attribute G_i, and the worst on all other. The assessment of the scaling constants can be done through the reference lottery approach, such that

$$k_i = U(u_1^-, u_2^-, ..., u_i^+, u_{i+1}^-, ..., u_n^-) \qquad (5)$$

where u_i^+ is the best utility for attribute G_i and u_j^- the worst utility for attribute G_j. In the case of two attributes with scaling constants k_1 and k_2, such that $k_1 + k_2 < 1$, it can be said that they complement each other. If $k_1 + k_2 > 1$, then the attributes can be considered as substitutes of each other.

Imprecise Domains

The Bayesian decision theory is widely employed in applications where agents choose a course of action under uncertainty involving trade-offs between multiple objectives. Because of the impracticality of the requirement for precise estimates and accurate measures of utilities, probabilities and weights, there is a strong need for models that can handle and evaluate imprecise information.

A requirement for the Bayesian framework is that there is a probability distribution *P(S)* over a set of states S_i, $i = \{1, \ldots, n\}$, which summarizes the beliefs of an agent about which state S_i obtains. The theory has been motivated by various axiom systems over the last five decades, and it usually boils down to two central results: $a \geq_p b \Leftrightarrow EU(a) \geq EU(b)$ and $a \sim_p b \Leftrightarrow EU(a) = EU(b)$, where $EU(x)$ denotes the *expected utility* of x.

The requirement to provide numerically precise information within this framework has often been considered too strong in practice, see, e.g., (Fischhoff et al. 1983), (Walley 1997), (Danielson and Ekenberg 1998), and (Ekenberg and Thorbiörnson 2001). In particular, even if there is a substantial amount of empirical data, finding one single true utility function (modulo linear mappings) is not an easy task and yet it is important for usable evaluation results. However, relaxing the Bayesian requirements and supporting evaluation of imprecise statements can circumvent these disadvantages.

Definition: Let X_i be the state space for attribute G_i, and let L_i be a set of mappings $L_i = \{u_i : X_i \rightarrow [0,1]\}$ where all u_i are increasing. Given a subset $U_i \subset L_i$, such that $V_{i,x_i} = \{u_i(x_i) : u_i \in U_i\}$ is a closed interval for all $x_i \in X_i$ and $\{0, 1\} \in \cup_{x_i \in X_i} V_{i,x_i}$, then the individual utilities are defined in terms of the closed intervals V_{i,x_i}.

The imprecise probabilities in this evaluation model are represented in terms of feasible interval probabilities (Weichselberger 1999), obtained through convex hull calculations on the input parameters.

Definition: Given n attributes, let X_i be the state space for attribute G_i, and let $\Omega \subseteq \Xi = X_1 \times X_2 \times \ldots \times X_n$ be the sample space. Given a σ-field Γ of random events in Ω and an interval valued set function $P(.)$ on Γ, then the probability of state $A = (x_1, \ldots, x_n)$ given the decision maker's strategy J, is

$$P(A|J) = [L(A|J), U(A|J)],$$
$$0 \leq L(A|J) \leq U(A|J) \leq 1,$$
$$U(A|J) = 1 - L(\neg A|J).$$

It is a feasible interval probability in the sense that there exists a single probability distribution with probabilities at one of the bounds of each interval, consistent with the (1-3) Kolmogorov axioms, for all $A \in \Omega$.

The Additive Utility Function

For assessing the weights, let x_i^+ be the best state with respect to attribute i. Further, let $u_i^{\min+}$ denote the lower bound of the utility interval corresponding to x_i^+. Similar to this, let x_i^- be the worst state of the i:th attribute and let $u_i^{\max-}$ denote the upper bound of the utility interval corresponding to x_i^-. $u_i^{\max+}$ and $u_i^{\min-}$ are then defined as follows.

$$u_i^{\max+} = \max\{V_{i,x_i^+}\},\ u_i^{\min+} = \min\{V_{i,x_i^+}\},$$
$$u_i^{\max-} = \max\{V_{i,x_i^-}\},\ u_i^{\min-} = \min\{V_{i,x_i^-}\} \qquad (6)$$

From this, $w_i^{\max}$ and $w_i^{\min}$ are defined as

$$w_i^{\max} = U(u_1^{\max-}, u_2^{\max-}, \ldots, u_i^{\max+}, u_{i+1}^{\max-}, \ldots, u_n^{\max-})$$
$$w_i^{\min} = U(u_1^{\min-}, u_2^{\min-}, \ldots, u_i^{\min+}, u_{i+1}^{\min-}, \ldots, u_n^{\min-}), \qquad (7)$$

from which $w_i^{\max} \geq w_i^{\min}$ follows. Further, $w_i^{\max} + \sum_{j \neq i} w_j^{\min} = 1$.

The Multiplicative Utility Function

Allowing interval-valued utilities, this has implications on the procedure of assessing the scaling constants suggested in (3). Let $u_i^{\min-}$, $u_i^{\max+}$, $u_i^{\min-}$, and $u_i^{\max+}$ be defined according to (6). From this, $k_i^{\max}$ and $k_i^{\min}$ are defined as

$$k_i^{\max} = U(u_1^{\max-}, u_2^{\max-}, \ldots, u_i^{\max+}, u_{i+1}^{\max-}, \ldots, u_n^{\max-})$$
$$k_i^{\min} = U(u_1^{\min-}, u_2^{\min-}, \ldots, u_i^{\min+}, u_{i+1}^{\min-}, \ldots, u_n^{\min-}), \qquad (8)$$

from which $k_i^{\max} \geq k_i^{\min}$ follows. The interval-valued utilities then allow imprecision in the scaling constants k_i in terms of the intervals $[k_i^{\min}, k_i^{\max}]$. We find the two nonzero solutions to (4) by using $k_i^{\min}$ and $k_i^{\max}$ to obtain $K^{\max}$ and $K^{\min}$ respectively, where $K^{\min} \leq K^{\max}$. Thus,

$$1 + K^{\min} = \prod_{i=1}^{n} \left(1 + K^{\min} k_i^{\max}\right) \text{ and}$$
$$1 + K^{\max} = \prod_{i=1}^{n} \left(1 + K^{\max} k_i^{\min}\right). \qquad (9)$$

Definition: Given two attributes G_i and G_j, and if $k_i^{\max} + k_j^{\max} < 1$, then they are *strong complements* of each other. If $k_i^{\min} + k_j^{\min} > 1$, then they are *strong substitutes* of each other.

Thus, in the case of strong complements – the maximum of the scaling constants $k_i^{\max} + k_j^{\max}$ representing the global utility where attribute G_i is the best and G_j is the worst for $k_i^{\max}$ and vice versa for $k_j^{\max}$ – an increase in both utilities will impose a significantly higher increase in global utility than an increase in one utility independent of the other, even when we consider the upper bounds for all utility intervals. The implication of this is that these two attributes are always complementary. Similarly, when two attributes are strong substitutes, by considering the lower utility bounds and when they are substitutes in these extreme points, these two attributes are always substitutes of each other.

Modeling Decision Situations

One problem with interval statements is that the utilities may overlap. A consequence may even be ranked as being both the best and the worst for one attribute. Thus, we need to evaluate the data according to different evaluation techniques beside the pointwise maximization of the expected utility. For this, the framework presented in (Danielson and Ekenberg 1998, 1999) is suggested.

In this framework, given a decision tree T, a decision node D can be considered a set $S = \{S_1, \ldots, S_r\}$ of strategies, i.e., the directed edges from D leading to S_i:s. Two collections of variable assignments and constraints (bases) are associated with D, one containing the probabilities of the edges from each S_i (probability base), and one containing the utility variables corresponding to possible consequence nodes (leaves) emanating from each S_i (utility base). The probability and utility bases then comprise the local decision frame corresponding to D and attribute G, $\langle P^G(D), V^G(D), T\rangle$. The user (human / machine) may give probability and utility assessments with respect to the tree, so that the probability and utility bases are local to each attribute.

The decision maker's probability and utility measures are represented by linear constraints in these variables. Interval statements are translated into inequality pairs, e.g., $P(A) \in [L(A), U(A)]$ is turned into the pair $P(A) \geq L(A)$ and $P(A) \leq U(A)$, and comparative statements are translated into inequalities such as $u_i(A) > u_i(B)$. Using this structure, numerically imprecise assessments can be represented and evaluated. The inequalities containing utility variables are included in a utility base $V(D)$, and inequalities containing probability variables are included in a probability base $P(D)$.

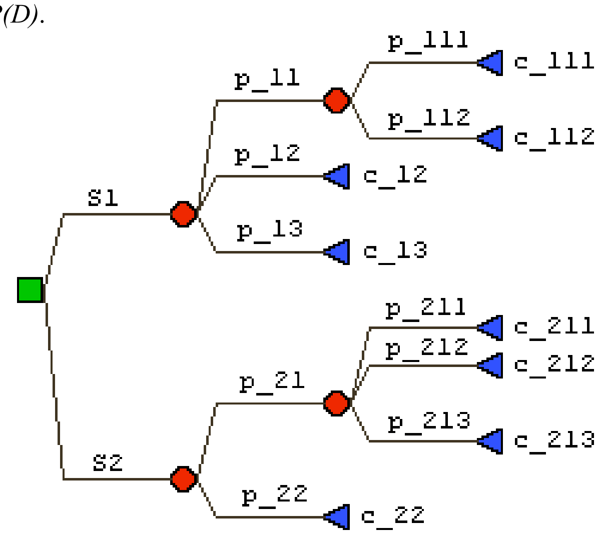

Figure 1: A decision tree with two strategies.

Definition: Given a decision tree, a sequence of edges $[S_1, \ldots, S_r]$ is a *strategy* if for all elements in the set, S_{i-1} is a directed edge from a decision node to a chance node C_{i-1}, and there is a directed edge from C_{i-1} to a decision node from which S_i is a directed edge.

Without loss of generality, assume in the following definition that the tree contains at most one decision node (labeled with the highest index) in each alternative (consequence set) having s_i number of consequences.

Definition: Given an attribute G, a decision tree associated with G, and a strategy $[S_1, \ldots, S_r]$, where each S_i is an alternative $\{c_{i1}, \ldots, c_{is_i}, D_{i(s_i+1)}\}$ associated with chance node C_i. The *expected utility of* $[S_1, \ldots, S_r]$ *with respect to attribute* G, $E^G(S_1, \ldots, S_r)$, is defined by the following:

(i) $E^G(S_i) = \sum_{k \leq s_i} p_{ik} \cdot u_{ik}$, when S_i is an alternative $\{c_{i1}, \ldots, c_{is_i}\}$ and

(ii) $E^G(S_1, \ldots, S_r) = (\sum_{k \leq s_i} p_{ik} \cdot u_{ik}) + p_{i(s_i+1)} \cdot E^G(S_{i+1}, \ldots, S_r)$, when S_i is an alternative $\{c_{i1}, \ldots, c_{is_i}, D_{i(s_i+1)}\}$, where u_{ij} denotes the individual utility of the consequence c_{ij}, and p_{ij} denotes the probability of the consequence c_{ij} (or D_{ij}), with respect to the attribute G.

Definition: Given a set of attributes $\{G_1, \ldots, G_n\}$, n decision trees (each associated with exactly one attribute) and a strategy $[S_1, \ldots, S_r]$, the *additive expected utility of* $[S_1, \ldots, S_r]$, $U_A(S_1, \ldots, S_r)$, is:

$$U_A(S_1, \ldots, S_r) = \sum_{i \leq n} E^{G_i}(S_1, \ldots, S_r) \cdot w_i \ , \qquad (10)$$

where w_i is the weight of attribute G_i. Further, the *multiplicative global expected utility of* $[S_1, \ldots, S_r]$, $U_M(S_1, \ldots, S_r)$, is defined as:

$$U_M(S_1, \ldots, S_r) = \frac{\left(\prod_{i \leq n} (1 + K k_i E^{G_i}(S_1, \ldots, S_r))\right) - 1}{K} \qquad (11)$$

where K and k_i are scaling constants as defined in (4) and (5).

By applying (11) and using the intervals $[K^{\min}, K^{\max}]$, and $[k_i^{\min}, k_i^{\max}]$, $1 \leq i \leq n$, together with $V(D)$ and $P(D)$, the computational framework presented in (Danielson and Ekenberg 1998) can be employed for the evaluation of different strategies by converting the statements into linear constraints represented in a system of inequalities.

Evaluation of Decision Situations

When allowing intervals for all decision parameters, it is not clear how to employ the principle of maximizing the expected utility as decision criteria. Furthermore, when dependencies between strategies are present, e.g., as variable identities or comparative statements, pairwise comparisons must be performed. For example, two strategies may alone show similar expected utilities, however a pairwise comparison may reveal that due to dependencies one strategy may be strictly preferred.

Hull Cut

The selection problem is easy under circumstances where $\min(U(S_{11}, \ldots, S_{1r}) - U(S_{21}, \ldots, S_{2s})) > 0$ holds, however this is not always the case. A way to refine the analysis is therefore to investigate how much the different interval widths can be decreased before an expression such as $U(S_{11},...,S_{1r}) - U(S_{21},...,S_{2s}) > 0$ ceases to be consistent. For this purpose, the *hull cut* is introduced in the framework. The hull cut can be seen as generalized sensitivity analyses to be carried out to determine the stability of the relation between the consequence sets under consideration. The hull cut avoids the complexity in combinatorial analyses, but it is still possible to study the stability of a result by gaining a better understanding of how important the interval boundary points are.

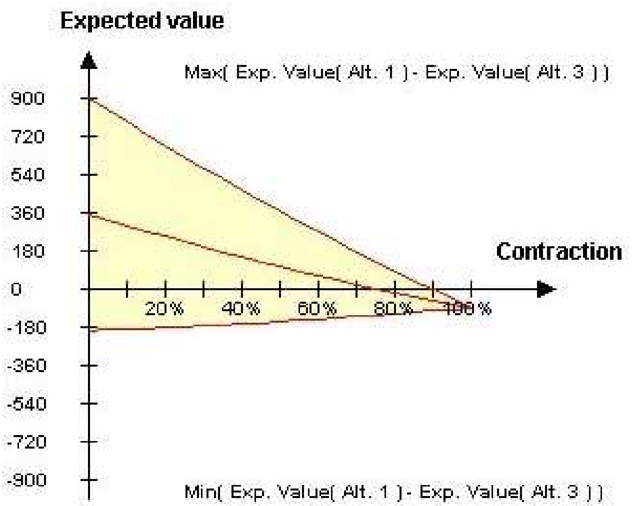

Figure 2: Graphical illustration of a pairwise comparison using *Decide*IT (Danielson et al. 2003). The upper line depicts $\max(U(S_{11},...,S_{1r}) - U(S_{21},...,S_{2s}))$, the lower line is $\min(U(S_{11},...,S_{1r}) - U(S_{21},...,S_{2s}))$, the middle line is the average of the upper and lower values. Along the horizontal axis is the level of hull cut ("contraction").

By co-varying the cutting of an arbitrary set of intervals, it is possible to gain much better insight into the influence of the structure of the information frame on the solutions. Intuitively, these hull cuts are based on values closer to the center of the interval being more reliable than boundary points, i.e., there is an underlying assumption that the second-order belief distributions over the intervals have a mass concentrated to the center. These concepts are more thoroughly described in (Danielson and Ekenberg 1998), (Danielson et al. 2003), and (Ekenberg and Thorbiörnson 2001).

Evaluation

The computational evaluation of the decision problem requires optimization of expressions such as $\max(U(S_{i1},...,S_{ir}) - U(S_{j1},...,S_{js}))$. This puts differing demands on the computational procedure depending on the utility function employed. The difference has two sources, variable identity and multi-linearity.

The variable identity in the evaluation derives from the identity of the weights and scaling constants.

Proposition: Let $\max({}^{A}\delta_{ij}) = \max(U_A(S_{i1},...,S_{ir}) - U_A(S_{j1},...,S_{js}))$ denote the maximized difference in additive global expected utility for strategy $[S_{i1},...,S_{ir}]$ over strategy $[S_{j1},...,S_{js}]$. Then, $\max({}^{A}\delta_{ij})$ is correctly calculated only if the variable identity of the weights is taken into account.

$\max({}^{A}\delta_{ij})$ is $\max(U_A(S_{i1},...,S_{ir}) - U_A(S_{j1},...,S_{js}))$, thus it can be written:

$$\max(\sum_{l\le n} E^{G_l}(S_{i1},...,S_{ir})\cdot w_l - \sum_{l\le n} E^{G_l}(S_{j1},...,S_{jr})\cdot w_l) =$$

$$\max(\sum_{l\le n} E^{G_l}(S_{i1},...,S_{ir}) - E^{G_l}(S_{j1},...,S_{jr}))\cdot w_l \,. \qquad (12)$$

This shows the importance of the variable identity. Without considering it, the same weight (being a variable) could obtain differing numerical values in the calculation of $U_A(S_{i1},...,S_{ir})$ and $U_A(S_{j1},...,S_{js})$. Similarly, let $\max({}^{M}\delta_{ij}) = \max(U_M(S_{i1},...,S_{ir}) - U_M(S_{j1},...,S_{js}))$ denote the maximized difference in multiplicative global expected utility for strategy $[S_{i1},...,S_{ir}]$ over strategy $[S_{j1},...,S_{js}]$. Analogous reasoning yields the variable identity of the scaling constants.

Without these identities, the optimization procedure of finding $\max({}^{A}\delta_{ij})$ and $\max({}^{M}\delta_{ij})$ would yield incorrect evaluation results. The sub-results from calculating $U(S_{i1},...,S_{ir})$ and $U(S_{j1},...,S_{js})$ would be incomparable since they derive from different solution vectors in the solution space. This is due to the necessary pairwise comparisons of strategies, where the imprecision otherwise will allow for the same weights to assume different values. Thus, variable identity is a necessity when calculating maximum differences in global expected utilities, and it is also required when using other decision rules based on such differences, e.g., minimax-regret.

Optimization

Optimization algorithms are required to find the maximum and minimum differences in global expected utility between two alternatives.

Proposition: Finding the max of the multiplicative utility function $\max({}^{M}\delta_{ij}) = \max(U_M(S_{i1},...,S_{ir}) - U_M(S_{j1},...,S_{js}))$ requires multi-linear optimization.

To see this, $\max({}^{M}\delta_{ij})$ can be written as

$$\max(\frac{\prod_{l\le n}(1+Kk_l E^{G_l}(S_{i1},...,S_{ir}))-1}{K} - \frac{\prod_{l\le n}(1+Kk_l E^{G_l}(S_{j1},...,S_{js}))-1}{K}) =$$

$$\frac{1}{K}\max(\prod_{l\le n}(1+Kk_l E^{G_l}(S_{i1},...,S_{ir})) - \prod_{l\le n}(1+Kk_l E^{G_l}(S_{j1},...,S_{js})))$$

which cannot be transformed into a bilinear form. This makes it less desirable than the additive approach from a computational viewpoint. Max($^{M}\delta_{ij}$) can only be found with non-linear programming algorithms, which are too slow and timewise unpredictable for real-time interactive use. Further, such algorithms only guarantee a local optimum, which is not appropriate for decision evaluation. Using the additive utility function, as can be seen from (12) the problem can be reduced into a bilinear problem, rendering it possible to directly employ the LP based algorithms suggested in (Danielson and Ekenberg 1998) to find each max($^{A}\delta_{ij}$). LP algorithms such as Simplex are fast and predictable. The solution is global, i.e. the appropriate result is always obtained. This enables the inclusion of decision algorithms in interactive applications.

Concluding Remarks

We have presented a method for multi-attribute decision evaluation, integrated into a common framework for analyzing decision under risk where the information is vague and numerically imprecise. The use of decision trees as a component in the decision model makes it appealing from a user perspective. Using such a framework, we have demonstrated how decision problems can be modeled and evaluated taking into account the attributes/criteria, weights, probabilities and utilities involved. We have also pointed out that the optimization procedure must be extended to support variable identities for obtaining accurate and reliable evaluation results when performing pairwise comparisons between strategies, which are necessary for correct results. Furthermore, employing the multiplicative utility function found in MAUT requires multi-linear optimization, while employing the additive utility function only requires optimization of bilinear expressions. The latter can be performed using LP methods. Thus, there is an inherent trade-off between computational aspects (the additive approach) and power of expression (the multiplicative approach).

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