Ranking Sets of Possibly Interacting Objects Using Shapley Extensions

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Abstract

We deal with the problem of how to extend a preference relation over a set X of "objects" to the set of all subsets of X. This problem has been carried out in the tradition of the literature on extending an order on a set to its power set with the objective to analyze the axiomatic structure of families of rankings over subsets. In particular, most of these approaches make use of axioms aimed to prevent any kind of interaction among the objects in X.

In this paper, we apply coalitional games to study the problem of extending preferences over a finite set X to its power set 2^X. A coalitional game can be seen as a numerical representation of a preference extension on 2^X. We focus on a particular class of extensions on 2^X such that the ranking induced by the Shapley value of each coalitional game representing an extension in this class, coincides with the original preference on X.

Some properties of Shapley extensions are discussed, with the objective to justify and contextualize the application of Shapley extensions to the problem of ranking sets of possibly interacting objects.

1 Introduction

A lot of problems in individual and collective decision making involve the comparison of *sets of objects*, where objects may have very different meanings (e.g., alternatives, opportunities, candidates, etc.). Consider, for instance, the comparison of the stability of groups in coalition formation theory, or the ranking of likely sets of events in the axiomatic analysis of subjective probability, or the evaluation of equity of sets of rights inside a society, or the comparison of assets in portfolio analysis, etc. In all of those situations, a ranking of the single elements of a (finite) universal set X is not sufficient to compare the subsets of X. On the other hand, for many practical problems, only the information about preferences among single objects is available. Consequently, a central question is: how to derive a ranking over the set of all subsets of X in a way that is "compatible" with the primitive ranking over the single elements of X?

This question has been carried out in the tradition of the literature on extending an order on a set to its power set with the objective to axiomatically characterize families of ordinal preferences over subsets (Barberà, Barrett, and Pattanaik 1984; Barberà, Bossert, and Pattanaik 2004; Bossert 1995; Bossert et al. 1994; Geist and Endriss 2011; Fishburn 1992; Kannai and Peleg 1984; Kreps 1979). In this context, an order $\sqsupseteq$ on the power set 2^X is an *extension* of a primitive order $\succcurlyeq$ on X if and only if the relative ranking of any two singleton sets according to $\sqsupseteq$ is the same as the relative ranking of the corresponding alternatives according to $\succcurlyeq$.

The interpretation of the properties used to characterize extensions is deeply interconnected to the meaning that is attributed to sets. According to the survey of Barberà, Bossert, and Pattanaik (2004), the main contributions from the literature on ranking sets of objects may be grouped in three main classes of problems: 1) *complete uncertainty*, where a decision maker is asked to rank sets which are considered as formed by mutually exclusive objects (i.e., only one object from a set will materialize), and taking into account that he cannot influence the selection of an object from a set (Barberà, Barrett, and Pattanaik 1984; Kannai and Peleg 1984; Nitzan and Pattanaik 1984); 2) *opportunity sets*, where sets contain again mutually exclusive objects but, in this case, a decision maker compares sets taking into account that he can select a single element (and only one) from a set (Bossert et al. 1994; Kreps 1979; Puppe 1996); 3) *sets as final outcomes*, where each set contains objects that are assumed to materialize simultaneously, if that set is selected (Bossert 1995; Fishburn 1992; Roth 1985).

This paper is devoted to the analysis of extensions for problems of the third class, where sets are formed by objects that are assumed to materialize at the same time. This situation can be observed in many different contexts like, for example, the *college admission problem* (Gale and Shapley 1962; Roth 1985), where different colleges need to rank sets of students based on their ranking of individual applicants. For these kind of problems, most of the axiomatic approaches from the literature focused on properties suggesting that the *interaction* among single objects should not play a relevant role in establishing the ranking among subsets (Bossert 1995; Roth 1985). For instance, the property of *responsiveness*, introduced by Roth (1985), says that a set $S \subseteq X$ is preferred to a set $T \subseteq X$ whenever S is obtained from T by replacing some object $t \in T$ with another $n \in X$ not in T which is preferred to t (according to the primitive ranking on the universal set X). As a consequence, the responsiveness property prevents *complementarity* effects among objects within sets of the same cardinality.

Another property that excludes a certain kind of interaction among objects of the universal set X is the property of *monotonicity* (with respect to set inclusion) (Barberà, Bossert, and Pattanaik 2004; Kreps 1979; Puppe 1996). This property states that a set $T \subseteq X$ is preferred to a set $S \subseteq X$ whenever S is a subset of T. Therefore, monotonicity excludes the possibility that the evaluation of a set could be deteriorated by the addition of a new object that is *incompatible* or *redundant* with objects already contained in the set.

On the other hand, in many practical problems, the attitude to interact among elements of X cannot be excluded *a priori*. To be more specific, consider a well known problem in computational biology, where statistical testing procedures are used to detect genes which are "strongly" differentially expressed between two conditions (e.g., case-control studies). Following this approach, genes with the highest (individual) discriminative power are often selected for further study (Moretti et al. 2008). But genes in a set can interact in ways that increase (via complementarity), or decrease (via incompatibility or redundancy), the overall valuation of a set of genes in characterizing a certain condition. Therefore, a procedure aimed to consider the effects of interaction among genes in each possible subset, with respect to the analysis of individual behaviors, is demanded.

In this paper, we introduce a new class of extensions for the problem of ranking sets of objects, and we call the elements of this class *Shapley extensions*, for their attitude to preserve the ranking provided by the *Shapley value* (Shapley 1953; Shapley and Shubik 1954) of associated *coalitional games* (Owen 1995). We show that Shapley extensions are able to keep into account possible interaction effects of different nature (Section 3). Moreover, we analyze the behavior of Shapley extensions with respect to properties which are aimed to prevent the interaction among objects and we axiomatically characterize a subclass of monotonic Shapley extensions (Section 4). Finally, in Section 5, we provide some sufficient conditions for extensions to be Shapley extensions in the presence of interactions among objects, and we compare Shapley extensions with other extensions based on *regular semivalues* (Carreras and Freixas 1999; 2000). From the game theoretic perspective, as a side-product of Sections 4 and 5 we also provide some results concerning the ordinal equivalence of regular semivalues (Tomiyama 1987; Carreras and Freixas 2008; Freixas 2010).

We start in the next section with some notations for binary relations and some basic definitions from cooperative game theory.

2 Preliminaries and notations

Let X be a finite set of objects. We denote by 2^X the *power set* (the set of all subsets) of X and by $|X|$ the *cardinality* (the number of elements) of X. To denote a subset S of X we indifferently use the notation $S \subseteq X$ or $S \in 2^X$; $S \subset X$ means $S \subseteq X$ and $S \neq X$. A *binary relation* on X is denoted by $\succcurlyeq \subseteq X \times X$. For each $x, y \in X$, the notation $x \succcurlyeq y$ will frequently be used instead of $(x, y) \in \succcurlyeq$ in order to simplify the exposition.

The following are some standard properties for a binary relation $\succcurlyeq \subseteq X \times X$: (*reflexivity*:) for each $x \in X$, $x \succcurlyeq x$; (*transitivity*:) for each $x, y, z \in X$, $x \succcurlyeq y$ and $y \succcurlyeq z \Rightarrow x \succcurlyeq z$; (*completeness*:) for each $x, y \in X$, $x \neq y \Rightarrow x \succcurlyeq y$ or $y \succcurlyeq x$; (*antysymmetry*:) for each $x, y \in X$, $x \succcurlyeq y$ and $y \succcurlyeq x \Rightarrow x = y$. A *total preorder* on X is a reflexive, transitive and complete binary relation $\succcurlyeq \subseteq X \times X$. A reflexive, transitive, complete and antisymetric binary relation is called *total order* or *linear order*.

We interpret a total preorder $\succcurlyeq$ on X as a *preference relation* on X (that is, for each $x, y \in X$, $x \succcurlyeq y$ stands for 'x is considered at least as good as y according to $\succcurlyeq$'). The strict preference relation $\succ$ and the indifference relation $\sim$ are defined by letting, for all $x, y \in X$, $x \succ y$ if and only if $x \succcurlyeq y$ and not $y \succcurlyeq x$; $x \sim y$ if and only if $x \succcurlyeq y$ and $y \succcurlyeq x$.

A map $u : X \to I\!R$ is a numerical representation of the preference relation $\succcurlyeq$ on X if for every $i, j \in X$ we have that

$$u(i) \geq u(j) \Leftrightarrow i \succcurlyeq j.$$

In order to rank the elements of 2^X, we use a total preorder $\sqsupseteq$ on 2^X, with strict preference relation denoted by $\sqsupset$ and indifference relation denoted by $\backsimeq$. Given a total preorder $\succcurlyeq$ on X, we say that a total preorder $\sqsupseteq$ on 2^X is an *extension* of $\succcurlyeq$ if and only if the relative ranking of any two singleton sets according to $\sqsupseteq$ is the same as the relative ranking of the corresponding alternatives according to $\succcurlyeq$ (i.e., for each $x, y \in X$, $\{x\} \sqsupseteq \{y\} \Leftrightarrow x \succcurlyeq y$).

Let $S \in 2^X \setminus \{\emptyset\}$. The set of *best* elements in S according to a binary relation $\succcurlyeq$ on X is given by $\mathbf{B}(S, \succcurlyeq) = \{x \in S | x \succcurlyeq y \; \forall y \in S\}$, and the set of *worst* elements in S according to $\succcurlyeq$ is given by $\mathbf{W}(S, \succcurlyeq) = \{x \in S | y \succcurlyeq x \; \forall y \in S\}$.

Perhaps the simplest extension rule is the *maxi-max criterion* on $\succcurlyeq$, which is defined as a binary relation $\sqsupseteq^{\max}$ on 2^X such that $(S \sqsupseteq^{\max} T) \Leftrightarrow (b_S \succcurlyeq b_T)$, where $b_S \in \mathbf{B}(S, \succcurlyeq)$ and $b_T \in \mathbf{B}(T, \succcurlyeq)$ for each $S, T \in 2^X \setminus \{\emptyset\}$. Similarly, the *maxi-min criterion*, is defined as a binary relation $\sqsupseteq^{\min}$ on 2^X such that $(S \sqsupseteq^{\min} T) \Leftrightarrow (w_S \succcurlyeq w_T)$, where $w_S \in \mathbf{W}(S, \succcurlyeq)$ and $w_T \in \mathbf{W}(T, \succcurlyeq)$ for each $S, T \in 2^X \setminus \{\emptyset\}$.

Now, let us introduce some basic game theoretical notations. A *coalitional game* or *characteristic-form game* is a pair (N, v), where N denotes a finite set of *players* and $v : 2^N \to I\!R$ is the *characteristic function*, with $v(\emptyset) = 0$. If the set N of players is fixed, we identify a coalitional game (N, v) with the corresponding characteristic function v. A group of players $T \subseteq N$ is called a *coalition* and $v(T)$ is called the *value* of this coalition.We will denote by $\mathcal{G}$ the class of all coalitional games. Let $\mathcal{C} \subseteq \mathcal{G}$ be a subclass of coalitional games. Given a set of players N, we denote by $\mathcal{C}^N \subseteq \mathcal{G}$ the class of coalitional games in $\mathcal{C}$ with N as the set of players.

A *payoff vector* or *allocation* $(x_1, \ldots, x_n)$ of a coalitional game (N, v) is an $|N|$-dimensional vector describing the payoffs of the players, such that each player $i \in N$ receives x_i. A *one-point solution* (or simply a *solution*) for a class

$\mathcal{C}^N$ of coalitional games is a function ψ that assigns a payoff vector $\psi(v)$ to every coalitional game in the class, that is $\psi : \mathcal{C}^N \to I\!R^N$.

The most famous solution in the theory of coalitional games is the *Shapley value*, introduced by (Shapley 1953; Shapley and Shubik 1954). The Shapley value of a coalitional game is an effective tool to convert information about the worth that subsets of the player set can achieve into a personal attribution (of payoff) to each of the players (see also Moretti and Patrone (2008) for an overview on different axiomatic characterizations of the Shapley value), so that players can evaluate ex-ante the convenience to participate to the game. The Shapley value may be computed according to the following formula (Shapley 1953; Shapley and Shubik 1954)

$$\phi_i(v) = \sum_{S \subset N : i \notin S} \frac{|S|!(|N| - |S| - 1)!}{|N|!} m_i^S(v) \qquad (1)$$

for each $i \in N$, where the quantity $m_i^S(v) = v(S \cup \{i\}) - v(S)$ is the *marginal contribution* of player i to coalition S, for each $S \subset N$ with $i \notin S$.

3 Shapley extensions

In this section, we introduce a new family of extensions, with the purpose to take account of possible effects of interaction among objects of the universal set. To clarify what we mean for effects of interaction, consider the following simple example. Let $X = \{x, y, z\}$ be a set of three objects (e.g., goods) and suppose that an agent's preference is such that $x \succcurlyeq y$, $x \succcurlyeq z$ and $y \succcurlyeq z$. Trying to extend $\succcurlyeq$ to 2^X, one could guess that set $\{x, y\}$ is better than $\{y, z\}$, because the agent will receive both y and x instead of y and z (and x is preferred to z). However, due to possible effects of complementarity between x and z (or because of incompatibility between x and y) the relative preference between the two sets could be reversed. As a "practical" instance of this issue, you may consider the problem of what to take on a backpacking trip on the mountains: a 'bottle of water' can be more essential (thus ranked higher) than a 'bottle of orange juice' or than a 'sandwich', and a 'bottle of orange juice' could be preferred (e.g., for dietary reasons) to a 'sandwich'. But if the problem is now which pair of the three to put in the backpack, a 'bottle of water' and a 'bottle of orange juice' together may be less preferred (because of the backpack weight) than a 'bottle of water' and a 'sandwich' together.

Moreover, an effect of incompatibility between two or more objects, for instance, between x and y, could produce a ranking where the single object $\{y\}$ is preferred to $\{x, y\}$, even if the latter includes the most preferred object x, other than y. For instance, this could be the case where x and y represent two distinct therapies for a disease: the combination of two treatments does not always improve the chances of success, and may provoke more serious side effects, with respect to each single treatment.

As we already said, the attitude to interact among elements of X cannot be excluded a priori. Nevertheless, the only information available in the model is the primitive ranking $\succcurlyeq$ on X, and the effects of interaction that can be considered should be compatible with this information. Therefore, our objective is to characterize those extensions which are able to take into account possible interaction effects without distorting the information provided by the primitive ranking on X.

Note that, given a total preorder $\sqsupseteq$ on 2^X, any numerical representation $v : 2^X \to I\!R$ of $\sqsupseteq$ (with the convention that $v(\emptyset) = 0$) is a coalitional game $v \in \mathcal{G}^X$ (we denote by $\mathcal{G}^X_{\sqsupseteq}$ the class of coalitional games that numerically represent $\sqsupseteq$). Consequently, by relation (1), the Shapley value $\phi(v)$, for each game $v \in \mathcal{G}^X_{\sqsupseteq}$, can be interpreted as a personal attribution of the importance of elements of X accounting for their (weighted) average marginal contributions over all possible coalitions (Shapley 1953; Shapley and Shubik 1954). Clearly, such an attribution of importance depends on the relative worths of coalitions in the game v, and different numerical representations of the same total preorder on 2^X may induce completely different Shapley values. In order to preserve the original information concerning the preference relation $\succcurlyeq$ over the elements of X, we focus our attention to those extensions where the Shapley value of the corresponding numerical representations provides the same ranking of $\succcurlyeq$ on X (independently from the numerical representation selected in $\mathcal{G}^X_{\sqsupseteq}$).

The following definition formally introduces the notion of Shapley extension. Note that every total preorder $\sqsupseteq$ on 2^X can be seen as an extension of a total preorder $\succcurlyeq$ on X such that

$$i \succcurlyeq j \Leftrightarrow \{i\} \sqsupseteq \{j\} \qquad (2)$$

for each $i, j \in X$. Therefore, in the remaining of the paper we will implicitly refer to a total preorder $\sqsupseteq$ on 2^X as an extension of the preference relation $\succcurlyeq$ on X induced by $\sqsupseteq$ according to relation (2).

Definition 1. *A total preorder $\sqsupseteq$ on 2^X is a Shapley extension iff for each numerical representation $v \in \mathcal{G}^X_{\sqsupseteq}$ of $\sqsupseteq$ we have that*

$$\{i\} \sqsupseteq \{j\} \Leftrightarrow \phi_i(v) \geq \phi_j(v)$$

for all $i, j \in X$.

The next example, providing a Shapley extension on a set $X = \{x, y, z\}$, shows that some effects of interaction may be represented by a Shapley extension.

Example 1. *Let $X = \{1, 2, 3\}$ and let $\sqsupset^a$ by a linear order on 2^X such that $\{1, 2, 3\} \sqsupset^a \{3\} \sqsupset^a \{2\} \sqsupset^a \{1, 3\} \sqsupset^a \{2, 3\} \sqsupset^a \{1\} \sqsupset^a \{1, 2\} \sqsupset^a \emptyset$ (note that, according to relation (2), $\sqsupset^a$ is an extension of $3 \succ 2 \succ 1$). Note also that, individually, element 2 is preferred to element 1, but when combined to element 3, we have that $\{1, 3\}$ is preferred to $\{2, 3\}$ (as a consequence of complementarity effects between elements 3 and 1, or of incompatibility effects between 3 and 2). In addition, there are other comparisons that may be ascribed to the effects of interaction among elements: for instance, the fact that the singleton $\{1\}$ is preferred to set $\{1, 2\}$ could be explained as an incompatibility between 1 and 2.*

Using relation (1) (or relation (3) below), it easy to check that for every numerical representation v of $\sqsupseteq^a$ in $\mathcal{G}^X_{\sqsupseteq^a}$ we have that

$$\phi_2(v)-\phi_1(v) = \frac{1}{2}\big(v(2)-v(1)\big)+\frac{1}{2}\big(v(2,3)-v(1,3)\big) > 0,$$

where the inequality follows from the fact that $v(2)-v(1) > v(1,3)-v(2,3)$ for every $v \in \mathcal{G}^X_{\sqsupseteq^a}$.

Following similar calculations, we also have that for every $v \in \mathcal{G}^X_{\sqsupseteq^a}$

$$\phi_3(v)-\phi_2(v) = \frac{1}{2}\big(v(3)-v(2)\big)+\frac{1}{2}\big(v(1,3)-v(1,2)\big) > 0,$$

where the inequality follows from the fact that $v(3) > v(2)$ and $v(1,3) > v(1,2)$ for each $v \in \mathcal{G}^X_{\sqsupseteq^a}$.

The following proposition is essential for the analysis of Shapley extensions that will be provided in Section 4 and 5.

Proposition 1. *Let (X, v) be a coalitional game. For each $i, j \in X$ the following relation holds:*

$$\phi_i(v) - \phi_j(v) = \sum_{S\subset N: i,j\notin S} \frac{|S|!(|N|-|S|-2)!}{(|N|-1)!}\, d^S_{ij}(v), \tag{3}$$

where $d^S_{ij}(v) = v(S\cup\{i\}) - v(S\cup\{j\})$, for each $S \in 2^X$ with $i, j \notin S$.

Proof. First note that

$$\begin{aligned} & m^S_i(v) - m^S_j(v) = \\ & v(S\cup\{i\}) - v(S) - (v(S\cup\{j\}) - v(S)) = \\ & d^S_{ij}(v), \end{aligned} \tag{4}$$

and

$$\begin{aligned} & m^{S\cup\{j\}}_i(v) - m^{S\cup\{i\}}_j(v) = \\ & v(S\cup\{i,j\}) - v(S\cup\{j\}) - \\ & (v(S\cup\{i,j\}) - v(S\cup\{i\})) = \\ & d^S_{ij}(v). \end{aligned} \tag{5}$$

Then, we have that

$$\begin{aligned} & \phi_i(v) - \phi_j(v) = \\ & \textstyle\sum_{S\subset X: i,j\notin S} \Big(\frac{|S|!(|N|-|S|-1)!}{|N|!}\, m^S_i(v) + \frac{(|S|+1)!(|N|-|S|-2)!}{|N|!}\, m^{S\cup\{j\}}_i(v)\Big) - \\ & \textstyle\sum_{S\subset X: i,j\notin S} \Big(\frac{|S|!(|N|-|S|-1)!}{|N|!}\, m^S_i(v) + \frac{(|S|+1)!(|N|-|S|-2)!}{|N|!}\, m^{S\cup\{j\}}_i(v)\Big) = \\ & \textstyle\sum_{S\subset X: i,j\notin S} \Big(\big(\frac{|S|!(|N|-|S|-1)!}{|N|!} + \frac{(|S|+1)!(|N|-|S|-2)!}{|N|!}\big)\, d^S_{ij}(v)\Big) = \\ & \textstyle\sum_{S\subset X: i,j\notin S} \frac{|S|!(|N|-|S|-2)!}{(|N|-1)!}\, d^S_{ij}(v), \end{aligned}$$

where the first equality follows from relation (1), and the second one from relations (4) and (5). □

Remark 1. *Let (X, v) be a coalitional game. As a consequence of Proposition 1, we have that if $v(S\cup\{i\}) \geq v(S\cup\{j\})$ for each $S \in 2^X$ with $i, j \notin S$, then we have that $\phi_i(v) \geq \phi_j(v)$. It immediately follows that an extension $\sqsupseteq$ on 2^X such that*

$$\{i\} \sqsupseteq \{j\} \Rightarrow (S\cup\{i\}) \sqsupseteq (S\cup\{j\})$$

for each $i, j \in X$ and each $S \in 2^X$, with $i, j \notin S$, is a Shapley extension, since for every numerical representation v of $\sqsupseteq$ in $\mathcal{G}^X_{\sqsupseteq}$, we also have that $v(S\cup\{i\}) \geq v(S\cup\{j\})$.

By Remark 1, it follows that the maxi-min extension and the maxi-max extension are Shapley extensions, since according to those procedures, adding an element i to S with $\{i\} \sqsupseteq \{j\}$, always improves the relative ranking of the best (worst) elements of $S\cup\{i\}$ with respect to the best (worst) elements of $S\cup\{j\}$. Same considerations apply to the *lexi-min* and the *lexi-max* extensions (Barberà, Bossert, and Pattanaik 2004; Bossert 1995; Pattanaik and Peleg 1984), which are obtained, respectively, as the lexicographical generalizations of the maxi-min and the maxi-max extensions. Other examples of Shapley extensions are provided by certain *median-based* extensions (Nitzan and Pattanaik 1984), where the relative ranking of the median alternatives is used as the criterion for comparing two sets.

On the other hand, note that extensions are not necessarily Shapley extensions, as it is shown by the following example.

Example 2. *Consider the set $X = \{1, 2, 3\}$ and take the linear order $\sqsupseteq^{a'}$ such that $\{1,2,3\} \sqsupset^{a'} \{2\} \sqsupset^{a'} \{3\} \sqsupset^{a'} \{1,3\} \sqsupset^{a'} \{2,3\} \sqsupset^{a'} \{1\} \sqsupset^{a'} \{1,2\} \sqsupset^{a'} \emptyset$. Note that only the relative ranking between $\{2\}$ and $\{3\}$ is changed with respect to $\sqsupseteq^a$. So, according to relation (2), $\sqsupseteq^{a'}$ is an extension of $2 \succ' 3 \succ' 1$, but it is not a Shapley extension. In fact, a game that represents $\sqsupseteq^{a'}$ is $v(\{1,2,3\}) = 16$, $v(\{2\}) = 9$, $v(\{3\}) = 8$, $v(\{1,3\}) = 7$, $v(\{2,3\}) = 6$, $v(\{1\}) = 5$, $v(\{1,2\}) = 4$, $v(\emptyset) = 0$ and, by relation (1), the corresponding Shapley value is $\phi_1(v) = 4$, $\phi_2(v) = 5.5$, $\phi_3(v) = 6.5$.*

4 Shapley extensions without interaction

A standard application of the problem of deriving a ranking $\sqsupseteq$ on 2^X from a preference relation over the single elements of X is the *college admission problem* (see, for instance, (Roth 1985; Gale and Shapley 1962)), where colleges need to rank sets of students based on their ranking of individual applicants.

For the analysis of the college admission problem, Roth (1985) introduced the property of *responsiveness*, which requires that if one element x in a set A is replaced by another element y, then the ranking between the new set $A \setminus \{x\} \cup \{y\}$ and the original set A according to $\sqsupseteq$ is determined by the ranking between x and y according to the preference over singletons. Formally, this axiom can be formulated as follows.

Property 1 (Responsiveness, RESP). *A total preorder $\sqsupseteq$ on 2^X satisfies the* responsiveness *property on 2^X iff for all $A \in 2^X \setminus \{X, \emptyset\}$, for all $x \in A$ and for all $y \in X \setminus A$ the* *following condition holds*[1]

$$A \sqsupseteq (A \setminus \{x\}) \cup \{y\} \Leftrightarrow \{x\} \sqsupseteq \{y\}. \tag{6}$$

Clearly, the RESP property is aimed at preventing complementarity effects. Restricted to sets A of fixed cardinality $q \in \mathbb{N}$, representing the maximum number of students the college can admit, the RESP property was used by Bossert (1995) to characterize[2] the family of *lexicographic rank-ordered extensions*, which generalize the idea of lexi-min and lexi-max orderings. An equivalent way to formulate the RESP property is given in the following proposition.

Proposition 2. *A total preorder $\sqsupseteq$ on 2^X satisfies the RESP property on 2^X iff for all $i, j \in X$ and all $S \in 2^X$, $i, j \notin S$ we have that*

$$S \cup \{i\} \sqsupseteq S \cup \{j\} \Leftrightarrow \{i\} \sqsupseteq \{j\}. \tag{7}$$

Proof. The proof is straightforward. In order to prove that conditions (6) and (7) are equivalent, take $A = S \cup \{i\}$, $x = i$ and $y = j$. □

An immediate consequence of Proposition 2 is the following.

Proposition 3. *Let $\sqsupseteq$ be a total preorder on 2^X. If $\sqsupseteq$ satisfies the RESP property, then $\sqsupseteq$ is a Shapley extension.*

Proof. We want to prove that if $\sqsupseteq$ satisfies the RESP property, then $\{i\} \sqsupseteq \{j\} \Leftrightarrow \phi_i(v) \geq \phi_j(v)$ for every $v \in \mathcal{G}^X_{\sqsupseteq}$.

proof of $RESP \Rightarrow [\{i\} \sqsupseteq \{j\} \Rightarrow \phi_i(v) \geq \phi_j(v)]$:
First note that if $\{i\} \sqsupseteq \{j\}$ then by RESP $S \cup \{i\} \sqsupseteq S \cup \{j\}$ for each $S \in 2^X$, with $i, j \notin S$, and by Remark 1 we have that $\phi_i(v) \geq \phi_j(v)$ for each $v \in \mathcal{G}^X_{\sqsupseteq}$ numerical representation of $\sqsupseteq$.

proof of $RESP \Rightarrow [\{i\} \sqsupseteq \{j\} \Leftarrow \phi_i(v) \geq \phi_j(v)]$:
Now, take $i, j \in X$ such that $\phi_i(v) \geq \phi_j(v)$ for each v numerical representation of $\sqsupseteq$. We want to prove that if $\sqsupseteq$ is an extension which satisfies RESP, then $\{i\} \sqsupseteq \{j\}$. Suppose $\{j\} \sqsupset \{i\}$. Then by the RESP property of $\sqsupseteq$ we have that $v(S \cup \{j\}) > v(S \cup \{i\})$ for each $S \in 2^X$, with $i, j \notin S$, but by Proposition 1 this implies that $\phi_j(v) > \phi_i(v)$, which yields a contradiction. □

Shapley extensions do not need to satisfy the RESP property. For example, the linear order $\sqsupseteq^a$ introduced in Example 1 does not satisfies the RESP property (see the relative ranking between $\{1, 3\}$ and $\{2, 3\}$), but it is a Shapley extension. We now introduce another property for extensions, namely, the *monotonicity* property (Kreps 1979; Puppe 1996).

Property 2 (Monotoncity, MON). *A total preorder $\sqsupseteq$ on 2^X satisfies the* monotonicity *property iff for each $S, T \in 2^X$ we have that*

$$S \subseteq T \Rightarrow T \sqsupseteq S.$$

The MON property states that each set of objects is weakly preferred to each of its subsets. In other words, the MON property excludes the possibility that some objects in a set $S \in 2^X$ may be incompatible with some others not in S. The extension $\sqsupseteq^a$ introduced in Example 1 does not satisfy neither the MON property nor the RESP property. An example of extension that does not satisfy the MON property, but that satisfies the RESP property, is the maxi-min criterion $\sqsupseteq^{\min}$ (for instance, if $X = \{1, 2\}$ and the original ranking on X is such that $2 \succ 1$, then we have that $\{2\} \sqsupset^{\min} \{1, 2\}$).

Let $\sqsupseteq$ be a total preorder on 2^X. For each $S \in 2^X \setminus \{\emptyset\}$, a *sub-extension* $\sqsupseteq_S$ is a relation on 2^S such that for each $U, V \in 2^S$,

$$U \sqsupseteq V \Leftrightarrow U \sqsupseteq_S V.$$

We may now introduce the last property of this section, namely the *sub-extendibility* property for Shapley extensions.

Property 3 (Sub-Extendibility, SE). *A Shapley extension $\sqsupseteq$ on 2^X satisfies the* sub-extendibility *property iff for each $S \in 2^X \setminus \{\emptyset\}$ we have that $\sqsupseteq_S$ is a Shapley extension on 2^S.*

The SE property states that the effects of interaction among objects must be "compatible" not only with the information provided by the original preference on single elements of X, but also with the information provided by all restrictions of such a preference to each non-empty subset S of X. This means that the personal attribution of importance assigned to objects, and taking into account the effects of interaction, must be consistent with the primitive ranking, independently from the size of the universal set considered.

Example 3. *Let $X = \{1, 2, 3, 4\}$ and let $\sqsupseteq^b$ be a total preorder such that $\{1, 2, 3, 4\} \sqsupset^b \{2, 3, 4\} \sqsupset^b \{1, 3, 4\} \sqsupset^b \{1, 2, 4\} \sqsupset^b \{3, 4\} \sqsupset^b \{1, 2, 3\} \sqsupset^b \{1, 3\} \simeq^b \{2, 4\} \sqsupset^b \{2, 3\} \simeq^b \{1, 4\} \sqsupset^b \{4\} \sqsupset^b \{3\} \sqsupset^b \{1, 2\} \sqsupset^b \{2\} \sqsupset^b \{1\} \sqsupset^b \emptyset$. By relation (3), it is easy to check that for every v numerical representation of $\sqsupseteq^b$ in $\mathcal{G}^X_{\sqsupseteq^b}$, the following relations hold.*

$$\begin{aligned} &\phi_2(v) - \phi_1(v) = \\ &\tfrac{1}{3}\big(v(2) - v(1)\big) + \tfrac{1}{3}\big(v(2,3,4) - v(1,3,4)\big) + \\ &\tfrac{1}{6}\big(v(2,3) - v(1,3)\big) + \tfrac{1}{6}\big(v(2,4) - v(1,4)\big) = \\ &\tfrac{1}{3}\big(v(2) - v(1)\big) + \tfrac{1}{3}\big(v(2,3,4) - v(1,3,4)\big) > 0 \end{aligned} \tag{8}$$

$$\begin{aligned} &\phi_3(v) - \phi_2(v) = \\ &\tfrac{1}{3}\big(v(3) - v(2)\big) + \tfrac{1}{3}\big(v(1,3,4) - v(1,2,4)\big) + \\ &\tfrac{1}{6}\big(v(1,3) - v(1,2)\big) + \tfrac{1}{6}\big(v(3,4) - v(2,4)\big) > 0 \end{aligned} \tag{9}$$

$$\begin{aligned} &\phi_4(v) - \phi_3(v) = \\ &\tfrac{1}{3}\big(v(4) - v(3)\big) + \tfrac{1}{3}\big(v(1,2,4) - v(1,2,3)\big) + \\ &\tfrac{1}{6}\big(v(1,4) - v(1,3)\big) + \tfrac{1}{6}\big(v(2,4) - v(2,3)\big) = \\ &\tfrac{1}{3}\big(v(4) - v(3)\big) + \tfrac{1}{3}\big(v(1,2,4) - v(1,2,3)\big) > 0 \end{aligned} \tag{10}$$

So, $\sqsupseteq^b$ is a Shapley extension. Note that $\sqsupseteq^b$ satisfies the MON property but it does not satisfies the RESP property. In fact, we have that $\{2\} \sqsupseteq \{1\}$, but $\{1, 3\} \sqsupset^b (\{1, 3\} \setminus \{1\} \cup \{2\}) = \{2, 3\}$.

[1]Note that in Bossert (1995), condition (6) was introduced together with the condition: $(A \setminus \{x\}) \cup \{y\} \sqsupseteq A \Leftrightarrow \{y\} \sqsupseteq \{x\}$. The last one, as also remarked by Bossert (1995), is redundant with respect to condition (6), if $\sqsupseteq$ is a total preorder; so we omit it.

[2]Together with another property called *fixed-Cardinality neutrality*, saying that the labelling of the alternatives is irrelevant in establishing the ranking among sets of fixed cardinality q.

Moreover, relation $\sqsupseteq^b$ does not satisfies the SE property too. In fact consider the sub-extension $\sqsupseteq^b_{\{1,2,3\}}$. Note that

$$\{1,2,3\} \sqsupset^b_{\{1,2,3\}} \{1,3\} \sqsupset^b_{\{1,2,3\}} \{2,3\} \sqsupset^b \{3\} \sqsupset^b_{\{1,2,3\}} \{1,2\} \sqsupset^b_{\{1,2,3\}} \{2\} \sqsupset^b_{\{1,2,3\}} \{1\} \sqsupset^b_{\{1,2,3\}} \emptyset.$$

Consider a coalitional game $(\{1,2,3\}, v)$ such that $v(\{1,2,3\}) = 10$, $v(\{1,3\}) = 9$, $v(\{2,3\}) = 5$, $v(\{3\}) = 4$, $v(\{1,2\}) = 3$, $v(\{2\}) = 2$, $v(\{1\}) = 1$ (and, of course, $v(\emptyset) = 0$).

Note that that v represents $\sqsupset^b_{\{1,2,3\}}$, but the corresponding Shapley value is $\phi_1(v) = 3$, $\phi_2(v) = 1.5$, $\phi_3(v) = 5.5$. Therefore, $\sqsupset^b_{\{1,2,3\}}$ is not a Shapley extension on $2^{\{1,2,3\}}$.

The following proposition is important in establishing the connection between MON, SE and RESP properties.

Proposition 4. *Let $\sqsupseteq$ be a Shapley extension on 2^X. If $\sqsupseteq$ satisfies both MON and SE properties, then it also satisfies the RESP property.*

Proof. Let $i, j \in X$. Since $\sqsupseteq$ is a Shapley extension, we have immediately that for all $S \in 2^X$ with $i, j \notin S$,

$$S \cup \{i\} \sqsupseteq S \cup \{j\} \Rightarrow \{i\} \sqsupseteq \{j\}.$$

It remains to prove that $\{i\} \sqsupseteq \{j\} \Rightarrow S \cup \{i\} \sqsupseteq S \cup \{j\}$ for all $S \in 2^X$ with $i, j \notin S$.

Let $i, j \in X$ be such that $\{i\} \sqsupseteq \{j\}$. Suppose there exists $S \in 2^X$, with $i, j \notin S$ and such that $S \cup \{j\} \sqsupset S \cup \{i\}$.

Consider the sub-extension $\sqsupseteq_{S^*}$, where $S^* = S \cup \{i, j\}$. Let $v_{S^*} \in \mathcal{G}^X_{\sqsupseteq_{S^*}}$ be a numerical representation of $\sqsupseteq_{S^*}$ and define $\delta^{v_{S^*}} = \phi_i(v_{S^*}) - \phi_j(v_{S^*})$. By the SE property, $\sqsupseteq_{S^*}$ is a Shapley extension on 2^{S^*}. Then $\delta^{v_{S^*}} \geq 0$.

Now, let $\Delta^{v_{S^*}} > \delta^{v_{S^*}}$ and consider another game $\hat{v}_{S^*}$ with player set S^*, such that $\hat{v}_{S^*}(U) = v_{S^*}(U) + |S^*|\Delta^{v_{S^*}}$ for each $U \in 2^{S^*}$ with $U \sqsupseteq_{S^*} S \cup \{j\}$, and $\hat{v}_{S^*}(T) = v_{S^*}(T)$ for all remaining coalitions T[3]. Note that $\hat{v}_{S^*}$ is still a numerical representation of $\sqsupseteq_{S^*}$, so $\phi_i(\hat{v}_{S^*}) \geq \phi_j(\hat{v}_{S^*})$.

By the MON property, for each $U \in 2^{S^*}$ such that $U \sqsupseteq_{S^*} S \cup \{j\} \sqsupset_{S^*} S \cup \{i\}$ we have that $\{i, j\} \subseteq U$. Then, by equation (1), the Shapley value of i in $\hat{v}_{S^*}$ is

$$\begin{aligned}
&\phi_i(\hat{v}_{S^*}) = \\
&\sum_{\substack{U \subset S\cup\{j\}:\\ S\cup\{j\} \sqsupset_{S^*} U\cup\{i\}}} w(U) m_i^U(v_{S^*}) + \\
&\sum_{\substack{U \subset S\cup\{j\}:\\ U\cup\{i\} \sqsupseteq_{S^*} S\cup\{j\}}} w(U)\big(m_i^U(v_{S^*}) + |S^*|\Delta^{v_{S^*}}\big) + \\
&\tfrac{1}{|S^*|} m_i^{S\cup\{j\}}(v_{S^*}) = \\
&\phi_i(v_{S^*}) + \sum_{\substack{U \subset S\cup\{j\}:\\ U\cup\{i\} \sqsupseteq_{S^*} S\cup\{j\}}} w(U)|S^*|\Delta^{v_{S^*}},
\end{aligned} \tag{11}$$

where $w(U) = \frac{(|U|)!(|S^*|-|U|-1)!}{(|S^*|)!}$ for each $U \in 2^{S^*}$ and the last term of the sum follows from the fact that

$$\begin{aligned}
&w(S \cup \{j\}) m_i^{S\cup\{j\}}(\hat{v}_{S^*}) = \\
&\tfrac{(|S\cup\{j\}|)!(|S^*|-|S\cup\{j\}|-1)!}{(|S^*|)!}\big(\hat{v}_{S^*}(S^*) - \hat{v}_{S^*}(S \cup \{j\})\big) = \\
&\tfrac{1}{|S^*|}\Big(v_{S^*}(S^*) + |S^*|\Delta^{v_{S^*}} - \\
&\big(v_{S^*}(S \cup \{j\}) + |S^*|\Delta^{v_{S^*}}\big)\Big) = \\
&\tfrac{1}{|S^*|} m_i^{S\cup\{j\}}(v_{S^*}).
\end{aligned} \tag{12}$$

In a similar way, it is possible to calculate the Shapley value of player j in $\hat{v}_{S^*}$:

$$\begin{aligned}
&\phi_j(\hat{v}_{S^*}) = \\
&\sum_{\substack{U \subset S\cup\{i\}:\\ S\cup\{j\} \sqsupset_{S^*} U\cup\{j\}}} w(U) m_j^U(v_{S^*}) + \\
&\sum_{\substack{U \subset S\cup\{i\}:\\ U\cup\{j\} \sqsupseteq_{S^*} S\cup\{j\}}} w(U)\big(m_j^U(v_{S^*}) + |S^*|\Delta^{v_{S^*}}\big) + \\
&\tfrac{1}{|S^*|}\big(m_j^{S\cup\{i\}}(v_{S^*}) + |S^*|\Delta^{v_{S^*}}\big) = \\
&\phi_j(v_{S^*}) + \Delta^{v_{S^*}} + \sum_{\substack{U \subset S\cup\{i\}:\\ U\cup\{j\} \sqsupseteq_{S^*} S\cup\{j\}}} w(U)|S^*|\Delta^{v_{S^*}}
\end{aligned} \tag{13}$$

As we already said, by the MON property, $U \sqsupseteq_{S^*} S \cup \{j\}$ implies that $\{i, j\} \subseteq U$. Therefore, we have that:

$$\begin{aligned}
&\sum_{\substack{U \subset S\cup\{j\}:\\ U\cup\{i\} \sqsupseteq_{S^*} S\cup\{j\}}} w(U)|S^*|\Delta^{v_{S^*}} = \\
&\sum_{\substack{U \subset S\cup\{i\}:\\ U\cup\{j\} \sqsupseteq_{S^*} S\cup\{j\}}} w(U)|S^*|\Delta^{v_{S^*}}.
\end{aligned} \tag{14}$$

Finally, if we compute the difference between relations (11) and (13) and keeping into account relation (14), we obtain:

$$\begin{aligned}
&\phi_i(\hat{v}_{S^*}) - \phi_j(\hat{v}_{S^*}) = \\
&\phi_i(v_{S^*}) - \phi_j(v_{S^*}) - \Delta^{v_{S^*}} = \\
&\delta^{v_{S^*}} - \Delta^{v_{S^*}} < 0,
\end{aligned} \tag{15}$$

which yields a contradiction. □

Note that if a total preorder $\sqsupseteq$ on 2^X satisfies the RESP property, then every sub-extension $\sqsupseteq_S$, for each $S \in 2^X$, $S \neq \emptyset$, satisfies the RESP property, and then by Proposition 3, $\sqsupseteq_S$ is a Shapley extension. The following theorem is then an immediate consequence of Propositions 3 and 4.

Theorem 1. *Let $\sqsupseteq$ be a total preorder on 2^X which satisfies the MON property. The following two statements are equivalent:*

- *(i) $\sqsupseteq$ satisfies the RESP property.*
- *(ii) $\sqsupseteq$ is a Shapley extension and satisfies the SE property.*

[3] The choice of the term $|S^*|\Delta^{v_{S^*}}$ follows from the fact that $\frac{1}{|S^*|} = w(S \cup \{j\}) = \frac{(|S\cup\{j\}|)!(|S^*|-|S\cup\{j\}|-1)!}{(|S^*|)!}$ is the weight assigned by the Shapley value to the marginal contribution $m_i^{S\cup\{j\}}(\hat{v}_{S^*})$ (see relation (12))

5 Shapley extensions with interaction

In the last section, we have characterized a class of Shapley extensions aimed to rank subsets of objects in absence of complementarity effects. The objective of this section is to analyze properties of Shapley extensions that, due to the effects of interaction among objects, may "invert" (with respect to the conditions imposed by the RESP property) the relative ranking of a limited number of subsets. First, we need some extra notations. Let $2^{X^k} = \{S \in 2^X : |S| = k\}$ be the set of all subsets of X of cardinality k, $k = 0, 1, \ldots, |X|$ (with the convention that $|\emptyset| = 0$). Moreover, for each $i, j \in X$ and each $k = 0, 1, \ldots, |X|-2$, we denote by $\mathcal{S}_{ij}^k$ the set of all subsets of X of cardinality k which do not contain i and j, i.e. $\mathcal{S}_{ij}^k = \{S \in 2^{X^k} : i, j \notin S\}$.

Property 4 (Permutational Responsiveness, PR). *A total preorder $\sqsupseteq$ on 2^X satisfies the* permutational responsiveness *property iff for all $i, j \in X$ and all $S \in 2^X$, with $i, j \notin S$, there exists a bijection $F_{ij}^k : \mathcal{S}_{ij}^k \to \mathcal{S}_{ij}^k$ such that*

$$S \cup \{i\} \sqsupseteq F_{ij}^k(S) \cup \{j\} \Leftrightarrow \{i\} \sqsupseteq \{j\} \tag{16}$$

for each $k = 0, 1, \ldots, |X| - 2$

Remark 2. *Note that if a total preorder $\sqsupseteq$ on 2^X satisfies the RESP property then it also satisfies the PR property (simply, take $F_{ij}^k(S) = S$ for each $i, j \in X$, each $k = 0, 1, \ldots, |X| - 2$, and each $S \in \mathcal{S}_{ij}^k$).*

Remark 3. *For all $i, j \in X$, if $k = 0$, then $\mathcal{S}_{ij}^k = \{\emptyset\}$ and $F_{ij}^k(\emptyset) = \emptyset$; so, condition (16) is always satisfied for $k = 0$.*

The PR property moderates the interaction effects of those relative rankings which violate the RESP property. Let $i, j \in X$ be such that $\{i\} \sqsupseteq \{j\}$. Saying that each set $S \in 2^X$, with $i, j \notin S$, can be matched with a set $T \in 2^X$, with $i, j \notin T$ and with the same cardinality of S, such that $T \cup \{i\} \sqsupseteq S \cup \{j\}$, the PR property implies that the effects of the interaction with i (over the distribution of subsets with a fixed cardinality) should ("globally") dominate the effects of the interaction with j.

From a slightly different perspective, note that the PR property for a total preorder $\sqsupseteq$ on 2^X can be introduced by comparing the elements of the sets $\Sigma_{ij}^k = \{S \cup \{i\} : S \in \mathcal{S}_{ij}^k\}$ and $\Sigma_{ji}^k = \{S \cup \{j\} : S \in \mathcal{S}_{ij}^k\}$ when they are arranged in descending order of preference, from the most preferred to the least preferred according to $\sqsupseteq$, for each $k = 0, 1, \ldots, |X| - 2$. Precisely, define two bijections $\sigma_{ij}^k, \sigma_{ji}^k : \{1, \ldots, |\mathcal{S}_{ij}^k|\} \to \mathcal{S}_{ij}^k$ such that

$$\sigma_{ij}^k(l) \cup \{i\} \sqsupseteq \sigma_{ij}^k(m) \cup \{i\}$$

and

$$\sigma_{ji}^k(l) \cup \{j\} \sqsupseteq \sigma_{ji}^k(m) \cup \{j\},$$

for each $l, m \in \{1, \ldots, |\mathcal{S}_{ij}^k|\}$ with $l < m$. Then a total preorder $\sqsupseteq$ on 2^X such that

$$\sigma_{ij}^k(l) \cup \{i\} \sqsupseteq \sigma_{ji}^k(l) \cup \{j\},$$

for each $i, j \in X$ such that $\{i\} \sqsupseteq \{j\}$, each $k = 0, 1, \ldots, |X| - 2$ and each $l \in \{1, \ldots, |\mathcal{S}_{ij}^k|\}$, satisfies the PR property. In other terms, for each $i, j \in X$ such that $\{i\} \sqsupseteq \{j\}$ and for each $k = 0, 1, \ldots, |X| - 2$, the PR property admits the possibility of relative rankings which violate the conditions imposed by the RESP property (i.e., $S \cup \{j\}$ is preferred to $S \cup \{i\}$) due to the effect of interaction with the objects in S. Nevertheless, such an interaction should be compatible with the requirement that, between sets of the same cardinality, the original relative ranking between $\{i\}$ and $\{j\}$ should be preserved with respect to the position of subsets in Σ_{ij}^k and Σ_{ji}^k when they are arranged in descending order of preference (i.e., the most preferred subsets in Σ_{ij}^k should be preferred to the most preferred subsets in Σ_{ji}^k; the second most preferred subsets in Σ_{ij}^k should be preferred to the second most preferred subsets in Σ_{ji}^k, etc.).

The following proposition establishes the connection between the PR property and Shapley extensions.

Proposition 5. *Let $\sqsupseteq$ be a total preorder on 2^X. If $\sqsupseteq$ satisfies the PR property, then $\sqsupseteq$ is a Shapley extension.*

Proof. We want to prove that if $\sqsupseteq$ satisfies the PR property, then $\{i\} \sqsupseteq \{j\} \Leftrightarrow \phi_i(v) \geq \phi_j(v)$ for every $v \in \mathcal{G}_{\sqsupseteq}^X$.

By PR, for each $k = 0, 1, \ldots, |X| - 2$ and each $i, j \in X$ such that $\{i\} \sqsupseteq \{j\}$ and $\mathcal{S}_{ij}^k \neq \emptyset$ there exists a bijection $F_{ij}^k : \mathcal{S}_{ij}^k \to \mathcal{S}_{ij}^k$ satisfying conditions (16); it immediately follows that

$$\begin{aligned}&\textstyle\sum_{S \in \mathcal{S}_{ij}^k} p(S)\, d_{ij}^S(v) = \\ &\textstyle\sum_{S \in \mathcal{S}_{ij}^k} p(S)\big(v(S \cup \{i\}) - (v(S \cup \{j\})\big) = \\ &\textstyle\sum_{S \in \mathcal{S}_{ij}^k} p(S)\big(v(S \cup \{i\}) - (v(F_{ij}^k(S) \cup \{j\})\big) \geq 0,\end{aligned} \tag{17}$$

where $p(S) = \frac{|S|!(|N|-|S|-2)!}{(|N|-1)!}$, for each $S \in 2^X$.

proof of PR $\Rightarrow [\{i\} \sqsupseteq \{j\} \Rightarrow \phi_i(v) \geq \phi_j(v)]$:
By relation (3) and (17), for each $i, j \in X$ such that $\{i\} \sqsupseteq \{j\}$, we have that

$$\phi_i(v) - \phi_j(v) = \textstyle\sum_{k=0}^{|X|-2} \sum_{S \in \mathcal{S}_{ij}^k} p(S)\, d_{ij}^S(v) \geq 0, \tag{18}$$

where the inequality directly follows from relation (17).

proof of PR $\Rightarrow [\{i\} \sqsupseteq \{j\} \Leftarrow \phi_i(v) \geq \phi_j(v)]$:
Now, take $i, j \in X$ such that $\phi_i(v) \geq \phi_j(v)$ for each $v \in \mathcal{G}_{\sqsupseteq}^X$ numerical representation of $\sqsupseteq$. Suppose $\{j\} \sqsupset \{i\}$. By relations (3) and (17) we then obtain the following contradiction

$$\phi_j(v) - \phi_i(v) = \textstyle\sum_{k=0}^{|X|-2} \sum_{S \in \mathcal{S}_{ij}^k} p(S)\, d_{ji}^S(v) > 0. \tag{19}$$

□

The following examples show that a Shapley extension which does not satisfy the RESP property, does not need to satisfy the PR property. In particular, the total preorder presented in Example 5 will be also useful, at the end of this section, to stress the difference between Shapley extensions and other extensions based on alternative game theoretic indices.

Example 4. *Let $X = \{1, 2, 3\}$. Consider the Shapley extension $\sqsupseteq^a$ of Example (1). Note that $\{2\} \sqsupset \{1\}$, but*

$\{1,3\} \sqsupset \{2,3\}$. *Consequently, a bijection F^1_{12} which satisfies condition (16) does not exist.*

Example 5. *Let $X = \{1,2,3,4,5\}$ and let $G = \{\{1\}, \{1,2\}, \{2,3\}, \{2,4\}, \{2,5\}, \{1,3,4\}, \{2,3,5\}, \{2,4,5\}, \{1,3,4,5\}, \{1,2,3,4,5\}\}$. Consider a total preorder $\sqsupseteq^d$ on 2^X such that G and $B = 2^X \setminus G$ form two indifference classes with respect to $\sqsupseteq^d$, and where $S \sqsupset^d T$ for each $S \in G$ and each $T \in B$.*

Note that for every numerical representation v of $\sqsupseteq^d$, $v \in \mathcal{G}^X_{\sqsupseteq^d}$, elements $3,4,5$ are symmetric players (i.e., for each $S \in 2^X$, with $i,j \notin S$, $v(S\cup\{i\}) = v(S\cup\{j\})$), and therefore $\phi_3(v) = \phi_4(v) = \phi_5(v)$.

Let $K_v = v(S) - v(T) > 0$ for each $S \in G$ and $T \in B$. Using relation (3), we have that

$$\begin{aligned}
&\phi_1(v) - \phi_2(v) = \tfrac{1}{4}\big[v(1) - v(2)\big] + \\
&\tfrac{1}{12}\big[v(1,3) - v(2,3) + v(1,4) - v(2,4) + \\
&v(1,5) - v(2,5)\big] + \\
&\tfrac{1}{12}\big[v(1,3,4) - v(2,3,4) + v(1,3,5) - v(2,3,5) + \\
&v(1,4,5) - v(2,4,5)\big] + \\
&\tfrac{1}{4}\big[v(1,3,4,5) - v(2,3,4,5)\big] = \\
&\tfrac{1}{4}K_v + \tfrac{1}{12}(-3K_v) + \tfrac{1}{12}(K_v - 2K_v) + \tfrac{1}{4}K_v = \\
&\tfrac{1}{6}K_v > 0,
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\phi_2(v) - \phi_3(v) = \tfrac{1}{4}\big[v(2) - v(3)\big] + \\
&\tfrac{1}{12}\big[v(1,2) - v(1,3) + v(2,4) - v(3,4) + \\
&v(2,5) - v(3,5)\big] + \\
&\tfrac{1}{12}\big[v(1,2,4) - v(1,3,4) + v(2,4,5) - v(3,4,5) + \\
&v(1,2,5) - v(1,3,5)\big] + \\
&\tfrac{1}{4}\big[v(1,2,4,5) - v(1,3,4,5)\big] = \\
&\tfrac{1}{4}0 + \tfrac{1}{12}(3K_v) + \tfrac{1}{12}(-K_v + K_v) + \tfrac{1}{4}(-K_v) = 0.
\end{aligned} \tag{21}$$

However, $\sqsupseteq^d$ does not satisfy the PR property. In fact, $\{1\} \sqsupseteq^d \{2\}$ but $S\cup\{2\} \sqsupset^d S\cup\{1\}$ for each $S \in \mathcal{S}^1_{12} = \{\{3\},\{4\},\{5\}\}$. Consequently, a bijection F^1_{12} which satisfies condition (16) does not exist.

The PR property can be useful to find classes of Shapley extensions which do not satisfy the RESP property. This is the case, for example, of a family of trichotomous total preorders (Brams and Fishburn 1994; Ju 2005) on 2^{X^k}, for each $k = 2,\ldots,|X|$, where (non-singleton) subsets of X of cardinality k are partitioned into three indifference classes. This kind of preferences are particularly realistic in voting situations (Brams and Fishburn 2002), where voters face the problem to choose from a set of candidates a non-empty subset of committee members and they not have much information about the quality of possible committees, but they still are able to distinguish “best”, “worst” and sometimes “medium” committees.

Precisely, we say that a total preorder on 2^X is *trichotomous (per cardinality k)* if, for each $k = 2,\ldots,|X|$, the elements of 2^{X^k} can be partitioned into three indifference classes, which are called the set of *good subsets* $G^k = \{G \in 2^{X^k} | G \sqsupseteq S \text{ for each } S \in 2^{X^k}\}$, the set of *bad subsets* $B^k = \{B \in 2^{X^k} | S \sqsupseteq B \text{ for each } S \in 2^{X^k}\}$ and the set of *null subsets* $N^k = \{N \in 2^{X^k} | G \sqsupset N \sqsupset B \text{ for each } G \in G^k \text{ and each } B \in B^k\}$.

Proposition 6. *Let $\sqsupseteq$ be a trichotomous (per cardinality k) total preorder on 2^X. For each $i \in X$ and each $k = 2,\ldots,|X|-1$, let $G^k_i = \{S \in G^k : i \in S\}$ be the set of good subsets of cardinality k containing i, and let $B^k_i = \{S \in B^k : i \in S\}$ be the set of bad subsets of cardinality k containing i. If $|G^k_i| \geq |G^k_j|$ and $|B^k_i| \leq |B^k_j|$ for each $k = 2,\ldots,|X|-1$ and each $i,j \in X$ with $\{i\} \sqsupseteq \{j\}$, then $\sqsupseteq$ is a Shapley extension.*

Proof. By Proposition 5, it is sufficient to prove that $\sqsupseteq$ satisfies the PR property. First, for each $i,j \in X$ and each $k = 1,\ldots,|X|-2$, define the set of good subsets of cardinality k containing i but not j as the set $G^k_{ij} = \{S \in \mathcal{S}^k_{ij} : (S\cup\{i\}) \in G^{k+1}_i\}$; in a similar way, define the set of bad subsets of cardinality k containing i but not j as the set $B^k_{ij} = \{S \in \mathcal{S}^k_{ij} : (S\cup\{i\}) \in B^{k+1}_i\}$.

Note that, for each $i,j \in X$ and each $k = 1,\ldots,|X|-2$,

$$|G^{k+1}_i| \geq |G^{k+1}_j| \Leftrightarrow |G^k_{ij}| \geq |G^k_{ji}|, \tag{22}$$

and

$$|B^{k+1}_i| \geq |B^{k+1}_j| \Leftrightarrow |B^k_{ij}| \geq |B^k_{ji}| \tag{23}$$

Now, let $i,j \in X$ be such that $\{i\} \sqsupseteq \{j\}$. Define a bijection $F^k_{ij} : \mathcal{S}^k_{ij} \to \mathcal{S}^k_{ij}$ such that, for each $k = 1,\ldots,|X|-2$, $G^k_{ji} \subseteq F^k_{ij}[G^k_{ij}]$ and $F^k_{ij}[B^k_{ij}] \subseteq B^k_{ji}$, where $F^k_{ij}[G^k_{ij}]$ and $F^k_{ij}[B^k_{ij}]$ are, respectively, the images of G^k_{ij} and B^k_{ij} under F^k_{ij} (by relations (22) and (23) such a bijection F^k_{ij} exists, for each $k = 1,\ldots,|X|-2$). So, for each $S \in G^k_{ij} \cup B^k_{ij}$ and for each $k = 1,\ldots,|X|-2$, condition (16) is satisfied. On the other hand, for each $S \in \mathcal{S}^k_{ij} \setminus (G^k_{ij} \cup B^k_{ij})$ and for each $k = 1,\ldots,|X|-2$, we have that $(S\cup\{i\}) \in N^k_i$, whereas $(F^k_{ij}(S)\cup\{j\}) \in N^k_j$ or $(F^k_{ij}(S)\cup\{j\}) \in B^k_j$, and, consequently, $S\cup\{i\} \sqsupseteq F^k_{ij}(S)\cup\{j\}$. It remains proved that $\sqsupseteq$ satisfies the PR property. □

Remark 4. *A total preorder $\sqsupseteq$ on 2^X such that each (non-singleton) subset of X of cardinality k, for each $k = 2,\ldots,|X|-1$, is classified either as “good” (i.e., it belongs to the indifference class G^k) or as “bad” (i.e., it belongs to the indifference class B^k), is said dichotomous (per cardinality k). As a direct consequence of Proposition 6, a dichotomous (per cardinality k) total preorder $\sqsupseteq$ on 2^X such that $|G^k_i| \geq |G^k_j|$ for each $k = 2,\ldots,|X|-1$ and each $i,j \in X$ with $\{i\} \sqsupseteq \{j\}$, is a Shapley extension.*

Example 6. *Let $X = \{1,2,3,4\}$ and let $\sqsupseteq^e$ be a total preorder such that $\{1,2,3\} \simeq^e \{1,3,4\} \simeq^e \{1,2,4\} \sqsupset^e \{1,2,3,4\} \sqsupset^e \{2,3,4\} \sqsupset^e \{1,2\} \simeq^e \{1,3\} \simeq^e \{2,3\} \sqsupset^e \{1\} \sqsupset^e \{2\} \sqsupset^e \{3\} \sqsupset^b \{4\} \sqsupset^e \{1,4\} \simeq^e \{2,4\} \simeq^e \{3,4\} \sqsupset^e \emptyset$. Note that $\sqsupseteq^e$ is a dichotomous total preorder on 2^X. Moreover, we have that*

$$G_1^2 = \{\{1,2\},\{1,3\}\}, G_2^2 = \{\{1,2\},\{2,3\}\},$$
$$G_3^2 = \{\{2,3\},\{1,3\}\}, G_4^2 = \emptyset;$$
$$G_1^3 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\},$$
$$G_2^3 = \{\{1,2,3\},\{1,2,4\}\}, G_3^3 = \{\{1,2,3\},\{1,3,4\}\},$$
$$G_4^3 = \{\{1,2,4\},\{1,3,4\}\}.$$

By Remark 4, it follows that $\sqsupseteq^e$ is a Shapley extension.

We conclude this section with some remarks about the comparison of Shapley extensions with extensions corresponding to other *probabilistic values* (Dubey, Neyman, and Weber 1981; Weber 1988; Monderer and Samet 2002), that are a natural generalization of the Shapley value where players possess their own probabilistic distribution over all coalitions that can be formed. Let X be a finite set and let $\mathbf{p} = ((p_i^S)_{S\subset X, i\notin S})_{i\in X}$ be a collection of probability distributions (so, for all $i \in X$, $p_i^S \geq 0$ for each $S \in 2^X$ with $i \notin S$ and $\sum_{S\subset X, i\notin S} p_i^S = 1$).

The probabilistic value $\pi^{\mathbf{p}} : \mathcal{C}^X \to I\!R^X$ is then defined as the vector of expected payoffs of each player with respect to $\mathbf{p}$, that is

$$\pi_i^{\mathbf{p}}(v) = \sum_{S\subset X: i\notin S} p_i^S \, m_i^S(v), \tag{24}$$

for each $v \in \mathcal{C}^X$ and each $i \in X$. Note that the Shapley value $\phi(v)$ equals the probabilistic value $\pi^{\hat{\mathbf{p}}}(v)$, where, for each $i \in X$, $\hat{p}_i^S = \frac{1}{|X|\binom{|X|-1}{|S|}} = \frac{|S|!(|X|-|S|-1)!}{|X|!}$ for each $S \subset X$ such that $i \notin S$ (i.e., coalitions of the same cardinality k, for each $k = 0, 1, \ldots, |X|-1$, have equal probability, and each cardinality is selected with the same probability). Another very well studied probabilistic value is the *Banzhaf value* (Banzhaf III 1964), which is defined according to relation (24) as the function $\pi_i^{\tilde{\mathbf{p}}}(v)$, where, for each $i \in X$, $\tilde{p}_i^S = \frac{1}{2^{|X|-1}}$ for each $S \subset X$ such that $i \notin S$ (i.e., each coalition has an equal probability to be chosen).

Consequently, other "probabilistic extensions" can be defined according to Definition 1, with $\pi^{\mathbf{p}}$, for some collection of probability distributions $\mathbf{p}$, in the role of ϕ. Formally, given a probabilistic value $\pi^{\mathbf{p}}$, where $\mathbf{p}$ is a collection of probability distributions as defined above, a *$\pi^{\mathbf{p}}$-based extension* is defined as a total preorder $\sqsupseteq$ on 2^X such that

$$\{i\} \sqsupseteq \{j\} \Leftrightarrow \pi_i^{\mathbf{p}}(v) \geq \pi_j^{\mathbf{p}}(v)$$

for each numerical representation $v \in \mathcal{G}_{\sqsupseteq}^X$ of $\sqsupseteq$ and all $i, j \in X$. For instance, adopting the collection $\tilde{\mathbf{p}}$ defined above, a *Banzhaf extension* is defined.

Now, a probabilistic value $\pi^{\mathbf{p}}$ corresponding to a collection of probability distributions $\mathbf{p}$ such that coalitions of the same size have equal probability, i.e. $\mathbf{p}$ satisfies the condition $p_i^S = p^k$ for each $k = 0, 1, \ldots, |X|-1$ and each $S \subset X$ such that $i \notin S$ and $|S| = k$, is a *semivalue* (Dubey, Neyman, and Weber 1981; Weber 1988). In addition, a semivalue with all positive probabilities is a *regular semivalue* (Carreras and Freixas 1999; 2000; Lucchetti, Radrizzani, and Munarini 2010) as, for instance, the Shapley value or the Banzhaf value.

According to Remark 2, for every regular semivalue $\pi^{\mathbf{p}}$ we have that

$$\pi_i^{\mathbf{p}}(v) - \pi_j^{\mathbf{p}}(v) = \sum_{S\subset X: i,j\notin S} (p^{|S|} + p^{|S+1|})\, d_{ij}^S(v). \tag{25}$$

Following the same arguments used in the proof of Proposition 3 for Shapley extensions, with $\pi^{\mathbf{p}}$ in the role of ϕ, it can be easily verified that an extension which satisfies the RESP property is also a $\pi^{\mathbf{p}}$-based extension, for every regular semivalue $\pi^{\mathbf{p}}$. Moreover, it is also possible to reformulate the SE property for every regular semivalue $\pi^{\mathbf{p}}$ in the following way.

Property 5 (Sub-Extendibility*, SE*). *Let $\pi^{\mathbf{p}}$ be a regular semivalue. A $\pi^{\mathbf{p}}$-based extension $\sqsupseteq$ on 2^X satisfies the* sub-extendibility* *property iff for each $S \in 2^X \setminus \{\emptyset\}$ we have that $\sqsupseteq_S$ is a $\pi^{\mathbf{p}}$-based extension on 2^S.*

Then, it is straightforward to adapt the arguments of Proposition 4, with a regular semivalue $\pi^{\mathbf{p}}$ in the role of ϕ and probabilities $p^{|S|}$ in the role of $w(S)$, for each $S \in 2^X$ with $i \notin S$. Consequently, we can state the following result, which extends Theorem 1.

Theorem 2. *Let $\pi^{\mathbf{p}}$ be a regular semivalue. Let $\sqsupseteq$ be a total preorder on 2^X which satisfies the MON property. The following two statements are equivalent:*

- *(i) $\sqsupseteq$ satisfies the RESP property.*
- *(ii) $\sqsupseteq$ is a $\pi^{\mathbf{p}}$-based extension and satisfies the SE* property.*

As a side-product of Theorem 2 we have that for a large family of coalitional games (precisely, those coalitional games who are a numerical representation of a total preorder that satisfies the RESP property) regular semivalues are *ordinal equivalent* (Tomiyama 1987; Carreras and Freixas 2008; Freixas 2010), i.e. the rankings on the set of players induced by regular semivalues coincide. For instances, *airport games* (Littlechild and Owen 1973; Littlechild and Thompson 1977) fall in this large family of coalitional games. In airport games, the objective is to divide the costs of a landing strip of an airport among the landings that occur during the lifetime of the airport. Since not all planes will need a landing strip of the same length, the cost imputed to a coalition of landings S is the cost associated with a landing strip long enough to accommodate all of the landings in S. Assuming that the cost of the landing strip increases with its length, it follows that the marginal contribution (to every possible coalition) of a landing that need a longer strip is always higher than the marginal contribution of a landing that need a shorter one. In other words, an airport game is a numerical representation of a total preorder that satisfies the RESP property and, consequently, all regular semivalues applied to an airports games are ordinal equivalent.

However, in general, an extension based on a certain regular semivalue does not need to coincide with another extension based on a different regular semivalue. For example, a

Shapley extension is not, in general, a Banzhaf extension, as it is shown by the following example.

Example 7. *Let $X = \{1,2,3,4,5\}$. Consider the total preorder $\sqsupseteq^d$ on 2^X of Example 5. First, note that*

$$\tilde{p}^{|S|} + \tilde{p}^{|S+1|} = \frac{1}{2^{|X|-1}} + \frac{1}{2^{|X|-1}} = \frac{1}{8}$$

for each $S \subset X$ such that $i, j \notin S$. Then, by relation (25), we have that

$$\begin{aligned} &\pi_1^{\tilde{\mathbf{p}}}(v) - \pi_2^{\tilde{\mathbf{p}}}(v) = \\ &\tfrac{1}{8}K_v + \tfrac{1}{8}(-3K_v) + \tfrac{1}{8}(K_v - 2K_v) + \tfrac{1}{8}K_v = \\ &-\tfrac{1}{4}K_v < 0, \end{aligned} \tag{26}$$

for every numerical representation v of $\sqsupseteq^d$ in $\mathcal{G}^X_{\sqsupseteq^d}$, and where $K_v = v(S) - v(T) > 0$ for each $S \in G$ and $T \in B$. On the other hand, $\{1\} \sqsupset^d \{2\}$. Therefore, $\sqsupseteq^d$ is not a $\pi^{\tilde{\mathbf{p}}}$-based extension, i.e. it is not a Banzhaf extension.

Note that the extension $\sqsupseteq^d$ used in Examples 5 and 7 does not satisfy the PR property. In fact, by the proof of Proposition 5, with $\pi^{\mathbf{p}}$ in the role of ϕ and $(p^{|S|} + p^{|S+1|})$ in the role of $p(S)$ for each $S \subset X$ such that $i, j \notin S$, it can be also verified that an extension which satisfies the PR property is also a $\pi^{\mathbf{p}}$-based extension, for every regular semivalue $\pi^{\mathbf{p}}$.

6 Conclusions

In this paper we introduced the class of *Shapley extensions* for the problem of ranking sets of objects of a universal set X when only a primitive ranking on X is given. We showed that this family of extensions is flexible enough to represent possible effects of interaction among objects. We also discussed some properties for extensions with the purpose of studying under which conditions Shapley extensions may be applied to represent preferences over sets of objects excluding or including interaction effects among the objects.

We want to remark that the the family of *lexicographic rank-ordered extensions* introduced by Bossert (1995), and which generalize rankings like lexi-min and lexi-max, satisfy the RESP property (Bossert 1995), and indeed are Shapley extensions. More precisely, since lexicographic rank-ordered extensions have been originally defined for ranking sets of objects with a fixed cardinality (Bossert 1995), we may conclude that each ranking on the power set of a universal set X that ranks subsets of a fixed cardinality q, for each $q = 2, \ldots, |X|$, according to a lexicographic rank-ordered extension, is a Shapley extension.

The direction of our future research is the analysis of subfamilies of Shapley extensions, driven by properties which are aimed at represent intermediate levels of interactions, possibly specified together with the information concerning the primitive ranking on X.

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