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Compactness and Its Implications for Qualitative Spatial and Temporal Reasoning

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Abstract

A constraint satisfaction problem has *compactness* if any infinite set of constraints is satisfiable whenever all its finite subsets are satisfiable. We prove a sufficient condition for compactness, which holds for a range of problems including those based on the well-known Interval Algebra (IA) and RCC8. Furthermore, we show that compactness leads to a useful necessary and sufficient condition for the recently introduced *patchwork* property, namely that patchwork holds exactly when every satisfiable finite network (i.e., set of constraints) has a *canonical* solution, that is, a solution that can be extended to a solution for any satisfiable finite extension of the network. Applying these general theorems to qualitative reasoning, we obtain important new results as well as significant strengthenings of previous results regarding IA, RCC8, and their fragments and extensions. In particular, we show that all the maximal tractable fragments of IA and RCC8 (containing the base relations) have patchwork and canonical solutions as long as networks are algebraically closed.

Introduction

A constraint satisfaction problem (CSP) has *compactness* if any infinite set of constraints is satisfiable whenever all its finite subsets are satisfiable. While its anologue in first-order logic is well known, compactness for CSPs does not hold in general. For example, if h is a natural number, then every finite subset of $\{h > 1, h > 2, \ldots\}$ is clearly satisfiable, yet the whole infinite set is not: There is no natural number that is greater than all natural numbers.

It is known, however, that compactness holds in the case of finite domains (Cowen 1998); hence we shall focus on CSPs over infinite domains, as is typical, for example, in qualitative spatial and temporal reasoning where domains are infinite sets of spatial or temporal entities.

The present paper establishes two major results on compactness of CSPs, with a number of useful corollaries. Our approach is the model-theoretic one that formulates each CSP as satisfiability of a set of first-order sentences in some relational structure $\langle \mathbf{A}, \mathbf{R} \rangle$, called a *template*, where $\mathbf{A}$ and $\mathbf{R}$ correspond to the domain and relations of the CSP. For example, $\mathfrak{P} = \langle \mathbb{Q}, \{=, <, >\} \rangle$ is the rational numbers $\mathbb{Q}$ with the usual comparison relations, and hence CSP($\mathfrak{P}$) is the problem of deciding whether a set of equalities and inequalities between rational variables is satisfiable, which corresponds to the well-known CSP for the Point Algebra (Vilain and Kautz 1986).

Our first result shows that CSP($\mathfrak{A}$) has compactness if the template $\mathfrak{A}$ is *ω-categorical*, that is, if $\mathfrak{A}$ is the unique countable model, up to isomorphism, of the set of all first-order sentences that are true in $\mathfrak{A}$. The structure $\mathfrak{P}$ mentioned above, for example, is ω-categorical (Hirsch 1996), implying that the CSP for the Point Algebra has compactness.

While compactness involves infinite constraint networks, our second result shows that it has useful consequences where we are dealing only with finite networks. Specifically, we show that where compactness holds, the recently introduced *patchwork* property (Lutz and Miličić 2007) is equivalent to the existence of a *canonical* solution for every finite satisfiable network, which is a solution that can be extended into a solution for any finite satisfiable extension of the network.

These two results for general CSPs have important corollaries that significantly strengthen and expand previous results in the field of qualitative reasoning:

- A range of problems, most notably those over the Interval Algebra (IA) (Allen 1983) and RCC8 (Randell, Cui, and Cohn 1992), are known to have formulations with ω-categorical templates (Hirsch 1996; Bodirsky and Wölfl 2011); hence we conclude that these CSPs all have compactness, which strengthens previous results showing the compactness of the CSPs for atomic IA and RCC8 networks (Li and Wang 2006; Lutz and Miličić 2007).
- It is known that the CSPs for the atomic fragments of IA and RCC8, respectively, both have patchwork (Lutz and Miličić 2007); hence we conclude that every satisfiable atomic IA or RCC8 network has a canonical solution, which strengthens a previous result showing the existence of *one-shot extensible* solutions for satisfiable atomic RCC8 networks (Li and Wang 2006).
- Ligozat (1996) gave a result that implies the existence of canonical solutions for the CSP over the *algebraically closed* ORD-Horn (Nebel and Bürckert 1995) fragment of IA; hence we conclude that networks in this fragment of IA have patchwork, which strengthens the previous result (Lutz and Miličić 2007) that networks over atomic IA

relations (a subset of ORD-Horn) have patchwork.

- We show that the three known maximal tractable fragments of RCC8 (Renz and Nebel 1999; Renz 1999)—$\hat{\mathcal{H}}_8, \mathcal{C}_8, \mathcal{Q}_8$—all have canonical solutions when networks are algebraically closed. This further implies that they all have patchwork. Thus we have shown that all the maximal tractable fragments of IA and RCC8 (containing the base relations) have patchwork when networks are algebraically closed, which as we shall discuss can have a significant positive impact on the efficiency and scalability of practical reasoning algorithms.
- Finally, we extend our analysis and results to the n-dimensional Block Algebra (Balbiani, Condotta, and del Cerro 1999b), RCC5 (Bennett 1994), and Cardinal Direction Calculus (CDC) (Ligozat 1998).

The remainder of the paper is structured as follows. We (1) describe the notational setup to be used throughout the paper; (2) define compactness and prove a sufficient condition for it; (3) define the notion of canonical solutions and prove its equivalence to the patchwork property in the presence of compactness; (4) discuss implications of these results for qualitative reasoning, particularly as regards IA, RCC8, and their fragments and extensions; and (5) finally conclude the paper. Related work will be discussed throughout the paper in comparison with the new results we present.

Preliminaries

We start by providing the formal background that will allow us to cast a CSP as satisfiability of a set of first-order sentences in a given structure, and then to define and reason about compactness.

Relational Structures

A relational structure $\mathfrak{A}$ is a pair $\langle \mathbf{A}, \mathbf{R} \rangle$ where $\mathbf{A}$, the *domain*, is a nonempty set and $\mathbf{R}$ is a set of relations on $\mathbf{A}$. For each $R \in \mathbf{R}$, let P_R be a predicate symbol of the same arity in some first-order language $\mathcal{L}_\mathbf{R}$. Further, let $\mathcal{L}_\mathbf{R}$ contain no other predicates, and no function symbols. Let $\mathcal{I}_\mathbf{R}$ be an interpretation for the language $\mathcal{L}_\mathbf{R}$ that maps P_R to R for all $R \in \mathbf{R}$. In the following, we may refer to $\mathfrak{A} = \langle \mathbf{A}, \mathbf{R} \rangle$ as models for $\mathcal{L}_\mathbf{R}$, where the actual models meant are $\langle \mathbf{A}, \mathcal{I}_\mathbf{R} \rangle$.

Two models $\langle \mathbf{A_1}, \mathcal{I}_1 \rangle$ and $\langle \mathbf{A_2}, \mathcal{I}_2 \rangle$ for the language $\mathcal{L}_\mathbf{R}$ are *isomorphic* if they are essentially the same up to renaming of domain elements, that is, if there is a bijection $f : \mathbf{A_1} \mapsto \mathbf{A_2}$ such that

$$\mathcal{I}_1(P_R)(a_1, \ldots, a_r) \Leftrightarrow \mathcal{I}_2(P_R)(f(a_1), \ldots, f(a_r))$$

for all $R \in \mathbf{R}$ and for all $a_1, \ldots, a_r \in \mathbf{A_1}$ where r is the arity of relation R.

The theory of a relational structure $\mathfrak{A}$, denoted Th$\mathfrak{A}$, is the set of all first-order sentences of $\mathcal{L}_\mathbf{R}$ that are true in $\mathfrak{A}$. $\mathfrak{A}$ is said to be *ω-categorical* if all countable models of Th$\mathfrak{A}$ are isomorphic to $\mathfrak{A}$.

Constraint Satisfaction

Given a relational structure $\mathfrak{A} = \langle \mathbf{A}, \mathbf{R} \rangle$ where $\mathbf{A}$ is countable and $\mathbf{R}$ is finite, let $\mathbf{C}$ be a countable set of constant (nullary function) symbols, and expand $\mathcal{L}_\mathbf{R}$ to $\mathcal{L} = \mathcal{L}_\mathbf{R} \cup \mathbf{C}$.

A *constraint* is an atomic formula $P_R(c_1, \ldots, c_r)$ where P_R is a predicate symbol in $\mathcal{L}_\mathbf{R}$ with arity r, and $c_1, \ldots, c_r \in \mathbf{C}$; $pred(\Psi)$ denotes the predicate symbol in constraint Ψ. A *constraint network* is a set of constraints. For a constraint network $\mathbf{\Psi}$, $\mathcal{V}_\mathbf{\Psi}$ denotes the set of constant symbols that appear in $\mathbf{\Psi}$; sometimes we also refer to these as the *variables* of the constraint network. For a set of constant symbols $\mathbf{V}$, $\mathbf{\Psi_V}$ denotes the maximal subset of $\mathbf{\Psi}$ whose constant symbols are all from $\mathbf{V}$. Finally, we say that $\mathbf{\Psi}$ is a *subnetwork* of $\mathbf{\Phi}$, and $\mathbf{\Phi}$ is an *extension* of $\mathbf{\Psi}$, denoted $\mathbf{\Psi} \prec \mathbf{\Phi}$, if $\mathbf{\Psi} = \mathbf{\Phi}_{\mathcal{V}_\mathbf{\Psi}}$.

An *instantiation* of a constraint network $\mathbf{\Psi}$ is a map $\mathcal{I}_\mathbf{\Psi} : \mathcal{V}_\mathbf{\Psi} \mapsto \mathbf{A}$ that assigns an element of the domain to every variable of the network. For instantiations $\mathcal{I}$ and $\mathcal{I}'$, we say that $\mathcal{I}'$ *extends* $\mathcal{I}$, and $\mathcal{I}$ is the *projection* of $\mathcal{I}'$ on $dom(\mathcal{I})$, if $dom(\mathcal{I}) \subseteq dom(\mathcal{I}')$ and $\mathcal{I}(c) = \mathcal{I}'(c)$ for all $c \in dom(\mathcal{I})$, where dom gives the domain of the map.

CSP($\mathfrak{A}$) is the problem of deciding the existence of an instantiation $\mathcal{I}_\mathbf{\Psi}$, called a *solution*, such that $\langle \mathbf{A}, \mathcal{I}_\mathbf{R}, \mathcal{I}_\mathbf{\Psi} \rangle$ is a model of a given constraint network $\mathbf{\Psi}$ (regarded as a set of first-order sentences). When a solution exists, we say that the constraint network $\mathbf{\Psi}$ is *satisfiable* (in $\mathfrak{A}$).

It is sometimes useful to place a restriction X on the constraint networks of a CSP($\mathfrak{A}$), so that one may study properties of the restricted class of networks that might not hold in general; in that case we will refer to the restricted problem as CSP$(\mathfrak{A})_X$. A particular restriction we will use is *algebraic closure* (Ligozat and Renz 2004), which we shall introduce in a later section where it becomes relevant.

Properties of First-Order Logic

In establishing the first main result of this paper, we shall make use of two well-known theorems in first-order logic. The first is the compactness theorem:

Theorem 1 (Compactness of First-Order Logic) *An infinite set of first-order sentences is satisfiable if all its finite subsets are satisfiable.*

As we have in fact formulated CSPs as satisfiability of sets of first-order sentences, one might wonder why CSPs do not automatically have compactness. The reason is the following: Given an infinite set of constraints $\mathbf{\Psi}$ for a CSP($\mathfrak{A}$), if all its finite subsets are satisfiable (in $\mathfrak{A}$), then Theorem 1 only guarantees the satisfiability of $\mathbf{\Psi}$ in *some* structure, which may not be $\mathfrak{A}$. In the following section we will see that under a certain condition we *will* be able to guarantee the satisfiability of $\mathbf{\Psi}$ in $\mathfrak{A}$, with the help of a second well-known theorem in first-order logic, a version of which sufficient for our purposes is given below (here a "countable first-order theory" means a set of first-order sentences using a countable set of predicate and function symbols):

Theorem 2 (Downward Löwenheim-Skolem Theorem) *A countable first-order theory has a countable model if it has an infinite model.*

We are now ready to formulate a notion of compactness for CSPs, and prove a sufficient condition for it.

Compactness

Given our first-order formulation of CSPs, we define compactness for CSPs in a way completely analogous to its counterpart in first-order logic (Lutz and Miličić, 2007, used a slightly different but essentially equivalent definition):

Definition 1 (Compactness) *CSP($\mathfrak{A}$) has* compactness *if any infinite set of constraints over $\mathfrak{A}$ is satisfiable whenever all its finite subsets are satisfiable.*

We now proceed directly to establish a simple sufficient condition for compactness. In a later section we shall see that this condition is in fact satisfied by a wide range of CSPs of common interest, particularly in the field of qualitative reasoning. The intuitive idea behind the proof of this condition is the following: Given that Theorem 1 guarantees the existence of *some* model satisfying the infinite set of atomic formulas, we use two techniques, respectively, to ensure that this model (i) is countable, and (ii) can be turned into a model based on $\mathfrak{A}$, the template of the CSP, thus showing the satisfiability of the infinite constraint network.

Proposition 1 (Sufficient Condition for Compactness) *CSP($\mathfrak{A}$) has* compactness *if $\mathfrak{A}$ is ω-categorical.*

Proof: Let $\mathbf{\Psi}$ be an infinite set of constraints over $\mathfrak{A} = \langle \mathbf{A}, \mathbf{R} \rangle$, where $\mathbf{A}$ is countable and $\mathbf{R}$ is finite. Suppose that every finite subset of $\mathbf{\Psi}$ is satisfiable in $\mathfrak{A}$, and that $\mathfrak{A}$ is ω-categorical. We wish to prove that $\mathbf{\Psi}$ is satisfiable in $\mathfrak{A}$.

Let $\mathbf{\Psi}' = \mathbf{\Psi} \cup \mathrm{Th}\mathfrak{A} \cup \{\neg(c'_i = c'_j) \mid i, j \in \mathbb{N}, i \neq j\}$ for a set of fresh constant symbols c'_i, one for each natural number. Clearly, every finite subset of $\mathbf{\Psi}'$ is satisfiable (in $\mathfrak{A}$); hence $\mathbf{\Psi}'$ has a model by the compactness of first-order logic. This model must be infinite, because the infinite set of constants $\{c'_i \mid i \in \mathbb{N}\}$ must be interpreted as distinct objects. Further, the language of $\mathbf{\Psi}'$ is $\mathcal{L} \cup \{c'_i \mid i \in \mathbb{N}\}$, and hence countable. Therefore, by the Downward Löwenheim-Skolem Theorem, $\mathbf{\Psi}'$ has a countable model.

Let $\langle \mathbf{A}', \mathcal{I}' \rangle$ denote this countable model of $\mathbf{\Psi}'$. Since $\mathrm{Th}\mathfrak{A} \subseteq \mathbf{\Psi}'$, $\langle \mathbf{A}', \mathcal{I}' \rangle$ is also a model of $\mathrm{Th}\mathfrak{A}$; hence it is isomorphic to $\mathfrak{A}$,[1] that is, to $\langle \mathbf{A}, \mathcal{I}_{\mathbf{R}} \rangle$, by the ω-categoricity of $\mathfrak{A}$. Let f be the bijection from $\mathbf{A}'$ to $\mathbf{A}$ that witnesses the isomorphism. For every predicate $P_R \in \mathcal{L}$ with arity r and any $a'_1, \ldots, a'_r \in \mathbf{A}'$, the isomorphism stipulates that

$$\mathcal{I}'(P_R)(a'_1, \ldots, a'_r) \Leftrightarrow \mathcal{I}_{\mathbf{R}}(P_R)(f(a'_1), \ldots, f(a'_r)).$$

But $\mathcal{I}_{\mathbf{R}}(P_R) = R$; hence

$$\mathcal{I}'(P_R)(a'_1, \ldots, a'_r) \Leftrightarrow R(f(a'_1), \ldots, f(a'_r)). \tag{1}$$

Let $\mathcal{I}$ be a new interpretation such that $\mathcal{I} = \mathcal{I}_{\mathbf{R}}$ for all predicates and $\mathcal{I} = f \circ \mathcal{I}'$ for all constants. Equation (1), together with the fact that $\langle \mathbf{A}', \mathcal{I}' \rangle$ is a model of $\mathbf{\Psi}$ (another subset of $\mathbf{\Psi}'$), implies that $\langle \mathbf{A}, \mathcal{I} \rangle$ is also a model of $\mathbf{\Psi}$; in other words, the infinite set of constraints $\mathbf{\Psi}$ is satisfiable in $\mathfrak{A}$. $\square$

In light of Proposition 1, we may now revisit the counterexample given in the first paragraph of the paper, a version of which can be formulated as an instance of CSP($\mathfrak{N}$) where $\mathfrak{N} = \langle \mathbb{N}, \{<\} \rangle$ is the natural numbers with the usual "$<$" relation. Specifically, take a total order among countably many variables $\{c_1 < c_2 < \ldots\}$, and add the constraints $\{c_i < h \mid i \in \mathbb{N}\}$ for a fresh variable h. The infinite network obtained witnesses the lack of compactness for CSP($\mathfrak{N}$), implying that the template $\mathfrak{N}$ is not ω-categorical.

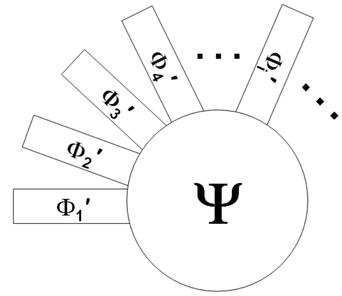

Figure 1: An infinite number of networks (each $\mathbf{\Phi}'_\mathbf{i}$ includes $\mathbf{\Psi}$) having a common subnetwork but otherwise disjoint.

Patchwork Property and Canonical Solutions

We show next that compactness has an important consequence in the form of the equivalence between the *patchwork* property (Lutz and Miličić 2007) and existence of what we shall define as *canonical* solutions.

Intuitively, patchwork ensures that the combination of two satisfiable constraint networks that agree on their common variables continues to be satisfiable. Below is a definition of patchwork (Lutz and Miličić 2007) phrased in our notation.

Definition 2 (Patchwork) *A CSP has* patchwork *if for any finite satisfiable constraint networks $\mathbf{\Psi}$ and $\mathbf{\Phi}$ of the CSP such that $\mathbf{\Psi}_{\mathcal{V}_\Psi \cap \mathcal{V}_\Phi} = \mathbf{\Phi}_{\mathcal{V}_\Psi \cap \mathcal{V}_\Phi}$, the constraint network $\mathbf{\Psi} \cup \mathbf{\Phi}$ is satisfiable.*

Note that the CSP in the above definition can be any CSP($\mathfrak{A}$) or CSP($\mathfrak{A}$)$_X$ (i.e., a restricted class of networks), and in the latter case, the network $\mathbf{\Psi} \cup \mathbf{\Phi}$ is not required to satisfy the restriction X.

As has been noted by Lutz and Miličić (2007), compactness and patchwork together potentially allow an infinite number of finite satisfiable networks to be "patched" together into an infinite satisfiable network. We now exploit this idea to establish a necessary and sufficient condition for patchwork in the presence of compactness. This condition will be based on the following notion of *canonical* solutions:

Definition 3 (Canonical Solutions) *For a given CSP, a solution $\mathcal{I}_\mathbf{\Psi}$ for a finite satisfiable constraint network $\mathbf{\Psi}$ is* canonical *if every finite satisfiable extension of $\mathbf{\Psi}$ has a solution that extends $\mathcal{I}_\mathbf{\Psi}$. We say that a CSP has* canonical solutions *if every finite satisfiable network of the CSP has a canonical solution.*

Let us proceed to present the second general result of this paper, where we show, interestingly, that compactness, which concerns infinite networks, enables us to prove a result on patchwork and canonical solutions, both of which

[1] More precisely, the *reduct* of $\langle \mathbf{A}', \mathcal{I}' \rangle$ to $\mathcal{L}$ is a model of $\mathrm{Th}\mathfrak{A}$, and is isomorphic to $\mathfrak{A}$.

concern only finite networks. The basic idea there is to patch together an infinite number of finite satisfiable extensions of a network (with variables suitably renamed), use compactness to establish a solution for the infinite network, and project the solution back on finite subsets of its variables to obtain solutions to its finite subnetworks.

To simplify notation, the CSP in the proof of the following proposition is represented as CSP($\mathfrak{A}$); however, the proof works as well for CSPs with restrictions, i.e., CSP$(\mathfrak{A})_X$ for some restriction X, where compactness is taken to mean that of the unrestricted CSP($\mathfrak{A}$).

Proposition 2 (Patchwork and Canonical Solutions) *A CSP* (i) *has canonical solutions if it has compactness and patchwork, and* (ii) *has patchwork if it has canonical solutions.*

Proof: In what follows we shall have occasion to implicitly appeal to two facts about countable sets: The set of all finite strings over a countable alphabet is countable; the set of all finite subsets of a countable set is countable.

Given a relational structure $\mathfrak{A} = \langle \mathbf{A}, \mathbf{R} \rangle$ where $\mathbf{A}$ is countable and $\mathbf{R}$ is finite, recall that each constraint in CSP($\mathfrak{A}$) is a predicate over constants in the first-order language $\mathcal{L} = \mathcal{L}_\mathbf{R} \cup \mathbf{C}$ where $\mathbf{C} = \{c_1, c_2, \ldots\}$ is a countable set of constant symbols. Let $\Delta(\mathfrak{A})$ be the set of all constraints that can be written for CSP($\mathfrak{A}$). We have that $\Delta(\mathfrak{A})$ is countable.

(i) Suppose that CSP($\mathfrak{A}$) has compactness and patchwork. Let $\mathbf{\Psi}$ be any finite satisfiable network of CSP($\mathfrak{A}$); without loss of generality let the constant symbols in $\mathbf{\Psi}$ be $c_1, \ldots, c_{|\mathcal{V}_\mathbf{\Psi}|}$. We shall show that $\mathbf{\Psi}$ has a canonical solution. Let $\mathbf{\Phi_1}, \mathbf{\Phi_2}, \ldots$ be an enumeration of all finite satisfiable networks of CSP($\mathfrak{A}$) that are extensions of $\mathbf{\Psi}$, which is possible as each $\mathbf{\Phi_i}$ is a finite subset of $\Delta(\mathfrak{A})$ and hence $\{\mathbf{\Phi_i}\}$ is countable. We shall next rename the constant symbols of all $\mathbf{\Phi}$'s so that no constant symbol outside $\mathcal{V}_\mathbf{\Psi}$ is shared between any $\mathbf{\Phi}$'s, as follows: For each $i = 1, 2, \ldots$, let $g(i) = |\mathcal{V}_\mathbf{\Psi}| + \sum_{j=1}^{i-1} |\mathcal{V}_{\mathbf{\Phi_j}} \backslash \mathcal{V}_\mathbf{\Psi}|$, and let $\mathbf{\Phi'_i}$ be $\mathbf{\Phi_i}$ with the constant symbols $\mathcal{V}_{\mathbf{\Phi_i}} \backslash \mathcal{V}_\mathbf{\Psi}$ replaced, respectively, by $\{c_{g(i)+1}, \ldots, c_{g(i+1)}\}$. This transformation is well-defined as $g(i)$ is finite for each i. We now have $\{\mathbf{\Phi'_i}\}$ such that $\mathbf{\Psi} \prec \mathbf{\Phi'_i}$ for all i and $\mathcal{V}_{\mathbf{\Phi'_i}} \cap \mathcal{V}_{\mathbf{\Phi'_j}} = \mathcal{V}_\mathbf{\Psi}$ for all $i \neq j$, as illustrated in Figure 1. This implies that the union of any finite number of elements from $\{\mathbf{\Phi'_i}\}$ is satisfiable (by applying patchwork a finite number of times), which in turn implies that any finite subset of $\bigcup\{\mathbf{\Phi'_i}\}$ is satisfiable, and hence $\bigcup\{\mathbf{\Phi'_i}\}$ is satisfiable by compactness. Let $\mathcal{I}_\mathbf{C}$ be a solution for $\bigcup\{\mathbf{\Phi'_i}\}$, $\mathcal{I}_\mathbf{\Psi}$ its projection on $\mathcal{V}_\mathbf{\Psi}$, and $\mathcal{I}_i$ its projection on $\mathcal{V}_{\mathbf{\Phi'_i}}$. Clearly, $\mathcal{I}_\mathbf{\Psi}$ is a solution for $\mathbf{\Psi}$, and $\mathcal{I}_i$ is a solution for $\mathbf{\Phi'_i}$ and an extension of $\mathcal{I}_\mathbf{\Psi}$. As $\mathbf{\Phi_i}$ and $\mathbf{\Phi'_i}$ are essentially the same network apart from the renaming of constant symbols outside $\mathcal{V}_\mathbf{\Psi}$, $\mathcal{I}_i$ can be turned into a solution for $\mathbf{\Phi_i}$ that continues to be an extension of $\mathcal{I}_\mathbf{\Psi}$. This shows that $\mathcal{I}_\mathbf{\Psi}$ is a canonical solution for $\mathbf{\Psi}$.

(ii) Suppose that CSP($\mathfrak{A}$) has canonical solutions. Let $\mathbf{\Psi}$ and $\mathbf{\Phi}$ be any two finite satisfiable networks such that $\mathbf{\Psi}_{\mathcal{V}_\mathbf{\Psi} \cap \mathcal{V}_\mathbf{\Phi}} = \mathbf{\Phi}_{\mathcal{V}_\mathbf{\Psi} \cap \mathcal{V}_\mathbf{\Phi}} = \mathbf{\Gamma}$. We wish to show that $\mathbf{\Psi} \cup \mathbf{\Phi}$ is satisfiable. Let $\mathcal{I}_\mathbf{\Gamma}$ be a canonical solution for $\mathbf{\Gamma}$. Since $\mathbf{\Gamma} \prec \mathbf{\Psi}$, there is an extension $\mathcal{I}_\mathbf{\Psi}$ of $\mathcal{I}_\mathbf{\Gamma}$ that is a solution for $\mathbf{\Psi}$; similarly, there is an extension $\mathcal{I}_\mathbf{\Phi}$ of $\mathcal{I}_\mathbf{\Gamma}$ that is a solution for $\mathbf{\Phi}$. The combination of $\mathcal{I}_\mathbf{\Psi}$ and $\mathcal{I}_\mathbf{\Phi}$ (in the natural way), which is well-defined as they agree on the intersection of their domains, is a solution for $\mathbf{\Psi} \cup \mathbf{\Phi}$. $\square$

While the proof above illustrates the usefulness of compactness, one may wonder whether compactness is inherently necessary for part (i) of the proposition to hold. In particular, could patchwork somehow imply compactness? The answer is unfortunately no. For a counterexample, consider CSP$(\mathfrak{N})_C$, the class of networks over the template $\mathfrak{N}$ that are *complete* (i.e., where there is a constraint "$<$" between every two variables). This CSP does not have compactness as discussed at the end of the preceding section, but has patchwork as the reader may easily verify: Any network of CSP$(\mathfrak{N})_C$ effectively defines a total order over natural numbers, and one can always choose numbers sufficiently far apart so that the solution obtained for $\mathbf{\Psi}_{\mathcal{V}_\mathbf{\Psi} \cap \mathcal{V}_\mathbf{\Phi}}$ can be extended into solutions for both $\mathbf{\Psi}$ and $\mathbf{\Phi}$ (as in Definition 2), which are finite networks. On the other hand, CSP$(\mathfrak{N})_C$ does not have canonical solutions, because no matter how far apart the numbers are, an extended network can always pose the infeasible task of fitting too many distinct numbers between them.

In the rest of the paper we will use part (ii) more often than part (i) of Proposition 2 to derive new results. Nevertheless, we note that the significance of part (i) lies also in that it provides the assurance that the sufficient condition we have identified for patchwork is no stronger than necessary, which will be useful in future work that attempts to establish patchwork for new CSPs. Previous work (Li, Huang, and Renz 2009), by contrast, used the condition of *strong n-consistency*, which is much stronger and not satisfied by many CSPs that in fact have patchwork as we show later.

Qualitative Reasoning

Having established two general results (Propositions 1 and 2) involving compactness of CSPs, we proceed to discuss a number of their implications with respect to concrete problems in qualitative (spatial and temporal) reasoning, where infinite domains are typical. Interestingly, some of these implications turn out significantly strengthen several previous results, some of which can have a direct positive impact on the efficiency of practical reasoning algorithms.

These problems in qualitative reasoning can each be formulated as a CSP whose template is a *qualitative calculus*. In this work we consider only qualitative calculi with binary relations, which are relational structures $\langle \mathbf{A}, \mathbf{R} \rangle$ satisfying the following conditions: (i) All relations in $\mathbf{R}$ are binary. (ii) There exists $\mathbf{B} \subseteq \mathbf{R}$ such that the relations $\mathbf{B}$ form a partitioning of $\mathbf{A} \times \mathbf{A}$. In other words, over any two (ordered) elements of $\mathbf{A}$, exactly one relation in $\mathbf{B}$ holds. These are known as *base* or *atomic* relations. It is assumed that $\mathbf{B}$ contains the identity relation and is closed under converse. (iii) $\mathbf{R} = \{\bigcup \mathbf{S} \mid \mathbf{S} \subseteq \mathbf{B}\}$; that is, relations in $\mathbf{R}$ are generated by taking all possible unions of (zero or more) base relations. We denote such a set $\mathbf{R}$ by $\mathbf{B}^*$.

For qualitative reasoning we assume that any constraint network contains at most one constraint for any (unordered) pair of variables (constant symbols). For notational convenience, we introduce a function $\ell_{\mathbf{\Psi}} : \mathbf{C} \times \mathbf{C} \mapsto \mathbf{R}$ for every constraint network $\mathbf{\Psi}$ such that $\ell_{\mathbf{\Psi}}(c_i, c_j)$ gives the relation specified in $\mathbf{\Psi}$ over variables c_i and c_j (i.e., if $\Psi \in \mathbf{\Psi}$ is a constraint over c_i and c_j, then $\ell_{\mathbf{\Psi}}(c_i, c_j) = \mathcal{I}_{\mathbf{R}}(pred(\Psi))$; $\ell_{\mathbf{\Psi}}(c_i, c_j)$ is defined to be the converse of $\ell_{\mathbf{\Psi}}(c_j, c_i)$ if $\mathbf{\Psi}$ contains a constraint over (c_j, c_i), and defined to be the *universal relation*, i.e., $\mathbf{A} \times \mathbf{A}$, if $\mathbf{\Psi}$ does not contain a constraint between c_i and c_j at all. Sometimes we refer to $\ell_{\mathbf{\Psi}}(c_i, c_j)$ as the *label* of the *edge* (c_i, c_j) of the network.

Several standard concepts in qualitative reasoning may now be introduced. Let $\mathbf{\Psi}$ and $\mathbf{\Psi}'$ be constraint networks for qualitative CSP($\langle \mathbf{A}, \mathbf{R} = \mathbf{B}^* \rangle$).

- $\mathbf{\Psi}$ is *complete* if it includes a constraint for every (unordered) pair of its variables.
- $\mathbf{\Psi}$ is *atomic* if the label of every edge is a base relation, that is, if $\ell_{\mathbf{\Psi}}(c_i, c_j) \in \mathbf{B}$ for all $c_i, c_j \in \mathcal{V}_{\mathbf{\Psi}}$.
- $\mathbf{\Psi}'$ is a *refinement* of $\mathbf{\Psi}$ if $\mathbf{\Psi}'$ specifies a stronger relation than $\mathbf{\Psi}$ for every pair of variables, that is, if $\ell_{\mathbf{\Psi}'}(a, b) \subseteq \ell_{\mathbf{\Psi}}(a, b)$ for all $a, b \in \mathcal{V}_{\mathbf{\Psi}}, a \neq b$.
- The *weak composition* of relations $R_1, R_2 \in \mathbf{R}$, denoted $R_1 \diamond R_2$, is the minimal relation in $\mathbf{R}$ that contains the (standard set-theoretic) composition of R_1 and R_2, or formally, $R_1 \diamond R_2 = \bigcup\{R \mid R \in \mathbf{B}, R \cap (R_1 \circ R_2) \neq \emptyset\}$, where $\circ$ denotes composition ($\diamond$ and $\circ$ are identical if $\mathbf{R}$ is closed under composition).
- $\mathbf{\Psi}$ is *algebraically closed* (a-closed) [2] if $\emptyset \neq \ell_{\mathbf{\Psi}}(c_i, c_k) \subseteq \ell_{\mathbf{\Psi}}(c_i, c_j) \diamond \ell_{\mathbf{\Psi}}(c_j, c_k)$ for all $c_i, c_j, c_k \in \mathcal{V}_{\mathbf{\Psi}}$ (if weak composition is identical to composition, then a-closure is identical to the standard notion of *path consistency*).[3]

Let C denote the restriction that networks be complete, and AC denote the restriction that networks be a-closed. The CSP for the atomic networks of a calculus $\langle \mathbf{A}, \mathbf{R} = \mathbf{B}^* \rangle$ can thus be written as CSP($\langle \mathbf{A}, \mathbf{B} \rangle)_C$.

We are now ready to discuss implications of Propositions 1 and 2 for concrete qualitative CSPs. We will focus on IA and RCC8, two of the best-known examples of qualitative calculi, and briefly discuss several others afterwards. In presenting our results, we shall make use of a basic fact that first-order interpretations preserve ω-categoricity[4] (Hodges 1993), and in particular its direct consequence that if $\langle \mathbf{A}, \mathbf{B} \rangle$ is ω-categorical, then so is $\langle \mathbf{A}, \mathbf{B}^* \rangle$ (Bodirsky and Wölfl 2011).

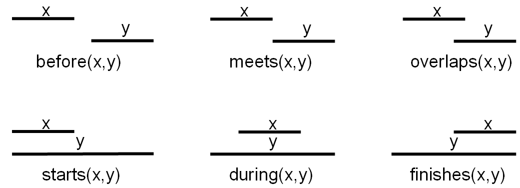

Figure 2: IA base relations: six shown above, their converses, and the identity relation.

Interval Algebra

IA (Allen 1983) is intended to model relations between intervals on the infinite time line. The domain of IA can be conveniently taken to be the set of intervals on the rational numbers:[5] $\mathbf{Intv} = \{(b, e) \mid b, e \in \mathbb{Q}, b < e\}$. The base relations of IA are $\mathbf{B_{IA}} = \{$before, after, meets, met-by, overlaps, overlapped-by, equals, during, includes, starts, started-by, finishes, finished-by$\}$. These capture the intuitive relations, illustrated in Figure 2, that hold between time intervals, and can be readily defined in a formal way; for example, before$= \{((b_1, e_1), (b_2, e_2)) \mid (b_1, e_1), (b_2, e_2) \in \mathbf{Intv}, e_1 < b_2\}$ (we shall not burden the reader with definitions of the rest).

The structure $\langle \mathbf{Intv}, \mathbf{B_{IA}} \rangle$ is known to be ω-categorical (Ladkin and Maddux 1994); hence $\mathbf{IA} = \langle \mathbf{Intv}, \mathbf{B}^*_{\mathbf{IA}} \rangle$ is ω-categorical as well, and by Proposition 1 we have:

Proposition 3 *CSP(***IA***) has compactness.*

Since $\mathbf{B}^*_{\mathbf{IA}}$ is much larger than $\mathbf{B_{IA}}$, Proposition 3 is a significant strengthening of the previous result (Lutz and Miličić 2007) that CSP($\langle \mathbf{Intv}, \mathbf{B_{IA}} \rangle)_C$, the CSP for atomic IA networks, has compactness.

CSP($\langle \mathbf{Intv}, \mathbf{B_{IA}} \rangle)_C$ is known to have patchwork (Lutz and Miličić 2007); hence by Proposition 2 it has canonical solutions. In fact, in this case one can establish a stronger result that *every* solution is a canonical solution, because satisfiable atomic IA networks over n variables are known to be strongly n-consistent (van Beek and Cohen 1990).

Our next result concerns **ORD-Horn**, a large subset of $\mathbf{B}^*_{\mathbf{IA}}$ containing 868 of the 2^{13} relations, identified as the maximal subset $\mathbf{R}'$ of $\mathbf{B}^*_{\mathbf{IA}}$ containing $\mathbf{B_{IA}}$ such that CSP($\langle \mathbf{Intv}, \mathbf{R}' \rangle)_C$ is tractable, i.e., solvable by a known polynomial-time algorithm (Nebel and Bürckert 1995). Since **ORD-Horn** contains the universal relation, the CSP for **ORD-Horn** networks is precisely CSP($\langle \mathbf{Intv}, \mathbf{ORD\text{-}Horn} \rangle$). Ligozat (1996) gave an alternative proof of tractability for this CSP by showing that a solution can be generated for every finite a-closed satisfiable network of the CSP by instantiating one variable at a

[2]Some authors use the term *transitively closed*.

[3]This definition of a-closure is a semantic one as it is relative to the template of the CSP. If one takes a qualitative calculus to be a relation algebra, which is syntactically equipped with the (weak) composition table, the same definition of a-closure would become a purely syntactic one. In Footnote 8 we will indeed have occasion to refer to a syntactic version of a-closure.

[4]That is, if $\mathfrak{A}$ can be interpreted in $\mathfrak{B}$, then $\mathfrak{A}$ is ω-categorical if $\mathfrak{B}$ is. While we omit a formal definition of *interpretation*, the following example should give the general idea: The (field of) rationals can be interpreted in the (ring of) integers as each rational can be defined by a pair of integers (hence the interpretation in this case is called *2-dimensional*), and each operation (addition, multiplication, etc.) on rationals can be defined by a formula on integers.

[5]Although IA has historically been introduced as a relation algebra, it is now standard practice to regard it as a qualitative calculus, hence a relational structure (which provides a *representation* for the algebra). Also, using real numbers results in an equivalent problem, since the set $\mathbf{C}$ of constant symbols of a CSP is assumed to be countable.

time without backtrack. The procedure, which is guaranteed to succeed (Theorem 2, Ligozat 1996), calls for assigning an interval to each variable, in any order, such that relevant constraints are respected and the endpoints of previously used intervals are avoided whenever possible. It follows that:

Proposition 4 *CSP(*$\langle \mathbf{Intv}, \mathbf{ORD\text{-}Horn} \rangle$*)*$_{AC}$ *has canonical solutions.*

Hence by Proposition 2, we have:

Proposition 5 *CSP(*$\langle \mathbf{Intv}, \mathbf{ORD\text{-}Horn} \rangle$*)*$_{AC}$ *has patchwork.*

Proposition 5 is a significant strengthening of the previous result that atomic IA networks have patchwork,[6] and can have an important positive impact on the efficiency of practical reasoning algorithms that often rely on an efficient a-closure test. Specifically, it has been shown that because atomic IA networks have patchwork, in testing the existence of an a-closed atomic refinement of an IA network $\mathbf{\Psi}$, one may identify a subset of the node triples, based on a decomposition of the network (into subnetworks that can be "patched" back together), that need *not* be examined, hence reducing the complexity of the test (Li, Huang, and Renz 2009).[7] This discovery has resulted in instances of the general CSP($\mathbf{IA}$) to be solved much more efficiently, and much larger instances to be solved, as CSP($\mathbf{IA}$) can be reduced to the existence of an a-closed atomic refinement of the network (Vilain and Kautz 1986).

A direct implication of Proposition 5 is that the above-described technique can be generalized to reasoners that rely on finding not just atomic refinements of networks, but also ORD-Horn refinements, such as GQR (Gantner, Westphal, and Wölfl 2008), one of the state-of-the-art qualitative reasoners. This will potentially have an impact similar to that observed by Li, Huang, and Renz (2009), but on a much broader class of algorithms.

We note here that the discussion above will become relevant again when we later provide a set of new results on the patchwork property for the tractable fragments of RCC8.

Region Connection Calculus RCC8

RCC8 (Randell, Cui, and Cohn 1992) is intended to model relations between topological regions, such as "disconnected" and "partially overlapping." Unfortunately, the concept of "regions," unlike that of time intervals, does not have a single simple definition. As a result, RCC8 is in fact not a single structure, but rather a class of structures satisfying a set of axioms that are intended to encode a concept of regions and their relations. These structures are referred to as *RCC models* and the set of axioms as a *RCC theory*.

In order to develop a mathematically precise context in which our formal Propositions may be directly applied, it is necessary to trace the history of RCC8 to the present point where the reasoning problem for RCC8 may be cast as a single CSP over an ω-categorical template. This we do next.

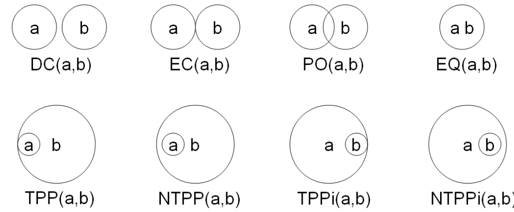

Figure 3: RCC8 base relations.

Historical Formulation of RCC8

There are different formulations of the RCC theory. In the formulation adopted by Bodirsky and Wölfl (2011), for example, a RCC model is built on top of a Boolean algebra, which is a structure $\langle \mathbf{G}, \cdot, +, -, 0, 1 \rangle$ satisfying a set of axioms $\mathbf{\Gamma_1}$ that govern the behavior of the three functions $\cdot, +, -$ and the two distinguished elements $0, 1$ of the domain $\mathbf{G}$ (RCC models further require that 0 and 1 not be identical). We may start to think of the elements of $\mathbf{G}$ as regions with 0 being the empty region and 1 being the union of all regions, and the functions $\cdot, +, -$ as intersection, union, and complement, respectively.

The structure $\mathfrak{G}$ is then extended into a *Boolean contact algebra* (Bennett and Düntsch 2007) by adding a single relation C intended to encode the connectedness of two regions. Again this intention is captured by a set of axioms $\mathbf{\Gamma_2}$.

This new structure is now further augmented with a set of 8 base relations $\mathbf{B_{RCC8}}$ $=\{$DC,EC,PO,EQ,TPP,NTPP,TPPi,NTPPi$\}$, governed by a new set of axioms $\mathbf{\Gamma_3}$ defining them in terms of C and ensuring that they form a partitioning of $\mathbf{G} \times \mathbf{G}$ as well as capture the intuitive notions of the following spatial relations (illustrated in Figure 3): disconnected, externally connected, partially overlapping, equal, tangential proper part, nontangential proper part, and converses of the last two.

The set of axioms $\mathbf{\Gamma} = \mathbf{\Gamma_1} \cup \mathbf{\Gamma_2} \cup \mathbf{\Gamma_3}$ (contents omitted) is then a RCC theory, and for any model $\mathfrak{T} = \langle \mathbf{G}, \cdot, +, -, 0, 1, \{C\} \cup \mathbf{B_{RCC8}} \rangle$ of $\mathbf{\Gamma}$, there is a corresponding qualitative reasoning problem CSP($\langle \mathbf{G}, \mathbf{B^*_{RCC8}} \rangle$), which we shall also denote by CSP($\mathfrak{T}$) (with a slight abuse of notation).

From Semantic to Syntactic Formulation

This is a very different picture from that of IA, and the lack of a single simple formulation of the CSP for RCC8 would have caused much inconvenience in both theory and practice but for the following discoveries: (i) A constraint network is satisfiable in some RCC model iff it is satisfiable in a particular RCC model $\mathfrak{T}$ that corresponds to a certain topological space where regions are its regular closed subsets (Li 2006). (ii) A finite atomic network is satisfiable in $\mathfrak{T}$ iff it is a-closed (Nebel 1995). (iii) An infinite atomic network is satisfiable

[6] For atomic IA networks, a-closure is equivalent to satisfiability (Vilain and Kautz 1986); hence patchwork for CSP($\langle \mathbf{Intv}, \mathbf{B_{IA}} \rangle$)$_C$ is the same as patchwork for CSP($\langle \mathbf{Intv}, \mathbf{B_{IA}} \rangle$)$_{C,AC}$. The same holds for atomic networks of all the other calculi we shall discuss in this work.

[7] The cited work uses the *atomic network amalgamation property*, which is equivalent to patchwork for atomic networks.

in $\mathfrak{T}$ iff all its finite subsets are satisfiable in $\mathfrak{T}$ (Li and Wang 2006).

These three results combined bring us to a situation similar to that with IA. Specifically, reasoning over RCC8 can now be taken to be the following single problem: Given a constraint network of CSP($\mathfrak{T}$), does it have an a-closed atomic refinement?

This is a purely syntactic problem, given that the weak composition of every pair of base relations in $\mathfrak{T}$ is known, in the form of a *RCC8 composition table*, here denoted RCC8-CT(R, S), which gives $R \diamond S$, as a set of base relations (whose union gives $R \diamond S$), for every pair of relations $R, S \in \mathbf{B_{RCC8}}$. The full composition table may be found in Cui, Cohn, and Randell (1993); as examples we cite here two of its entries: RCC8-CT(DC,EC)={DC,EC,PO,TPP,NTPP}, RCC8-CT(EC,TPPi)={DC,EC}.

Back to Semantic Formulation

Very recently, Bodirsky and Wölfl (2011) gave an alternative formulation of the RCC theory, based not on a Boolean contact algebra but on the RCC8 composition table. This theory has only two simple parts (Definition 2, Bodirsky and Wölfl, 2011):

1. Axioms on the basic behavior of the base relations: (a) $\mathbf{B_{RCC8}}$ is a partitioning of $\mathbf{G} \times \mathbf{G}$; (b) EQ is the identity relation; (c) DC, EC, and PO are symmetric; (d) TPPi and NTPPi are respectively converses of TPP and NTPP.
2. $\forall xyz (P_R(x, y) \wedge P_S(y, z) \to P_{T_1}(x, z) \vee \ldots \vee P_{T_k}(x, z))$, for every pair of base relations $R, S \in \mathbf{B_{RCC8}}$, where RCC8-CT($R$, S)= $\{T_1, \ldots, T_k\}$.

Given that the second part of the theory above effectively "encodes" the RCC8 composition table, if $\mathfrak{R} = \langle \mathbf{G}, \mathbf{B_{RCC8}} \rangle$ is a model of the theory such that all finite and countable models of the theory embed into it,[8] then CSP($\mathfrak{G} = \langle \mathbf{G}, \mathbf{B^*_{RCC8}} \rangle$) is equivalent to the syntactic formulation of the RCC8 reasoning problem described earlier (Bodirsky and Wölfl 2011). Furthermore, the same paper presents such a model $\mathfrak{R}$ that is, in addition, ω-categorical, implying that $\mathfrak{G}$ is also ω-categorical.

New Results

The reasoning problem for RCC8 being now formulated as a single CSP($\mathfrak{G}$) with an ω-categorical template $\mathfrak{G}$, by Proposition 1, we finally arrive at the conclusion that:

Proposition 6 *The CSP for RCC8 has compactness.*

Again this is a significant strengthening of the previous result that the CSP for atomic RCC8 networks has compactness (Li and Wang 2006; Lutz and Miličić 2007), which has itself been an important resolution of an open question raised earlier (Balbiani, Challita, and Condotta 2003).

Note that in presenting our results on RCC8 (and later RCC5) we omit the explicit mention of the template $\mathfrak{G}$ and its restrictions to various subcalculi. This does not affect the validity of the comparison of these results with those in previous work, since satisfiability in $\mathfrak{G}$ is equivalent to satisfiability in the standard model $\mathfrak{T}$ as discussed earlier.

Since atomic RCC8 networks are also known to have patchwork (Lutz and Miličić 2007), we immediately obtain the following by Proposition 2:

Proposition 7 *The CSP for atomic RCC8 networks has canonical solutions.*

Proposition 7 may be contrasted with the previous result (Li and Wang 2006) that every satisfiable atomic RCC8 network has a *one-shot extensible* solution, which is a solution that can be extended into a solution for any satisfiable atomic extension of the network that has precisely one additional variable and two additional constraints linking that variable to two existing variables. It should be clear that the result presented here is much stronger as it applies to extensions of arbitrary size, not just *one-shot* extensions.

Our next set of results concerns $\hat{\mathcal{H}}_8, \mathcal{C}_8, \mathcal{Q}_8$, the (only) three maximal subsets of $\mathbf{B^*_{RCC8}}$ containing $\mathbf{B_{RCC8}}$ over which reasoning is tractable (Renz 1999). We have discussed earlier (under Proposition 5) that the patchwork property permits a decomposition of networks that can significantly improve the efficiency and scalability of practical reasoning. While this method applies to both IA and RCC8, so far the only known fragment of RCC8 that has patchwork is the atomic fragment $\mathbf{B_{RCC8}}$ containing just the 8 base relations, which limits the applicability of decomposition.

Here we show that the three large subsets $\hat{\mathcal{H}}_8, \mathcal{C}_8, \mathcal{Q}_8$ (containing respectively 148,158, and 160 of the 2^8 relations) all lead to fragments of RCC8 that have patchwork, thus greatly broadening the applicability of decomposition. We first show that these fragments have canonical solutions; again we require networks in these fragments to be a-closed:

Proposition 8 *The three CSPs for a-closed $\hat{\mathcal{H}}_8, \mathcal{C}_8$, and $\mathcal{Q}_8$ networks, respectively, all have canonical solutions.*

Proof: We first note that a-closed $\hat{\mathcal{H}}_8, \mathcal{C}_8$, or $\mathcal{Q}_8$ networks are guaranteed to be satisfiable (Renz and Nebel 1999; Renz 1999). Renz (Lemma 20, 1999) exhibited a map $f : \hat{\mathcal{H}}_8 \cup \mathcal{C}_8 \cup \mathcal{Q}_8 \mapsto \mathbf{B_{RCC8}}$ that gives a unique satisfiable atomic refinement for any a-closed $\hat{\mathcal{H}}_8, \mathcal{C}_8$, or $\mathcal{Q}_8$ network $\mathbf{\Psi}$, by replacing every predicate P_R in $\mathbf{\Psi}$ with $P_{f(R)}$. Let $f(\mathbf{\Psi})$ denote this atomic refinement. By Proposition 7, $f(\mathbf{\Psi})$ has a solution $\mathcal{I}_{\mathbf{\Psi}}$ that can be extended into a solution for any satisfiable atomic extension of $f(\mathbf{\Psi})$.

We claim that $\mathcal{I}_{\mathbf{\Psi}}$ is a canonical solution for each of the three CSPs mentioned in the Proposition: If $\mathbf{\Psi}'$ is an extension of $\mathbf{\Psi}$ in one of these CSPs, then $f(\mathbf{\Psi}')$ is an atomic extension of $f(\mathbf{\Psi})$, and hence $\mathcal{I}_{\mathbf{\Psi}}$ can be extended into a solution for $f(\mathbf{\Psi}')$, which is also a solution for $\mathbf{\Psi}'$. □

By Proposition 2, we have:

[8] $\mathfrak{A}$ *embeds* into $\mathfrak{B}$ if $\mathfrak{A}$ is isomorphic to a substructure of $\mathfrak{B}$ (i.e., a subset of the domain of $\mathfrak{B}$ with the same relations of $\mathfrak{B}$ restricted to the reduced domain). Hence this condition essentially means that $\mathfrak{R}$ satisfies every finite or countable atomic RCC8 network that is a-closed (with respect to the RCC8 composition table). Moreover, $\mathfrak{R}$ will not satisfy networks that are not a-closed by virtue of the second part of the theory above. Therefore, CSP($\mathfrak{R}$) is precisely the problem of deciding if an atomic RCC8 network is a-closed.

Proposition 9 *The three CSPs for a-closed $\hat{\mathcal{H}}_8, \mathcal{C}_8$, and $\mathcal{Q}_8$ networks, respectively, all have patchwork.*

Other Calculi

Having studied IA and RCC8 in some detail, we now further extend our results to a number of related calculi.

Block Algebra

IA has a natural generalization to the n-dimensional space $\mathbb{Q}^n$. The resulting algebra $\mathbf{BA}_n = \langle \mathbf{Intv}^n, \mathbf{B}_{\mathbf{IA}}^n{}^* \rangle$ is known as the n-dimensional Block Algebra (BA), or Rectangle Algebra in case $n = 2$ (Balbiani, Condotta, and del Cerro 1998; 1999b).

It is straightforward to verify that for each $n \geq 2$, $\mathbf{BA}_n$ has an n-dimensional (first-order) interpretation in $\langle \mathbf{Intv}, \mathbf{B_{IA}} \rangle$, and therefore is ω-categorical. Hence by Proposition 1 we have:

Proposition 10 *CSP($\mathbf{BA}_n$) has compactness.*

By decomposing each instance of CSP($\langle \mathbf{Intv}^n, \mathbf{X}^n \rangle$) into n independent instances of CSP($\langle \mathbf{Intv}, \mathbf{X} \rangle$), and noting that the n component instances must all be a-closed if the original instance is, it can be seen that the patchwork property (and existence of canonical solutions) for any subset $\mathbf{X}$ of IA relations will extend to the corresponding subset $\mathbf{X}^n$ of BA relations. Specifically, we have:

Proposition 11 *CSP($\langle \mathbf{Intv}^n, \mathbf{B}_{\mathbf{IA}}^n \rangle$)$_C$, the CSP for atomic $\mathbf{BA}_n$ networks, and CSP($\langle \mathbf{Intv}^n, \mathbf{ORD\text{-}Horn}^n \rangle$)$_{AC}$, the CSP for a-closed $\mathbf{ORD\text{-}Horn}^n$ networks,[9] both have patchwork and canonical solutions.*

RCC5

We now turn to RCC5 (Bennett 1994), which can be regarded as a subcalculus of RCC8 generated by 5 base relations $\mathbf{B_{RCC5}} = \{$DR,PO,EQ,PP,PPi$\}$, where DR (for "discrete") is DC $\cup$ EC, PP (for "proper part") is TPP $\cup$ NTPP, and PPi is the converse of PP; in other words, no significance is attached to boundaries of regions.

Since $\mathbf{B}^*_{\mathbf{RCC5}}$ is a subset of $\mathbf{B}^*_{\mathbf{RCC8}}$, Propositions 8 and 9 immediately carry over to $\hat{\mathcal{H}}_8 \cap \mathbf{B}^*_{\mathbf{RCC5}}, \mathcal{C}_8 \cap \mathbf{B}^*_{\mathbf{RCC5}}$, and $\mathcal{Q}_8 \cap \mathbf{B}^*_{\mathbf{RCC5}}$, the three corresponding tractable subsets of RCC5. Specifically, with $\hat{\mathcal{H}}_5, \mathcal{C}_5$, and $\mathcal{Q}_5$ denoting these three sets respectively, we have:

Proposition 12 *The three CSPs for a-closed $\hat{\mathcal{H}}_5, \mathcal{C}_5$, and $\mathcal{Q}_5$ networks, respectively, all have patchwork and canonical solutions.*

Since $\hat{\mathcal{H}}_5$ contains all the base RCC5 relations (Renz and Nebel 1999), we have in particular:

Proposition 13 *The CSP for atomic RCC5 networks has patchwork and canonical solutions.*

[9]The set $\mathbf{ORD\text{-}Horn}^n$ is also known as *saturated preconvex* relations (Balbiani, Condotta, and del Cerro 1999a; 1999b).

Cardinal Direction Calculus

CDC (Ligozat 1998) has 9 base relations modeling the four cardinal directions (N, E, S, W), the four intermediate directions (NE, SE, SW, NW), plus the identity relation, and can be conveniently formulated using the Point Algebra: $\mathbf{CDC} = \langle \mathbb{Q}^2, \{=,<,>\}^{2^*} \rangle$ (Ligozat 1998).

It is well-known that $\langle \mathbb{Q}, \{=,<,>\} \rangle$ is ω-categorical (Bodirsky 2008). As in the case for BA, it is straightforward to verify that $\mathbf{CDC}$ has a 2-dimensional (first-order) interpretation in $\langle \mathbb{Q}, \{=,<,>\} \rangle$, and therefore is ω-categorical. Hence by Proposition 1 we have:

Proposition 14 *CSP($\mathbf{CDC}$) has compactness.*

Ligozat (1998) identified a maximal tractable subset of the calculus, called the *preconvex* relations. Let this subset be denoted by $\mathcal{P}_{CDC}$, and as usual, let CSP($\langle \mathbb{Q}^2, \mathcal{P}_{CDC} \rangle$)$_{AC}$ be the CSP for a-closed CDC networks using only the $\mathcal{P}_{CDC}$ relations. It was shown (Theorem 2, Ligozat 1998) that a solution can be generated for any network of CSP($\langle \mathbb{Q}^2, \mathcal{P}_{CDC} \rangle$)$_{AC}$ by instantiating one variable at a time without backtrack, using a procedure analogous to the one for CSP($\langle \mathbf{Intv}, \mathbf{ORD\text{-}Horn} \rangle$)$_{AC}$ that we described earlier. This implies that the a-closed preconvex subcalculus of CDC has canonical solutions, and hence patchwork. Since this subcalculus includes all the base relations, we have in particular that atomic CDC networks have these same properties. Formally, that is to say:

Proposition 15 *CSP($\langle \mathbb{Q}^2, \{=,<,>\}^2 \rangle$)$_C$ and CSP($\langle \mathbb{Q}^2, \mathcal{P}_{CDC} \rangle$)$_{AC}$ both have patchwork and canonical solutions.*

Again, the results presented above, among other things, will allow the use of the decomposition-based method (Li, Huang, and Renz 2009) described earlier for BA, RCC5, and CDC in practical qualitative reasoners.

Conclusion

The first part of our contribution is two novel, general results regarding the compactness of CSPs: a sufficient condition which is satisfied by a range of CSPs of common interest, and a consequence that reveals an intimate connection between compactness and two other interesting properties of CSPs, namely the patchwork property and the existence of canonical solutions. The second part of our contribution stems from applying these results to qualitative reasoning, where we obtained a sequence of corollaries that significantly strengthen and expand previous results regarding the Interval Algebra, RCC8, and their fragments and extensions. A complete summary of all our results is conveniently available in the form of Propositions 1–15.

These results can have a significant positive impact on the performance of practical qualitative reasoning algorithms, as they will potentially allow us to extend to a wider range of qualitative CSPs a technique based on graph decomposition (Li, Huang, and Renz 2009) that has been shown to significantly enhance the efficiency and scalability of reasoning. They also have direct implications for other areas of theoretical study, particularly providing new methods for extending the decidability result of Lutz and Miličić (2007) to a wider range of relational structures (called “concrete domains” in the cited work).

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