A Bipolar Framework for Combining Beliefs about Vague Propositions

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Abstract

A bipolar framework is introduced for combining agents' beliefs so as to enable them to reach a common shared position or viewpoint. Our approach exploits the truth-gaps inherent to propositions involving vague concepts, by allowing agents to soften directly conflicting opinions. To this end we adopt a bipolar truth-model for propositional logic characterised by lower and upper valuations on the sentences of the language. According to this model sentences may be *absolutely true*, *absolutely false* or *borderline* (i.e. neither *absolutely true* nor *absolutely false*). The added flexibility of a possible truth-gap between absolutely true and absolutely false allows agents with inconsistent viewpoints, in which a proposition p is absolutely true according to one view and absolutely false according to the other, to reach a compromise position in which p is borderline. Within this framework four combination operators are proposed for combining different viewpoints as represented by different valuation pairs. Intuitively, these correspond to compromise positions with different levels of semantic precision (or vagueness). Kleene belief pairs are then introduced as lower and upper measures quantifying epistemic uncertainty about the sentences of the language when valuation pairs provide the underlying truth model. The combination operators on valuation pairs are then extended to belief pairs using a general schema incorporating a probabilistic model of the interaction between agents. The properties of the four operators are then investigated within this extended framework.

Introduction

In many decision making and negotiation scenarios intelligent agents need to arrive at a common shared position or viewpoint concerning a set of relevant propositions. One route to such a consensus is for each agent to adopt a more vague interpretation of underlying concepts so as to soften directly conflicting opinions. In this regard, a defining feature of vague concepts is their admittance of borderline cases which neither absolutely satisfy the concept nor its negation. For example, there are some height values which would neither be classified as being *absolutely short* nor *absolutely not short*. For propositions involving vague concepts this naturally results in truth-gaps. In other words, there are cases in which a proposition is neither *absolutely true* nor *absolutely false* but instead *borderline*. The fundamental idea of this paper is that such truth-gaps can be exploited to provide additional flexibility when combining different, and possibly inconsistent, viewpoints and beliefs in order to achieve consensus. In particular, two inconsistent points of view each giving different truth values to a certain proposition p might be combined into a compromise position in which p is a borderline case. We illustrate this idea with a simple example as follows:

Consider a scenario in which two agents a_1 and a_2 need to agree about the truth or falsity of the proposition $p =$'*UK inflation is currently low*'. Now the truth of p is principally dependent on two factors; the actual level of UK inflation f as a percentage, and the definition of the predicate low when applied to inflation rates. In this example we can view f as being objectively known to both a_1 and a_2 having been established by using generally accepted economic metrics, while the definition of *low* in this context is subjective and may differ between a_1 and a_2. For instance, one possible bipolar model of *low* could be as follows: Suppose each agent a_i defines lower and upper thresholds $\underline{f}_i \leq \overline{f}_i$ on percentages so that, for them, p is *absolutely true* if $f \leq \underline{f}_i$, *absolutely false* if $f > \overline{f}_i$ and *borderline* if $\underline{f}_i < f \leq \overline{f}_i$. Now further suppose that initially $\underline{f}_1 < \overline{f}_1 < f \leq \underline{f}_2 < \overline{f}_2$ and hence a_1 views p as being *absolutely false*, while a_2 views it as being *absolutely true*. One way in which a_1 and a_2 can adapt their viewpoints to a obtain a common position would be for a_1 to increase their upper threshold so that $\overline{f}_1 \geq f$ and for a_2 to decrease their lower threshold so that $\underline{f}_2 < f$ thus allowing both a_1 and a_2 to agree on a truth valuation of *borderline* for p.

The adequate representation of epistemic uncertainty is of central importance in any effective model of belief. Typically we think of uncertainty as arising because of insufficient information about the state of the world. However, in the presence of vagueness there may also be semantic uncertainty due to our partial knowledge of language conventions. For instance, in the above example each agent may be uncertain about the truth-value of p for the following reasons: There may be some doubt as to the actual current inflation rate f, perhaps because of inherent measurement errors or simply because the agent does not have access to the most up to date information. In this case, the agent would be uncertain as to the ordering of the actual inflation value f relative to the thresholds $\underline{f}_i$ and $\overline{f}_i$ and consequently be uncertain as to whether p was *absolutely true*, *absolutely false* or *borderline*. Alternatively, the agents may be uncertain about exactly what threshold values $\underline{f}_i$ and $\overline{f}_i$ they should adopt in their definition of the concept *low*. This is a form of *semantic uncertainty* which may arise from the empirical manner in which we learn the appropriate use of description labels in natural language. Clearly, even if the actual inflation rate is known exactly, this kind of semantic uncertainty can result in an agent still being uncertain as to the truth-value of p. It is worth noting that in our approach we do not propose to model epistemic uncertainty using truth-gaps, which are instead assumed to be a manifestation of the inherent flexibility in the underlying language conventions. Consequently in the absence of epistemic uncertainty an agent may be *certain* that a particular proposition is *borderline*, as would be the case for p if agent a_i had no doubt about the values of f, $\underline{f}_i$ and $\overline{f}_i$, and if $\underline{f}_i < f \leq \overline{f}_i$. The potential confusion resulting from using truth-gaps to model epistemic uncertainty is highlighted in (Dubois 2008).

In this paper we investigate a number of belief combination operators which exploit truth-gaps associated with vague propositions so as to reach a compromise viewpoint, but where agents may also have epistemic uncertainty about underlying truth-values. Furthermore, we will introduce belief combination operators where agents' beliefs are given by bipolar belief measures, recently proposed in (Lawry and González-Rodríguez 2011), and which combine probabilistic uncertainty with truth-gaps as represented in Kleene's strong three-valued logic (Kleene 1952). An outline of the paper is as follows: Section 2 gives an introduction to Kleene valuation pairs, relates them to Kleene's strong three-valued logic and gives a characterisation in terms of positive and negative sets of propositions. In addition, we shall propose a consistency condition between different valuation pairs. In section 3 we define a partial ordering on valuation pairs reflecting semantic precision. We then propose four operators for combining valuation pairs and consider their relative semantic precision in terms of this ordering. Section 4 introduces Kleene belief pairs to model epistemic uncertainty within this framework and section 5 extends the combination operators defined in section 3 to belief pairs. Section 6 briefly discusses the difference between the proposed bipolar model of vague propositions and superficially similar models of incomplete information. Section 7 then gives some conclusions.

Kleene Valuation Pairs

In this section, we introduce valuation pairs as a bipolar model of truth which allows for the explicit representation of borderline cases. Here we are assuming that all propositions admit truth-gaps in that they may be neither absolutely true nor absolutely false. Typical examples are declarative sentences containing vague adjectives e.g. *low*, *tall*, *fast* etc, although truth-gaps can of course result from other sources of vagueness such as from verbs and nouns. In the sequel a fundamental assumption will be that all propositions under consideration should not only have the potential to exhibit truth-gaps, but that agents should be willing and able to adapt their subjective definitions of relevant vague concepts so as to reach a compromise agreement with other agents by exploiting these truth-gaps. We now propose to model truth-gaps by replacing a single binary, true or false, valuation on propositions with distinct lower and upper valuations representing absolutely true and not absolutely false respectively. Borderline cases then correspond to those sentences in which the lower and upper valuation differ.

Let $\mathcal{L}$ be a language of propositional logic with connectives $\wedge$, $\vee$ and $\neg$ and propositional variables $\mathcal{P} = \{p_1, \ldots, p_n\}$. Let $S\mathcal{L}$ denote the sentences of $\mathcal{L}$. We also define $L\mathcal{L} = \mathcal{P} \cup \{\neg p_i : p_i \in \mathcal{P}\}$ to be the literals of $\mathcal{L}$. A valuation pair on $S\mathcal{L}$ consists of two binary functions $\underline{v}$ and $\overline{v}$ representing lower and upper truth-values. The underlying idea is that $\underline{v}$ represents the strong criterion of *absolutely true* while $\overline{v}$ represents the weaker criteria of *not absolutely false*. In accordance with (Parikh 1996), we might think of a sentence being absolutely true as meaning that it can be uncontroversially asserted without any risk of censure, while being not absolutely false only means that it is acceptable to assert i.e. one can get away with such an assertion. For example, consider a witness in a court of law describing a suspect as being *short*. Depending on the actual height of the suspect this statement may be deemed as clearly true or clearly false, in which latter case the witness could be accused of perjury. However, there will also be an intermediate height range for which, while there may be doubt and differing opinions concerning the use of the description *short*, it would not be deemed as definitely inappropriate and hence the witness would not be viewed as committing perjury. In other words, for certain height values of the suspect, it may be acceptable to assert the statement the suspect was short, even though this statement would not be viewed as being absolutely true.

Definition 1. *Kleene Valuation Pairs*
A Kleene valuation pair on $\mathcal{L}$ is a pair of functions $\vec{v} = (\underline{v}, \overline{v})$ where $\underline{v} : S\mathcal{L} \to \{0,1\}$ and $\overline{v} : S\mathcal{L} \to \{0,1\}$ such that $\underline{v} \leq \overline{v}$ and where $\forall \theta, \varphi \in S\mathcal{L}$ the following hold:

- $\underline{v}(\neg\theta) = 1 - \overline{v}(\theta)$ *and* $\overline{v}(\neg\theta) = 1 - \underline{v}(\theta)$
- $\underline{v}(\theta \wedge \varphi) = \min(\underline{v}(\theta), \underline{v}(\varphi))$ *and* $\overline{v}(\theta \wedge \varphi) = \min(\overline{v}(\theta), \overline{v}(\varphi))$
- $\underline{v}(\theta \vee \varphi) = \max(\underline{v}(\theta), \underline{v}(\varphi))$ *and* $\overline{v}(\theta \vee \varphi) = \max(\overline{v}(\theta), \overline{v}(\varphi))$

The link to three-valued logic is clear when we view the three possible values of a valuation pair for a sentence as truth values i.e. $\mathbf{t} = (1,1)$ as absolutely true, $\mathbf{b} = (0,1)$ as borderline and $\mathbf{f} = (0,0)$ as absolutely false. From definition 1 we can then determine truth-tables for the connectives $\wedge$, $\vee$ and $\neg$ in terms of the truth-values $\{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$ identical to those of Kleene's logic (Kleene 1952). Shapiro (Shapiro 2006) has recently proposed the use of Kleene's three-valued logic to model truth-gaps in vague predicates, arguing that Kleene's truth tables 'reflect the open-texture of vague predicates'. For example, if instead we were to adopt Lukasiewicz logic (Łukasiewicz 1920) this would mean that for two borderline propositional variables their conjunction would be absolutely false, even though neither conjunct was absolutely false. This would seem to be a totally unwarranted elimination of vagueness. One might of course consider a non-functional calculus for valuation pairs based, for example, on supervaluationist principles (Fine 1975). Indeed, this idea is explored in forthcoming work (Lawry and Tang 2011). Shapiro also emphasises that interpretations of predicates can be defined by sets of positive and negative cases. For example, an interpretation of *tall* might correspond to two disjoint subsets of heights identifying those heights which are *absolutely tall* and those which are *absolutely not tall*. The heights lying outside either of these two sets are then *borderline* cases both of tall and of not tall. In the following we shall show that for the propositional logic case this means that a valuation pair can be characterised by disjoint sets of positive and negative propositional variables identifying those propositions which are *absolutely true* and those which are *absolutely false* respectively.

A Positive and Negative Characterisation

In this sub-section we consider a characterisation of Kleene valuation pairs in terms of positive and negative propositions as represented by the sets of absolutely true propositional variables and negated propositional variables respectively. More formally, a Kleene valuation pair $\vec{v}$ can be characterised by an *orthopair* $(P,N) \in 2^{\mathcal{P}} \times 2^{\mathcal{P}}$ where $P = \{p_i \in \mathcal{P} : \underline{v}(p_i) = 1\}$ and $N = \{p_i \in \mathcal{P} : \underline{v}(\neg p_i) = 1\}$. Notice, that from definition 1 it holds immediately that $P \cap N = \emptyset$. Given an orthopair $(P,N) \in 2^{\mathcal{P}} \times 2^{\mathcal{P}}$ we denote the associated Kleene valuation pair by $\vec{v}_{(P,N)}$. The following results show how the value of $\vec{v}$ across $S\mathcal{L}$ can be determined directly from its associated orthopairs (P,N).

Definition 2. *λ-mapping*
Let $\lambda : S\mathcal{L} \to 2^{2^{\mathcal{P}} \times 2^{\mathcal{P}}}$ be defined recursively as follows: $\forall \theta, \varphi \in S\mathcal{L}$

- $\lambda(p_i) = \{(F,G) \in 2^{\mathcal{P}} \times 2^{\mathcal{P}} : p_i \in F\}$
- $\lambda(\theta \wedge \varphi) = \lambda(\theta) \cap \lambda(\varphi)$
- $\lambda(\theta \vee \varphi) = \lambda(\theta) \cup \lambda(\varphi)$
- $\lambda(\neg\theta) = \{(G^c, F^c) : (F,G) \in \lambda(\theta)\}^c$

Notice that the λ-mapping in definition 2 is not restricted solely to orthopairs but also includes pairs of sets of propositional variables with non-empty intersection. As described in (Lawry and González-Rodríguez 2011), such sets characterize a more general class of binary function pairs (v_1, v_2) which satisfy the duality and min-max combination rules of definition 1 but without the requirement that $v_1 \leq v_2$. This class of functions clearly includes Kleene valuation pairs as a special case. Consequently, many of the results in (Lawry and González-Rodríguez 2011) carry across to the current context including the following characterization theorem.

Theorem 3. *(Lawry and González-Rodríguez 2011) For a Kleene valuation pair $\vec{v}_{(P,N)} = (\underline{v}, \overline{v})$, $\forall \theta \in S\mathcal{L}$, $\underline{v}(\theta) = 1$ if and only if $(P,N) \in \lambda(\theta)$ and $\overline{v}(\theta) = 1$ if and only if $(P,N) \in \lambda(\neg\theta)^c$.*

Example 4. *Let $p_i, p_j \in \mathcal{P}$ then $\lambda(p_i) = \{(F,G) : p_i \in F\}$, $\lambda(\neg p_j) = \{(F,G) : p_j \in G\}$ and $\lambda(p_i \wedge \neg p_j) = \{(F,G) : p_i \in F, p_j \in G\}$. Hence, $\underline{v}(p_i) = 1$ iff $p_i \in P$ and $\overline{v}(p_i) = 1$ iff $p_i \notin N$. Similarly, $\underline{v}(\neg p_j) = 1$ iff $p_j \in N$ and $\overline{v}(\neg p_j) = 1$ iff $p_j \notin P$. Furthermore, $\underline{v}(p_i \wedge \neg p_j) = 1$ iff $p_i \in P$ and $p_j \in N$, and $\overline{v}(p_i \wedge \neg p_j) = 1$ iff $p_i \notin N$ and $p_j \notin P$.*

Theorem 5. *(Lawry and González-Rodríguez 2011) $\forall \theta \in S\mathcal{L}$ if $(P,N) \in \lambda(\theta)$ and $P' \supseteq P$ and $N' \supseteq N$ then $(P', N') \in \lambda(\theta)$*

We now propose a consistency condition between distinct valuation pairs which takes account of truth-gaps.

Definition 6. *Consistency*
Kleene valuation pairs $\vec{v}_1$ and $\vec{v}_2$ are consistent *if and only if $\forall \theta \in S\mathcal{L}$,*

$$\min(\max(\overline{v_1}(\neg\theta), \overline{v_2}(\theta)), \max(\overline{v_2}(\neg\theta), \overline{v_1}(\theta))) = 1$$

The underlying intuition behind definition 6 is that two valuations are consistent provided that, if a sentence is *absolutely true* according to one valuation it is *not absolutely false* according to the other, and vice versa. A consequence of this definition is that if a proposition is classified as being *borderline* in a given valuation then that proposition cannot be the source of conflict with any other valuation.

Theorem 7. *Valuation pairs $\vec{v_1}$ and $\vec{v_2}$ are consistent if and only if $P_1 \cap N_2 = P_2 \cap N_1 = \emptyset$.*

Proof. ($\Rightarrow$) Let $p_i \in P_1$, so that $\underline{v}_1(p_i) = 1$ and $\overline{v}_1(\neg p_i) = 0$. Hence, $\overline{v}_2(p_i) = 1$ which implies that $p_i \in N_2^c$. Therefore, $P_1 \subseteq N_2^c$. Similarly, if $p_i \in P_2$ then $\underline{v}_2(p_i) = 1$ and $\overline{v}_2(\neg p_i) = 0$ and hence $\overline{v}_1(p_i) = 1$. This implies that $p_i \in N_1^c$ and hence $P_2 \subseteq N_1^c$ as required.
($\Leftarrow$) Suppose $P_1 \cap N_2 = P_2 \cap N_1 = \emptyset$ but $\exists \theta \in S\mathcal{L}$, such that $\min(\max(\overline{v_1}(\neg\theta), \overline{v_2}(\theta)), \max(\overline{v_2}(\neg\theta), \overline{v_1}(\theta))) = 0$. In this case either $\max(\overline{v_1}(\neg\theta), \overline{v_2}(\theta)) = 0$ or $\max(\overline{v_2}(\neg\theta), \overline{v_1}(\theta)) = 0$ and w.l.o.g assume $\max(\overline{v_1}(\neg\theta), \overline{v_2}(\theta)) = 0$. In this case $\overline{v_1}(\neg\theta) = \overline{v_2}(\theta) = 0$. Hence, by definition 1 $\underline{v_1}(\theta) = 1 - \overline{v_1}(\neg\theta) = 1$ and similarly $\underline{v_2}(\neg\theta) = 1 - \overline{v_2}(\theta) = 1$. $\underline{v_1}(\theta) = 1$ implies that $(P_1, N_1) \in \lambda(\theta)$ by theorem 3. Also, by theorem 3 $\underline{v_2}(\neg\theta) = 1$ implies that $(P_2, N_2) \in \lambda(\neg\theta)$. Hence, by definition 2 $(P_2, N_2) \notin \{(G^c, F^c) : (F,G) \in \lambda(\theta)\}$. Therefore. $(N_2^c, P_2^c) \notin \lambda(\theta)$. However, since $P_1 \cap N_2 = \emptyset$ we have that $N_2^c \supseteq P_1$ and furthermore since $P_2 \cap N_1 = \emptyset$ then $P_2^c \supseteq N_1$. Hence, since $(P_1, N_1) \in \lambda(\theta)$ it follows by theorem 5 that $(N_2^c, P_2^c) \in \lambda(\theta)$ which is a contradiction. □

In this paper we view belief combination as a mechanism by which agents can adapt their beliefs in order to achieve a shared position or viewpoint. Consequently, from this perspective we would expect a valid combination operator to generated a new valuation which is always consistent (in the sense of definition 6) with both the original valuations. This is a minimal requirement which will hold for all the combination operators introduced in the following section.

Ordering and Combining Valuation Pairs

We now use the orthopair characterisation described above to inspire a number of operators for combining valuation pairs representing different viewpoints or opinions. These result in new valuation pairs of different semantic precision, where the relative semantic precision of valuation pairs is determined by the extent to which they tend to categorize sentences of $\mathcal{L}$ as being borderline cases. This is formalized by a partial ordering on the set of Kleene valuation pairs, denoted by $\mathbb{V}$, as follows: Let $\vec{v}_1$ and $\vec{v}_2$ be Kleene valuation pairs with associated orthopairs (P_1, N_1) and (P_2, N_2) respectively. Then we defined the ordering $\preceq$ on $\mathbb{V}$ according to:

Definition 8. *Semantic Precision*
$\vec{v}_1 \preceq \vec{v}_2$ *iff* $P_1 \subseteq P_2$ *and* $N_1 \subseteq N_2$.

Shapiro (Shapiro 2006) proposed essentially the same ordering of interpretations which he refers to as *sharpening* i.e. $\vec{v}_1 \preceq \vec{v}_2$ means that $\vec{v}_2$ extends or sharpens $\vec{v}_1$. Here we shall refer to $\preceq$ as the *semantic precision* ordering on valuation pairs whereby, as the following theorem shows, if $\vec{v}_1 \preceq \vec{v}_2$ then $\vec{v}_1$ tends to classify more sentences of $\mathcal{L}$ as *borderline* than $\vec{v}_2$. In other words, one might think of $\preceq$ as ordering valuation pairs according to their relative vagueness.

Theorem 9. $\vec{v}_1 \preceq \vec{v}_2$ *iff* $\forall \theta \in S\mathcal{L}$ $\underline{v_1}(\theta) \leq \underline{v_2}(\theta)$ *and* $\overline{v_1}(\theta) \geq \overline{v_2}(\theta)$.

Proof. ($\Rightarrow$) Suppose $\underline{v_1}(\theta) = 1$ then by theorem 3 $(P_1, N_1) \in \lambda(\theta)$. By theorem 5 this implies that $(P_2, N_2) \in \lambda(\theta)$ and hence $\underline{v_2}(\theta) = 1$. Similarly, suppose $\overline{v_2}(\theta) = 1$ then by theorem 3 $(P_2, N_2) \in \lambda(\neg\theta)^c$. Therefore, by theorem 5 $(P_1, N_1) \in \lambda(\neg\theta)^c$ and hence $\overline{v_1}(\theta) = 1$. ($\Leftarrow$) Trivial. □

Notice that it follows immediately from theorem 7 and definition 8 that if $\vec{v}_1 \preceq \vec{v}_2$ then $\vec{v}_1$ and $\vec{v}_2$ are consistent.

We now define four combination operators generated by taking different Boolean combinations of P_1 and P_2, and N_1 and N_2 to obtain new orthopairs.

Definition 10. *Conservative Combination Operator*
Given Kleene valuation pairs $\vec{v}_1$ and $\vec{v}_2$ with associated orthopairs (P_1, N_1) and (P_2, N_2) we define the conservative combination operator such that:

$$\vec{v}_1 \otimes \vec{v_2} = \vec{v}_{(P_1 \cap P_2, N_1 \cap N_2)}$$

$\otimes$ is a semantically conservative operator. Trivially, we have that $\vec{v}_1 \otimes \vec{v}_2 \preceq \vec{v}_1$ and $\vec{v}_1 \otimes \vec{v}_2 \preceq \vec{v}_2$. Together with theorem 9 this implies that $\forall \theta \in S\mathcal{L}$;

$$\underline{v_1 \otimes v_2}(\theta) \leq \min(\underline{v}_1(\theta), \underline{v}_2(\theta)) \text{ and}$$
$$\overline{v_1 \otimes v_2}(\theta) \geq \max(\overline{v}_1(\theta), \overline{v}_2(\theta)$$

Indeed, the following result shows that if we restrict attention to literals then the inequalities in the above formula can be replaced by equality.

Theorem 11. $\forall l \in L\mathcal{L}$, $\underline{v_1 \otimes v_2}(l) = \min(\underline{v}_1(l), \underline{v_2}(l))$ *and* $\overline{v_1 \otimes v_2}(l) = \max(\overline{v}_1(l), \overline{v_2}(l))$.

Proof. $\underline{v_1 \otimes v_2}(p_i) = 1$ iff $p_i \in P_1 \cap P_2$ iff $p_i \in P_1$ and $p_i \in P_2$ iff $\underline{v}_1(p_i) = 1$ and $\underline{v}_2(p_i) = 1$ as required. Also, $\overline{v_1 \otimes v_2}(p_i) = 1$ iff $p_i \notin N_1 \cap N_2$ iff $p_i \notin N_1$ or $p_i \notin N_2$ iff $\overline{v}_1(p_i) = 1$ or $\overline{v}_2(p_i) = 1$ as required. The results for negated propositional variables follow by duality. □

Notice, however, that theorem 11 does not extend to all sentences in $S\mathcal{L}$. Consider, for example, Kleene valuation pairs $\vec{v}_1$ and $\vec{v}_2$ such that $(P_1, N_1) = (\{p_i\}, \emptyset)$ and $(P_2, N_2) = (\emptyset, \{p_i\})$ and take $\theta = p_i \vee \neg p_i$. In this case $\vec{v}_1 \otimes \vec{v}_2$ has orthopairs $(\emptyset, \emptyset)$ and hence $\vec{v}_1(\theta) = (1, 1)$, $\vec{v}_2(\theta) = (1, 1)$ while $\vec{v}_1 \otimes \vec{v}_2(\theta) = (0, 1)$.

We can also see that $\vec{v}_1 \otimes \vec{v}_2$ is the greatest lower bound of $\vec{v_1}$ and $\vec{v_2}$ with respect to $\preceq$ as the following result shows.

Theorem 12. *If* $\vec{v}_3 \preceq \vec{v}_1$ *and* $\vec{v}_3 \preceq \vec{v}_2$ *then* $\vec{v}_3 \preceq \vec{v}_1 \otimes \vec{v}_2$

Proof. Let $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ have orthopairs (P_1, N_1), (P_2, N_2) and (P_3, N_3) respectively. Then $P_3 \subseteq P_1$ and $P_3 \subseteq P_2$ implies that $P_3 \subseteq P_1 \cap P_2$. Similarly, $N_3 \subseteq N_1 \cap N_2$. □

In other words, $\vec{v}_1 \otimes \vec{v}_2$ is the most semantically precise valuation pair which is less precise than both $\vec{v}_1$ and $\vec{v}_2$. The following theorem highlights the fact that $\otimes$ is a conservative operator in that, for any sentence θ for which $\vec{v}_1$ and $\vec{v}_2$ have different values, $\vec{v}_1 \otimes \vec{v}_2$ returns a borderline truth-value for θ.

Theorem 13. $\forall \theta \in S\mathcal{L}$, *If* $\vec{v}_1(\theta) \neq \vec{v}_2(\theta)$ *then*
$\vec{v}_1 \otimes \vec{v}_2(\theta) = (0, 1)$

Proof. If $\vec{v_1}(\theta) \neq \vec{v_2}(\theta)$ then either $\underline{v_1}(\theta) \neq \underline{v_2}(\theta)$ or $\overline{v_1}(\theta) \neq \overline{v_2}(\theta)$
Case 1: $\underline{v_1}(\theta) \neq \underline{v_2}(\theta)$
w.l.o.g suppose $\underline{v_1}(\theta) = 1$ and $\underline{v_2}(\theta) = 0$. Now $\underline{v_2}(\theta) = 0$ implies that $(P_2, N_2) \notin \lambda(\theta)$ and hence by theorem 5 $(P_1 \cap P_2, N_1 \cap N_2) \notin \lambda(\theta)$ and hence $\underline{v_1 \otimes v_2}(\theta) = 0$. Also, since $\underline{v_1}(\theta) = 1$ then by definition 1 $\overline{v_1}(\theta) = 1$ and hence $(P_1, N_1) \in \lambda(\neg\theta)^c$ and by theorem 5 this implies that $(P_1 \cap P_2, N_1 \cap N_2) \in \lambda(\neg\theta)^c$. Consequently $\overline{v_1 \otimes v_2}(\theta) = 1$
Case 2: $\overline{v_1}(\theta) \neq \overline{v_2}(\theta)$
w.l.o.g suppose $\overline{v_1}(\theta) = 1$ and $\overline{v_2}(\theta) = 0$. Now $\overline{v_1}(\theta) = 1$ implies that $(P_1, N_1) \in \lambda(\neg\theta)^c$ and hence by theorem 5 $(P_1 \cap P_2, N_1 \cap N_2) \in \lambda(\neg\theta)^c$ and hence $\overline{v_1 \otimes v_2}(\theta) = 1$. Also since $\overline{v_2}(\theta) = 0$ it follows by definition 1 that $\underline{v_2}(\theta) = 0$ and hence $(P_2, N_2) \notin \lambda(\theta)$. This implies by theorem 5 that $(P_1 \cap P_2, N_1 \cap N_2) \notin \lambda(\theta)$ and hence $\underline{v_1 \otimes v_2}(\theta) = 0$. □

In the case of two consistent valuation pairs it is possible to define an *optimistic* combination operator. This operator results in a semantically more precise valuation pair than either $\vec{v}_1$ or $\vec{v}_2$ i.e. a sharpening of $\vec{v}_1$ and $\vec{v}_2$

Definition 14. *Optimistic Combination Operator*
Given consistent Kleene valuation pairs $\vec{v}_1$ and $\vec{v}_2$ with associated orthopairs (P_1, N_1) and (P_2, N_2) then the optimistic combination of $\vec{v}_1$ and $\vec{v}_2$ is the valuation pair

$$\vec{v}_1 \oplus \vec{v}_2 = \vec{v}_{(P_1 \cup P_2, N_1 \cup N_2)}$$

In the case that $\vec{v}_1$ and $\vec{v}_2$ are inconsistent $\oplus$ is undefined.

For consistent valuation pairs $\vec{v}_1$ and $\vec{v}_2$, clearly $\vec{v}_1 \preceq \vec{v}_1 \oplus \vec{v}_2$ and $\vec{v}_2 \preceq \vec{v}_1 \oplus \vec{v}_2$ so that by theorem 9 $\forall \theta \in S\mathcal{L}$;

$$\underline{v_1 \oplus v_2}(\theta) \geq \max(\underline{v}_1(\theta), \underline{v}_2(\theta)) \text{ and}$$
$$\overline{v_1 \oplus v_2}(\theta) \leq \min(\overline{v}_1(\theta), \overline{v}_2(\theta))$$

Consequently, $\oplus$ increases semantic precision. Furthermore, if we restrict attention to literals then the inequalities in the above formula can be replaced with equalities.

Theorem 15. *If $\vec{v}_1$ and $\vec{v}_2$ are consistent then $\forall l \in L\mathcal{L}$;*

$$\underline{v_1 \oplus v_2}(l) = \max(\underline{v}_1(l), \underline{v}_2(l)) \text{ and}$$
$$\overline{v_1 \oplus v_2}(l) = \min(\overline{v}_1(l), \overline{v}_2(l))$$

Proof. $\underline{v_1 \oplus v_2}(p_i) = 1$ iff $p_i \in P_1 \cup P_2$ iff $p_i \in P_1$ or $p_i \in P_2$ iff $\overline{v}_1(p_i) = 1$ or $\overline{v}_2(p_i) = 1$ as required. Also, $\overline{v_1 \oplus v_2}(p_i) = 1$ iff $p_i \notin N_1 \cup N_2$ iff $p_i \notin N_1$ and $p_i \notin N_2$ iff $\overline{v}_1(p_i) = 1$ and $\overline{v}_2(p_i) = 1$ as required. The results for negated proposition variables then follow by duality. □

We can also see that $\vec{v}_1 \oplus \vec{v}_2$ is the least upper bound of $\vec{v}_1$ and $\vec{v}_2$ with respect to $\preceq$.

Theorem 16. *If $\vec{v}_1 \preceq \vec{v}_3$ and $\vec{v}_2 \preceq \vec{v}_3$ then $\vec{v}_1 \oplus \vec{v}_2 \preceq \vec{v}_3$*

Proof. Since $P_1 \subseteq P_3$, $P_2 \subseteq P_3$, $N_1 \subseteq N_3$ and $N_2 \subseteq N_3$ it follows immediately that $P_1 \cup P_2 \subseteq P_3$ and $N_1 \cup N_2 \subseteq N_3$ as required. □

The following asymmetric operator is motivated by the idea that in some circumstances an agent may need to minimally adapt their beliefs so that they become consistent with another agent's viewpoint.

Definition 17. *The Difference Operator*
For Kleene valuation pairs $\vec{v}_1$ and $\vec{v}_2$ we define:

$$\vec{v}_1 \ominus \vec{v}_2 = \vec{v}_{(P_1 \setminus N_2, N_1 \setminus P_2)}$$

Notice immediately that $\vec{v}_1 \ominus \vec{v}_2 \preceq \vec{v}_1$ and that $\vec{v}_1 \ominus \vec{v}_2$ is consistent with $\vec{v}_2$. Indeed, as the following result shows, it is also the most semantically precise valuation pair with both of these properties.

Theorem 18. *For Kleene valuation pairs $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$, if $\vec{v}_3 \preceq \vec{v}_1$ and $\vec{v}_3$ is consistent with $\vec{v}_2$ then $\vec{v}_3 \preceq \vec{v}_1 \ominus \vec{v}_2$.*

Proof. We have that $P_3 \subseteq P_1$, $N_3 \subseteq N_1$ and $P_3 \cap N_2 = P_2 \cap N_3 = \emptyset$. Hence, $P_3 \subseteq N_2^c$ and $N_3 \subseteq P_2^c$ so that $P_3 \subseteq P_1 \cap N_2^c$ and $N_3 \subseteq N_1 \cap P_2^c$ as required. □

Hence, we may think of $\vec{v}_1 \ominus \vec{v}_2$ is the minimal *softening* [1] of the viewpoint represented by $\vec{v}_1$ so as to make it consistent with $\vec{v}_2$.

Theorem 19. *Let $l \in L\mathcal{L}$ then*

$$\underline{v_1 \ominus v_2}(l) = \min(\underline{v}_1(l), \overline{v}_2(l)) \text{ and}$$
$$\overline{v_1 \ominus v_2}(l) = \max(\overline{v}_1(l), \underline{v}_2(l))$$

Proof. Let $l = p_i \in \mathcal{P}$: Then $\underline{v_1 \ominus v_2}(p_i) = 1$ iff $p_i \in P_1 - N_2$ iff $p_i \in P_1$ and $p_i \notin N_2$ iff $\underline{v_1}(p_i) = 1$ and $\overline{v_2}(p_i) = 1$ as required. Also, $\overline{v_1 \ominus v_2}(p_i) = 1$ iff $p_i \notin N_1 \setminus P_2$ iff $p_1 \notin N_1$ or $p_i \in P_2$ iff $\overline{v_1}(p_i) = 1$ or $\underline{v}_2(p_i) = 1$ as required.
Let $l = \neg p_i$ where $p_i \in \mathcal{P}$: Then $\underline{v_1 \ominus v_2}(\neg p_i) = 1$ iff $p_i \in N_1 \setminus P_2$ iff $p_i \in N_1$ and $p_i \notin P_2$ iff $\underline{v_1}(\neg p_i) = 1$ and $\overline{v_2}(\neg p_i) = 1$ as required. Also $\overline{v_1 \ominus v_2}(\neg p_i) = 1$ iff $p_i \notin P_1 \setminus N_2$ iff $p_i \notin P_1$ or $p_i \in N_2$ iff $\overline{v_1}(\neg p_i) = 1$ or $\underline{v}_2(\neg p_i) = 1$ as required. □

However, the following example shows that the rule:

$$\underline{v_1 \ominus v_2}(\theta) = \min(\underline{v_1}(\theta), \overline{v_2}(\theta)) \text{ and}$$
$$\overline{v_1 \ominus v_2}(\theta) = \max(\overline{v_1}(\theta), \underline{v_2}(\theta))$$

does not hold for all $\theta \in S\mathcal{L}$.

Example 20. *Let $\vec{v}_1$ and $\vec{v}_2$ have orthopairs $(\{p_i\}, \{p_j\})$ and $(\{p_j\}, \{p_i\})$ respectively. In this case $\vec{v}_1 \ominus \vec{v}_2$ has orthopair $(\emptyset, \emptyset)$. Hence, $\overline{v_1 \ominus v_2}(p_i \wedge p_j) = 1$ while $\max(\overline{v_1}(p_i \wedge p_j), \underline{v_2}(p_i \wedge p_j)) = 0$. Also, $\underline{v_1 \ominus v_2}(p_i \vee p_j) = 0$ while $\min(\underline{v}_1(p_i \vee p_j), \overline{v}_2(p_i \vee p_j)) = 1$.*

Instead we have the following bounds on $\vec{v}_1 \ominus \vec{v}_2$.

Theorem 21.

$$\forall \theta \in S\mathcal{L}, \underline{v_1 \ominus v_2}(\theta) \leq \min(\underline{v_1}(\theta), \overline{v_2}(\theta)) \text{ and}$$
$$\overline{v_1 \ominus v_2}(\theta) \geq \max(\overline{v}_1(\theta), \underline{v_2}(\theta)).$$

Proof. Suppose, $\underline{v_1 \ominus v_2}(\theta) = 1$ then $\underline{v_1}(\theta) = 1$ since $\vec{v}_1 \ominus \vec{v}_2 \preceq \vec{v}_1$. Now also suppose $\overline{v}_2(\theta) = 0$ then $(P_2, N_2) \in \lambda(\neg\theta)$ which implies that $(N_2^c, P_2^c) \notin \lambda(\theta)$ by definition 2. Hence, by theorem 5 it follows that $(P_1 \cap N_2^c, N_1 \cap P_2^c) \notin \lambda(\theta)$, which implies that $\underline{v_1 \ominus v_2}(\theta) = 0$. This is a contradiction and therefore $\min(\underline{v_1}(\theta), \overline{v_2}(\theta)) = 1$ as required.

Suppose, $\overline{v_1 \ominus v_2}(\theta) = 0$ then $\overline{v_1}(\theta) = 0$ since $\vec{v}_1 \ominus \vec{v}_2 \preceq \vec{v}_1$. Now also suppose that $\underline{v}_2(\theta) = 1$ then $(P_2, N_2) \in \lambda(\theta)$, which implies that $(N_2^c, P_2^c) \notin \lambda(\neg\theta)$ by definition 2. Hence, by theorem 5 it follows that $(P_1 \cap N_2^c, N_1 \cap P_2^c) \notin \lambda(\neg\theta)$, which implies that $\overline{v_1 \ominus v_2}(\theta) = 1$. This is a contradiction and therefore $\max(\overline{v}_1(\theta), \underline{v_2}(\theta)) = 0$ as required. □

We now use the difference operator in order to define a consensus operator which combines two valuations to obtain a new consensus viewpoint even when the original valuations are inconsistent, and which is more semantically precise than that obtained from the conservative operator. For example, when applied to literals this new operator only results in a borderline if the two original valuations are in direct conflict (i.e. $\mathbf{t}$ and $\mathbf{f}$ or $\mathbf{f}$ and $\mathbf{t}$), or if only borderline valuations are involved (i.e. $\mathbf{b}$ and $\mathbf{b}$). This is in contrast with the conservative operator which results in a borderline whenever the two original valuations differ (see theorem 13).

Definition 22. *Consensus Operator*
For Kleene valuation pairs $\vec{v}_1$ and $\vec{v}_2$ we define:

$$\vec{v}_1 \odot \vec{v}_2 = (\vec{v}_1 \ominus \vec{v}_2) \oplus (\vec{v}_2 \ominus \vec{v}_1)$$

[1] Here we are using the term *softening* as an opposite to Shapiro's term *sharpening* (Shapiro 2006) i.e. if $\vec{v}_1 \preceq \vec{v}_2$ then $\vec{v}_1$ is a softening of $\vec{v}_2$.

The underlying motivation for the $\odot$ operator can be seen as analogous to the idea in belief revision theory that revision operations can be decomposed into two sub-operations: *contraction* according to which two inconsistent sets of information are both weakened so that they become consistent, followed by *expansion* whereby the sets of information are then simply added together (Gardenförs 1988). In our context, the consensus operator consists of a form of contraction whereby the two agents minimally soften their viewpoints until they become consistent, followed by a form of expansion where the two viewpoints are added together using the optimistic combination operator. The following theorem now shows how $\odot$ can be defined directly in terms of Boolean operations on the orthopairs.

Theorem 23.

$$\vec{v}_1 \odot \vec{v}_2 = \vec{v}_{((P_1 \cup P_2) \setminus (N_1 \cup N_2), (N_1 \cup N_2) \setminus (P_1 \cup P_2))}$$

Proof. Trivial since $P_1 \subseteq N_1^c$ and $P_2 \subseteq N_2^c$ □

Theorem 24. *For $l \in L\mathcal{L}$*

$$\underline{v_1 \odot v_2}(l) = \max(\min(\underline{v}_1(l), \overline{v}_2(l)), \min(\underline{v}_2(l), \overline{v}_1(l)))$$
$$\overline{v_1 \odot v_2}(l) = \min(\max(\overline{v}_1(l), \underline{v}_2(l)), \max(\overline{v}_2(l), \underline{v}_1(l)))$$

Proof. Follows immediately from theorem 15 and theorem 19. □

Table 1 shows the operators $\otimes$, $\oplus$, $\ominus$ and $\odot$ when applied to literals. Also, figure 1 is a hasse diagram showing the relative semantic precision of different combined valuations in comparison with the original valuation pairs $\vec{v}_1$ and $\vec{v}_2$. $\vec{v}_1 \oplus \vec{v}_2$ is not shown on the diagram since if $\vec{v}_1$ and $\vec{v}_2$ are consistent then $\vec{v}_1 \ominus \vec{v}_2 = \vec{v}_1$, $\vec{v}_2 \ominus \vec{v}_1 = \vec{v}_2$ and $\vec{v}_1 \odot \vec{v}_2 = \vec{v}_1 \oplus \vec{v}_2$.

$\otimes$	**t**	**b**	**f**
t	**t**	**b**	**b**
b	**b**	**b**	**b**
f	**b**	**b**	**f**

$\oplus$	**t**	**b**	**f**
t	**t**	**t**	–
b	**t**	**b**	**f**
f	–	**f**	**f**

$\ominus$	**t**	**b**	**f**
t	**t**	**b**	**b**
b	**t**	**b**	**f**
f	**b**	**b**	**f**

$\odot$	**t**	**b**	**f**
t	**t**	**t**	**b**
b	**t**	**b**	**f**
f	**b**	**f**	**f**

Table 1: Truth tables for the operators $\otimes$, $\oplus$, $\ominus$, and $\odot$ when applied to literals.

Note that these combination operations do not coincide with any Boolean connective when the tables are restricted to **t, f**. Instead, $\otimes$ is the median $med(x, y, \mathbf{b})$ of the two inputs and **b**, $\oplus$ is a partially defined uninorm (Grabisch et al. 2009). Both are associative. On the other hand, $\odot$ is a type of commutative non-associative average $((\mathbf{f} \odot \mathbf{f}) \odot \mathbf{t} \neq \mathbf{f} \odot (\mathbf{f} \odot \mathbf{t}))$.

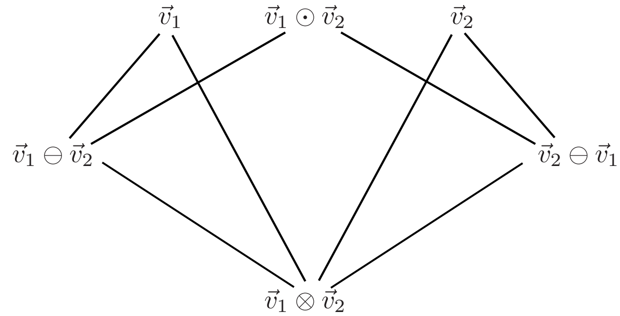

Figure 1: Hasse diagram showing the ordering (relative to $\preceq$) of the different valuation pairs resulting from applying the operators $\otimes$, $\ominus$ and $\odot$ to $\vec{v}_1$ and $\vec{v}_2$.

Kleene Belief Pairs

Within the proposed bipolar framework, uncertainty concerning the sentences of $\mathcal{L}$ effectively corresponds to uncertainty as to which is the correct Kleene valuation pair for $\mathcal{L}$. As outlined earlier we view uncertainty as being epistemic in nature, resulting from a lack of knowledge concerning either, the state of the world to which propositions refer, or the underlying definitions of concepts used in propositions. In the following we assume that this uncertainty is quantified by a probability measure w on the set of Kleene valuation pairs $\mathbb{V}$.

Definition 25. *Kleene Belief Pairs*
Let $\mathbb{V}$ be the set of all Kleene valuation pairs on $\mathcal{L}$ and let w be a probability distribution defined on $\mathbb{V}$ so that $w(\vec{v})$ is the agent's subjective belief that $\vec{v}$ is the true valuation pair for $\mathcal{L}$. Then $\vec{\mu}_w = (\underline{\mu}_w, \overline{\mu}_w)$ is a Kleene belief pair where $\forall \theta \in S\mathcal{L}$, $\underline{\mu}_w(\theta) = w(\{\vec{v} \in \mathbb{V} : \underline{v}(\theta) = 1\})$ and $\overline{\mu}_w(\theta) = w(\{\vec{v} \in \mathbb{V} : \overline{v}(\theta) = 1\})$.

Notice, trivially, that $\forall \theta \in S\mathcal{L}$, $\underline{\mu}_w(\theta) \leq \overline{\mu}_w(\theta)$ and also that we have the following duality relationship between the lower and upper measures:

There is a clear rationality argument for defining belief measures in this manner when Kleene valuation pairs are the underlying truth model for $\mathcal{L}$. From a general result due to Paris (Paris 2001), it follows that an agent can only avoid Dutch books where the outcomes of bets are dependent on lower (upper) Kleene valuations if their belief measures on $S\mathcal{L}$ correspond to lower (upper) belief measures as given in definition 25.

Theorem 26. *$\forall \theta \in S\mathcal{L}$, $\underline{\mu}_w(\neg\theta) = 1 - \overline{\mu}_w(\theta)$ and $\overline{\mu}_w(\neg\theta) = 1 - \underline{\mu}_w(\theta)$*

It is also interesting to note that a special case of Kleene belief pairs has the same calculus as the interval (or type 2) fuzzy membership functions proposed by Zadeh (Zadeh 1975). This is the case of Kleene belief pairs in which there is only uncertainty about the level of semantic precision of the valuation pair. More formally we have the following result:

Theorem 27. *(Lawry and González-Rodríguez 2011) Let w be a probability distribution on $\mathbb{V}$ for which $\{\vec{v} \in \mathbb{V} :$*

$w(\vec{v}) > 0\} = \{\vec{v}_1, \ldots, \vec{v}_k\}$ *can be ordered such that* $\vec{v}_1 \preceq \vec{v}_2 \ldots \preceq \vec{v}_k$*. In this case* $\vec{\mu}_w$ *satisfies the following properties;* $\forall \theta, \varphi \in S\mathcal{L}$,

$$\underline{\mu}_w(\theta \wedge \varphi) = \min(\underline{\mu}_w(\theta), \underline{\mu}_w(\varphi))$$
$$\overline{\mu}_w(\theta \wedge \varphi) = \min(\overline{\mu}_w(\theta), \overline{\mu}_w(\varphi))$$
$$\underline{\mu}_w(\theta \vee \varphi) = \max(\underline{\mu}_w(\theta), \underline{\mu}_w(\varphi))$$
$$\overline{\mu}_w(\theta \vee \varphi) = \max(\overline{\mu}_w(\theta), \overline{\mu}_w(\varphi))$$

Combination of Belief Pairs

In this section we consider the combination of Kleene belief pairs by extending the operators introduced above in order to allow for epistemic uncertainty. Suppose we have two agents with beliefs about $S\mathcal{L}$ quantified by Kleene belief pairs $\vec{\mu}_{w_1}$ and $\vec{\mu}_{w_2}$ respectively. Then the following definition proposes how the conservative operator can be extended to this case, as well as providing an exemplar of a general scheme which can then be employed to extend other combination operators for valuation pairs to belief pairs.

Definition 28. *Conservative Combination of Belief Pairs A conservative combination of Kleene belief pairs* $\vec{\mu}_{w_1}$ *and* $\vec{\mu}_{w_2}$ *is a belief pair* $\vec{\mu}_{w_1 \otimes_q w_2}$ *where* $w_1 \otimes_q w_2$ *is a probability distribution on* $\mathbb{V}$ *for which*

$$w_1 \otimes_q w_2(\vec{v}) = q(\{(\vec{v_1}, \vec{v_2}) : \vec{v}_1 \otimes \vec{v}_2 = \vec{v}\})$$

where q *is any 2-dimensional probability distribution on* $\mathbb{V} \times \mathbb{V}$ *with marginals* w_1 *and* w_2*.*

Alternatively, according to definition 28:

$$\underline{\mu}_{w_1 \otimes_q w_2}(\theta) = q(\{(\vec{v}_1, \vec{v}_2) : \underline{v_1 \otimes v_2}(\theta) = 1\}) \text{ and}$$
$$\overline{\mu}_{w_1 \otimes_q w_2}(\theta) = q(\{(\vec{v}_1, \vec{v}_2) : \overline{v_1 \otimes v_2}(\theta) = 1\})$$

The joint distribution q in definition 28 should be viewed as an integral part of the belief combination, potentially agreed as a result of negotiation between the two agents involved. Specifically, $q(\vec{v}_1, \vec{v}_2)$ is the probability weighting in the overall belief merging, which is allocated by the two agents specifically to the combination of $\vec{v}_1$ and $\vec{v}_2$ where $\vec{v}_1$ is drawn from w_1 and $\vec{v}_2$ from w_2. The fact that q has marginals w_1 and w_2 means that the total weight of importance in the belief combination which is allocated by an agent to a particular valuation $\vec{v}$, corresponds to that agent's overall belief that $\vec{v}$ is the true valuation. For example, $q = w_1 \times w_2$ corresponds to the case of minimal interaction between the two agents when allocating weightings to particular combinations of valuation pairs. Indeed, we might think of each agent as independently identifying valuation pairs to combine, by selecting them at random according to their respective distributions on $\mathbb{V}$.

The following theorem shows that conservative combination results in a new bipolar belief pair which is less precise than either $\vec{\mu}_{w_1}$ or $\vec{\mu}_{w_2}$.

Theorem 29. *If* $\vec{\mu}_{w_1 \otimes_q w_2}$ *is a conservative combination of* $\vec{\mu}_{w_1}$ *and* $\vec{\mu}_{w_2}$ *then* $\forall \theta \in S\mathcal{L}$*;*

$$\underline{\mu}_{w_1 \otimes_q w_2}(\theta) \leq \min(\underline{\mu}_{w_1}(\theta), \underline{\mu}_{w_2}(\theta)) \text{ and}$$
$$\overline{\mu}_{w_1 \otimes_q w_2}(\theta) \geq \max(\overline{\mu}_{w_1}(\theta), \overline{\mu}_{w_2}(\theta))$$

Proof. $\forall \theta \in S\mathcal{L}$, $\underline{\mu}_{w_1 \otimes_q w_2}(\theta) =$
$q(\{(\vec{v}_1, \vec{v}_2) : \underline{v_1 \otimes v_2}(\theta) = 1\}) \leq$
$q(\{(\vec{v}_1, \vec{v}_2) : \underline{\min(\underline{v}_1(\theta), \underline{v}_2(\theta))} = 1\}) \leq$
$q(\{(\vec{v}_1, \vec{v}_2) : \underline{v}_1(\theta) = 1\}) = w_1(\{\vec{v}_1 : \underline{v}_1(\theta) = 1\})$
$= \underline{\mu}_{w_1}(\theta)$. Similarly, $\underline{\mu}_{w_1 \otimes_q w_2}(\theta) \leq \underline{\mu}_{w_2}(\theta)$.
Also, by duality $\overline{\mu}_{w_1 \otimes_q w_2}(\theta) = 1 - \underline{\mu}_{w_1 \otimes_q w_2}(\neg\theta)$ and by the above argument,
$1 - \underline{\mu}_{w_1 \otimes_q w_2}(\neg\theta) \geq 1 - \min(\underline{\mu}_{w_1}(\neg\theta), \underline{\mu}_{w_2}(\neg\theta))$
$= \max(1 - \underline{\mu}_{w_1}(\neg\theta), 1 - \underline{\mu}_{w_2}(\neg\theta))$
$= \max(\overline{\mu}_{w_1}(\theta), \overline{\mu}_{w_2}(\theta))$. □

Notice that for conservative combinations of belief pairs the lower and upper bounds given in theorem 29 cannot always be reached. For example, let $\vec{v}_1$ and $\vec{v}_2$ be Kleene valuation pairs with associated orthopairs $(\{p_1\}, \emptyset)$ and $(\emptyset, \{p_1\})$ respectively. Also, let w_1 and w_2 be probability distributions on $\mathbb{V}$ such that $w_1(\vec{v}_1) = 1$ and $w_2(\vec{v}_2) = 1$ so that $q(\vec{v}_1, \vec{v}_2) = 1$. Hence, $\underline{\mu}_{w_1}(p_1 \wedge \neg p_1) = 1$ and $\underline{\mu}_{w_1}(p_1 \wedge \neg p_1) = 1$ while $\underline{\mu}_{w_1 \otimes_q w_2}(p_1 \wedge \neg p_1) = 0$.

In the specific case that $q = w_1 \times w_2$ then we obtain tighter bounds on conservative combination as is shown in the following result.

Theorem 30. *If* $\vec{\mu}_{w_1 \otimes_q w_2}$ *is a conservative combination of* $\vec{\mu}_{w_1}$ *and* $\vec{\mu}_{w_2}$ *where* $q = w_1 \times w_2$ *then* $\forall \theta \in S\mathcal{L}$*;*

$$\underline{\mu}_{w_1 \otimes_q w_2}(\theta) \leq \underline{\mu}_{w_1}(\theta) \times \underline{\mu}_{w_2}(\theta) \text{ and}$$
$$\overline{\mu}_{w_1 \otimes_q w_2}(\theta) \geq \overline{\mu}_{w_1}(\theta) + \overline{\mu}_{w_2}(\theta) - \overline{\mu}_{w_1}(\theta) \times \overline{\mu}_{w_2}(\theta)$$

Proof. $\overline{\mu}_{w_1 \otimes_q w_2}(\theta) \leq$
$q(\{(\vec{v}_1, \vec{v}_2) : \min(\underline{v}_1(\theta), \underline{v}_2(\theta)) = 1\}) =$
$q(\{(\vec{v}_1, \vec{v}_2) : \underline{v}_1(\theta) = 1,\ \underline{v}_2(\theta) = 1\}) =$
$w_1(\{\vec{v}_1 : \underline{v}_1(\theta) = 1\}) \times w_2(\{\vec{v}_2 : \underline{v}_2(\theta) = 1\}) =$
$\underline{\mu}_{w_1}(\theta) \times \underline{\mu}_{w_2}(\theta)$. Furthermore, by duality and the above argument, $\overline{\mu}_{w_1 \otimes_q w_2}(\theta) = 1 - \underline{\mu}_{w_1 \otimes_q w_2}(\neg\theta) \geq$
$1 - \underline{\mu}_{w_1}(\neg\theta) \times \underline{\mu}_{w_2}(\neg\theta) = 1 - (1 - \overline{\mu}_{w_1}(\theta)) \times (1 - \overline{\mu}_{w_2}(\theta))$
$= \overline{\mu}_{w_1}(\theta) + \overline{\mu}_{w_2}(\theta) - \overline{\mu}_{w_1}(\theta) \times \overline{\mu}_{w_2}(\theta)$. □

Furthermore, if we restrict ourselves to literals then, assuming an independent interaction model, the combined lower measure is the product of $\underline{\mu}_{w_1}$ and $\underline{\mu}_{w_2}$, while the upper measure is the algebraic sum of $\overline{\mu}_{w_1}$ and $\overline{\mu}_{w_2}$.

Theorem 31. *If* $\vec{\mu}_{w_1 \otimes_q w_2}$ *is a conservative combination of* $\vec{\mu}_{w_1}$ *and* $\vec{\mu}_{w_2}$ *where* $q = w_1 \times w_2$ *then* $\forall l \in L\mathcal{L}$*;*

$$\underline{\mu}_{w_1 \otimes_q w_2}(l) = \underline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l) \text{ and}$$
$$\overline{\mu}_{w_1 \otimes_q w_2}(l) = \overline{\mu}_{w_1}(l) + \overline{\mu}_{w_2}(l) - \overline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l)$$

Proof. Follows from theorem 11 □

In general we can adapt definition 28 so as to extend other combination operators to belief pairs, simply by replacing $\otimes$ by the relevant operator. For the case of the optimistic operator this requires an additional normalisation step so as to take account of the fact that $\oplus$ is undefined if the two valuation pairs are inconsistent.

Definition 32. *Optimistic Combination of Belief Pairs*
An optimistic combination of Kleene belief pairs $\vec{\mu}_{w_1}$ and $\vec{\mu}_{w_2}$ is a belief pair $\vec{\mu}_{w_1 \oplus_q w_2}$ where $w_1 \oplus_q w_2$ is a probability distribution on $\mathbb{V}$ for which

$$w_1 \oplus_q w_2(\vec{v}) = c \times q(\{(\vec{v_1}, \vec{v_2}) : \vec{v}_1 \oplus \vec{v}_2 = \vec{v}\})$$

where q is any 2-dimensional probability distribution on $\mathbb{V} \times \mathbb{V}$ with marginals w_1 and w_2, and $\frac{1}{c} = q(\{(\vec{v_1}, \vec{v_2}) : P_1 \cap N_2 = P_2 \cap N_1 = \emptyset\})$. In the case that $q(\{(\vec{v_1}, \vec{v_2}) : P_1 \cap N_2 = P_2 \cap N_1 = \emptyset\}) = 0$ then $\vec{\mu}_{w_1 \oplus_q w_2}$ is undefined.

Theorem 33. *If $\vec{\mu}_{w_1 \oplus_q w_2}$ is an optimistic combination of $\vec{\mu}_{w_1}$ and $\vec{\mu}_{w_2}$ then $\forall \theta \in S\mathcal{L}$;*

$$\underline{\mu}_{w_1 \oplus_q w_2}(\theta) \geq 1 - c + c\max(\underline{\mu}_{w_1}(\theta), \underline{\mu}_{w_2}(\theta))$$
$$\overline{\mu}_{w_1 \oplus_q w_2}(\theta) \leq c\min(\overline{\mu}_{w_1}(\theta), \overline{\mu}_{w_2}(\theta))$$

Proof. By theorem 15 $\overline{\mu}_{w_1 \oplus_q w_2}(\theta) = cq(\{(\vec{v}_1, \vec{v}_2) : \overline{v_1 \oplus v_2}(\theta) = 1\}) \leq cq(\{(\vec{v}_1, \vec{v}_2) : \min(\overline{v_1}(\theta), \overline{v_2}(\theta)) = 1, P_1 \cap N_2 = P_2 \cap N_1 = \emptyset\}) \leq cq(\{(\vec{v}_1, \vec{v}_2) : \min(\overline{v_1}(\theta), \overline{v_2}(\theta)) = 1\}) \leq cq(\{(\vec{v}_1, \vec{v}_2) : \overline{v_1}(\theta) = 1\}) = c\overline{\mu}_{w_1}(\theta)$. Similarly, $\overline{\mu}_{w_1 \oplus_q w_2}(\theta) \leq c\overline{\mu}_{w_2}(\theta)$. Also, by duality $\underline{\mu}_{w_1 \oplus_q w_2}(\theta) = 1 - \overline{\mu}_{w_1 \oplus_q w_2}(\neg\theta) \geq 1 - c\min(\overline{\mu}_{w_1}(\neg\theta), \overline{\mu}_{w_2}(\neg\theta)) = 1 - c\min(1 - \underline{\mu}_{w_1}(\theta), 1 - \underline{\mu}_{w_2}(\theta))$
$= 1 - c(1 - \max(\mu_{w_1}(\theta), \mu_{w_2}(\theta))) = 1 - c + c\max(\mu_{w_1}(\theta), \mu_{w_2}(\theta))$ □

Theorem 34. *If $\vec{\mu}_{w_1 \oplus_q w_2}$ is an optimistic combination of $\vec{\mu}_{w_1}$ and $\vec{\mu}_{w_2}$ where $q = w_1 \times w_2$ then $\forall l \in L\mathcal{L}$;*

$$\underline{\mu}_{w_1 \oplus_q w_2}(l) \leq c(\underline{\mu}_{w_1}(l) + \underline{\mu}_{w_2}(l) - \underline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l))$$
$$\text{and } \overline{\mu}_{w_1 \oplus_q w_2}(l) \leq c \times \overline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l)$$

Proof. Follows from theorem 15 □

Example 35. *Let $\mathcal{P} = \{p_1, p_2, p_3, p_4\}$ and let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \mathbb{V}$ be such that $(P_1, N_1) = (\{p_1\}, \{p_3\})$, $(P_2, N_2) = (\{p_1, p_2\}, \{p_3, p_4\})$, $(P_3, N_3) = (\{p_2\}, \{p_3\})$ and $(P_4, N_4) = (\{p_2, p_4\}, \{p_3\})$. Now suppose two agents have beliefs characterised by probability distributions w_1 and w_2 on $\mathbb{V}$ respectively, where $w_1(\vec{v}_1) = 0.3$, $w_1(\vec{v}_2) = 0.7$ and $w_2(\vec{v}_3) = 0.6$, $w_2(\vec{v}_4) = 0.4$. Hence, for this example q must take the following form: $q(\vec{v}_1, \vec{v}_3) = x$, $q(\vec{v}_1, \vec{v}_4) = 0.3 - x$, $q(\vec{v}_2, \vec{v}_3) = 0.6 - x$ and $q(\vec{v}_2, \vec{v}_4) = 0.1 + x$ where $x \in [0, 0.3]$. In the case that the two agents interact independently we have that $x = 0.18$. Furthermore, the cases where $x = 0$ or $x = 0.3$ model strong dependency between the agents. To see this notice that $\vec{v}_1 \preceq \vec{v}_2$ and $\vec{v}_3 \preceq \vec{v}_4$. Taking $x = 0$ then minimizes $q(\vec{v}_1, \vec{v}_3)$ and $q(\vec{v}_2, \vec{v}_4)$ while maximizing $q(\vec{v}_1, \vec{v}_4)$ and $q(\vec{v}_2, \vec{v}_3)$. This corresponds to the assumption that the two agents tend to make opposite judgments about semantic precision. In other words, if the valuation pair identified by agent one is relatively semantically precise then agent two tends to identify a relatively imprecise valuation pair, and vice versa. This might perhaps arise from cooperative interaction where the two agents are aiming to be as consistent as possible with each other so as to facilitate belief combination. In contrast, taking $x = 0.3$ assumes that the two agents tend to make similar judgments about semantic precision. Table 2 shows the conservative combination operator applied to pairs of valuations together with the associated q value. From this we can see that, for the propositional variables, the resulting belief pair values are then given by; $\vec{\mu}_{w_1 \otimes_q w_2}(p_1) = (0, 1)$, $\vec{\mu}_{w_1 \otimes_q w_2}(p_2) = (0.7, 1)$, $\vec{\mu}_{w_1 \otimes_q w_2}(p_3) = (0, 0)$ and $\vec{\mu}_{w_1 \otimes_q w_2}(p_4) = (0, 1)$.*

Table 2: Table showing the results of applying the conservative operator to the two bipolar belief measures in example 35.

$\vec{v_i} \otimes \vec{v_j}$ $q(\vec{v_i}, \vec{v_j})$	$(\{p_2\}, \{p_3\})$ 0.6	$(\{p_2, p_4\}, \{p_3\})$ 0.4
$(\{p_1\}, \{p_3\})$ 0.3	$(\emptyset, \{p_3\})$ x	$(\emptyset, \{p_3\})$ $0.3 - x$
$(\{p_1, p_2\}, \{p_3, p_4\})$ 0.7	$(\{p_2\}, \{p_3\})$ $0.6 - x$	$(\{p_2\}, \{p_3\})$ $0.1 + x$

Table 3 shows the optimistic combination operator applied to pairs of valuations together with the associated q value. Notice that for $\vec{v}_2$ and $\vec{v}_4$, $\oplus$ is not defined. Hence, $c = \frac{1}{1 - q(\vec{v}_2, \vec{v}_4)} = \frac{1}{0.9 - x}$. From this we can see that, for the propositional variables, the resulting belief pair values are then given by; $\vec{\mu}_{w_1 \oplus_q w_2}(p_1) = (1, 1)$, $\vec{\mu}_{w_1 \oplus_q w_2}(p_2) = (1, 1)$, $\vec{\mu}_{w_1 \oplus_q w_2}(p_3) = (0, 0)$ and $\vec{\mu}_{w_1 \oplus_q w_2}(p_4) = (\frac{0.3 - x}{0.9 - x}, \frac{0.3}{0.9 - x})$.

Table 3: Table showing the results of applying the optimistic operator to the two bipolar belief measures in example 35.

$\vec{v_i} \oplus \vec{v_j}$ $q(\vec{v_i}, \vec{v_j})$	$(\{p_2\}, \{p_3\})$ 0.6	$(\{p_2, p_4\}, \{p_3\})$ 0.4
$(\{p_1\}, \{p_3\})$ 0.3	$(\{p_1, p_2\}, \{p_3\})$ $\frac{x}{0.9 - x}$	$(\{p_1, p_2, p_4\}, \{p_3\})$ $\frac{0.3 - x}{0.9 - x}$
$(\{p_1, p_2\}, \{p_3, p_4\})$ 0.7	$(\{p_1, p_2\}, \{p_3, p_4\})$ $\frac{0.6 - x}{0.9 - x}$	undefined 0

We now apply the approach introduced in definition 28 to extend $\ominus$ and $\odot$ to belief pairs.

Definition 36. *Difference and Consensus Combination of Belief Pairs*
A difference combination of belief pairs $\vec{\mu}_{w_1}$ and $\vec{\mu}_{w_2}$ is a belief pair $\vec{\mu}_{w_1 \ominus_q w_2}$ where $w_1 \ominus_q w_2$ is a probability distribution on $\mathbb{V}$ given by:

$$w_1 \ominus_q w_2(\vec{v}) = q(\{(\vec{v}_1, \vec{v}_2) : \vec{v_1} \ominus \vec{v}_2 = \vec{v}\})$$

Similarly, a consensus combination of $\vec{\mu}_{w_1}$ and $\vec{\mu}_{w_2}$ is a belief pair $\vec{\mu}_{w_1 \odot_q w_2}$ where $w_1 \odot_q w_2$ is a probability distribution on $\mathbb{V}$ given by:

$$w_1 \odot_q w_2(\vec{v}) = q(\{(\vec{v}_1, \vec{v}_2) : \vec{v_1} \odot \vec{v}_2 = \vec{v}\})$$

As above q is any 2-dimensional probability distribution on $\mathbb{V} \times \mathbb{V}$ with marginals w_1 and w_2

The next result shows that, as expected, $\vec{\mu}_{w_1 \ominus_q w_2}$ is less precise than $\vec{\mu}_{w_1}$ but more precise than $\vec{\mu}_{w_1 \otimes_q w_2}$.

Theorem 37. $\forall \theta \in S\mathcal{L}$,

$$\underline{\mu}_{w_1 \ominus_q w_2}(\theta) \le \min(\underline{\mu}_{w_1}(\theta), \overline{\mu}_{w_2}(\theta))$$
$$\overline{\mu}_{w_1 \ominus_q w_2}(\theta) \ge \max(\overline{\mu}_{w_1}(\theta), \underline{\mu}_{w_2}(\theta))$$

Proof. By theorem 21 we have that:

$$\underline{\mu}_{w_1 \ominus_q w_2}(\theta) = q(\{(\vec{v}_1, \vec{v}_2) : \underline{v_1 \ominus v_2}(\theta) = 1\})$$
$$\le q(\{(\vec{v}_1, \vec{v}_2) : \min(\underline{v}_1(\theta), \overline{v}_2(\theta)) = 1\})$$
$$\le q(\{(\vec{v}_1, \vec{v}_2) : \underline{v}_1(\theta) = 1\}) = \underline{\mu}_{w_1}(\theta)$$

Similarly $\underline{\mu}_{w_1 \ominus_q w_2}(\theta) \le \overline{\mu}_{w_2}(\theta)$ as required. Also, by theorem 21 we have that:

$$\overline{\mu}_{w_1 \ominus_q w_2}(\theta) = q(\{(\vec{v}_1, \vec{v}_2) : \overline{v_1 \ominus v_2}(\theta) = 1\})$$
$$\ge q(\{(\vec{v}_1, \vec{v}_2) : \max(\overline{v}_1(\theta), \underline{v}_2(\theta)) = 1\})$$
$$\ge q(\{(\vec{v}_1, \vec{v}_2) : \overline{v}_1(\theta) = 1\}) = \overline{\mu}_{w_1}(\theta)$$

Similarly $\overline{\mu}_{w_1 \ominus_q w_2}(\theta) \ge \underline{\mu}_{w_2}(\theta)$ as required. □

If we assume an independent interaction model then we can obtain tighter bounds for $\vec{\mu}_{w_1 \ominus_q w_2}$ as follows:

Theorem 38. *If* $q = w_1 \times w_2$ *then* $\forall \theta \in S\mathcal{L}$,

$$\underline{\mu}_{w_1 \ominus_q w_2}(\theta) \le \underline{\mu}_{w_1}(\theta) \times \overline{\mu}_{w_2}(\theta)$$
$$\overline{\mu}_{w_1 \ominus_q w_2}(\theta) \ge \overline{\mu}_{w_1}(\theta) + \underline{\mu}_{w_2}(\theta) - \overline{\mu}_{w_1}(\theta) \times \underline{\mu}_{w_2}(\theta)$$

Proof. By theorem 21 we have that:

$$\underline{\mu}_{w_1 \ominus_q w_2}(\theta) \le q(\{(\vec{v}_1, \vec{v}_2) : \min(\underline{v}_1(\theta), \overline{v}_2(\theta)) = 1\})$$
$$= q(\{(\vec{v}_1, \vec{v}_2) : \underline{v}_1(\theta) = 1, \overline{v}_2(\theta) = 1\}) =$$
$$w_1(\{\vec{v}_1 : \underline{v}_1(\theta) = 1\}) \times w_2(\{\vec{v}_2 : \overline{v}_2(\theta) = 1\})$$
$$= \underline{\mu}_1(\theta) \times \overline{\mu}_2(\theta)$$

Furthermore, by duality and the above result we have that:

$$\overline{\mu}_{w_1 \ominus_q w_2}(\theta) = 1 - \underline{\mu}_{w_1 \ominus_q w_2}(\theta)$$
$$\ge 1 - \underline{\mu}_{w_1}(\neg\theta) \times \overline{\mu}_{w_2}(\neg\theta) =$$
$$1 - (1 - \overline{\mu}_{w_1}(\theta)) \times (1 - \underline{\mu}_{w_2}(\theta)) =$$
$$\overline{\mu}_{w_1}(\theta) + \underline{\mu}_{w_2}(\theta) - \overline{\mu}_{w_1}(\theta) \times \underline{\mu}_{w_2}(\theta)$$

□

Furthermore, by restricting ourselves to literals we can replace the inequalities in theorem 38 with equalities.

Theorem 39. *If* $q = w_1 \times w_2$ *then* $\forall l \in L\mathcal{L}$ *it holds that:*

$$\underline{\mu}_{w_1 \ominus_q w_2}(l) = \underline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l)$$
$$\overline{\mu}_{w_1 \ominus_q w_2}(l) = \overline{\mu}_{w_1}(l) + \underline{\mu}_{w_2}(l) - \overline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l)$$

Proof. Follows from theorem 19. □

Finally, we consider the case of the consensus operator applied to belief pairs. The following theorem shows that the application of $\odot_q$ to distributions w_1 and w_2 can be broken down into the subtraction operations $w_1 \ominus_q w_2$ and $w_2 \ominus w_1$ followed by the addition operation $\oplus_{q'}$. The latter then involves a different interaction probability q' derived from q.

Theorem 40.

$$w_1 \odot_q w_2 = (w_1 \ominus_q w_2) \oplus_{q'} (w_2 \ominus_q w_1) \text{ where}$$
$$q'(\vec{v}'_1, \vec{v}'_2) = q(\{(\vec{v}_1, \vec{v}_2) : \vec{v}_1 \ominus \vec{v}_2 = \vec{v}'_1, \vec{v}_2 \ominus \vec{v}_1 = \vec{v}'_2\})$$

Proof. Initially we show that the marginals of q' are $w_1 \ominus_q w_2$ and $w_2 \ominus_q w_1$.

$$\sum_{\vec{v}'_2} q'(\vec{v}'_1, \vec{v}'_2) =$$
$$\sum_{\vec{v}'_2} q(\{(\vec{v}_1, \vec{v}_2) : \vec{v}_1 \ominus \vec{v}_2 = \vec{v}'_1, \vec{v}_2 \ominus \vec{v}_1 = \vec{v}'_2\})$$
$$= q(\{(\vec{v}_1, \vec{v}_2) : \vec{v}_1 \ominus \vec{v}_2 = \vec{v}'_1\}) = w_1 \ominus_q w_2$$

Similarly we can show that the second marginal is $w_2 \ominus_q w_1$. Now,

$$w_1 \odot_q w_2(\vec{v}) = q(\{(\vec{v}_1, \vec{v}_2) : \vec{v}_1 \odot \vec{v}_2 = \vec{v}\})$$
$$= q(\{(\vec{v}_1, \vec{v}_2) : (\vec{v}_1 \ominus \vec{v}_2) \oplus (\vec{v}_2 \ominus \vec{v}_1) = \vec{v}\}) =$$
$$\sum_{(\vec{v}'_1, \vec{v}'_2) : \vec{v}'_1 \oplus \vec{v}'_2 = \vec{v}} q(\{(\vec{v}_1, \vec{v}_2) : \vec{v}_1 \ominus \vec{v}_2 = \vec{v}'_1, \vec{v}_2 \ominus \vec{v}_1 = \vec{v}'_2\})$$
$$= \sum_{(\vec{v}'_1, \vec{v}'_2) : \vec{v}'_1 \oplus \vec{v}'_2 = \vec{v}} q'(\vec{v}'_1, \vec{v}'_2) = (w_1 \ominus_q w_2) \oplus_{q'} (w_2 \ominus_q w_1)$$

as required. □

This immediately leads to the following bounds on $\vec{\mu}_{w_1 \odot_q w_2}$.

Corollary 41. $\forall \theta \in S\mathcal{L}$,

$$\underline{\mu}_{w_1 \odot_q w_2}(\theta) \ge \max(\underline{\mu}_{w_1 \ominus_q w_2}(\theta), \underline{\mu}_{w_2 \ominus_q w_1}(\theta))$$
$$\overline{\mu}_{w_1 \odot_q w_2}(\theta) \le \min(\overline{\mu}_{w_1 \ominus_q w_2}(\theta), \overline{\mu}_{w_2 \ominus_q w_1}(\theta))$$

Proof. Follows immediately from theorems 33 and 40. □

A more precise result can then be obtained for literals in the case when $q = w_1 \times w_2$.

Theorem 42. *If* $q = w_1 \times w_2$ *then* $\forall l \in L\mathcal{L}$,

$$\underline{\mu}_{w_1 \odot_q w_2}(l) = \underline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l) + \overline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l)$$
$$- \underline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l)$$
$$\overline{\mu}_{w_1 \odot_q w_2}(l) = \underline{\mu}_{w_1}(l) + \underline{\mu}_{w_2}(l) + \overline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l)$$
$$- \overline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l) - \underline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l)$$

Proof. From theorem 24 we have that,

$$\underline{\mu}_{w_1 \odot_q w_2}(l) = q(\{(\vec{v}_1, \vec{v}_2) : \underline{v_1 \odot v_2}(l) = 1\}) =$$
$$q(\{(\vec{v}_1, \vec{v}_2) : \max(\min(\underline{v}_1(l), \overline{v}_2(l)), \min(\underline{v}_2(l), \overline{v}_1(l)) = 1\})$$
$$= q(\{(\vec{v}_1, \vec{v}_2) : \min(\underline{v}_1(l), \overline{v}_2(l)) = 1 \text{ or } \min(\underline{v}_2(l), \overline{v}_1(l)) = 1\})$$
$$= q(\{(\vec{v}_1, \vec{v}_2) : \min(\underline{v}_1(l), \overline{v}_2(l)) = 1)\})$$
$$+ q(\{(\vec{v}_1, \vec{v}_2) : \min(\underline{v}_2(l), \overline{v}_1(l)) = 1\})$$
$$- q(\{(\vec{v}_1, \vec{v}_2) : \min(\underline{v}_1(l), \overline{v}_2(l)) = 1, \min(\underline{v}_2(l), \overline{v}_1(l)) = 1\})$$
$$= w_1(\{\vec{v}_1 : \underline{v}_1(l) = 1\}) \times w_2(\{\vec{v}_2 : \overline{v}_2(l) = 1\})$$
$$+ w_1(\{\vec{v}_1 : \overline{v}_1(l) = 1\}) \times w_2(\{\vec{v}_2 : \underline{v}_2(l) = 1\})$$
$$- w_1(\{\vec{v}_1 : \underline{v}_1(l) = 1\}) \times w_2(\{\vec{v}_2 : \underline{v}_2(l) = 1\})$$
$$= \underline{\mu}_{w_1}(l) \times \overline{\mu}_{w_2}(l) + \overline{\mu}_{w_2}(l) \times \underline{\mu}_{w_1}(l) - \underline{\mu}_{w_1}(l) \times \underline{\mu}_{w_2}(l)$$

The result for $\overline{\mu}_{w_1 \odot_q w_2}(\theta)$ follows similarly. □

Example 43. *Recall the scenario described in example 35. Table 4 shows the consensus combination operator applied to pairs of valuations together with the associated q value. From this we can see that, for the propositional variables, the resulting belief pair values are then given by;* $\vec{\mu}_{w_1 \otimes_q w_2}(p_1) = (1,1)$, $\vec{\mu}_{w_1 \otimes_q w_2}(p_2) = (1,1)$, $\vec{\mu}_{w_1 \otimes_q w_2}(p_3) = (0,0)$ *and* $\vec{\mu}_{w_1 \otimes_q w_2}(p_4) = (0.3 - x, 0.4 + x)$.

Table 4: Table showing the results of applying the consensus operator to the two bipolar belief measures in example 35.

$\vec{v}_i \odot \vec{v}_j$ $q(\vec{v}_i, \vec{v}_j)$	$(\{p_2\},\{p_3\})$ 0.6	$(\{p_2, p_4\},\{p_3\})$ 0.4
$(\{p_1\},\{p_3\})$ 0.3	$(\{p_1, p_2\},\{p_3\})$ x	$(\{p_1, p_2, p_4\},\{p_3\})$ $0.3 - x$
$(\{p_1, p_2\},$ $\{p_3, p_4\})$ 0.7	$(\{p_1, p_2\},\{p_3, p_4\})$ $0.6 - x$	$(\{p_1, p_2\},\{p_3\})$ $0.1 + x$

Vagueness vs. Incomplete Information

The above approach to vagueness based on Kleene logic and orthopairs may be puzzling for scholars who are following the original intuitions of (Kleene 1952) according to which the third truth-value **b** is interpreted as *unknown* instead of *borderline*. Indeed, in this paper we are interested in classifying precisely described objects with respect to vague categories represented by propositional variables $p_i \in \mathcal{P}$, and where the underlying truth model is three-valued. However, there is a one-to-one correspondence between Kleene valuations on a set $\mathcal{P}$ of propositions and incomplete Boolean valuations (also called partial models (Blamey 1998)). Specifically, an orthopair (P, N) can either represent a completely specified three-valued Kleene valuation $\mathcal{P} \to \{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$, or alternatively a partially defined Boolean truth-assignment $\tau : \mathcal{P} \to \{t, f\}$ [2] such that $\tau(p) = t$ if $p \in P$ and $\tau(p) = f$ if $p \in N$. The latter interpretation is very common in formalisms which aim to handle uncertainty about Boolean (crisp) propositions due to incomplete information (e.g. partial logic (Blamey 1998)). In contrast, this paper is concerned with the handling of vague propositions in the presence of complete information.

In this section we briefly compare these two distinct interpretations of the Kleene model in order to highlight the subtle but important differences between them. The similarity between the two settings carries over to valuation pairs, but differences emerge regarding truth-functionality. Consider the incomplete information setting. The idea is that while the Boolean truth-values of some propositions $p_i \in \mathcal{P}$ are known, the remaining propositions have unknown truth-values, not because such propositions are vague, but simply because there is no information about them. In this case, the corresponding orthopair (P, N) represents a state of information (or epistemic state) that can be described by means of a consistent conjunction of literals:

$$\phi_{(P,N)} = \bigwedge_{p_i \in P} p_i \wedge \bigwedge_{p_j \in N} \neg p_j,$$

Given $\phi_{(P,N)}$ we can naturally define a pair (N, Π) of functions $S\mathcal{L} \to \{0,1\}$ as follows (Dubois and Prade 1988): $\forall \theta \in S\mathcal{L}$,

- $N(\theta) = 1$ if $\phi_{(P,N)} \models \theta$ and 0 otherwise.
- $\Pi(\theta) = 1$ if $\phi_{(P,N)} \not\models \neg\theta$ and 0 otherwise.

N is called a *necessity measure* and Π a *possibility measure*. $N(\theta) = 1$ means that θ is *certainly true*, and $\Pi(\theta) = 1$ that θ is *possibly true*, if the epistemic state is described by (P, N). In particular, if $N(\theta) = 0$ and $\Pi(\theta) = 1$ it means that the truth of θ is unknown in epistemic state (P, N).

There is a striking similarity between Kleene valuation pairs $(\underline{v}, \overline{v})$ and necessity-possibility pairs (N, Π). In particular, the following properties can be compared to those for valuation pairs given in definition 1:

- $N(\neg\theta) = 1 - \Pi(\theta)$ and $\Pi(\neg\theta) = 1 - N(\theta)$
- $N(\theta \wedge \varphi) = \min(N(\theta), N(\varphi))$
- $\Pi(\theta \vee \varphi) = \max(\Pi(\theta), \Pi(\varphi))$

However there is also an important difference between them: while $\overline{v}(\theta \wedge \varphi) = \min(\overline{v}(\theta), \overline{v}(\varphi))$ and $\underline{v}(\theta \vee \varphi) = \max(\underline{v}(\theta), \underline{v}(\varphi))$, in general we only have that $\Pi(\theta \wedge \varphi) \leq \min(\Pi(\theta), \Pi(\varphi))$ and $N(\theta \vee \varphi) \geq \max(N(\theta), N(\varphi))$. In particular, $\Pi(\theta \wedge \neg\theta) = 0$ (non-contradiction law) and $N(\theta \vee \neg\theta) = 1$ (excluded middle law). In fact, (N, Π) is a pair of KD modalities in epistemic logic, which explains why they are not compositional. A Kleene valuation pair $(\underline{v}, \overline{v})$ would be trivial in a Boolean context, and this paper emphasizes that in the three-valued propositional setting accommodating borderline cases, such deviant modalities (where the lower valuation distributes over disjunctions) are not trivial. Such deviant modalities for more general Kleene algebras are also studied in (Cattaneo et al. 2011).

Conclusions

In this paper we have outlined a bipolar framework for combining potentially inconsistent beliefs which exploits the inherent vagueness of concepts in natural language. Four operators have been proposed for combining different viewpoints expressed in a language of propositional logic, where the underlying truth model is Kleene valuation pairs. These exploit the possibility of truth-gaps in which certain sentences are inherently borderline cases, to allow for different levels of compromise between the viewpoints resulting in new valuation pairs with differing levels of semantic precision.

Kleene belief pairs have been introduced as quantitative lower and upper measures of belief which incorporate both semantic indeterminacy and epistemic uncertainty. We have then proposed a schema for extending combination operators from valuation pairs to belief pairs and investigated the properties of the four operators within this extended framework.

[2] Here we are using t, f to denote true and false in the classical Boolean sense, in contrast to $\mathbf{t}, \mathbf{f}$ which denote *absolutely true* and *absolutely false* (alternatively *certainly true* and *certainly false* in an incomplete information setting).

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