Model Based Horn Contraction

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Abstract

Following the recent trend of adapting the AGM (Alchourrón and Makinson 1985) framework to propositional Horn logic, Delgrande and Peppas (Delgrande and Peppas 2011) give a model theoretic account for revision in the Horn logic setting. The current paper complements their work by studying the model theoretic approach for contraction. A model based Horn contraction is constructed and shown to give a model theoretic account to the transitively relational partial meet Horn contraction studied in (Zhuang and Pagnucco 2011). Significantly however, in contrast to (Delgrande and Peppas 2011), our model-based characterisation of Horn contraction does not require the property of Horn compliance and totality over preorders. The model based contraction, upon proper restriction, also gives a model theoretic account for the epistemic entrenchment based Horn contraction studied in (Zhuang and Pagnucco 2010a).

1 Introduction

The theory of belief change deals with the dynamics of an agent's beliefs. The change often involves removal of existing beliefs—the *contraction* operation—and incorporation of newly acquired beliefs—the *revision* operation. The AGM (Alchourrón and Makinson 1985) framework, named after the initials of its originators, is generally held to be the most compelling account of belief change and provides a common point of reference and comparison.

In the AGM framework, beliefs are represented by propositional sentences. The set of beliefs held by an agent is termed a *belief set* and is a logically closed set of belief representing sentences. AGM formalises rationality postulates for capturing the intuition behind contraction and revision. Construction methods are defined for accomplishing the change and are shown to be sound and complete with respect to their corresponding set of postulates; that is, a construction for contraction (revision) can be characterised by the set of contraction (revision) postulates.

The nature of belief change where the underlying logic is restricted to the Horn fragment of propositional logic (Horn logic) has recently attracted significant attention (Delgrande 2008; Booth, Meyer, and Varzinczak 2009; Booth et al. 2010; Delgrande and Wassermann 2010; Zhuang and Pagnucco 2010a; 2010b; 2011; Delgrande and Peppas 2011). The topic is interesting for several reasons. Horn logic is an important subset of propositional logic which has found use in many artificial intelligence and database applications. The study of belief change under Horn logic broadens the practical applicability of the AGM framework and in particular it provides a key step towards applying the AGM framework to non-classical logics with less expressive and reasoning power than propositional logic.

The classic construction for AGM contraction is based on the notion of *remainder sets*. Remainder sets of a belief set K with respect to a sentence ϕ are the maximal[1] subsets of K that fail to imply ϕ, denoted by $K \downarrow \phi$. In this construction, the resulting belief set is obtained by intersecting the most desirable remainder sets chosen by a selection function. If the selection function is transitive and relational, then the contraction constructed is called *transitively relational partial meet contraction* (TRPMC). Levi suggests that revision can be defined from contraction via the identity: $K * \phi = (K \dot{-} \neg\phi) \cup \{\phi\}$ for $*$ a revision operator and $\dot{-}$ a contraction operator. The revision obtained from TRPMC via the *Levi identity* is called *transitively relational partial meet revision* (TRPMR).

AGM revision can be constructed directly without referring to a contraction. Due to its logical closure, a belief set can be identified by its set of models which makes it possible to study change operations in terms of the models involved. Katsuno and Mendelzon (Katsuno and Mendelzon 1992) gave a model based approach for constructing revision directly. In this approach the input to the revision are models of the belief set and those of the new belief and the output is a set of revised models from which the resulting belief set is obtained. A preorder over those models is used to determine the revision. According to (Katsuno and Mendelzon 1992), if the preorder is total and it is faithful with respect to the original belief set, then the determined revision performs identically to TRPMR. So essentially it gives a model theoretic account for TRPMR. Although (Katsuno and Mendelzon 1992) deals with revision, its contraction counterpart can be easily derived which gives a model theoretic account

[1]The maximality property implies that if $X \in K \downarrow \phi$ and $\psi \in K \setminus X$ then $X \cup \{\psi\} \vdash \phi$.

for TRPMC.

With a less expressive logic such as Horn logic there are restrictions on how beliefs can be represented. A belief set under Horn logic contains only Horn formulas and is thus referred to as a *Horn belief set*. Accordingly, contraction and revision operations for a Horn belief set are termed *Horn contraction* and *Horn revision* which concern the removal and the incorporation of Horn formulas from and to the Horn belief set. In this paper we focus on Horn contraction.

The recent work by Delgrande and Peppas (Delgrande and Peppas 2011) proposed a model based Horn revision along with its characterisation. As in (Katsuno and Mendelzon 1992), the construction is based on a preorder of interpretations, however, they demonstrate that the preorder has to satisfy the *Horn compliance* condition (together with totality and faithfulness) for it to generate meaningful Horn revisions.

As a complement to their work we study the model theoretic approach of defining Horn contraction. Unlike (Delgrande and Peppas 2011), the preorder used for determining the Horn contraction is faithful but not necessarily total and Horn compliant. The Horn contraction thus constructed performs identically to the *transitively relational partial meet Horn contraction* (TRPMHC) in (Zhuang and Pagnucco 2011) which is based on a transitive relation over weak remainder sets. The equality stems from the correspondence between their determining preference relations. From this equality we immediately get a characterisation for the model based contraction and moreover we can conclude that Horn compliance and totality are not mandatory for defining meaningful model based Horn contractions.

Also by properly restricting the behaviour of the model based Horn contraction, it gives a semantic characterisation of two restricted forms of TRPMHC, that is the maximised TRPMHC in (Zhuang and Pagnucco 2011) and the *epistemic entrenchment based Horn contraction* (EEHC) in (Zhuang and Pagnucco 2010a).

2 Technical Preliminaries

We assume a fixed propositional language $\mathcal{L}$ over a finite set of atoms $\mathbf{P} = \{p, q, \ldots\}$. Classical logical consequence and logical equivalence are denoted by $\vdash$ and $\equiv$ respectively. Cn is the Tarskian consequence operator such that $Cn(X) = \{\phi : X \vdash \phi\}$. An interpretation of $\mathcal{L}$ is a function from $\mathbf{P}$ to $\{true, false\}$. Truth and falsity of a formula in $\mathcal{L}$ is determined by standard rules of propositional logic. We assume standard propositional semantics. The set of all interpretations of $\mathcal{L}$ is denoted by U. An interpretation I is a model of a formula ϕ if ϕ is true in I, written $I \models \phi$. Given a set of formulas X, $[X]$ denotes the set of models of X. To denote the set of models of a formula ϕ, we write $[\phi]$ instead of $[\{\phi\}]$. An interpretation is identified by the set of atoms assigned true, e.g., the interpretation bc indicates atoms b, c are assigned true and the others are assigned false.[2]

A *Horn clause* is a clause that contains at most one positive atom, e.g., $\neg p \vee \neg q \vee r$. A *Horn formula* is a conjunction of Horn clauses. The Horn language $\mathcal{L}_H$ is the subset of $\mathcal{L}$ that contains only Horn formulas. The Horn logic generated from $\mathcal{L}_H$ is just propositional logic acting on Horn formulas. A *Horn theory* H is a set of Horn formulas such that $H = Cn_H(H)$. We add the suffix H to logical operators under Horn logic. For example, Cn_H is the Horn consequence operator such that $Cn_H(X) = \{\phi : X \vdash \phi, \phi \in \mathcal{L}_H\}$. $Horn : 2^{\mathcal{L}} \to 2^{\mathcal{L}_H}$ is a function such that $Horn(X) = \{\phi : \phi \in X \text{ and } \phi \in \mathcal{L}_H\}$. Negation is not always available in Horn logic, for example, the negation of $\neg p \wedge \neg q$, which is $p \vee q$, is not a Horn formula. Let ϕ be a Horn formula, $[\neg\phi]$ is the set of interpretations in which ϕ is false, that is $[\neg\phi] = U \setminus [\phi]$.

The *intersection* of a pair of interpretations is the interpretation that assigns true to those atoms that are assigned true by both of the interpretations. We denote the intersection of interpretations m_1 and m_2 by $m_1 \cap m_2$, e.g., $ab \cap cb = b$, $ab \cap cd = \emptyset$. If $m_1 \cap m_2 = m_3$ then m_3 is the *induced interpretation* of m_1 and m_2. In the above example b is the induced interpretation of ab and bc, and $\emptyset$ is the induced interpretation of ab and cd. Given a set of interpretations M, the *closure* of M under intersection is denoted as $Cl_\cap(M)$. Models of any Horn theory are closed under intersection, that is for a Horn theory H, if $m_1, m_2 \in [H]$ then $m_1 \cap m_2 \in [H]$. Conversely any set of interpretations that are closed under intersection can be identified by a unique Horn theory. We will make frequent use of a function $t_H : 2^U \to 2^{\mathcal{L}_H}$ that, given a set of interpretations M, $t_H(M)$ returns the set of Horn formulas consistent with all interpretations in M, i.e. $t_H(M) = \{\phi \in \mathcal{L}_H \mid m \models \phi \text{ for every } m \in M\}$. Let $t_H(M) = H$ then it is easy to see that $H = Cn_H(H)$ and $[H] = Cl_\cap(M)$. For example, $t_H(\{ab, cb\}) = Cn_H(\{b, \neg a \vee \neg c\})$. Note that $[Cn_H(\{b, \neg a \vee \neg c\})] = \{ab, cb, b\} = Cl_\cap(\{ab, cb\})$.

3 Model Based Contraction and Revision

A model theoretic account of AGM revision is given in (Katsuno and Mendelzon 1992) and equivalently in (Grove 1988) in terms of system of spheres. In the account of (Katsuno and Mendelzon 1992), a *preorder* $\preceq$ is a reflexive and transitive binary relation over the set of all interpretations U. The strict relation $\prec$ is defined as $u \prec v$ if and only if $u \preceq v$ and $v \not\preceq u$. The equivalence relation $=_\preceq$ is defined as $u =_\preceq v$ if and only if $u \preceq v$ and $v \preceq u$. A preorder is *total* if for every pair of $u, v \in U$, either $u \preceq v$ or $v \preceq u$. In this paper, a preorder over U is regarded as an *I-relation*. Each belief set K is assigned an I-relation $\preceq_K$[3] which represents a measure of closeness between models of the belief set K and an interpretation such that $u \preceq_K v$ means u is at least as close to $[K]$ as v is. Intuitively, models of K are always the closest to themselves. I-relations with this property are called *faithful*. Formally an I-relation $\preceq_K$ is faithful with respect to K if it satisfies:

1). If $u, v \in [K]$, then $u =_{\preceq_K} v$, and

2). If $u \in [K]$ and $v \notin [K]$, then $u \prec_K v$.

[2] Here bc is a shorthand for $\{b, c\}$ for the sake of simplicity.

[3] (Katsuno and Mendelzon 1992) deals with formulas rather than belief sets. As we are working with a finite language, the difference vanishes.

Let M be a set of interpretations, $\mathit{Min}(M, \preceq_K)$ is the set of interpretations in M that is closest to $[K]$ by means of $\preceq_K$. Formally,

$$\mathit{Min}(M, \preceq_K) = \{u \in M \mid \nexists v \in M \text{ such that } v \prec_K u\}.$$

Let $*$ be a revision for K such that

$$[K * \phi] = \mathit{Min}([\phi], \preceq_K)$$

for all ϕ. Katsuno and Mendelzon showed that $*$ satisfies all the AGM revision postulates whenever $\preceq_K$ is a faithful total I-relation. Thus the revision corresponds exactly to TRPMR.

The model theoretic account can also be applied to contraction. Through the so-called *Harper identity*, a contraction $\dot{-}$ can be obtained from a revision $*$ by putting $K \dot{-} \phi = (K * \neg\phi) \cap K$. So, model theoretically we have $[K \dot{-} \phi] = \mathit{Min}([\neg\phi], \preceq_K) \cup [K]$. The model based contraction thus obtained satisfies the full set of contraction postulates and corresponds exactly to TRPMC. In this paper, we abbreviate the model based contraction as MC.

4 Model Based Horn Revision

In accordance with (Katsuno and Mendelzon 1992), Delgrande and Peppas (Delgrande and Peppas 2011) studied model based revision under Horn logic. In (Delgrande and Peppas 2011), the *model based Horn revision* (MHR) $*$ for a Horn belief set H is defined as

$$H * \phi = t_H(\mathit{Min}([\phi], \preceq_H)$$

for all $\phi \in \mathcal{L}_H$. $\preceq_H$ is a total I-relation that is faithful with respect to H. Furthermore, $\preceq_H$ is Horn compliant. A set of interpretations M is *Horn elementary* if and only if $M = Cl_\cap(M)$. An I-relation $\preceq$ is *Horn compliant* if and only if for every $\phi \in \mathcal{L}_H$, $\mathit{Min}([\phi], \preceq)$ is Horn elementary.

A set of characterising postulates for MHR is identified.

Theorem 1. *(Delgrande and Peppas 2011) Let $*$ be a Horn revision for a Horn belief set H, then $*$ is a MHR iff it satisfies the following postulates:*

$(H * 1)$ $H * \phi = Cn_H(H * \phi)$.
$(H * 2)$ $\phi \in H * \phi$.
$(H * 3)$ $H * \phi \subseteq H + \phi$.
$(H * 4)$ *If* $\bot \notin H + \phi$, *then* $H + \phi \subseteq H * \phi$.
$(H * 5)$ *If* ϕ *is consistent then* $\bot \notin H * \phi$.
$(H * 6)$ *If* $\phi \equiv \psi$, *then* $H * \phi = H * \psi$.
$(H * 7)$ $H * (\phi \wedge \psi) \subseteq (H * \psi) + \phi$.
$(H * 8)$ *If* $\bot \notin (H * \psi) + \phi$ *then* $(H * \psi) + \phi \subseteq H * (\phi \wedge \psi)$.
$(Acyc)$ *If for* $0 \leq i < n$ *we have* $(H * \mu_{i+1}) + \mu_i \nvdash \bot$, *and* $(H * \mu_0) + \mu_n \nvdash \bot$, *then* $(H * \mu_n) + \mu_0 \nvdash \bot$.

The newly proposed Horn compliant condition restricts the allowable I-relations for generating MHR. The condition guarantees the set of resulting models of MHR (i.e. $\mathit{Min}([\phi], \preceq_H)$) is always Horn elementary. More importantly, the condition is mandatory for the generated revision to satisfy $(H * 7)$ and $(H * 8)$. Since $(H * 7)$ and $(H * 8)$ are well motivated postulates, revisions violating them are hardly considered as rational. In this aspect, the Horn compliant condition is well justified as it rules out the irrational revisions.

Apart from Horn analogues of the AGM revision postulates (i.e. $(H*1)$–$(H*8)$), postulate $(Acyc)$ is also required to characterise MHR. It is shown that there are Horn revisions satisfying $(H*1)$–$(H*8)$ that cannot be generated by I-relations. The postulate $(Acyc)$ rules out such revisions. According to (Delgrande and Peppas 2011), $(Acyc)$ is derivable from the AGM revision postulates (where the underlying logic contains propositional logic), thus it is compatible with AGM revision. To the contrary, $(Acyc)$ is independent of $(H * 1)$–$(H * 8)$ when we work with Horn logic.

From the study of MHR, we may conclude that totality and Horn compliance are mandatory conditions for constructing meaningful Horn revisions. It will be clear from the subsequent sections that this is not the case for constructing model based Horn contraction. The next section is devoted to the introduction of TRPMHC which is closely related to the model based Horn contraction we are going to present.

5 Transitively Relational Partial Meet Horn Contraction

TRPMHC is based on a transitive relation over *weak remainder sets* (Delgrande and Wassermann 2010). A weak remainder set is an adaptation of a standard remainder set used in the construction of TRPMC. The set of weak remainder sets of a Horn belief set H with respect to a Horn formula ϕ is denoted as $H \downarrow_w \phi$. And $X \in H \downarrow_w \phi$ if and only if $X = Cn_H(X)$ and $[X] = Cl_\cap(H \cup \{u\})$ for some $u \in [\neg\phi]$. In this paper, a relation over weak remainder sets is regarded as a *R-relation*. A selection function γ for a Horn belief set H is such that $\gamma(H \downarrow_w \phi)$ returns a non-empty subset of $H \downarrow_w \phi$ whenever $H \downarrow_w \phi$ is non-empty and returns H otherwise. We say that γ is *transitively relational* over H if and only if a transitive R-relation $\leqslant$ for H is used to generate γ via the *marking off* identity:

$$\gamma(H \downarrow_w \phi) = \{X \in H \downarrow_w \phi \,|\, Y \leqslant X \text{ for all } Y \in H \downarrow_w \phi\}.$$

Let γ be a transitively relational selection function for H, then a TRPMHC $\dot{-}$ is defined as $H \dot{-} \phi = \bigcap \gamma(H \downarrow_w \phi)$ for all $\phi \in \mathcal{L}_H$.

A representation theorem is provided.

Theorem 2. *(Zhuang and Pagnucco 2011) Let H be a Horn belief set. $\dot{-}$ is a TRPMHC for H iff $\dot{-}$ satisfies the following postulates:*

$(H \dot{-} 1)$ $H \dot{-} \phi = Cn_H(H \dot{-} \phi)$.
$(H \dot{-} 2)$ $H \dot{-} \phi \subseteq H$.
$(H \dot{-} 3)$ *If* $\phi \notin H$, *then* $H \dot{-} \phi = H$.
$(H \dot{-} 4)$ *If* $\nvdash \phi$, *then* $\phi \notin H \dot{-} \phi$.
$(H \dot{-} f)$ *If* $\vdash \phi$, *then* $H \dot{-} \phi = H$.
$(H \dot{-} 6)$ *If* $\phi \equiv \psi$, *then* $H \dot{-} \phi = H \dot{-} \psi$.
$(H \dot{-} wr)$ *If* $\psi \in H \setminus (H \dot{-} \phi)$, *then there is some* H' *such that* $H \dot{-} \phi \subseteq H'$, $\phi \notin Cn_H(H')$ *and* $\phi \in Cn_H(H' \cup \{\psi\})$
$(H \dot{-} pa)$ $(H \dot{-} \phi) \cap Cn_H(\phi) \subseteq H \dot{-} \phi \wedge \psi$.
$(H \dot{-} 8)$ *If* $\phi \notin H \dot{-} \phi \wedge \psi$ *then* $H \dot{-} \phi \wedge \psi \subseteq H \dot{-} \phi$.

$(H \dot{-} 1)$–$(H \dot{-} 4)$ and $(H \dot{-} 6)$ are Horn analogues of the corresponding AGM postulates. $(H \dot{-} f)$ captures the *failure property* which states that the contraction of a tautology leaves the belief set unchanged. $(H \dot{-} wr)$ is a weaker version of the *relevance* postulate (Hansson 1999). $(H \dot{-} pa)$

and $(H\dot{-}8)$ are Horn analogues of *partial antitony* and *conjunctive inclusion* respectively. An AGM contraction satisfies partial antitony if and only if it satisfies *conjunctive overlapping*, however, this is not the case for Horn contractions. Following the AGM tradition, $(H\dot{-}pa)$ and $(H\dot{-}8)$ are regarded as supplementary postulates and the rest as basic postulates.

The R-relation $\leqslant$ is intended to capture the intuition that $X \leqslant Y$ if and only if Y is at least as worth retaining as X. It is reasonable to enforce connectedness on the relation $\leqslant$ so that all pairs of X, Y are comparable by $\leqslant$. A selection function γ is *connectively relational* over a Horn belief set H if and only if it is generated from a connected relation of H via the marking off identity. A TRPMHC is connected if its determining selection function is connected. Connectedness is a redundant condition for TRPMHC, as for each TRPMHC $\dot{-}$, there is a connected TRPMHC that is identical to $\dot{-}$, and vice versa.

Theorem 3. *(Zhuang and Pagnucco 2011) Let H be a Horn belief set and $\dot{-}$ be a Horn contraction over H, then $\dot{-}$ is a TRPMHC iff it is a connected TRPMHC.*

According to the principle of minimal change, R-relations should put more value on a set than any of its proper subsets. A R-relation $\leqslant$ for H is *maximised* if for all weak remainder sets X, Y of H, $X \subset Y$ implies $X < Y$. A selection function is called *maximisingly relational* if and only if it is generated from a maximised relation via the marking off identity. A TRPMHC is maximised if its determining R-relation is maximised. The maximality condition is not redundant; it articulates TRPMHC in terms of retaining old information. The postulate $(H\dot{-}wr)$ captures some minimal change behaviour of TRPMHC, however, with a maximised TRPMHC, a stronger postulate, namely relevance (Hansson 1992), is satisfied.

Theorem 4. *(Zhuang and Pagnucco 2011) Let H be a Horn belief set. $\dot{-}$ is a maximised TRPMHC for H iff $\dot{-}$ satisfies $(H\dot{-}1)$–$(H\dot{-}4)$, $(H\dot{-}f)$, $(H\dot{-}6)$, $(H\dot{-}pa)$, $(H\dot{-}8)$, and the following relevance postulate:*
$(H\dot{-}r)$ If $\psi \in H \setminus (H\dot{-}\phi)$, then there is some $H^{'}$ such that $H\dot{-}\phi \subseteq H^{'} \subseteq H$ and $\phi \notin Cn_H(H^{'})$ but $\phi \in Cn_H(H^{'} \cup \{\psi\})$.

The appropriateness of TRPMHC is highlighted through a comparison with the AGM contraction TRPMC, demonstrating that TRPMHC and TRPMC perform identically in terms of Horn formulas. Thus we say that TRPMHC is *Horn equivalent* to TRPMC.

Theorem 5. *(Zhuang and Pagnucco 2011)*[4] *Let $\dot{-}_H$ be a TRPMHC for a Horn belief set H. Then there is a TRPMC $\dot{-}$ for a belief set $K = Cn(H)$ such that $H\dot{-}_H\phi = (K\dot{-}\phi) \cap H$ for all $\phi \in \mathcal{L}_H$.*

Let $\dot{-}$ be a TRPMC for a belief set K. Then there is a TRPMHC $\dot{-}_H$ for the Horn belief set $H = Horn(K)$ such that $H\dot{-}_H\phi = (K\dot{-}\phi) \cap H$ for all $\phi \in \mathcal{L}_H$.

[4]We hide the technical details in the original theorem to make it more readable.

The notion of Horn equivalence is central to several of our main results. In the subsequent sections, we will make use of the Horn equivalence results for TRPMHC and for the epistemic entrenchment based Horn contraction in (Zhuang and Pagnucco 2010a). Also we will explore the result for the model based Horn contraction to be defined in the next section.

6 Model Based Horn Contraction

6.1 Constructing Model Based Horn Contraction

In this section we give the construction of a *model based Horn contraction* (MHC). In contrast to (Delgrande and Peppas 2011), the determining I-relation is not necessarily total and Horn compliant.

Definition 1. *Let $\preceq$ be a faithful I-relation for a Horn belief set H, a MHC $\dot{-}$ for H is defined as:*

$$H\dot{-}\phi = t_H(min([\neg\phi], \preceq) \cup [H])$$

for all $\phi \in \mathcal{L}_H$.

From the definition, it is easy to see that $[H\dot{-}\phi] = Cl_\cap(min([\neg\phi], \preceq) \cup [H])$. We may require the set of resulting models (i.e. $min([\neg\phi], \preceq_H) \cup [H]$) to be Horn elementary (i.e. $min([\neg\phi], \preceq) \cup [H] = Cl_\cap(min([\neg\phi], \preceq) \cup [H])$) as for the resulting models of MHR. However, as illustrated in Example 1, this is not always possible. There exists a Horn belief set H and Horn formula ϕ such that $min([\neg\phi], \preceq) \cup [H]$ is not Horn elementary for every possible I-relation $\preceq$.

Example 1. *Let $\mathcal{L}_H$ contain Horn formulas consist of atoms in $\{a, b, c\}$ only, let $H = Cn_H(a \wedge \neg b)$, and let $\phi = a \vee \neg b$. Then $[H] = \{a\}$ and $[\neg\phi] = \{b\}$. Let's consider $H\dot{-}\phi$ for $\dot{-}$ a MHC. Since $[\neg\phi]$ is a singleton set, it does not matter which $\preceq$ is used, we always have $min([\neg\phi], \preceq) = \{a\}$, that is $min([\neg\phi], \preceq) \cup [H] = \{a, b\}$. But $Cl_\cap(\{a, b\}) = \{a, b, \emptyset\} \neq \{a, b\}$ which means $min([\neg\phi], \preceq) \cup [H]$ is not Horn elementary.*

A MHC may return a set of models that is not Horn elementary, however, by applying t_H we can obtain a unique Horn belief set.

MHC is a Horn adaptation of MC with a minor difference. It is easily seen from the construction that MHC performs identically to MC at the knowledge level. If we look at the mechanism of change over the involved models then there is no difference between MHC and the classic approach; the two approaches obtain an identical set of resulting models. MHC differs from the classic approach in that it returns the Horn belief set that corresponds to the resulting set of models. Consequently, MHC is Horn equivalent to MC.

Theorem 6. *Let $\dot{-}_H$ be a MHC for a Horn belief set H. Then there is the MC $\dot{-}$ for a belief set $K = Cn(H)$ such that $H\dot{-}_H\phi = (K\dot{-}\phi) \cap H$ for all $\phi \in \mathcal{L}_H$.*

Let $\dot{-}$ be a MC for a belief set K. Then there is a MHC $\dot{-}_H$ for the Horn belief set $H = Horn(K)$ such that $H\dot{-}_H\phi = (K\dot{-}\phi) \cap H$ for all $\phi \in \mathcal{L}_H$.

In the next subsection, we show that MHC and TRPMHC perform identically, thus the characterisation for MHC is the same as that for TRPMHC. It follows from the characterisation that Horn compliance is not mandatory for Horn contraction as it is not required, as in the case of MHR, for the determined MHC to satisfy the supplementary postulates for Horn contraction.

6.2 Characterising Model Based Horn Contraction

In this subsection we obtain a representation theorem for MHC by revealing the mapping between MHC and TRPMHC. In the context of AGM contraction, a close connection has been demonstrated in (Grove 1988)[5] between the faithful total I-relation $\preceq$ of a belief set K and the transitive R-relation $\leqslant$ of K. The connection lies in the bijection between elements of $K \downarrow \phi$ (i.e., remainder sets of K with respect to ϕ) and those of $[\neg\phi]$ (i.e., models of the negation of ϕ). Due to the maximal nature of remainder sets, $X \in K \downarrow \phi$ if and only if $X = Cn(X)$ and $[X] = [K] \cup \{u\}$ for some $u \in [\neg\phi]$, thus a remainder set X of K can be identified by an interpretation u not in $[K]$, in which case we say that X is determined by u.

Based on this bijection, a connected and transitive R-relation $\leqslant$ over remainder sets of K induces an I-relation $\preceq$ by defining $u \preceq v$ if $X, Y \in K \downarrow \mathcal{L}$[6] are such that $Y \leq X$ and $[X] = [K] \cup \{u\}$ and $[Y] = [K] \cup \{v\}$. The induced I-relation $\preceq$ is total and faithful with respect to K. Moreover the (model based) contraction determined by $\preceq$ and the (transitively relational partial meet) contraction determined by $\leqslant$ are identical. The reverse construction is similar; a faithful total I-relation induces an equivalent R-relation for constructing contractions.

From the definition of weak remainder set, we can easily see that there is also a bijection between elements of $H \downarrow_w \phi$ and those of $[\neg\phi]$.

Lemma 1. *Let H be a Horn belief set and ϕ be a Horn formula such that $\phi \in H$ and $\phi \not\vdash \bot$. Then there is a bijection between elements of $H \downarrow_w \phi$ and those of $[\neg\phi]$.*

Moreover, R-relations and I-relations over a Horn belief set can be obtained from one another in a similar manner.

Definition 2. *Let H be a Horn belief set. Let $\leqslant$ be a transitive and connected R-relation (over $H \downarrow_w \mathcal{L}_H$[7]) for H. The relation $\preceq$ over U, denoted as $I(\leqslant)$, is defined as*
1). $u \prec v$ if $u \in [H]$ and $v \notin [H]$,
2). $u =_{\preceq} v$ if $u, v \in [H]$, and
3). $u \preceq v$ if $u, v \notin [H]$ and $X, Y \in H \downarrow_w \mathcal{L}_H$ are such that $[X] = Cl_\cap([H] \cup \{u\})$, $[Y] = Cl_\cap([H] \cup \{v\})$, and $Y \leqslant X$.

As expected, the induced $\preceq$ is a faithful (with respect to H) total I-relation.

[5](Grove 1988) investigates the notion of a system of spheres instead of preorders over possible worlds but the two notions are equivalent. We assume here the modelling in (Grove 1988) is in terms of preorders over possible worlds.

[6]$K \downarrow \mathcal{L} = \bigcup\{K \downarrow \phi : \phi \in \mathcal{L}\}$.

[7]$H \downarrow_w \mathcal{L}_H = \bigcup\{H \downarrow_w \phi : \phi \in \mathcal{L}_H\}$.

Lemma 2. *Let H be a Horn belief set. If $\leqslant$ is a connected and transitive R-relation for H, then the relation $\preceq = I(\leqslant)$ is a faithful (with respect to H) total I-relation.*

The converse translation is as follows.

Definition 3. *Let H be a Horn belief set. Let $\preceq$ be a faithful (with respect to H) total I-relation. The R-relation $\leqslant$ (over $H \downarrow_w \mathcal{L}_H$) for H, denoted as $R(\preceq)$, is defined as*
1). $Y \leqslant X$ if $X = H$,
2). $Y \leqslant X$ if $X \neq H$ and there are $u, v \in U$ such that $[X] = Cl_\cap([H] \cup \{u\})$, $[Y] = Cl_\cap([H] \cup \{v\})$, and $u \preceq v$.

As expected, the induced R-relation $\leqslant$ is connected and transitive.

Lemma 3. *Let H be a Horn belief set. If $\preceq$ is a total faithful (with respect to H) I-relation, then the R-relation $\leqslant = R(\preceq)$ for H is connected and transitive.*

Due to the tight connection between I-relations and R-relations we are expecting a tight connection between constructions of contraction that are based on the two relations. For one direction, let $\dot{-}_{\leqslant}$ be a TRPMHC for a Horn belief set H that is determined by a R-relation $\leqslant$ and let $\dot{-}_{\preceq}$ be a MHC that is determined by an I-relation $\preceq = I(\leqslant)$, then the two contractions are identical.

Lemma 4. *Let H be a Horn belief set, then:*
1). If $\dot{-}_{\leqslant}$ is a TRPMHC for H, then there is a MHC $\dot{-}_{\preceq}$ for H such that $H \dot{-}_{\leqslant} \phi = H \dot{-}_{\preceq} \phi$ for all $\phi \in \mathcal{L}_H$.
2). If $\dot{-}_{\preceq}$ is a MHC for H, then there is a TRPMHC for H such that $H \dot{-}_{\leqslant} \phi = H \dot{-}_{\preceq} \phi$ for all $\phi \in \mathcal{L}_H$.

We then conclude from Lemma 4 that the characterisation for MHC is the same as that for TRPMHC.

Theorem 7. *A Horn contraction $\dot{-}$ is a MHC iff it satisfies $(H\dot{-}1)$–$(H\dot{-}4)$, $(H\dot{-}f)$, $(H\dot{-}wr)$, $(H\dot{-}6)$, $(H\dot{-}pa)$ and $(H\dot{-}8)$.*

A restricted form of TRPMHC, namely maximised TRPMHC, in general, preserves more beliefs than TRPMHC. In the next subsection we give a model theoretic account for maximised TRPMHC.

6.3 Maximised Model Based Horn Contraction

Maximised TRPMHC, as mentioned in Section 5, is based on a maximised selection function whose generating R-relation is maximised. A maximised R-relation gives precedence to a weak remainder set over its proper subsets (which are weak remainder sets). An example is given in (Zhuang and Pagnucco 2011) where a Horn belief set H may have weak remainder sets X, Y such that $X \subset Y$. With a maximised R-relation $\leqslant$, Y is strictly more preferred than X; that is $X < Y$. Before giving a model theoretic account, we make explicit some properties of maximised TRPMHC that are not shown in (Zhuang and Pagnucco 2011).

Standard remainder sets are maximal, that is if $X \in K \downarrow \phi$ and $\psi \in K \setminus X$, then $X \cup \{\psi\} \vdash \phi$. This is not always the case for weak remainder sets where each non-maximal weak remainder set has a proper superset that is a maximal weak remainder set.

Lemma 5. *Let H be a Horn belief set. If $X \in H \downarrow_w \phi$ and $\psi \in H \setminus X$, then either $X \cup \{\psi\} \vdash \phi$ or there is $Y \in H \downarrow_w \phi$ such that $X \subset Y$ and $Y \vdash \psi$.*

A maximised selection function always returns maximal weak remainder sets, which is crucial for a maximised TRPMHC to satisfy *relevance*.

Lemma 6. *Let H be a Horn belief set, and γ be a maximised selection function for H. If $X \in \gamma(H \downarrow_w \phi)$ and $\psi \in H \setminus X$ then $X \cup \{\psi\} \vdash \phi$.*

Since selection functions are generated from a R-relation, it is the maximality of the generating R-relation that matters. We need to find out the condition on I-relations for generating MHC that corresponds to the maximality requirement on R-relations. By Lemma 1, for each Horn belief set H, there is a bijection between interpretations not in $[H]$ and weak remainder sets of H. Thus it is sufficient to restrict the I-relations such that the interpretations corresponding to the maximal weak remainder sets are strictly more preferred than those corresponding to the non-maximal ones. The restricted I-relations are regarded as *maximised I-relations*.

Definition 4. *Let $\preceq$ be an I-relation. $\preceq$ is maximised with respect to a Horn belief set H iff it satisfies the following condition:*

If $u \in [H]$, $v, w \notin [H]$ and $u \cap v = w$, then $w \prec v$.

Section 6.2 describes methods for translating between R-relations and I-relations. Lemma 7 shows that by applying the translation methods, the R-relation for a Horn belief set H, induced by a maximised I-relation for H, is maximised; and conversely the I-relation for H induced by a maximised R-relation for H, is maximised.

Lemma 7. *Let H be a Horn belief set, then:*
1). If $\leqslant$ is a transitive maximised R-relation for H and $\preceq = I(\leqslant)$, then $\preceq$ is a faithful (with respect to H) maximised I-relation.
2). If $\preceq$ is a faithful (with respect to H) maximised I-relation and $\leqslant = R(\preceq)$, then $\leqslant$ is a transitive maximised R-relation for H.

Just like maximised TRPMHC is a restricted form of TRPMHC, the *maximised MHC* defined below is a restricted form of MHC.

Definition 5. *Let H be a Horn belief set. $\dot{-}$ is a maximised MHC for H iff $\dot{-}$ is a MHC for H and its determining I-relation is maximised.*

Lemma 7 guarantees the correspondence between maximised R-relation and maximised I-relation. The two relations generate, respectively, a maximised TRPMHC and a maximised MHC. The following equivalence result between maximised MHCs and maximised TRPMHCs is a corollary of Lemma 4 and Lemma 7.

Theorem 8. *Let H be a Horn belief set, then:*
1). If $\dot{-}_{\leqslant}$ is a maximised TRPMHC for H, then there is a maximised MHC $\dot{-}_{\preceq}$ for H such that $H \dot{-}_{\leqslant} \phi = H \dot{-}_{\preceq} \phi$ for all $\phi \in \mathcal{L}_H$.
2). If $\dot{-}_{\preceq}$ is a maximised MHC for H, then there is a maximised TRPMHC for H such that $H \dot{-}_{\leqslant} \phi = H \dot{-}_{\preceq} \phi$ for all $\phi \in \mathcal{L}_H$.

As guaranteed by Theorem 8, we provide, via maximised MHC, a model theoretic account for maximised TRPMHC.

Besides TRPMHC, the *epistemic entrenchment Horn contraction* (EEHC) in (Zhuang and Pagnucco 2010a) is another way of constructing Horn contractions that involves preference information. We have shown that a Katsuno and Mendelzon style construction of model based Horn contraction, corresponds exactly to TRPMHC. Subsequently, we will demonstrate how the Horn contraction can be restricted to perform like an EEHC.

6.4 Horn Strengthened Model Based Horn Contraction

The EEHC of (Zhuang and Pagnucco 2010a) is the Horn analogue of the *epistemic entrenchment contraction* (EEC) (Gärdenfors and Makinson 1988). EEC is based on a preference relation over $\mathcal{L}$, which we regard as an *EE-relation*. An EE-relation reflects the relative entrenchment of formulas in $\mathcal{L}$. The more entrenched, the more preferred a formula. Given a belief set K and its associated EE-relation $\leq$, the EEC $\dot{-}$ for K is defined as $K \dot{-} \phi = K \cap \{\psi \mid \phi < \phi \vee \psi\}$ when $\phi \notin Cn(\emptyset)$ and $K \dot{-} \phi = K$ otherwise. Furthermore the determining EE-relation $\leq$ satisfies the following conditions:

$(EE1)$	If $\phi \leq \psi$ and $\psi \leq \chi$, then $\phi \leq \chi$
$(EE2)$	If $\phi \vdash \psi$, then $\phi \leq \psi$
$(EE3)$	$\phi \leq \phi \wedge \psi$ or $\psi \leq \phi \wedge \psi$
$(EE4)$	If $K \not\vdash \bot$, then $\phi \notin K$ iff $\phi \leq \psi$ for every ψ
$(EE5)$	If $\phi \leq \psi$ for every ϕ, then $\vdash \psi$

$(EE1)$ requires the EE-relation to be transitive. $(EE2)$ requires logically stronger formulas are at most as entrenched as a logically weaker ones. By $(EE2)$ we have that a conjunction is at most as entrenched as its conjuncts, together with $(EE3)$, we then have that a conjunction is equally entrenched to its least entrenched conjunct. $(EE1)$, $(EE2)$, and $(EE3)$ put together imply that the EE-relation is connected. $(EE4)$ and $(EE5)$ take care of the limiting cases, according to which formulas not in the given belief set are least entrenched and tautologies are most entrenched. It is shown that EEC is identical to TRPMC.

The construction of EEC involves arbitrary disjunctions which may not be Horn formulas. To cope with this expressivity problem, EEHC makes use of the notion of *Horn strengthening* (Selman and Kautz 1991). Given a non-Horn formula ψ, its Horn strengthenings are the logically weakest Horn formulas that entail ψ.

Definition 6. *(Selman and Kautz 1991) Given a clause ϕ, its set of Horn-strengthenings, denoted by $\mathcal{HS}(\phi)$, is such that $\phi^h \in \mathcal{HS}(\phi)$ iff $\phi^h \in \mathcal{L}_H$, $[\phi^h] \subseteq [\phi]$ and there is no $\phi' \in \mathcal{L}_H$ such that $[\phi^h] \subset [\phi'] \subseteq [\phi]$.*

A Horn-strengthening of a conjunction of clauses $\phi_1 \wedge \cdots \wedge \phi_n$ is such that $\phi_1^h \wedge \cdots \wedge \phi_n^h \in \mathcal{HS}(\phi_1 \wedge \cdots \wedge \phi_n)$ iff $\phi_i^h \in \mathcal{HS}(\phi_i)$ for $1 \leq i \leq n$.

Given a Horn belief set H and its associated EE-relation $\leq$ that satisfies $(EE1)$–$(EE5)$, an EEHC $\dot{-}$ for H is defined as $H \dot{-} \phi = H \cap \{\psi \mid \phi < \chi$ for all $\chi \in \mathcal{HS}(\phi \vee \psi)\}$ when $\phi \notin Cn(\emptyset)$ and $H \dot{-} \phi = H$ otherwise.

From its definition, the condition for retaining a formula in EEHC is stricter than that for EEC. With the same EE-relation, a Horn formula retained by EEC may not be retained by the corresponding EEHC. EEHC is therefore not Horn equivalent to EEC, however, (Zhuang and Pagnucco 2010a) shows that the equivalence holds if the EE-relation for determining EEC satisfies one extra condition, namely $(EE6)$. $(EE6)$ enforces any non-Horn formula to be equally entrenched to its most entrenched Horn strengthening.

$(EE6)$: If $\phi \notin \mathcal{L}_H$, then there is a $\psi \in \mathcal{HS}(\phi)$ such that $\phi \leq \psi$.

Roughly speaking, $(EE6)$ encodes the extra strictness of EEHC into the EE-relations for determining EEC so that the EEC is as strict as EEHC in retaining formulas. We call an EEC whose determining EE-relation satisfies $(EE1)$–$(EE6)$, a *Horn strengthened EEC*.

(Zhuang and Pagnucco 2010b) shows that for each EEHC, there is a TRPMHC that performs identically to it but not vice versa, thus EEHC is a restricted form of TRPMHC. The result is strengthened in (Wassermann and Delgrande 2011) which shows that EEHC satisfies all the characterising postulates of TRPMHC. As MHC is identical to TRPMHC, thus there is a restricted form of MHC that is identical to EEHC. The key for specifying the restricted form, as we will show, is the condition $(EE6)$.

(Meyer, Labuschagne, and Heidema 2000) mentioned a method for inducing EE-relations from I-relations.

Definition 7. *(Meyer, Labuschagne, and Heidema 2000) Let $\preceq$ be an I-relation. The EE-relation $\leq$ induced by $\preceq$ is defined as: $\phi \leq \psi$ iff for every $y \in [\neg\psi]$ there is an $x \in [\neg\phi]$ such that $x \preceq y$.*

It is shown that the induced EE-relation satisfies $(EE1)$–$(EE5)$ if and only if the I-relation is faithful and total. For our purpose, we specify a *Horn strengthened* condition on I-relations. The condition guarantees the induced EE-relation also satisfies $(EE6)$. Accordingly the MC whose determining I-relation is Horn strengthened is called a *Horn strengthened MC*.

Definition 8. *An I-relation $\preceq$ is Horn strengthened iff it satisfies the following condition:*

$$\textit{If } x \cap y = w, \textit{ then either } w \preceq y \textit{ or } w \preceq x.$$

A Horn strengthened I-relation is such that for all pairs of interpretations x and y, if their intersection is w, then either x or y is at least as preferred as w. The following lemma reveals the connection between the Horn strengthened condition and $(EE6)$.

Lemma 8. *An EE-relation satisfies $(EE1)$–$(EE6)$ iff it is induced by a faithful total and Horn strengthened I-relation.*

Lemma 8 guarantees that a total faithful I-relation is Horn strengthened if and only if its induced EE-relation satisfies $(EE6)$. The Horn strengthened I-relation determines a Horn strengthened MC and the induced EE-relation determines a Horn strengthened EEC. (Meyer, Labuschagne, and Heidema 2000) shows that the MC determined by an I-relation $\preceq$ is identical to the EEC determined by the EE-relation $\leq$ induced from $\preceq$. As a Horn strengthened MC is a MC and a Horn strengthened EEC is an EEC, a Horn strengthened MC is identical to a Horn strengthened EEC.

Theorem 9. *Let K be a belief set. Then $\dot{-}$ is a Horn strengthened MC for K iff it is a Horn strengthened EEC for K.*

The restricted form of MHC whose determining I-relation is Horn strengthened is called a *Horn strengthened MHC*.

Definition 9. *Let H be a Horn belief set, $\dot{-}$ is a Horn strengthened MHC for H iff $\dot{-}$ is a MHC for H and its determining I-relation is Horn strengthened.*

Similar to MHC and MC, on the model theoretic side, the construction of Horn strengthened MHC is identical to that of Horn strengthened MC, therefore Horn strengthened MHC is Horn equivalent to Horn strengthened MC.

Finally, it follows from Theorem 9, **Proposition 3** of (Zhuang and Pagnucco 2010a), and the Horn equivalence between Horn strengthened MHC and Horn strengthened MC that Horn strengthened MC is identical to EEHC. Thus we have obtained a semantic characterisation for EEHC.

Theorem 10. *Let H be a Horn belief set. Then $\dot{-}$ is a Horn strengthened MHC for H iff it is an EEHC for H.*

7 Related Work

Existing work on Horn contraction falls roughly into two groups: those focussing on the basic set of AGM contraction postulates (Booth, Meyer, and Varzinczak 2009; Delgrande and Wassermann 2010; Booth et al. 2011) and those focussing on the full set of postulates by taking into account preference information over remainder sets (Delgrande 2008; Zhuang and Pagnucco 2011), over formulas (Zhuang and Pagnucco 2010a), and over interpretations (the current paper).

Horn contractions focussing on the basic postulates are incomparable with MHC as they lack the preference relation which is crucial in the construction of MHC. The contraction in (Delgrande and Wassermann 2010) is extended in (Zhuang and Pagnucco 2011) of which the resulting contraction is TRPMHC. We have shown that MHC is identical to TRPMHC thus identifying a semantic characterisation for TRPMHC. (Zhuang and Pagnucco 2010b) showed that the EEHC of (Zhuang and Pagnucco 2010a) is subsumed by TRPMHC. We have shown that a restricted form of MHC, namely Horn strengthened MHC, is identical to EEHC.

The *maxichoice e-contraction* (MEC) in (Delgrande 2008) is based on a relation over a variation of standard remainder sets that is less expressive than weak remainder sets (Delgrande and Wassermann 2010). In addition to transitivity, the relation is required to be reflexive, connected and antisymmetric. The MEC thus generated is a Horn analogue of a special case of TRPMC which is usually regarded as *orderly maxichoice partial meet contraction*. (Wassermann and Delgrande 2011) shows that MEC does not satisfy one of the supplementary postulates for Horn contraction. However, MHC satisfies all supplementary postulates and so all its restricted forms, therefore we can not restrict MHC so that it gives a semantic characterisation for MEC.

Apart from contraction, Delgrande and Peppas (Delgrande and Peppas 2011) studied revision under Horn logic. Their approach relies on the notion of Horn compliance. The Horn compliant condition restricts the allowable I-relations which in turn restricts the possible ways of forming preference information. Horn compliance can be formalised as follows:

Lemma 9. *An I-relation $\preceq$ is Horn compliant iff it satisfies the following condition:*

$$\text{If } x \cap y = w \text{ and } x =_{\preceq} y, \text{ then } w \preceq x.$$

From the formalisation, any I-relation with a single element in each rank is Horn compliant. In MHC the I-relation does not have to be Horn compliant so, vaguely speaking, MHC is more comprehensive than MHR as it allows more ways of forming preference information through I-relations.

8 Conclusion and Future Work

We have given a model theoretical account of Horn contraction by defining a model based contraction under Horn logic, namely MHC. The construction is based on a faithful preorder that is not necessarily total. Our MHC succeeds in giving a semantic characterisation for TRPMHCs. We also identified two restricted forms of MHC, namely maximised MHC and Horn strengthened MHC, which give, respectively, semantic characterisations for maximised TRPMHC and EEHC. One important aspect of MHC is that, on the model theoretic side, it is entirely identical to the construction of MC. The only distinction is that MHC obtains a Horn belief set from the set of resulting models. This implies that MHC performs identically to MC in terms of Horn formulas.

So far a fair amount of work has been done on constructing Horn contractions. The ability to incorporate new beliefs into a Horn belief set is no less important than that of removing old beliefs. The work of (Delgrande and Peppas 2011) gives a model theoretic approach for defining Horn revisions. In the AGM framework, a revision can also be defined indirectly from a contraction via the *Levi identity*. For further work, we aim to investigate this way of defining Horn revisions. The main obstacle for defining Horn revisions in this way is the lack of full negation ($\neg(\neg\phi \wedge \neg\psi) \notin \mathcal{L}_H$) in Horn logic. The notion of Horn strengthening used in defining EEHC is useful here for obtaining Horn approximations of non-Horn formulas, thus allowing us to obtain the Horn revision.

Appendix

Proof of **Lemma 2**

Suppose $\leqslant$ is a connected and transitive R-relation over $H \downarrow_w \mathcal{L}_H$ and $\preceq = I(\leqslant)$. We need to show $\preceq$ is a faithful total I-relation.

total: It suffices to show that for all pairs of $u, v \in U$ either $u \preceq v$ or $v \preceq u$. Let $[X] = Cl_\cap([H] \cup \{u\})$, $X = Cn^h(X)$, $[Y] = Cl_\cap([H] \cup \{v\})$ and $Y = Cn^h(Y)$, then $X, Y \in H \downarrow_w \mathcal{L}_H$. Since $\leqslant$ is connected, either $X \leqslant Y$ or $Y \leqslant X$, we then have by the derivation of $\preceq$, either $u \preceq v$ or $v \preceq u$.

transitive: Suppose $u \preceq v$ and $v \preceq w$, we need to show $u \preceq w$. We can construct, as in the proof of totality, weak remainder sets X, Y and Z such that $[X] = Cl_\cap([H] \cup \{u\})$, $[Y] = Cl_\cap([H] \cup \{v\})$, and $[Z] = Cl_\cap([H] \cup \{w\})$. By the generation of $\preceq$ we have $Y \leqslant X$ and $Z \leqslant Y$. Then by transitivity of $\leqslant$, we have $Z \leqslant X$. It follows from $Z \leqslant X$ and the the derivation of $\preceq$ that $u \preceq w$.

faithful: Follows directly from *1)*, *2)* of Definition 2.

□

Proof of **Lemma 3**

Suppose $\preceq$ is a faithful total I-relation and $\leqslant = R(\preceq)$. We need to show $\leqslant$ is a connected transitive R-relation over $H \downarrow_w \mathcal{L}_H$.

connected: Suppose $X, Y \in H \downarrow_w \mathcal{L}_H$ we need to show either $X \leqslant Y$ or $Y \leqslant X$. By the definition of weak remainder sets we have $[X] = Cl_\cap([H] \cup \{u\})$ and $[Y] = Cl_\cap([H] \cup \{v\})$ for some $u, v \in \mathcal{M}$. Since $\preceq$ is total, either $u \preceq v$ or $v \preceq u$. Then by the construction of $\leqslant$, we have $X \leqslant Y$ or $Y \leqslant X$.

transtive: Suppose $X, Y, Z \in H \downarrow_w \mathcal{L}_H$, $X \leqslant Y$ and $Y \leqslant Z$ we need to show $X \leqslant Z$. By the definition of weak remainder sets there are $u, v, w \in U$ such that $[X] = Cl_\cap([H] \cup \{u\})$, $[Y] = Cl_\cap([H] \cup \{v\})$, and $[Z] = Cl_\cap([H] \cup \{w\})$. From the construction of $\leqslant$, $X \leqslant Y$ and $Y \leqslant Z$ it follows that $v \preceq u$ and $w \preceq v$. Since $\preceq$ is transitive, $v \preceq u$ and $w \preceq v$ implies $w \preceq u$. Again by the construction of $\leqslant$, it follows from $w \preceq u$, $[Z] = Cl_\cap([H] \cup \{w\})$ and $[X] = Cl_\cap([H] \cup \{u\})$ that $X \leqslant Z$.

□

Proof of **Lemma 4**

Let H be a Horn belief set, $\leqslant$ a transitive R-relation for H, and $\preceq$ a faithful I-relation for H.

1). Let $\dot{-}_{\leqslant}$ be a TRPMHC determined by $\leqslant$. By Theorem 2, $\dot{-}_{\leqslant}$ satisfies $(H\dot{-}1)$–$(H\dot{-}4)$, $(H\dot{-}f)$, $(H\dot{-}wr)$, $(H\dot{-}6)$, $(H\dot{-}pa)$ and $(H\dot{-}8)$. Let $\preceq = I(\leqslant)$ and $\dot{-}_{\preceq}$ be a MHC determined by $\preceq$. By Lemma 2, $\preceq$ is a faithful total I-relation. We need to show $H\dot{-}_{\leqslant}\phi = H\dot{-}_{\preceq}\phi$ for all $\phi \in \mathcal{L}_H$.

Case $\phi \notin H$: By $(H\dot{-}3)$, $H\dot{-}_{\leqslant}\phi = H$. Since $\phi \notin H$, $[\neg\phi] \cap [H] \neq \emptyset$. By the faithfulness of $\preceq$, we have for all $u \in min([\neg\phi], \preceq)$, $u \in [H]$. Thus $H\dot{-}_{\preceq}\phi = t_H(min([\neg\phi], \preceq) \cup [H]) = t_H([H]) = H$.

Case $\phi \in Cn(\emptyset)$: By $(H\dot{-}f)$, $H\dot{-}_{\leqslant}\phi = H$. Since ϕ is a tautology, $[\neg\phi] = \emptyset$ which implies $min([\neg\phi], \preceq) = \emptyset$. Thus $H\dot{-}_{\preceq}\phi = t_H(min([\neg\phi], \preceq) \cup [H]) = t_H([H]) = H$.

Case $\phi \in H$: Then $H \downarrow_w \phi \neq \emptyset$. Let $X_1, \ldots, X_n \in max(H \downarrow_w \phi, \leqslant)$ then $H\dot{-}_{\leqslant}\phi = \bigcap(max(H \downarrow_w \phi, \leqslant)) = X_1 \cap \cdots \cap X_n$. By the derivation of $\preceq$, we have $u_1, \ldots, u_n \in min([\neg\phi], \preceq)$ such that $[X_i] = Cl_\cap([H] \cup \{u_i\})$ for $1 \leq i \leq n$. Thus $H\dot{-}_{\preceq}\phi = t_H(min([\neg\phi], \preceq) \cup [H]) = t_H(\{u_1, \ldots, u_n\} \cup [H]) = t_H(Cl_\cap([H] \cup \{u_1\}) \cup \cdots \cup Cl_\cap([H] \cup \{u_n\})) = t_H([X_1] \cup \cdots \cup [X_n]) = X_1 \cap \cdots \cap X_n$.

2). Let $\dot{-}_{\preceq}$ be a MHC determined by $\preceq$. Let $\leqslant = R(\preceq)$ and $\dot{-}_{\leqslant}$ be a TRPMHC determined by $\leqslant$. By Lemma 3, $\leqslant$ is a transitive R-relation. We need to show $H\dot{-}_{\leqslant}\phi = H\dot{-}_{\preceq}\phi$ for all $\phi \in \mathcal{L}_H$. We can easily show that MHC satisfies $(H\dot{-}3)$ and $(H\dot{-}f)$, thus for the special cases $\phi \notin H$ and $\phi \in Cn(\emptyset)$ the proof is similar to those in Lemma 4.

So suppose $\phi \in H$. By the construction of MHC, $H \dot{-}_{\preceq} \phi = t_H(min([\neg\phi], \preceq) \cup [H])$. Let $min([\neg\phi], \preceq) = \{u_1, \ldots, u_n\}$ then $H \dot{-}_{\preceq} \phi = t_H(\{u_1, \ldots, u_n\} \cup [H])$. By the definition of weak remainder set, we have $X_1, \ldots, X_n \in H \downarrow_w \phi$ such that $[X_i] = Cl_\cap([H] \cup \{u_i\})$ for $1 \leq i \leq n$. By the derivation of $\leqslant$, since $min([\neg\phi], \preceq) = \{u_1, \ldots, u_n\}$, $\{X_1, \ldots, X_n\} = max(H \downarrow \phi, \leqslant)$. Then by the construction of TRPMHC, we have $H \dot{-}_{\leqslant} \phi = \bigcap(max(H \downarrow \phi, \leqslant)) = X_1 \cap \cdots \cap X_n = t_H([X_1 \cap \cdots \cap X_n]) = t_H(Cl_\cap([H] \cup \{u_1\}) \cup \cdots \cup Cl_\cap([H] \cup \{u_n\})) = t_H(Cl_\cap(\{u_1, \ldots, u_n\} \cup [H])) = t_H(\{u_1, \ldots, u_n\} \cup [H])$.

□

Proof of **Lemma 5**

Suppose $X \in H \downarrow_w \phi$, $\psi \notin X$, and $X \cup \{\psi\} \nvdash \phi$, it suffices to show there is $Y \in H \downarrow_w \phi$ such that $X \subset Y$ and $Y \vdash \psi$. By the definition of weak remainder set, $[X] = Cl_\cap([H] \cup \{u\})$ for some $u \in [\neg\phi]$. $\psi \in H$ implies $[H] \subseteq [\psi]$. Assume $u \in [\psi]$, then $[H] \cup \{u\} \subseteq [\psi]$. Since ψ is a Horn formula, $Cl_\cap([H] \cup \{u\}) \subseteq [\psi]$. It then follows from $[X] = Cl_\cap([H] \cup \{u\})$ that $[X] \subseteq [\psi]$ which implies $X \vdash \psi$. But $X \vdash \psi$ contradicts the original assumption, hence $u \notin [\psi]$. $X \cup \{\psi\} \nvdash \phi$ implies $[X] \cap [\psi] \nsubseteq [\phi]$, thus there is $v \in [X] \cap [\psi]$ such that $v \in [\neg\phi]$. Also we have $[H] \cup \{v\} \subseteq [X]$ and $[H] \cup \{v\} \subseteq [\psi]$. By the Horn closure of $[\psi]$ and [X], we have $Cl_\cap([H] \cup \{v\}) \subseteq [\psi]$ and $Cl_\cap([H] \cup \{v\}) \subseteq [X]$. Let $[Y] = Cl_\cap([H] \cup \{v\})$ then by the definition of weak remainder set, $Y \in H \downarrow_w \phi$. $[Y] \subseteq [\psi]$ implies $Y \vdash \psi$. Since $Cl_\cap([H] \cup \{v\}) \subseteq [X]$, we have $[Y] \subseteq [X]$. Since $u \notin [\psi]$ and $[Y] \subseteq [\psi]$, we have $u \notin [Y]$. It then follows from $[Y] \subseteq [X]$, $u \notin [Y]$, and $u \in [X]$ that $[Y] \subset [X]$ which implies $X \subset Y$.

□

Proof of **Lemma 6**

Let H be a Horn belief, and γ be a maximised selection function for H. Suppose $X \in \gamma(H \downarrow_w \phi)$ and $\psi \in H \setminus X$ we need to show $X \cup \{\psi\} \vdash \phi$. Suppose to the contrary that $X \cup \{\psi\} \nvdash \phi$, then by Lemma 5 there is $Y \in H \downarrow_w \phi$ such that $X \subset Y$ and $\psi \in Y$. But γ is maximised and $X \subset Y$, thus γ should return Y instead of X, a contradiction.

□

Proof of **Lemma 7**

Transitivity and faithfulness are handled by Lemma 2 and Lemma 3, we only need to prove maximality. Let H be a Horn belief set.

1). Let $\leqslant$ be a transitive maximised R-relation over $H \downarrow_w \mathcal{L}_H$, and let $\preceq = I(\leqslant)$. We need to show $\preceq$ is a maximised I-relation for H. Suppose $u \in [H]$, $v, w \notin [H]$, and $u \cap v = w$, it suffices to show $w \prec v$. By the definition of weak remainder set, there are $X, Y \in H \downarrow_w \mathcal{L}_H$ such that $[X] = Cl_\cap([H] \cup \{v\})$ and $[Y] = Cl_\cap([H] \cup \{w\})$. It follows from $u \in [H]$ and $u \cap v = w$ that $w \in Cl_\cap([H] \cup \{v\})$ which implies $[Y] \subseteq [X]$. By the property of the closure operator and the fact that $u \cap v = w$ there is no $x \in U$ such that $x \cap w = u$, thus $v \notin Cl_\cap([H] \cup \{w\})$ which implies $[X] \nsubseteq [Y]$. Hence we have $[Y] \subset [X]$ which implies $X \subset Y$. It follows from the maximality of $\leqslant$ and $X \subset Y$ that $X < Y$. By the derivation of $\preceq$, we have $w \prec v$.

2). Let $\preceq$ be a faithful maximised I-relation for H and let $\leqslant = R(\preceq)$. We need to show $\leqslant$ is a maximised R-relation over $H \downarrow_w \mathcal{L}_H$. Suppose $X, Y \in H \downarrow_w \mathcal{L}_H$ and $X \subset Y$, it suffices to show $X < Y$. Without loss of generality, let's assume $[X] = Cl_\cap([H] \cup \{v\})$ and $[Y] = Cl_\cap([H] \cup \{w\})$. Then we have two cases:

Case $Y = H$: $w \in [H]$. It must be the case that $v \notin [H]$ for otherwise $X = H$ which contradicts $X \subset Y$. By the faithfulness of $\preceq$, we have $w \prec v$. Furthermore, by the derivation of $\leqslant$, we have $X < Y$.

Case $Y \neq H$: $w \notin [H]$. Since $X \subset Y$, we have $[Y] \subset [X]$ which implies $w \in Cl_\cap([H] \cup \{v\})$. Thus there is $u \in [H]$ such that $u \cap v = w$. By the maximality of $\preceq$, we have $w \prec v$. It then follows from the derivation of $\leqslant$ that $X < Y$.

□

Proof of **Lemma 8**

Let H be a Horn belief set.

For one direction, suppose $\preceq$ is a faithful, total and Horn strengthened I-relation for H and $\leq$ is an EE-relation induced by $\preceq$. By Theorem 3 of (Meyer, Labuschagne, and Heidema 2000) $\leq$ satisfies $(EE1)$–$(EE5)$. It remains to show $\leq$ satisfies $(EE6)$. It follows from $(EE3)$ that the entrenchment relation for a conjunction of non-Horn clauses is determined by its least entrenched conjunct. It is trivial to show that if the entrenchment relations for non-Horn clauses are compatible with $(EE6)$ then so is conjunctions of non-Horn clauses. Thus we only consider the entrenchment relations for single non-Horn clauses.

Let ϕ be a non-Horn clause. Since $[\phi] \neq Cl_\cap([\phi])$ there are $x_i, y_i \in [\phi]$ and $w_i \notin [\phi]$ such that $x_i \cap y_i = w_i$ for $1 \leq i \leq n$. Since $\preceq$ is Horn strengthened, either $w_i \preceq x_i$ or $w_i \preceq y_i$. Assume w.l.o.g. $w_i \preceq x_i$.

Let ψ be such that $[\psi] \subset [\phi]$ and $[\phi] \setminus [\psi] = \{x_1, \ldots, x_n\}$. We first show that $\psi \in \mathcal{HS}(\phi)$. $[\psi] = Cl_\cap([\psi])$ as there is no $x, y \in [\psi]$ and $w \notin [\psi]$ such that $x \cap y = w$, thus $\psi \in \mathcal{L}_H$. Let $\psi^{'}$ be such that $[\psi] \subset [\psi^{'}] \subseteq [\phi]$. Then there is a $x_i \in [\psi^{'}]$. As $x_i \cap y_i = w_i$ and $w_i \notin [\psi^{'}]$, $[\psi^{'}] \neq Cl_\cap([\psi^{'}])$ which implies $\psi^{'} \notin \mathcal{L}_H$. Thus $\psi \in \mathcal{HS}(\phi)$. It remains to show $\phi \leq \psi$.

By Definition 7, we have to show for all $y \in [\neg\psi]$ there is $x \in [\neg\phi]$ such that $x \preceq y$. $[\psi] \subset [\phi]$ implies $[\neg\phi] \subset [\neg\psi]$. There are two cases:

Case $y \in [\neg\phi]$: $y \preceq y$ follows from the reflexivity of $\preceq$.

Case $y \notin [\neg\phi]$: $y \in [\phi] \setminus [\psi]$. By our assumption, for each $y \in [\phi] \setminus [\psi]$ there is $x \in [\phi]$ such that $x \cap y = w$, $w \notin [\phi]$ and $w \preceq y$.

For the other direction, suppose $\leq$ is an EE-relation that satisfies $(EE1)$–$(EE6)$ and $\preceq$ is induced by $\leq$. We need to show $\preceq$ is a faithful, total and Horn strengthened I-relation for H. By Theorem 3 of (Meyer, Labuschagne, and Heidema 2000) $\preceq$ is a faithful total I-relation, it remains to show $\preceq$ is Horn strengthened.

Suppose $x \cap y = w$, we need to show either $w \preceq x$ or $w \preceq y$. Let ϕ be a non-Horn clause that contains all atoms in $\mathcal{L}$ and only two atoms are positive. Then $\neg\phi$ has only one model and we let it to be w. Since ϕ has two positive atoms, it has two Horn strengthenings. So suppose $\mathcal{HS}(\phi) = \{\psi, \mu\}$ and $\mu \leq \psi$. Due to $(EE6)$, we have $\phi \leq \psi$. By the property of Horn strengthening and the nature of ϕ, there are x, y such that $[\psi] \cup \{x\} = [\phi]$, $[\mu] \cup \{y\} = [\phi]$ and

$x \cap y = w$. So we have $[\neg\psi] = \{x, w\}$. Since $[\neg\phi] = \{w\}$, $w \preceq x$ follows from Definition 7 and $\phi \leq \psi$.

$\square$

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