Only-Knowing Meets Nonmonotonic Modal Logic

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Abstract

Only-knowing was originally introduced by Levesque to capture the beliefs of an agent in the sense that its knowledge base is all the agent knows. When a knowledge base contains defaults Levesque also showed an exact correspondence between only-knowing and autoepistemic logic. Later these results were extended by Lakemeyer and Levesque to also capture a variant of autoepistemic logic proposed by Konolige and Reiter's default logic. One of the benefits of such an approach is that various nonmonotonic formalisms can be compared within a single monotonic logic leading, among other things, to the first axiom system for default logic. In this paper, we will bring another large class of nonmonotonic systems, which were first studied by McDermott and Doyle, into the only-knowing fold. Among other things, we will provide the first possible-world semantics for such systems, providing a new perspective on the nature of modal approaches to nonmonotonic reasoning.

Introduction

When considering a knowledge-based agent, it seems natural to think of the beliefs[1] of the agent to be those that follow from the assumption that its knowledge base (KB) is *all* that is believed. Levesque [1990] was the first to capture this notion explicitly in his logic of *only-knowing*. One of the advantages of this approach is that beliefs can be analyzed in terms of the valid sentences of a logic without requiring additional meta-logical notions like fixed points or partial orders. This is done by using two modal operators in the language, $\boldsymbol{K}$ for belief, and $\boldsymbol{O}$ for only knowing. For example, in the logic proposed by Levesque, the sentence

$$\boldsymbol{O}(P(a) \vee P(b)) \ \supset \ \boldsymbol{K}(\exists x. P(x) \wedge \neg \boldsymbol{K} P(x))$$

is valid, which can be read as "if we only know that $P(a)$ or $P(b)$, then we know that something is a P, but not what."

Levesque also showed that, when the KB itself is allowed to mention $\boldsymbol{K}$, then $\boldsymbol{O}$ captures the autoepistemic logic (AEL) proposed by Moore [1985], in the sense that the beliefs entailed by only-knowing KB are precisely those which are in all stable expansions of KB. This connection made it possible to study autoepistemic reasoning within a classical monotonic logic, leading, among other things, to an axiomatic characterization of the logic in the propositional case, and a first-order account that handles quantifying-in.

In subsequent work by Lakemeyer and Levesque, henceforth called LL, [2005; 2006], only-knowing was extended to capture other forms of nonmonotonic reasoning, and in particular, the default logic (DL) proposed by Reiter [1980] and a variant of AEL due to Konolige [1988]. As described by Reiter, a default rule $\alpha : \beta / \gamma$ has an intuitive reading of "if α is believed and it is consistent to believe β then infer that γ is true." Hence Konolige proposed translating the default rule into a sentence of AEL of the form

$$\boldsymbol{K}\alpha \wedge \boldsymbol{M}\beta \ \supset \ \gamma.$$

In the simplest case, $\boldsymbol{M}$ is understood as the dual of $\boldsymbol{K}$ in the sense that $\boldsymbol{M}\beta$ stands for $\neg\boldsymbol{K}\neg\beta$. To properly characterize DL, however, a more complex treatment of the $\boldsymbol{M}$ is needed. Nonetheless, LL were able to present a variant of only-knowing that did the job and allowed the properties of DL to be understood in terms of an underlying model of belief in a classical monotonic logic: a model theory based on possible worlds, and later, a proof theory based on axioms and rules of inference [Lakemeyer and Levesque, 2005; 2006].[2]

In this paper, we continue this work and bring another large class of nonmonotonic systems into the only-knowing fold. We investigate the so-called *nonmonotonic modal systems* (NMS) first introduced by McDermott and Doyle [1980], and reconstructed by Marek *et al.* [1993]. Roughly speaking, an NMS starts with a classical modal system of belief (like the system **K** or **K45** or **T**, in the terminology of Chellas [1980]), and declares a set of formulas to be an expansion of α in the NMS if it consists of the formulas that can be derived in the modal system from α together with the assumptions $\neg\boldsymbol{K}\beta$ for those β that cannot be derived. Marek *et al.* show various properties of these NMS based on a variety of modal logics, including how different modal systems λ_1 and λ_2 can sometimes give rise to the same NMS (that is, where the λ_1-expansions coincide with the λ_2-expansions).

[1] In this paper, we use the terms "knowledge" and "belief" interchangeably to mean belief.

[2] We remark that other nonmonotonic logics with two distinct modalities were proposed that also capture DL such as [Lin and Shoham, 1990; Lifschitz, 1994]. See [Lakemeyer and Levesque, 2005] for a discussion how these relate to LL's work.

These NMS are certainly less popular in the research community than DL or AEL (not to mention answer set programming). They are also much more difficult to work with, with the possible exception of the one based on **K45** that aligns exactly with AEL. However, like the variant of AEL proposed by Konolige, they help shed light on nonmonotonic reasoning as a whole, and in seeing what is at issue in the various approaches to default reasoning. In particular, we will see that the NMS based on the modal system **K** is a virtually unstudied nonmonotonic system that has much to recommend it: it is very close to Reiter's DL, but arguably avoids one of the main drawbacks of DL.

The rest of the paper is organized as follows. We begin by reviewing nonmonotonic modal logic following [Marek et al., 1993]. We then introduce the logic $\mathcal{OKL}$, which provides a possible-world semantics for nonmonotonic logic **K** in terms of only-knowing. This is followed by a reformulation and generalization of the semantics, which has as a parameter a monotonic modal logic λ. After that we discuss in detail how only-knowing based on nonmonotonic logic **K** relates to only-knowing based on a variant of AEL proposed by Konolige and Reiter's default logic. Then we conclude.

Nonmonotonic Modal Logic

Marek *et al.* [1993] re-examined the nonmonotonic modal logics first introduced by McDermott and Doyle [1980]. The idea is that for any classical modal logic of belief λ, there is a notion of λ derivability (written $\vdash_\lambda$) from which a nonmonotonic logic can then be defined. By λ here we mean a standard modal logic which we name using names like **K**, **K4** and **K45** following the terminology of Chellas [1980]. By $\Gamma \vdash_\lambda \gamma$, we mean that γ can be derived in a classical way from Γ using the axioms and rules of propositional (or first-order) logic, the special axioms of the system λ, and the rule of necessitation: from α, infer $\boldsymbol{K}\alpha$.

For any modal system λ, Marek *et al.* define the nonmonotonic expansions of a formula as follows:

Definition 1 *For any modal system λ, a consistent set of formulas E is called a nonmonotonic λ-expansion of a formula α iff for every formula γ, the following condition holds:*

$$\gamma \in E \quad \textit{iff} \quad \{\alpha\} \cup \{\neg \boldsymbol{K}\beta \,|\, \beta \notin E\} \vdash_\lambda \gamma.$$

So a nonmonotonic expansion of a formula includes everything that can be derived from the formula and from the assumptions $\neg\boldsymbol{K}\beta$ for every β that cannot be derived. (It is this non-derivability that makes the definition nonmonotonic.) Within this framework, the autoepistemic logic of Moore [1985] can be characterized succinctly: what he calls the consistent stable expansions of a formula α are precisely the **K45**-expansions of α.

Marek *et al.* show various properties of nonmonotonic modal systems based on a variety of modal logics, including how different modal systems λ_1 and λ_2 can sometimes give rise to the same nonmonotonic system (that is, where the λ_1-expansions coincide with the λ_2-expansions).

One modal logic that *does* differ nonmonotonically from **K45** is the system **K**. The logic **K** is a minimal logic of belief in that it requires only that belief be closed under implication:

K : $\boldsymbol{K}\alpha \wedge \boldsymbol{K}(\alpha \supset \beta) \supset \boldsymbol{K}\beta$.

It says nothing about introspection. In fact, the system **K45** can be obtained from **K** by adding two axioms:

4 : $\boldsymbol{K}\alpha \supset \boldsymbol{K}\boldsymbol{K}\alpha$.
5 : $\neg\boldsymbol{K}\alpha \supset \boldsymbol{K}\neg\boldsymbol{K}\alpha$.

Marek *et al.* prove that the **K**-expansions are a proper subset of the **K45** ones (the stable expansions of AEL).

For example, let α be the formula $(\boldsymbol{K}p \supset p)$. This is the modal encoding of a (vacuous) rule that says that if p is already known then it can be concluded to be true. There is only one **K**-expansion of α: the set whose objective subset is just the classical tautologies. So nothing can be concluded in nonmonotonic **K** about p. This is what Reiter's default logic would do as well: there is only one extension of this rule in default logic and it consists of just the tautologies.

When we turn to **K45**, however, we get a second **K45**-expansion of α, one that includes p. Arguably, this is an undesirable property of nonmonotonic **K45**, but it is a direct consequence of its introspection: for the set E in question, we have that $\neg\boldsymbol{K}p \notin E$, and so $\neg\boldsymbol{K}\neg\boldsymbol{K}p \in E$, and therefore by introspection $\boldsymbol{K}p \in E$, which together with $(\boldsymbol{K}p \supset p)$ allows us to conclude p. The key step, from $\neg\boldsymbol{K}\neg\boldsymbol{K}p$ to $\boldsymbol{K}p$, is not available in the system **K**.

So it appears that nonmonotonic **K** is a more restrained version of **K45**, perhaps better suited as a basis for general default reasoning. In fact, we will show that it is actually very close to Reiter's default logic! But our first job is to establish a connection between it and only-knowing.

A Semantics for Nonmonotonic System K

Let us now turn to a logic which we call $\mathcal{OKL}$ and which gives meaning to a new only-knowing operator which will be shown to capture nonmonotonic **K**. The symbols of the language of $\mathcal{OKL}$ are the usual logical connectives, punctuation, a countably infinite set of propositional variables (or atomic propositions), and the modal operators $\boldsymbol{K}$,$\boldsymbol{O}_M$, and $\boldsymbol{O}^K$. Here, to be consistent with LL, we use $\boldsymbol{O}_M$ rather than $\boldsymbol{O}$ to refer to only-knowing in the sense of Moore. The $\boldsymbol{O}^K$ operator is our new variant of only-knowing.[3] The *formulas* of $\mathcal{OKL}$ are defined by the following:

1. every propositional variable is a formula;
2. if α and β are formulas, then $\neg\alpha$, $(\alpha \wedge \beta)$ are formulas, as are the *modal* formulas, $\boldsymbol{K}\alpha$, $\boldsymbol{O}_M\alpha$, and $\boldsymbol{O}^K\alpha$ with the restriction that $\boldsymbol{O}_M$ and $\boldsymbol{O}^K$ are only applied to formulas whose only modal operator is $\boldsymbol{K}$.

As usual, we treat $(\alpha \vee \beta)$, $(\alpha \supset \beta)$, and $(\alpha \equiv \beta)$ as abbreviations. We will also write $\boldsymbol{M}\alpha$ instead of $\neg\boldsymbol{K}\neg\alpha$. Formulas without modal operators are called *objective*, and those where all the propositional variables appear in the scope of a modal operator are called *subjective*. Formulas whose only modal operator is $\boldsymbol{K}$ are called *basic*. A basic formula without nested occurrences of $\boldsymbol{K}$ is called *flat*.

[3]In the coming sections, we will introduce a number of other only-knowing operators $\boldsymbol{O}^{\boldsymbol{\lambda}}$ for various modal systems λ, but $\boldsymbol{O}^K$ (for system **K**) and $\boldsymbol{O}_M$ (for Moore) will do for now.

The semantics of $\mathcal{OKL}$ builds directly on the semantics of $\mathcal{OL}$ from LL [2001]. The starting point is the notion of a *valuation* (or world), which is a mapping from the set of propositional variables into $\{0, 1\}$, and an epistemic state, which consists of a set of valuations. We let e_0 be the set of all valuations.

Let v be a valuation, and e an epistemic state. Then the satisfaction relation $e, v \models \alpha$ is defined as follows:

1. $e, v \models p$ iff $w[p] = 1$ for atomic propositions p;
2. $e, v \models \neg\alpha$ iff $e, v \not\models \alpha$;
3. $e, v \models (\alpha \wedge \beta)$ iff $e, v \models \alpha$, and $e, v \models \beta$;
4. $e, v \models \boldsymbol{K}\alpha$ iff for every v', if $v' \in e$, then $e, v' \models \alpha$;
5. $e, v \models \boldsymbol{O}_M\alpha$ iff for every v', $v' \in e$ iff $e, v' \models \alpha$.

Notice that the meaning of $\boldsymbol{O}_M$ is simply a strengthening of $\boldsymbol{K}$, replacing *if* by an *iff*.

To give meaning to the remaining operator $\boldsymbol{O}^K$, we need to define a superset of an epistemic state e for a given basic formula α.

Definition 2 *Let α be a basic formula. Then*

$e_\alpha = \bigcup e'$, *where*
$e' \supseteq e$ *and* $\forall v \in e' \exists e^*$ *s.t.* $e \subseteq e^* \subseteq e'$ *and* $e^*, v \models \alpha$.

Roughly, the purpose of e_α is to capture those beliefs which are forced when α is believed and when the beliefs are closed only under **K** and necessitation. The semantics of $\boldsymbol{O}^K$ is then simply this:

6. $e, v \models \boldsymbol{O}^K\alpha$ iff $e, v \models \boldsymbol{K}\alpha$ and $e = e_\alpha$.

To complete the specification of the logic, we say that α is *valid* (which we write as $\models \alpha$) iff $e, v \models \alpha$ for every e and v. If α is objective, we sometimes write $v \models \alpha$; if α is subjective, we sometimes write $e \models \alpha$.

$\mathcal{OKL}$ and nonmonotonic system K

When the language is restricted to propositional variables, Boolean connectives, and the modalities $\boldsymbol{K}$ and $\boldsymbol{O}_M$, the logic coincides precisely with $\mathcal{OL}$ of LL [2001]. In particular, $\boldsymbol{K}$ is a **K45**-operator and $\boldsymbol{O}_M$ precisely captures **K45**-expansions.

Definition 3 *Given an epistemic state e, the set of formulas*

$$E(e) = \{\gamma \mid \gamma \text{ is basic and } e \models \boldsymbol{K}\gamma\}$$

is called the belief set *of e.*

Theorem 1 (Levesque) *The consistent* **K45**-*expansions of α are precisely the belief sets of those epistemic states $e \neq \{\}$ satisfying $e \models \boldsymbol{O}_M\alpha$.*

Let us now turn to the properties of $\boldsymbol{O}^K$. We begin with a useful lemma which states that e_α satisfies a simple fix-point equation:

Lemma 1 *For every basic α, if $e \models \boldsymbol{K}\alpha$ then*

$$e_\alpha = \{v \mid \exists e^* \text{ s.t. } e \subseteq e^* \subseteq e_\alpha \text{ and } e^*, v \models \alpha\}.$$

Proof: Let $\widetilde{e} = \{v \mid \exists e^* \text{ s.t. } e \subseteq e^* \subseteq e_\alpha \text{ and } e^*, v \models \alpha\}$. We need to show that $e_\alpha = \widetilde{e}$.

To prove $e_\alpha \subseteq \widetilde{e}$, let $v \in e_\alpha$. Then $v \in e'$ for some $e' \supseteq e$ and there is an e^* such that $e \subseteq e^* \subseteq e'$ and $e^*, v \models \alpha$. Since $e' \subseteq e_\alpha$, we also have $e \subseteq e^* \subseteq e_\alpha$ and, hence, $v \in \widetilde{e}$.

To prove $\widetilde{e} \subseteq e_\alpha$, let $\widetilde{v} \in \widetilde{e}$. Now consider $e_\alpha \cup \{\widetilde{v}\}$. By the first part of the proof and since $\widetilde{v} \in \widetilde{e}$ we have that for all $v \in e_\alpha \cup \{\widetilde{v}\}$ there is an e^* such that $e \subseteq e^* \subseteq e_\alpha \cup \{\widetilde{v}\}$ and $e^*, \widetilde{v} \models \alpha$. Then, by Definition 2, $e_\alpha \cup \{\widetilde{v}\} \subseteq e_\alpha$, that is, $\widetilde{v} \in e_\alpha$. ∎

The following theorem, together with Theorem 1, says that the consistent belief sets sanctioned by $\boldsymbol{O}^K$ are also **K45**-expansions.

Theorem 2 $\models \boldsymbol{O}^K\alpha \supset \boldsymbol{O}_M\alpha$.

Proof: Let $e \models \boldsymbol{O}^K\alpha$. We need to show that $e \models \boldsymbol{O}_M\alpha$. Since $e \models \boldsymbol{K}\alpha$ by assumption, it suffices to show that for all v, if $e, v \models \alpha$, then $v \in e$. So let $e, v \models \alpha$. Then, by Lemma 1, $v \in e_\alpha$. Since $e_\alpha = e$ by assumption, $v \in e$. ∎

When α is objective it is easy to see that the two notions of only-knowing coincide.

Theorem 3 *For objective α, $\models \boldsymbol{O}^K\alpha \equiv \boldsymbol{O}_M\alpha$.*

Proof: It suffices to show that $\models \boldsymbol{O}_M\alpha \supset \boldsymbol{O}^K\alpha$. So let $e \models \boldsymbol{O}_M\alpha$. Since α is objective, $e = \{v \mid v \models \alpha\}$. Since no world outside of e satisfies α, we clearly have $e = e_\alpha$ and we are done. ∎

To see that the two differ for non-objective α, we consider two examples.

Example 1 *Let $\alpha = \boldsymbol{K}p \supset p$ for some atomic proposition p. There are exactly two* **K45**-*expansions of α, one which believes p and one which does not. With Theorem 1, we have that the only epistemic states e such that $e \models \boldsymbol{O}_M\alpha$ are $e = e_0$ (the set of all words) and $e = e_p$, where $e_p = \{v \mid v \models p\}$. It is easy to see that $e_\alpha = e_0$ since $e_0 \models \neg\boldsymbol{K}p$ and hence, $e_0, v \models \alpha$ for all v. Thus $e_0 \models \boldsymbol{O}^K\alpha$ yet $e_p \not\models \boldsymbol{O}^K\alpha$, that is, $\not\models \boldsymbol{O}_M\alpha \supset \boldsymbol{O}^K\alpha$.*

The following example demonstrates that, even in the case when there is a unique epistemic state which only-knows α according to Moore, this epistemic state may be rejected by $\boldsymbol{O}^K$.

Example 2 *Let α be the conjunction of these formulas:*

$$\boldsymbol{K}p \supset q$$
$$\boldsymbol{K}q \supset p$$
$$\boldsymbol{M}\neg p \supset p$$
$$\boldsymbol{M}\neg q \supset q$$

First note that any epistemic state which believes α also believes both p and q, that is, $\models \boldsymbol{K}\alpha \supset \boldsymbol{K}(p \wedge q)$. This is a direct consequence of introspection since $\boldsymbol{K}(\boldsymbol{M}\neg p \supset p)$, which is the same as $\boldsymbol{K}(\boldsymbol{K}p \vee p)$, is logically equivalent to $\boldsymbol{K}p$ in **K45**, *and similarly for $\boldsymbol{K}(\boldsymbol{M}\neg q \supset q)$. As a consequence, it is not hard to show that $e_{pq} = \{v \mid v \models p \wedge q\}$ is the unique epistemic state such that $e_{pq} \models \boldsymbol{O}_M\alpha$.*

But intuitively, when reading α as defaults, there really is no good reason to believe p or to believe q. (Why should *one believe p on the basis of $\neg p$ being consistent?) And indeed, this epistemic state is rejected under $\boldsymbol{O}^K$ because* $e_\alpha = \{v \mid v \models (p \vee q)\}$.

As we need them for the upcoming main result of this section and the next, we now briefly review Kripke structures. A Kripke structure M is a triple $\langle W, R, \pi\rangle$, where W is a non-empty set (of worlds), R is an accessibility relation, and π is a mapping from W into the set of valuations. Given an $M = \langle W, R, \pi\rangle$ and $w \in W$, (M, w) is called a (pointed) Kripke model. The truth of basic formulas is then defined in the usual way:

1. $M, w \models p$ iff $\pi(w)(p) = 1$;
2. $M, w \models \neg\alpha$ iff $M, w \not\models \alpha$;
3. $M, w \models \alpha \wedge \beta$ iff $M, w \models \alpha$ and $M, w \models \beta$;
4. $M, w \models \boldsymbol{K}\alpha$ iff $\forall w'$, if $(w, w') \in R$ then $M, w' \models \alpha$.

Note that the main difference between Kripke structures and the much simpler, valuation-based semantics introduced earlier is that Kripke structures allow for different worlds whose valuations are the same and, of course, rather than having a globally accessible set of worlds e, R can be arbitrary. Indeed, different modal logics are obtained by restricting the accessibility relation in various ways. For example, without any restrictions, we obtain the logic **K**; if R is reflexive, we obtain **T**; if R is transitive and Euclidean, we obtain **K45**. For a given modal logic λ, logical implication between a set of basic formulas S and a basic formula α is denoted as $S \models_\lambda \alpha$. Kripke models of a modal logic λ are also referred to as λ-models.

In the following we will need to relate models of our evaluation-based semantics and Kripke structures. In both cases we will use $\models$ for satisfaction. It will be clear from the context which type of semantics is meant.

For the following result relating $\boldsymbol{O}^K$ to **K**-expansions, we need to restrict ourselves to flat basic formulas as arguments of $\boldsymbol{O}^K$. Note that flat formulas include the translations of default theories as discussed in the introduction. Hence they arguably cover the most important class of theories from a nonmonotonic-reasoning perspective. The main result is that with this restriction, our definition of $\boldsymbol{O}^K$ correctly captures the nonmonotonic modal system **K** in a way that is exactly parallel to Theorem 1 above:

Theorem 4 *For any flat basic α, the consistent* **K**-*expansions of α are precisely the belief sets of those epistemic states $e \neq \{\}$ satisfying $e \models \boldsymbol{O}^K\alpha$.*

Proof: The proof needs to appeal to Kripke structures and is quite involved. Here we only outline the main argument. We need to prove two things: (1) for any non-empty e such that $e \models \boldsymbol{O}^K\alpha$, its belief set $E(e)$ is a $\boldsymbol{K}$-expansion of α, and (2) for any **K**-expansion E of α, there is an e such that $e \models \boldsymbol{O}^K\alpha$ and $E = E(e)$.

To prove (1), let $e \models \boldsymbol{O}^K\alpha$. Since it is well known that belief sets and expansions are stable sets [Stalnaker, 1993], which are uniquely determined by their objective subsets, it suffices to show that for all objective ϕ, $e \models \boldsymbol{K}\phi$ iff $\{\alpha\} \cup \{\neg\boldsymbol{K}\beta \mid e \models \neg\boldsymbol{K}\beta\} \vdash_{\mathbf{K}} \phi$. This can be shown to be equivalent to $e \models \boldsymbol{K}\phi$ iff $F(\alpha, e) \models_{\mathbf{K}} \phi$, where $F(\alpha, e) = \{\boldsymbol{K}^*\gamma \mid \gamma = \alpha \text{ or } \gamma \in \{\neg\boldsymbol{K}\beta \mid \beta \notin E(e)\}\}$, $\boldsymbol{K}^*\gamma$ means γ preceded by an arbitrary number of $\boldsymbol{K}$ operators, and $\models_{\mathbf{K}}$ is entailment in logic **K**. If $e \not\models \boldsymbol{K}\phi$ then $\neg\boldsymbol{K}\phi \in F(\alpha, e)$ and, hence, $F(\alpha, e) \not\models_{\mathbf{K}} \phi$. Now suppose $e \models \boldsymbol{K}\phi$ and let $M = \langle W, R, \pi\rangle$ be an arbitrary **K**-Kripke structure such that $M, w^* \models F(\alpha, e)$ for some $w^* \in W$. It is then possible to show that $M, w^* \models \phi$ using the assumption that $e \models \boldsymbol{O}^K\alpha$ and the fact that $\pi(w^*) \in e_\alpha$ and, hence, $\pi(w^*) \in e$.

To prove (2), let E be a **K**-expansion. Since **K**-expansions are also **K45**-expansions [Marek et al., 1993], by Theorem 1, there must be an e such that $e \models \boldsymbol{O}_M\alpha$ and, in particular, $e \models \boldsymbol{K}\alpha$. To show that $e \models \boldsymbol{O}^K\alpha$, it then suffices to prove that $e_\alpha \subseteq e$. So let $v^* \in e_\alpha$. Then there is some e^* such that $e \subseteq e^* \subseteq e_\alpha$ and $e^*, v^* \models \alpha$. It is then possible to construct a Kripke structure $\langle W, R, \pi\rangle$ with a world $w^* \in W$ such that $M, w^* \models F(\alpha, e)$ and $v^* = \pi(w^*)$. By assumption, for all objective ϕ, $e \models \boldsymbol{K}\phi$ iff $F(\alpha, e) \models_{\mathbf{K}} \phi$. Thus for all objective ϕ, if $e \models \boldsymbol{K}\phi$ then $M, w^* \models \phi$ and, therefore, $v^* \models \phi$. Since $e \models \boldsymbol{O}_M\alpha$, e is the set of all worlds which satisfy all objective formulas in $E(e)$. Hence $v^* \in e$. ∎

A Unified Semantic Framework for $\boldsymbol{O}^\lambda$

An appealing feature of the original proposal for nonmonotonic modal systems is that different NMS are obtained by varying only the underlying monotonic modal logic and keeping the definition of expansions otherwise fixed. In our semantic reconstruction, this modularity seems to be lost somehow since the definitions for $\boldsymbol{O}_M$ and $\boldsymbol{O}^K$ are quite different in nature. In this section we will propose an alternative semantics of only-knowing which also appeals to an underlying modal logic λ and is uniform otherwise, thus giving rise to different forms of only-knowing $\boldsymbol{O}^\lambda$ only by varying λ. As we will see, we again get a correspondence with the respective NMS in the case of $\lambda = \mathbf{K}$ and **K45**, which also extends to **K5**. The investigation of other modal systems is left for future work.

As we will see in a moment, the semantics of $\boldsymbol{O}^\lambda$ appeals to both the evaluation-based semantics of $\mathcal{OKL}$ and Kripke structures. In particular, for basic formulas, $e, v \models \alpha$ refers to satisfaction in $\mathcal{OKL}$ for a given valuation v and set of valuations e; $M, w \models \alpha$ refers to satisfaction in the modal logic λ for a given λ-model (M, w).

We will write $R(w)$ to mean $\{w' \mid (w, w') \in R\}$. If $S \subseteq W$ then $R(S)$ stands for $\bigcup_{w' \in S} R(w')$. As before, we assume that $\boldsymbol{O}^\lambda$ is only applied to basic formulas.

The following definition of Kripke structures *based on* a given set of valuations is key to our upcoming definition of $\boldsymbol{O}^\lambda$.

Definition 4 *Let e be a set of valuations. A λ-model (M, w) with $M = \langle W, R, \pi\rangle$ and $w \in W$ is called based on e iff the following conditions are satisfied:*

1. $W = \{w\} \cup R(w)$;
2. *for all* $w' \in W$, $\pi(R(w')) \supseteq e$;
3. *for all* $w' \in R(w)$, *if* $\pi(w') \in e$ *then* $\pi(R(w')) = e$;
4. $\pi(R(R(w))) \subseteq \pi(R(w))$.

Some Remarks: (1) is a rather tight constraint on the set of worlds considered in a λ-model. It is possible that models with less constrained sets of worlds will work as well, but the current constraint simplifies proofs and suffices for the purposes of this paper. Conditions (2) and (3) are inspired by our definition of e_α, where worlds in e_α can only access a set of worlds which is a superset of e and worlds in e can only access all and only worlds in e. Condition (4) is needed to have the same effect as necessitation in NMS.

The semantics of $\boldsymbol{O}^{\boldsymbol{\lambda}}\alpha$ for a given valuation v and set of valuations e is then defined as follows.

- $e, v \models \boldsymbol{O}^{\boldsymbol{\lambda}}\alpha$ iff $e, v \models \boldsymbol{K}\alpha$ and
  for all λ-models (M, w) based on e s.t. $M, w \models \boldsymbol{K}\alpha$,
  for all $w' \in W$, if $M, w' \models \alpha$ then $\pi(w') \in e$.

We will now establish the correspondence with NMS for several λ starting with $\lambda = \mathbf{K}$. As we want to compare the new definition of $\boldsymbol{O}^{K}$ with the old one of the previous section, we will write $\boldsymbol{O}^{K'}$ whenever referring to the old definition in this section.

Three lemmas are needed for the main theorem. The first lemma is straightforward and says that a **K**-model and its valuation-based counterpart, where only the directly accessible worlds are considered, satisfy exactly the same flat basic formulas. (All proofs are relegated to the appendix.)

Lemma 2 *For a given* **K***-model* (M, w)*. Then for every flat basic* α*,* $M, w \models \alpha$ *iff* $\pi(R(w)), \pi(w) \models \alpha$*.*

The following two lemmas establish the crucial connection between e_α on the one hand and **K**-models based on e on the other: given a **K**-model (M, w) based on e, the worlds accessible from w are members of e_α, and there is a model where the accessible worlds are exactly e_α.

Lemma 3 *Let* α *be flat and* (M, w) *based on* e *such that* $e \models \boldsymbol{K}\alpha$ *and* $M, w \models \boldsymbol{K}\alpha$*. Then* $\pi(R(w)) \subseteq e_\alpha$*.*

Lemma 4 *Let* α *be flat and* $e \models \boldsymbol{K}\alpha$*. Then there exists a* (M, w) *based on* e *s.t.* $\pi(R(w)) = e_\alpha$ *and* $M, w \models \boldsymbol{K}\alpha$*.*

With these properties it is not hard to establish the main result (see appendix).

Theorem 5 *For any basic flat* α*,* $e \models \boldsymbol{O}^{K'}\alpha$ *iff* $e \models \boldsymbol{O}^{K}\alpha$*.*

Next we show the connection between $\boldsymbol{O}^{K45}$ and $\boldsymbol{O}_M$. It turns out that they coincide for all e but the empty set. The following two lemmas are key to proving the result. They essentially say that a set of valuations e and any **K45**-model based on e agree on all basic beliefs.

Lemma 5 *Let* $e \neq \{\}$ *and let* (M, w) *be a* **K45***-model based on e. Then for all* $w' \in W$*,* $\pi(R(w')) = e$*.*

Lemma 6 *Let* (M, w) *be a* **K45***-model based on* $e \neq \{\}$ *and let* $w' \in W$*. Then* $M, w' \models \alpha$ *iff* $e, \pi(w') \models \alpha$ *for all basic* α*.*

Theorem 6 *If* $e \neq \{\}$ *then* $e \models \boldsymbol{O}_M\alpha$ *iff* $e \models \boldsymbol{O}^{K45}\alpha$*.*

With Theorem 1, we then immediately get

Corollary 1 *The consistent* **K45***-expansions of* α *are precisely the belief sets of those epistemic states* $e \neq \{\}$ *satisfying* $e \models \boldsymbol{O}^{K45}\alpha$*.*

In other words, our definition of $\boldsymbol{O}^{\boldsymbol{\lambda}}$ also does the right thing for $\lambda = \mathbf{K45}$. Interestingly, $\boldsymbol{O}_M$ and $\boldsymbol{O}^{K45}$ differ in the case of $e = \{\}$.

Example 3
Let $e = \{\}$*. Then* $e \models \boldsymbol{O}_M\neg\boldsymbol{K}p$ *yet* $e \not\models \boldsymbol{O}^{K45}\neg\boldsymbol{K}p$*.*

$e \models \boldsymbol{O}_M\neg\boldsymbol{K}p$ holds because clearly $e \models \boldsymbol{K}\neg\boldsymbol{K}p$ and there is no valuation v such that $e, v \models \neg\boldsymbol{K}p$, since the empty epistemic state believes every formula.

To show that $e \not\models \boldsymbol{O}^{K45}\neg\boldsymbol{K}p$, consider the following **K45**-model (M, w):

- Let $W = \{w\} \cup e_0$ such that $\pi(v) = v$ for all $v \in e_0$, the set of all valuations and $\pi(w)$ arbitrary.
- Let $R = (\{w\} \times e_0) \cup (e_0 \times e_0)$.

It is easy to verify that (M, w) is based on e. We also clearly have $M, w \models \neg\boldsymbol{K}p$ as w considers all valuations possible, yet $w \notin e$.

Note though that $e_0 \models \boldsymbol{O}^{K45}\neg\boldsymbol{K}p$, which follows from Theorem 6 and the fact that $e_0 \models \boldsymbol{O}_M\neg\boldsymbol{K}p$. In a sense then, for examples like these, $\boldsymbol{O}^{K45}$ seems already a little better behaved than $\boldsymbol{O}_M$ by ruling out some unwanted expansions.

As a final result of this section, we obtain that $\boldsymbol{O}^{K5}$ and $\boldsymbol{O}^{K45}$ agree completely.

Theorem 7 $\models \boldsymbol{O}^{K45}\alpha \equiv \boldsymbol{O}^{K5}\alpha$*.*

The proof is actually trivial since, given the conditions of Definition 4, it is easy to see that **K45**-models based on e are the same as **K5**-models based on e.

Again, this result is reassuring since Marek et al. [1993] already showed that nonmonotonic **K45** and **K5** are identical.

Relating $\boldsymbol{O}^{K}$ to Konolige and Reiter

Two other variants of only-knowing, $\boldsymbol{O}_{Ko}$ and $\boldsymbol{O}_R$, were considered by LL [2005; 2006]. $\boldsymbol{O}_{Ko}$ captures Konolige's moderately grounded **K45**-expansions [Konolige, 1988] and $\boldsymbol{O}_R$ captures default extensions in the sense of Reiter [1980]. As we will see, our new variant of only-knowing $\boldsymbol{O}^{K}$ relates to these in interesting ways.

As was shown in [Lakemeyer and Levesque, 2005], $\boldsymbol{O}_R\alpha$ lines up with default extensions only in case α is in so-called Reiter Normal Form (RNF), which is a conjunction of formulas of the form

$$\boldsymbol{K}\phi \wedge \boldsymbol{M}\psi_1 \wedge \cdots \wedge \boldsymbol{M}\psi_n \supset \chi,$$

where ϕ, ψ_i, and χ are themselves objective. (Note that α in RNF is a flat basic formula.)

For this reason and for the purpose of comparison we assume, unless noted otherwise, that both $\boldsymbol{O}_{Ko}$ and $\boldsymbol{O}_R$ are only applied to formulas α in RNF, that is modal variants of default theories. Let us begin with the semantics of $\boldsymbol{O}_{Ko}$:[4]

- $e, v \models \boldsymbol{O}_{Ko}\alpha$ iff $e, v \models \boldsymbol{O}_M\alpha$ and for every e'
  such that $e \subseteq e'$, if $e', v \models \boldsymbol{O}_M\alpha$ then $e = e'$.

[4] $\boldsymbol{O}_{Ko}$ and $\boldsymbol{O}_R$ are defined as extensions of the evaluation-based semantics we introduced initially for $\mathcal{OKL}$.

We clearly have $\models \boldsymbol{O}_{Ko}\alpha \supset \boldsymbol{O}_{M}\alpha$, but in addition, if $e \models \boldsymbol{O}_{Ko}\alpha$ then no proper superset only-knows α in the sense of Moore. In other words, the $e's$ in question are the most ignorant ones which only-know α according to Moore. For example, if $\alpha = (\boldsymbol{K}p \supset p)$ then we have that $e_0 \models \boldsymbol{O}_{Ko}\alpha$ but $e_p \not\models \boldsymbol{O}_{Ko}\alpha$ since $e_p \subseteq e_0$.

While $\boldsymbol{O}_{Ko}$ does a reasonable job in eliminating some ungrounded **K45**-expansions, as the previous example demonstrates, it still has problems. For consider

$$\alpha = (\boldsymbol{K}p \supset p) \wedge (\boldsymbol{M}\neg p \supset q).$$

Here both $e_p \models \boldsymbol{O}_{Ko}\alpha$ and $e_q \models \boldsymbol{O}_{Ko}\alpha$. Because of cases like these, Konolige called the resulting expansions *moderately grounded.* He then went on to define *strongly grounded* expansions, which dealt with this example correctly, but the definition was cumbersome as it was syntax-dependent. As was already observed by Shvarts [1990] and Marek et al. [1993], **K**-expansions and, hence, $\boldsymbol{O}^{K}$ also do the right thing, but in a much more elegant way.

We have just seen an example showing that $(\boldsymbol{O}_{Ko}\alpha \supset \boldsymbol{O}^{K}\alpha)$ is not valid in general. However, in case α does not mention $\boldsymbol{M}$, the implication holds.

Theorem 8 *Let α be in RNF not mentioning $\boldsymbol{M}$. Then $\models \boldsymbol{O}_{Ko}\alpha \supset \boldsymbol{O}^{K}\alpha$.*

Proof: Let $e \models \boldsymbol{O}_{Ko}\alpha$ and suppose $e \not\models \boldsymbol{O}^{K}\alpha$. Then $e \neq e_\alpha$, that is, $e_\alpha \supsetneq e$. We show that $e_\alpha \models \boldsymbol{O}_{M}\alpha$, contradicting the assumption that $e \models \boldsymbol{O}_{Ko}\alpha$. Wlog let $\alpha = \bigwedge \delta_i$ with $\delta_i = (\boldsymbol{K}\phi_i \supset \chi_i)$. Let $v \in e_\alpha$. Then for some e^*, $e \subseteq e^* \subseteq e_\alpha$ and $e^*, v \models \delta_i$ for all i. Since $e^* \subseteq e_\alpha$, we also obtain that $e_\alpha, v \models \delta_i$ for all i, that is, $e_\alpha \models \boldsymbol{K}\alpha$. Now let $e_\alpha, v \models \alpha$ for an arbitrary v. By Lemma 1, $v \in e_\alpha$, from which $e_\alpha \models \boldsymbol{O}_{M}\alpha$ follows. ∎

The converse, on the other hand, holds in general.

Theorem 9 *Let α be in RNF. Then $\models \boldsymbol{O}^{K}\alpha \supset \boldsymbol{O}_{Ko}\alpha$.*

Proof: Suppose otherwise. Then for some e, $e \models \boldsymbol{O}^{K}\alpha$ and $e \not\models \boldsymbol{O}_{Ko}\alpha$. Since $e \models \boldsymbol{O}_{M}\alpha$ by Theorem 2, there is an $e' \supsetneq e$ such that $e' \models \boldsymbol{O}_{M}\alpha$. Then for all $v' \in e'$, $e', v' \models \alpha$. Thus for all $v' \in e'$ there is an e^* such that $e \subseteq e^* \subseteq e'$ and $e^*, v' \models \alpha$. Hence $e' \subseteq e_\alpha$, contradicting the assumption that $e = e_\alpha$. ∎

Let us now turn to Reiter's default logic. To define the semantics for the corresponding only-knowing operator $\boldsymbol{O}_{R}$, let α/e denote α with all occurrences of every subformula $\boldsymbol{M}\beta$ (ie. $\neg\boldsymbol{K}\neg\beta$) replaced by TRUE if $e \models \boldsymbol{M}\beta$ and by FALSE otherwise. For example, let $\alpha = p \wedge (\boldsymbol{K}p \wedge \boldsymbol{M}q \supset q) \wedge (\boldsymbol{M}\neg q \supset \neg q)$ and $e = e_{pq}$. Then $\alpha/e = p \wedge (\boldsymbol{K}p \wedge \text{TRUE} \supset q) \wedge (\text{FALSE} \supset \neg q)$.

Given an epistemic state e, we can then define $\boldsymbol{O}_{R}$ as follows:

- $e, v \models \boldsymbol{O}_{R}\alpha$ iff $e, v \models \boldsymbol{O}_{Ko}\alpha/e$.

In other words, only-knowing α according to Reiter reduces to only-knowing in the sense of Konolige after replacing all occurrences of $\boldsymbol{M}\beta$ by their respective truth values. We remark that the original definition of $\boldsymbol{O}_{R}$ by LL [2006] differs from this one in that they introduce $\boldsymbol{M}$ not as a shorthand for $\neg\boldsymbol{K}\neg$ but as a separate modal operator with $\boldsymbol{K}$ being interpreted over e_1 and $\boldsymbol{M}$ over e_2. In the interpretation of $\boldsymbol{O}_{R}$, they keep e_2 fixed while e_1 can vary. In other words, $\boldsymbol{K}$ and $\boldsymbol{M}$ are no longer duals as far as $\boldsymbol{O}_{R}$ is concerned. It is an easy exercise to show that for formulas in RNF our way of substituting $\boldsymbol{M}$-subformulas by truth values amounts to the same as LL's approach. (In particular, we inherit the precise connection proven by LL between $\boldsymbol{O}_{R}$ and default extensions as originally defined.)

To see how $\boldsymbol{O}_{R}$ relates to $\boldsymbol{O}^{K}$ and $\boldsymbol{O}_{Ko}$, let us again reconsider some of our previous examples:

- Let $\alpha = (\boldsymbol{K}p \supset p)$. Since this α contains no $\boldsymbol{M}$'s we have $\alpha/e = \alpha$ and thus $\boldsymbol{O}_{R}$ agrees with $\boldsymbol{O}_{Ko}$ and $\boldsymbol{O}^{K}$ here.
- Let $\alpha = (\boldsymbol{K}p \supset p) \wedge (\boldsymbol{M}\neg p \supset q)$. Then we get that $e_q \models \boldsymbol{O}_{R}\alpha$ since α/e_q is logically equivalent to $(\boldsymbol{K}p \supset p) \wedge q$ and e_q is the only epistemic state which only knows this. However, $e_p \not\models \boldsymbol{O}_{R}\alpha$ because α/e_p is equivalent to $(\boldsymbol{K}p \supset p)$ and $\boldsymbol{O}_{Ko}$ rejects e_p for this formula. Hence $\boldsymbol{O}_{R}$ differs from $\boldsymbol{O}_{Ko}$ but agrees with $\boldsymbol{O}^{K}$ here.
- Let α be the formula of Example 2. As we saw, e_{pq} is the unique epistemic state such that $e_{pq} \models \boldsymbol{O}_{M}\alpha$. But $\boldsymbol{O}_{R}$ rejects e_{pq} because α/e_{pq} replaces both $\boldsymbol{M}$-subformulas by FALSE and is thus equivalent to $(\boldsymbol{K}p \supset q) \wedge (\boldsymbol{K}q \supset p)$, which has no stable expansion. Again, $\boldsymbol{O}_{R}$ agrees with $\boldsymbol{O}^{K}$ but differs from $\boldsymbol{O}_{Ko}$.

In general, we have:

Theorem 10 *Let α be in RNF. Then $\models \boldsymbol{O}_{R}\alpha \supset \boldsymbol{O}^{K}\alpha$.*

Proof: Suppose otherwise, that is, let $e \models \boldsymbol{O}_{R}\alpha$ and $e \not\models \boldsymbol{O}^{K}\alpha$. Since $e \models \boldsymbol{K}\alpha$, this means that $e \neq e_\alpha$. Since $e \models \boldsymbol{O}_{R}\alpha$, we have $e \models \boldsymbol{O}_{Ko}\alpha'$ for $\alpha' = \alpha/e$. Since α' no longer mentions $\boldsymbol{M}$, by Theorem 8, $e \models \boldsymbol{O}^{K}\alpha'$, from which $e = e_{\alpha'}$ follows. Since, by assumption, $e \neq e_\alpha$, $e_{\alpha'} \subsetneq e_\alpha$. We will show that $e_\alpha \subseteq e_{\alpha'}$, a contradiction.

It suffices to show that for all $v \in e_\alpha$ there is an e^* such that $e \subseteq e^* \subseteq e_\alpha$ and $e^*, v \models \alpha'$. Wlog let $\alpha = \bigwedge \delta_i$ with $\delta_i = (\boldsymbol{K}\phi_i \wedge \boldsymbol{M}\psi_i \supset \chi_i)$ and let $\delta_i' = \delta/e$. By Lemma 1 we have that for all $v \in e_\alpha$ there is an e^* such that $e \subseteq e^* \subseteq e_\alpha$ and $e^*, v \models \alpha$. Let $\tau(v)$ be such an e^* corresponding to v. We will show that $\tau(v), v \models \alpha'$, that is, $\tau(v), v \models \delta_i'$ for all i. There are three cases:

- $\tau(v) \models \neg\boldsymbol{K}\phi$. Then clearly $\tau(v), v \models \delta_i'$.
- $\tau(v) \models \boldsymbol{K}\phi$ and $e \models \boldsymbol{M}\psi_i$. Then $\tau(v) \models \boldsymbol{M}\psi_i$. Since $\tau(v), v \models \delta_i$, $v \models \chi_i$ and, therefore, $\tau(v), v \models \delta_i'$.
- $\tau(v) \models \boldsymbol{K}\phi$ and $e \not\models \boldsymbol{M}\psi_i$. Then $\delta_i/e \equiv \text{TRUE}$, that is, $\tau(v), v \models \delta_i'$ holds vacuously.

Thus $\tau(v), v \models \alpha'$ for all $v \in e_\alpha$, that is, $e_\alpha \subseteq e_{\alpha'}$. ∎

We remark that the restriction to RNF is necessary. As was shown by LL [2005], $\boldsymbol{O}_{R}$ coincides with $\boldsymbol{O}_{M}$ on formulas which mention only $\boldsymbol{M}$. For example, in the case of $\alpha = (\neg\boldsymbol{M}\neg p \supset p)$ we have $e_p \models \boldsymbol{O}_{R}\alpha$ and yet $\boldsymbol{O}^{K} \not\models \boldsymbol{O}\alpha$. But note that this α is not a translation of a Reiter default, that is, it is not in RNF.

It is not hard to see that in case a formula in RNF does not mention any $\boldsymbol{M}$, then $\boldsymbol{O}_{R}$, $\boldsymbol{O}^{K}$, and $\boldsymbol{O}_{Ko}$ agree completely.

Corollary 2 *Let α be in RNF such that α does not mention any $\boldsymbol{M}$-subformulas. Then $\models \boldsymbol{O}_R\alpha \equiv \boldsymbol{O}^K\alpha \equiv \boldsymbol{O}_{Ko}\alpha$.*

Proof: This follows immediately from Theorem 9, Theorem 10, and the fact that $e \models \boldsymbol{O}_R\alpha$ iff $e \models \boldsymbol{O}_{Ko}\alpha$ in case $\alpha = \alpha/e$. ∎

The question then remains whether there are examples where $\boldsymbol{O}_R$ and $\boldsymbol{O}^K$ differ. Indeed there are, and this has been noticed long ago, for example, [Marek and Truszczynski , 1989]. For consider $\alpha = (\boldsymbol{K}p \supset p) \wedge (\boldsymbol{M}\neg p \supset p)$. It is easy to see that α is logically equivalent to p and, therefore, $e_p \models \boldsymbol{O}^K\alpha$. Yet $e_p \not\models \boldsymbol{O}_R\alpha$ because α/e_p is equivalent to $(\boldsymbol{K}p \supset p)$. Essentially, the difference arises because $\boldsymbol{O}^K$ is able to apply standard modal reasoning to defaults such as α, while $\boldsymbol{O}_R$ cannot look for conclusions that depend on reasoning with the defaults themselves. Arguably, this is a limitation of Reiter's logic not shared by the NMS **K**.

However the two systems are very close. Indeed, we conjecture that $\boldsymbol{O}_R$ and $\boldsymbol{O}^K$ will agree after closing a formula α in RNF under the logic $\boldsymbol{K}$, that is after conjoining α with every "minimal" objective formula or default that follows from it under logic $\boldsymbol{K}$.

To sum up, we have shown that only-knowing in the sense of NMS **K** lies just between only-knowing in the sense of Reiter and only-knowing in the sense of Konolige. Indeed, $\boldsymbol{O}^K$ seems very close to $\boldsymbol{O}_R$ except that it allows for some additional expansions that result from modal reasoning.

Conclusions

The work of LL has opened up the possibility of providing simple monotonic characterizations (including axiomatizations) of certain nonmonotonic logics. Their work on the nonmonotonic logics of Moore, Konolige, and Reiter was based on a simple possible-world characterization.

In this paper, we continued this line of work by considering versions of only-knowing for nonmonotonic systems in the sense of McDermott and Doyle, with a focus on NMS **K**. We first provided a simple possible-world characterization of $\boldsymbol{O}^K$ and then provided a uniform semantics for $\boldsymbol{O}^{\boldsymbol{\lambda}}$, where λ is a parameter for a monotonic modal logic. We were able to show the correspondence between $\boldsymbol{O}^{\boldsymbol{\lambda}}$ and NMS λ in the case of **K**, **K45**, and **K5** for the uniform semantics. As $\boldsymbol{O}^K$ seems the most interesting of all in terms of nonmonotonic reasoning, we then went on comparing it to Konolige's variant of autoepistemic logic and Reiter's default logic. We found that $\boldsymbol{O}^K$ lies just between Reiter and Konolige.

Three areas of future work suggest themselves. First, because of the popularity of default logic, it would be worthwhile to investigate precisely those cases where $\boldsymbol{O}_R$ and $\boldsymbol{O}^K$ coincide. Also, an axiomatization of **K**-expansions in terms of only-knowing may shed additional light on some of its properties. Finally, as was already mentioned, it remains to be seen whether our definition of $\boldsymbol{O}^{\boldsymbol{\lambda}}$ or some variant carries over to other nonmonotonic modal systems such as **S4F** [Schwarz and Truszczynski, 1992].

Appendix

Lemma 2 For a given **K**-model (M, w). Then for every flat basic α, $M, w \models \alpha$ iff $\pi(R(w)), \pi(w) \models \alpha$.

Proof: The proof is by induction on α.
The lemma clearly holds for atomic propositions and, by induction, for $\neg$ and $\wedge$. Hence the lemma holds for all objective formulas.

$M, w \models \boldsymbol{K}\alpha$ iff for all $w' \in R(w)$, $M, w' \models \alpha$ iff for all $w' \in R(w)$, $\pi(R(w')), \pi(w') \models \alpha$ by induction iff for all $v' \in \pi(R(w))$, $\pi(R(w)), v' \models \alpha$ because α is objective iff $\pi(R(w)), \pi(w) \models \boldsymbol{K}\alpha$. ∎

Lemma 3 Let α be flat and (M, w) based on e such that $e \models \boldsymbol{K}\alpha$ and $M, w \models \boldsymbol{K}\alpha$. Then $\pi(R(w)) \subseteq e_\alpha$.

Proof: Recall that

$e_\alpha = \bigcup e'$, where
$e' \supseteq e$ and $\forall v \in e' \exists e^*$ s.t. $e \subseteq e^* \subseteq e'$ and $e^*, v \models \alpha$.

Let $e' = \pi(R(w))$. In order to show that $e' \subseteq e_\alpha$, it suffices to establish that for all $v' \in e'$ there exists a e^* such that $e \subseteq e^* \subseteq e'$ and $e^*, v' \models \alpha$.

Let $v' \in e'$. Then there exists a $w' \in R(w)$ such that $\pi(w') = v'$. Let $e^* = \pi(R(w'))$. Since (M, w) is based on e, $\pi(R(w')) \supseteq e$ by Property 2. By Property 4, $\pi(R(w')) \subseteq \pi(R(w)) = e'$. Hence $e \subseteq e^* \subseteq e'$. Since $M, w \models \boldsymbol{K}\alpha$ by assumption, $M, w' \models \alpha$. Since α is flat, $e^*, v' \models \alpha$ follows by Lemma 2. ∎

Lemma 4 Let α be flat and $e \models \boldsymbol{K}\alpha$. Then there exists a (M, w) based on e s.t. $\pi(R(w)) = e_\alpha$ and $M, w \models \boldsymbol{K}\alpha$.

Proof: Recall that, by Lemma 1,

$$(*) \qquad e_\alpha = \{v \mid \exists e' \text{ s.t. } e \subseteq e' \subseteq e_\alpha \text{ and } e', v \models \alpha\}.$$

Construct M, w with $M = \langle W, R, \pi \rangle$ as follows:

- Let $W = \{w\} \cup e_\alpha$ s.t. $\forall v \in e_\alpha$, $\pi(v) = v$ and $\pi(w)$ arbitrary.
- $\forall w' \in W$, let $w' \in R(w)$ iff $w' \in e_\alpha$ and $\forall w' \in e_\alpha$, let $w'' \in R(w')$ iff $w'' \in \tau(w')$, where $\tau(w')$ is a set of valuations such that $e \subseteq \tau(w') \subseteq e_\alpha$ and $\tau(w'), w' \models \alpha$. (Such $\tau(w')$ must exist by $(*)$ above and the Axiom of Choice.) In particular, if $w' \in e$ then let $\tau(w') = e$.

We need to show that (M, w) is based on e such that $\pi(R(w)) = e_\alpha$ and $M, w \models \boldsymbol{K}\alpha$. Note that $\pi(R(w)) = e_\alpha$ holds by construction. To show that (M, w) is based on e, we establish that all four conditions of Definition 4 hold.

1. $W = \{w\} \cup R(w)$ holds by construction.
2. Since $e \subseteq \tau(w') \subseteq e_\alpha$ holds for all $w' \in R(w)$ by construction and $R(w) = e_\alpha$, it follows that for all $w' \in W$, $\pi(R(w')) \supseteq e$.
3. If $w' \in e$ then $\tau(w') = e$ by construction and thus $\pi(R(w')) = e$.
4. $\pi(R(R(w)) = \bigcup \tau(w') \subseteq e_\alpha = \pi(R(w))$.

Hence (M, w) is based on e.

Next we need to show that $M, w \models \boldsymbol{K}\alpha$, that is, for all $w' \in R(w)$, $M, w' \models \alpha$. So let $w' \in R(w)$. By construction, $R(w') = \tau(w')$ such that $\tau(w'), w' \models \alpha$. Since α is flat, $M, w' \models \alpha$ by Lemma 2. ∎

Theorem 5 For any basic flat α, $e \models \boldsymbol{O}^{K'}\alpha$ iff $e \models \boldsymbol{O}^{K}\alpha$.

Proof: To prove the only-if direction, let $e \models \boldsymbol{O}^{K'}\alpha$. Then $e \models \boldsymbol{K}\alpha$ and $e = e_\alpha$. We need to show that for all (M, w) based on e such that $M, w \models \boldsymbol{K}\alpha$, for all $w' \in W$, if $M, w' \models \alpha$ then $\pi(w') \in e$. So let (M, w) be based on e such that $M, w \models \boldsymbol{K}\alpha$ and let $M, w' \models \alpha$ for an arbitrary $w' \in W$. We first show that $\pi(R(w')) = e$. If $w' = w$ then, by Lemma 3, $\pi(R(w')) \subseteq e_\alpha$. Since $e = e_\alpha$, we obtain $\pi(R(w')) \supseteq e_\alpha$ by Property 2 of Definition 4 and thus $\pi(R(w')) = e$. If $w' \neq w$, that is $w' \in R(w)$, then the fact that $\pi(R(w)) = e$ together with Properties 2 and 4 of Definition 4 establishes that $\pi(R(w')) = e$. Since $M, w' \models \alpha$ and α is flat, $e, \pi(w') \models \alpha$ follows now by Lemma 2. Therefore, $\pi(w') \in e_\alpha$ and thus $\pi(w') \in e$.

Conversely, let $e \models \boldsymbol{O}^{K}\alpha$. Then $e \models \boldsymbol{K}\alpha$. We need to show that $e = e_\alpha$. By Lemma 4, there is a (M, w) based on e such that $M, w \models \boldsymbol{K}\alpha$ and $\pi(R(w)) = e_\alpha$. Since $e \models \boldsymbol{O}^{K}\alpha$, $\pi(R(w)) \subseteq e$. Since by Property 2 of Definition 4, $\pi(R(w)) \supseteq e$, we obtain $\pi(R(w)) = e$. Thus $e_\alpha = e$. ∎

Lemma 5 Let $e \neq \{\}$ and let (M, w) be a **K45**-model based on e. Then for all $w' \in W$, $\pi(R(w')) = e$.

Proof: By Property 2 of Definition 4, we have that for all $w' \in W$, $\pi(R(w')) \supseteq e$. Suppose there is a $w^* \in R(w')$ such that $\pi(w^*) \notin e$. Since $e \neq \{\}$, by Property 2 of Definition 4, there is a $w^{**} \in R(w')$ such that $\pi(w^{**}) \in e$. Since M is a **K45**-model, $w^* \in R(w^{**})$, contradicting Property 3 of Definition 4. ∎

Lemma 6 Let (M, w) be a **K45**-model based on $e \neq \{\}$ and let $w' \in W$. Then $M, w' \models \alpha$ iff $e, \pi(w') \models \alpha$ for all basic α.

Proof: The proof is by induction on α.
The lemma clearly holds for atomic propositions and, by induction, for $\neg$ and $\wedge$.

$M, w' \models \boldsymbol{K}\alpha$ iff for all $w^* \in R(w')$, $M, w^* \models \alpha$ iff for all $w^* \in R(w')$, $e, \pi(w^*) \models \alpha$ by induction iff for all $v \in e$, $e, v \models \alpha$ by the previous lemma iff $e, \pi(w') \models \boldsymbol{K}\alpha$. ∎

Theorem 6 If $e \neq \{\}$ then $e \models \boldsymbol{O}_M\alpha$ iff $e \models \boldsymbol{O}^{K45}\alpha$.

Proof: To prove the only-if direction, let $e \models \boldsymbol{O}_M\alpha$. Then $e \models \boldsymbol{K}\alpha$. Let (M, w) be a **K45**-model based on e such that $M, w \models \boldsymbol{K}\alpha$ and let $M, w' \models \alpha$ for $w' \in W$. We need to show that $\pi(w') \in e$. By Lemma 6, $e, \pi(w') \models \alpha$. Since $e \models \boldsymbol{O}_M\alpha$, $\pi(w') \in e$.

For the if direction, let $e \models \boldsymbol{O}^{K45}\alpha$. Then $e \models \boldsymbol{K}\alpha$. Suppose $e, v \models \alpha$ for some valuation v. We need to show that $v \in e$. Consider the following $M = \langle W, R, \pi \rangle$:

- Let $W = e \cup \{w\}$ such that $\pi(w) = v$.
- Let $R = (\{w\} \times e) \cup (e \times e)$.
- Let $\pi(v) = v$ for all $v \in e$.

Then M is clearly a **K45**-model and it is easy to verify that (M, w) is based on e. Since $e, v \models \alpha$ by assumption, $M, w \models \alpha$ follows by Lemma 6. Since $e \models \boldsymbol{O}^{K45}\alpha$, $\pi(w) \in e$ follows. Since $v = \pi(w)$ by assumption, $v \in e$ follows. ∎

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