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An Automata-Theoretic Approach to Uniform Interpolation and Approximation in the Description Logic $\mathcal{EL}$

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Abstract

We study (i) uniform interpolation for TBoxes that are formulated in the lightweight description logic $\mathcal{EL}$ and (ii) $\mathcal{EL}$-approximations of TBoxes formulated in more expressive languages. In both cases, we give model-theoretic characterizations based on simulations and cartesian products, and we develop algorithms that decide whether interpolants and approximants exist. We present a uniform approach to both problems, based on a novel amorphous automaton model called $\mathcal{EL}$ automata (EA). Using EAs, we also establish a simpler proof of the known result that conservative extensions of $\mathcal{EL}$-TBoxes can be decided in EXPTIME.

1 Introduction

Formal ontologies provide a conceptual model of a domain of interest by describing the vocabulary of that domain in terms of a logical language, such as a description logic (DL). To cater for different applications and uses of ontologies, DLs and other ontology languages vary significantly regarding expressive power and computational complexity (Baader et al. 2003). For example, lightweight DLs such as the OWL2 profile OWL2EL and its underlying core logic $\mathcal{EL}$ are relatively inexpressive, focussing mainly on conjunction and existential quantification, but admit PTIME reasoning. In contrast, expressive DLs such as OWL2DL, $\mathcal{ALC}$, and $\mathcal{SHIQ}$ are equipped with all Boolean operators and existential as well as universal quantification, and may additionally include a variety of other features; consequently, the complexity of reasoning is between EXPTIME and 2NEXPTIME.

When an ontology is used for a new purpose, it is often desirable or even unavoidable to *customize* the ontology in a suitable way. In this paper, we look at two rather common such customizations: (i) restricting the signature (set of vocabulary items) that is covered by the ontology and (ii) restricting the logical language that the ontology is formulated in.

The basic operation for restricting the signature of an ontology is uniform interpolation, also called forgetting and variable elimination (Reiter and Lin 1994; Lang, Liberatore, and Marquis 2003; Eiter et al. 2006; Konev, Walther, and Wolter 2009; Kontchakov, Wolter, and Zakharyaschev 2010; Wang et al. 2010a). Specifically, uniform interpolation is the problem of constructing, given a TBox $\mathcal{T}$ and a signature Σ, a new ontology $\mathcal{T}'$ that uses only symbols from Σ and has the same logical consequences as $\mathcal{T}$ as far as the signature Σ is concerned. In other words, $\mathcal{T}'$ is obtained from $\mathcal{T}$ by forgetting all non-Σ-symbols. We then call $\mathcal{T}'$ a *uniform* Σ*-interpolant* of $\mathcal{T}$. There are various applications of uniform interpolation, of which we mention three. *Ontology reuse*. When reusing an existing ontology in a new application, then typically only a small number of the symbols is relevant. Instead of reusing the whole ontology, one can thus use the potentially much smaller ontology that results from forgetting the extraneous symbols. *Predicate hiding*. When an ontology is to be published, but some part of it has to be concealed from the public, then this part can be removed by forgetting the symbols that belong to it (Grau and Motik 2010). *Ontology summary.* The result of forgetting often provides a smaller and more focussed ontology that summarizes what the original ontology says about the retained symbols, potentially facilitating ontology comprehension.

The basic operation for restricting the logical language of an ontology is approximation: given an ontology $\mathcal{T}$ formulated in some expressive DL $\mathcal{L}$, construct a new ontology $\mathcal{T}'$ in a less expressive DL $\mathcal{L}'$ such that $\mathcal{T}'$ is logically entailed by $\mathcal{T}$ and is most specific with this property, i.e., any $\mathcal{L}'$-ontology $\mathcal{T}''$ entailed by $\mathcal{T}$ is also entailed by $\mathcal{T}'$. We then call $\mathcal{T}'$ an $\mathcal{L}'$*-approximant of* $\mathcal{T}$. An introduction to this type of *semantic* approximation is given in (Selman and Kautz 1996). We mention two relevant applications of approximation. *Ontology reuse*. Approximation is required when an ontology that is formulated in a DL $\mathcal{L}$ is reused in an application which requires reasoning, but where reasoners for $\mathcal{L}$ are not available or not sufficiently efficient. For example, if ontologies are used to access instance data, then scalable query answering is currently only available for lightweight DLs such as $\mathcal{EL}$, but not for more expressive ones such as $\mathcal{ALC}$. Indeed, we show that the uniform interpolants and approximants studied in this paper can both be exploited for answering queries over instance data. *Ontology summary*. The comprehension of an ontology may be hindered not only by a too large vocabulary, but also by a too complex and detailed modeling. To get to grips with understanding an ontology, it can thus be useful to approximate it in a less expressive DL, in this way concealing all modeling details that can only be expressed in more powerful languages.

A general problem with both uniform interpolation and approximation is that the result of these operations need not be expressible in the desired language. For example, the uniform Σ-interpolant of the $\mathcal{EL}$-TBox

$$\{A \sqsubseteq \exists r.B, B \sqsubseteq \exists r.B\}$$

with $\Sigma = \{A, r\}$ is not expressible as a (finite) $\mathcal{EL}$-TBox, and the $\mathcal{EL}$-approximant of the $\mathcal{ELU}$-TBox

$$\{A \sqsubseteq \exists r.(B_1 \sqcup B_2), B_1 \sqcup B_2 \sqsubseteq \exists r.(B_1 \sqcup B_2)\}$$

is not expressible as a (finite) $\mathcal{EL}$-TBox either. Thus, when working with uniform interpolation and approximation, a fundamental task is to determine whether the desired result exists in the first place.

In this paper, we concentrate on the description logic $\mathcal{EL}$ and consider (i) uniform interpolants of $\mathcal{EL}$-TBoxes and (ii) $\mathcal{EL}$-approximants of TBoxes formulated in more expressive languages, with an emphasis on the extension $\mathcal{ELU}$ of $\mathcal{EL}$ with disjunction. Our main aims are, on the one hand, to provide semantic characterizations of uniform interpolants and approximants, based on model-theoretic notions such as (equi)simulations and products. On the other hand, we develop algorithms for deciding the existence of uniform interpolants and approximants and analyze the computational complexity of these problems. While we do not directly study the actual computation of uniform interpolants and approximants, we believe that the machinery developed in this paper provides an important technical foundation also for this task.

We use a uniform approach that allows us to decide the existence of both uniform interpolants and $\mathcal{EL}$-approximants and highlights the commonalities and differences between uniform interpolation and approximation. Our main technical tool is a novel type of automaton, called $\mathcal{EL}$ automaton (EA), which bears some similarity to the 'amorphous' automata models introduced in (Janin and Walukiewicz 1995; Wilke 2001). In particular, EAs are tree automata in spirit, but run directly on (not necessarily tree-shaped) DL interpretations. In contrast to existing automata models, they are tailored towards Horn-like logics and, in particular, $\mathcal{EL}$-TBoxes. Consequently, the languages that they accept are closed under intersection and projection, but not under union and complementation. While the expressive power of EAs is only moderately larger than that of $\mathcal{EL}$-TBoxes, we show that it is *always* possible to express uniform interpolants of $\mathcal{EL}$-TBoxes and $\mathcal{EL}$-approximants of classical $\mathcal{ELU}$-TBoxes as EAs (a classical TBox is a set of statements $A \equiv C$ and $A \sqsubseteq C$; definitorial cycles are allowed). This enables us to decompose the aforementioned existence problems into two separate steps: first compute the desired uniform interpolant or approximant, represented as an EA, and then decide whether the EA accepts a language that can be defined using an $\mathcal{EL}$-TBox. Note that the latter step is a pure EA problem and does not involve reference to uniform interpolation or approximation. Using this machinery, we show that deciding the existence of uniform interpolants of $\mathcal{EL}$-TBoxes is ExpTime-complete and deciding the existence of $\mathcal{EL}$-approximants of classical $\mathcal{ELU}$-TBoxes is in 2ExpTime (and ExpTime-hard). The precise complexity of the latter problem remains open. We also use EAs to give a new and arguably simpler proof of the known result that deciding conservative extensions of $\mathcal{EL}$-TBoxes is in ExpTime (Lutz and Wolter 2010).

The existence problems considered in this paper are known to be challenging. For the expressive DL $\mathcal{ALC}$, a number of results for uniform interpolation were obtained in (Wang et al. 2009; 2010b), but the first correct algorithm for deciding the existence of uniform interpolants for TBoxes was only recently given in (Lutz and Wolter 2011). The first paper concerned with uniform interpolation in $\mathcal{EL}$ is (Konev, Walther, and Wolter 2009), but unlike the current paper it considers only *classical* $\mathcal{EL}$-TBoxes instead of general ones, and it does not prove any decidability results for the existence of uniform interpolants. A method for computing interpolants of general $\mathcal{EL}$-TBoxes and deciding their existence in ExpTime was recently claimed in (Nikitina 2011), but while the approach is promising we found it difficult to fully verify the claimed results based on the currently available material. Approximation has been considered on the level of concepts in (Brandt, Küsters, and Turhan 2002). In contrast, no results appear to be known for deciding the existence of approximants on the TBox level, with the notable exception of DL-Lite (Botoeva, Calvanese, and Rodriguez-Muro 2010). Instead, research has focussed on heuristic (and incomplete) approaches to computing TBox approximants, see for example (Pan and Thomas 2007; Ren, Pan, and Zhao 2010). While the results for $\mathcal{EL}$-approximation of classical $\mathcal{ELU}$-TBoxes obtained in this paper may not seem too impressive on first sight, already this restricted case requires intricate technical machinery and we view it as an important first step towards sound and complete TBox approximation beyond DL-Lite.

Most proofs are deferred to the long version of this paper available at http://www.csc.liv.ac.uk/~frank/publ/publ.html.

2 Preliminaries

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be countably infinite and mutually disjoint sets of *concept* and *role names*. $\mathcal{EL}$*-concepts* C are built according to the rule

$$C \quad := \quad A \quad | \quad \top \quad | \quad \bot \quad | \quad C \sqcap D \quad | \quad \exists r.C$$

where A ranges over $\mathsf{N_C}$, r over $\mathsf{N_R}$, and C, D over $\mathcal{EL}$-concepts.[1] An $\mathcal{EL}$*-concept inclusion (CI)* is an expression $C \sqsubseteq D$ with C, D $\mathcal{EL}$-concepts, and a *(general)* $\mathcal{EL}$*-TBox* is a finite set of $\mathcal{EL}$-CIs. In some cases, we drop the finiteness condition on TBoxes and then explicitly speak of *infinite TBoxes*.

The semantics of $\mathcal{EL}$ is given by *interpretations* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where the *domain* $\Delta^{\mathcal{I}}$ is a non-empty set and $\cdot^{\mathcal{I}}$ is an *interpretation function* that maps each concept name

[1]Note that $\mathcal{EL}$ is sometimes defined without "$\bot$". The results in this paper are independent of whether or not $\bot$ is present.

A to a subset $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ and each role name r to a binary relation $r^{\mathcal{I}}$ over $\Delta^{\mathcal{I}}$. We extend $\cdot^{\mathcal{I}}$ as follows:

$$\begin{array}{rclrcl} \top^{\mathcal{I}} & := & \Delta^{\mathcal{I}} & \qquad \bot^{\mathcal{I}} & := & \emptyset \\ (C \sqcap D)^{\mathcal{I}} & := & C^{\mathcal{I}} \cap D^{\mathcal{I}} & & & \\ (\exists r.C)^{\mathcal{I}} & := & \{d \in \Delta^{\mathcal{I}} \mid \exists e \in C^{\mathcal{I}} : (d,e) \in r^{\mathcal{I}}\} & & & \end{array}$$

An interpretation $\mathcal{I}$ *satisfies* a CI $C \sqsubseteq D$, written $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$; $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$ if $\mathcal{I}$ satisfies all CIs in $\mathcal{T}$. We use $\mathsf{mod}(\mathcal{T})$ to denote the class of all models of $\mathcal{T}$ and write $\mathcal{T} \models C \sqsubseteq D$ if every model of $\mathcal{T}$ satisfies $C \sqsubseteq D$ and $\mathcal{T} \models \mathcal{T}'$ if $\mathcal{T} \models C \sqsubseteq D$ for all $C \sqsubseteq D \in \mathcal{T}'$.

A *signature* is a set $\Sigma \subseteq \mathsf{N_C} \cup \mathsf{N_R}$ of concept and role names, which we uniformly call *symbols* in this context. The *signature* $\mathsf{sig}(C)$ *of a concept* C is the set of symbols that occur in C, and likewise for $\mathsf{sig}(C \sqsubseteq D)$ and $\mathsf{sig}(\mathcal{T})$ for CIs $C \sqsubseteq D$ and TBoxes $\mathcal{T}$. An $\mathcal{EL}_\Sigma$*-concept* is an $\mathcal{EL}$-concept C with $\mathsf{sig}(C) \subseteq \Sigma$ and likewise for $\mathcal{EL}_\Sigma$*-CIs* and $\mathcal{EL}_\Sigma$*-TBoxes*.

We sometimes use TBoxes formulated in more expressive logics than $\mathcal{EL}$, including full first-order logic (FO). More specifically, an *FO-TBox* is a finite set of FO sentences using unary predicates from $\mathsf{N_C}$ and binary predicates from $\mathsf{N_R}$ (plus equality), and without any function symbols or constants. It is well-known that any $\mathcal{EL}$-TBox can be viewed as an FO-TBox, and the same is true for many other DLs such as $\mathcal{ELU}$ and $\mathcal{ALC}$ (see (Baader et al. 2003) for more details).

The definitions of our main notions, uniform interpolation and approximation, which are given in the two subsequent sections, both rely on the logical consequences of a TBox. In the case of uniform interpolation, there is an emphasis on the signature in which such consequences are formulated while approximation emphasises the logic in which they are formulated. We treat these different aspects in a uniform way based on the notions of $\mathcal{EL}_\Sigma$-entailment and $\mathcal{EL}_\Sigma$-inseparability. Let Σ be a signature. An FO-TBox $\mathcal{T}$ $\mathcal{EL}_\Sigma$*-entails* an FO-TBox $\mathcal{T}'$ if for all $\mathcal{EL}_\Sigma$-CIs $C \sqsubseteq D$, $\mathcal{T}' \models C \sqsubseteq D$ implies $\mathcal{T} \models C \sqsubseteq D$. Note that when $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$ and $\mathcal{T}'$ is an $\mathcal{EL}$-TBox, then $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ if and only if $\mathcal{T} \models \mathcal{T}'$. Two FO-TBoxes $\mathcal{T}$ and $\mathcal{T}'$ are $\mathcal{EL}_\Sigma$*-inseparable*, written $\mathcal{T}_1 \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}_2$, if $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ and vice versa. If $\mathcal{T}$ and $\mathcal{T}'$ are $\mathcal{EL}$-TBoxes and $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are logically equivalent if they are $\mathcal{EL}_\Sigma$-inseparable. In all these notations, we drop Σ when $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$.

3 Uniform Interpolation and Conservative Extensions

We introduce uniform interpolation and conservative extensions and establish semantic characterisations based on a certain kind of simulation between interpretations.

Definition 1. For $\mathcal{EL}$-TBoxes $\mathcal{T}$ and $\mathcal{T}'$, we say that

- $\mathcal{T}$ is a *uniform* $\mathcal{EL}_\Sigma$*-interpolant* of $\mathcal{T}'$ if $\mathsf{sig}(\mathcal{T}) \subseteq \Sigma \subseteq \mathsf{sig}(\mathcal{T}')$, $\mathcal{T}' \models \mathcal{T}$, and $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$;
- $\mathcal{T}'$ is an $\mathcal{EL}$-conservative extension of $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ for $\Sigma = \mathsf{sig}(\mathcal{T})$.

Both notions can be characterized using $\mathcal{EL}_\Sigma$-inseparability: it is not hard to verify that an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ is a uniform $\mathcal{EL}_\Sigma$-interpolant of an $\mathcal{EL}$-TBox $\mathcal{T}'$ if and only if $\mathcal{T} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}'$, and $\mathcal{T}'$ is an $\mathcal{EL}$-conservative extension of $\mathcal{T}$ if and only if $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}'$. Also note that uniform $\mathcal{EL}_\Sigma$-interpolants are unique up to logical equivalence, when they exist.

Forgetting is dual to uniform interpolation in the following sense: a TBox $\mathcal{T}'$ is *the result of forgetting about a signature* Σ *in a TBox* $\mathcal{T}$ if $\mathcal{T}'$ is a uniform $\mathsf{sig}(\mathcal{T}) \setminus \Sigma$-interpolant of $\mathcal{T}$. Therefore, the results presented in this paper for uniform interpolation also apply to forgetting. For more information, see e.g. (Wang et al. 2010b; Lutz and Wolter 2011).

As an example, consider the following TBox $\mathcal{T}_1$:

$$\begin{array}{rcl} \mathsf{Patient} \sqcap \exists \mathsf{finding.MajorFinding} & \sqsubseteq & \mathsf{InPatient} \\ \mathsf{Finding} \sqcap \exists \mathsf{status.PotentiallyLethal} & \sqsubseteq & \mathsf{MajorFinding} \\ \mathsf{MajorFinding} & \sqsubseteq & \mathsf{Finding} \end{array}$$

For $\Sigma = \mathsf{sig}(\mathcal{T}_1) \setminus \{\mathsf{MajorFinding}\}$, a uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}_1$ (equivalently: the result of forgetting $\mathsf{MajorFinding}$) consists of the single CI

$$\begin{array}{l} \mathsf{Patient} \sqcap \exists \mathsf{finding}.(\mathsf{Finding} \sqcap \exists \mathsf{status.PotentiallyLethal}) \\ \qquad\qquad \sqsubseteq \mathsf{InPatient}. \end{array}$$

As another example, consider the TBox

$$\mathcal{T}_2 = \{A \sqsubseteq \exists r.B, B \sqsubseteq \exists r.B\}$$

mentioned in the introduction and let $\Sigma = \{A, r\}$. It can be shown that the infinite TBox $\mathcal{T}' = \{A \sqsubseteq \exists r^n.\top \mid n \geq 1\}$ satisfies $\mathcal{T}_2 \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}'$ and thus, except for its infinity, qualifies as a uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}_2$. However, uniform interpolants are required to be *finite* TBoxes and, in this case, it can be proved that there is no (finite) uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}_2$ (see appendix of the long version of this paper).

Note that uniform $\mathcal{EL}_\Sigma$-interpolants are weaker than uniform $\mathcal{ALC}_\Sigma$-interpolants as studied in (Lutz and Wolter 2011), even when the original TBox $\mathcal{T}$ is formulated in $\mathcal{EL}$. The reason is that $\mathcal{ALC}_\Sigma$-interpolants are required to preserve all $\mathcal{ALC}$-consequences of $\mathcal{T}$ formulated in Σ, instead of all $\mathcal{EL}$-consequences. In fact, we show in the appendix of the long version that there is an $\mathcal{EL}$-TBox that has a uniform $\mathcal{EL}_\Sigma$-interpolant, but no uniform $\mathcal{ALC}_\Sigma$-interpolant. It follows that the results and algorithms in (Lutz and Wolter 2011) do not apply to the framework studied in this paper.

We have defined uniform interpolants based on subsumption. To illustrate their utility beyond subsumption, we show that they can be used for instance query answering. DL instance data is represented by an *ABox*, which is a finite set of assertions of the form $A(a)$ and $r(a,b)$ where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and a, b are *individual names*. An *instance query* takes the form $C(a)$, where C is an $\mathcal{EL}$-concept and a an individual name; we speak of a Σ*-instance query* when C is an $\mathcal{EL}_\Sigma$-concept and write $\mathcal{A}, \mathcal{T} \models C(a)$ if every model of $\mathcal{A}$ and $\mathcal{T}$ satisfies $C(a)$, see e.g. (Baader et al. 2003) for more details. *Instance query answering* is the problem to decide, given an ABox $\mathcal{A}$, TBox $\mathcal{T}$, and instance query $C(a)$, whether $\mathcal{A}, \mathcal{T} \models C(a)$. The following result, proved in (Lutz and Wolter 2010) in terms of $\mathcal{EL}_\Sigma$-inseparability, demonstrates the utility of uniform interpolants for instance query answering: when only a few of the symbols in a TBox $\mathcal{T}$ are relevant for instance query answering, then $\mathcal{T}$ can be replaced with a potentially much smaller uniform interpolant.

Theorem 2. *For any $\mathcal{EL}$-TBox $\mathcal{T}$, signature Σ, and $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}'$, $\mathcal{T}'$ is a uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$ iff for all Σ-ABoxes $\mathcal{A}$ and Σ-instance queries $C(a)$, $\mathcal{A}, \mathcal{T} \models C(a)$ iff $\mathcal{A}, \mathcal{T}' \models C(a)$.*

For the announced semantic characterization, we introduce simulations. A *pointed interpretation* is a pair $(\mathcal{I}, d)$ with $\mathcal{I}$ an interpretation and $d \in \Delta^\mathcal{I}$.

Definition 3. Let Σ be a signature and $(\mathcal{I}_1, d_1)$, $(\mathcal{I}_2, d_2)$ be pointed interpretations. A relation $S \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a *Σ-simulation between $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$*, written $S : (\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$, if $(d_1, d_2) \in S$ and the following conditions hold:

(base) for all $A \in \Sigma \cap \mathsf{N_C}$ and $(e_1, e_2) \in S$, if $e_1 \in A^{\mathcal{I}_1}$ then $e_2 \in A^{\mathcal{I}_2}$;

(forth) for all $r \in \Sigma \cap \mathsf{N_R}$, $(e_1, e_2) \in S$, and $(e_1, e_1') \in r^{\mathcal{I}_1}$, there is an $(e_2, e_2') \in r^{\mathcal{I}_2}$ with $(e_1', e_2') \in S$.

If such an S exists, then we write $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$. We say that *$(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ are Σ-equisimilar* if $(\mathcal{I}_1, d_1) \leq_\Sigma (\mathcal{I}_2, d_2)$ and $(\mathcal{I}_2, d_2) \leq_\Sigma (\mathcal{I}_1, d_1)$, written $(\mathcal{I}_1, d_1) \approx_\Sigma (\mathcal{I}_2, d_2)$.

There is a close relationship between (equi)simulations and the truth of $\mathcal{EL}$-concepts in pointed interpretations, see (Lutz and Wolter 2010). To make this precise, we say that $(\mathcal{I}_1, d_1)$ and $(\mathcal{I}_2, d_2)$ are *$\mathcal{EL}_\Sigma$-equivalent*, written $(\mathcal{I}_1, d_1) \equiv_\Sigma^{\mathcal{EL}} (\mathcal{I}_2, d_2)$, if for all $\mathcal{EL}_\Sigma$-concepts C, we have $d_1 \in C^{\mathcal{I}_1}$ iff $d_2 \in C^{\mathcal{I}_2}$. An interpretation $\mathcal{I}$ has *finite outdegree* if $\{d' \mid (d, d') \in \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I}\}$ is finite, for all $d \in \Delta^\mathcal{I}$.

Lemma 4. *For all pointed interpretations $(\mathcal{I}_1, d_1)$, $(\mathcal{I}_2, d_2)$ and signatures Σ, $(\mathcal{I}_1, d_1) \approx_\Sigma (\mathcal{I}_2, d_2)$ implies $(\mathcal{I}_1, d_1) \equiv_\Sigma^{\mathcal{EL}} (\mathcal{I}_2, d_2)$. The converse holds if $\mathcal{I}_1, \mathcal{I}_2$ are of finite outdegree.*

Now for the semantic characterizations. For a class $\mathfrak{C}$ of interpretations and a signature Σ, we use $\mathsf{cl}_\approx^\Sigma(\mathfrak{C})$ to denote the *closure under global Σ-equisimulations* of $\mathfrak{C}$, i.e., the class of all interpretations $\mathcal{I}$ such that for all $d \in \Delta^\mathcal{I}$, there is a $\mathcal{J} \in \mathfrak{C}$ and a $d' \in \Delta^\mathcal{J}$ such that $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, d')$. When $\Sigma = \mathsf{N_C} \cup \mathsf{N_R}$, we simply write $\mathsf{cl}_\approx(\mathfrak{C})$ instead of $\mathsf{cl}_\approx^\Sigma(\mathfrak{C})$. It was shown in (Lutz, Piro, and Wolter 2011) that for each $\mathcal{EL}$-TBox $\mathcal{T}$, $\mathsf{mod}(\mathcal{T})$ is closed under global equisimulations.

Theorem 5. *Let $\mathcal{T}, \mathcal{T}'$ be $\mathcal{EL}$-TBoxes and Σ a signature.*

1. *$\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ iff $\mathsf{mod}(\mathcal{T}) \subseteq \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$;*
2. *Let $\mathcal{T} \subseteq \mathcal{T}'$. Then $\mathcal{T}'$ is an $\mathcal{EL}$-conservative extension of $\mathcal{T}$ iff $\mathsf{mod}(\mathcal{T}) \subseteq \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$ for $\Sigma = \mathsf{sig}(\mathcal{T})$;*
3. *Let $\mathsf{sig}(\mathcal{T}) \subseteq \Sigma \subseteq \mathsf{sig}(\mathcal{T}')$. Then $\mathcal{T}$ is a uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}'$ iff $\mathsf{mod}(\mathcal{T}) = \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$.*

Proof. (Sketch) The challenging part is to establish Point 1. Once this is done, Points 2 and 3 follow by definition of conservative extensions/uniform interpolants and by Lemma 4. In the proof of Point 1, the "$\Leftarrow$" direction essentially consists of an application of Lemma 4: assume $\mathsf{mod}(\mathcal{T}) \subseteq \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$ and $\mathcal{T}' \models C \sqsubseteq D$ with $\mathsf{sig}(C), \mathsf{sig}(D) \subseteq \Sigma$, and to the contrary of what is to be shown, $\mathcal{T} \not\models C \sqsubseteq D$; then there is an $\mathcal{I} \in \mathsf{mod}(\mathcal{T})$ with $d \in C^\mathcal{I} \setminus D^\mathcal{I}$ and we have $\mathcal{I} \in \mathsf{cl}_\approx^\Sigma(\mathsf{mod}(\mathcal{T}'))$, which yields a model $\mathcal{J}$ of $\mathcal{T}'$ and a $d' \in \Delta^\mathcal{J}$ with $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, d')$; by (the first part of) Lemma 4, $d' \in C^\mathcal{J} \setminus D^\mathcal{J}$ in contradiction to $\mathcal{J}$ being a model of $\mathcal{T}'$. The "$\Rightarrow$" direction can be proved applying (the second part of) Lemma 4, but only for interpretations $\mathcal{I}$ of finite outdegree. To obtain"$\Rightarrow$" in its full generality, we rely on automata-theoretic result obtained later on. The proof is given in Section 7. ❑

4 TBox Approximation

We introduce TBox approximation and establish a semantic characterization based on global equisimulations and products of interpretations. The characterization is rather general and applies to the $\mathcal{EL}$-approximation of FO-TBoxes.

Definition 6. Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox and $\mathcal{T}'$ an FO-TBox. Then $\mathcal{T}$ is an *$\mathcal{EL}$-approximant of* $\mathcal{T}'$ if $\mathcal{T}' \models \mathcal{T}$ and for any $\mathcal{EL}$-TBox $\mathcal{T}''$ with $\mathcal{T}' \models \mathcal{T}''$, we have $\mathcal{T} \models \mathcal{T}''$.

It is easy to verify that an $\mathcal{EL}$-TBox $\mathcal{T}$ is an *$\mathcal{EL}$-approximant of* an FO-TBox $\mathcal{T}'$ if and only if $\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}'$. Also, $\mathcal{EL}$-approximants of an FO-TBox are unique up to logical equivalence, if they exist. Note the similarity to the remark after Definition 1.

When establishing algorithms and complexity results for approximation, we will have a much more modest aim than the approximation of full FO-TBoxes. Specifically, we will consider the approximation of classical $\mathcal{ELU}$-TBoxes by general $\mathcal{EL}$-TBoxes. Here, $\mathcal{ELU}$ is the extension of $\mathcal{EL}$ with the union constructor $\sqcup$ whose semantics is $(C \sqcup D)^\mathcal{I} = C^\mathcal{I} \cup D^\mathcal{I}$, and a *classical TBox* is a finite set of $\mathcal{ELU}$-CIs of the form $A \sqsubseteq C$ or $A \equiv C$ where A is a concept name and left-hand sides of CIs are unique within the TBox. Subsumption is ExpTime-complete in the presence of classical $\mathcal{ELU}$-TBoxes while it is PTime-complete for general $\mathcal{EL}$-TBoxes, thus approximation pays off in terms of computational complexity. As an example, consider the $\mathcal{ELU}$-TBox $\mathcal{T}$ that consists of

$$\begin{aligned} \mathsf{Node} &\sqsubseteq \mathsf{Left} \sqcup \mathsf{Right} \\ \mathsf{Left} &\sqsubseteq \exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right}) \\ \mathsf{Right} &\sqsubseteq \exists \mathsf{succ}.(\mathsf{Left} \sqcup \mathsf{Right}) \end{aligned}$$

The infinite TBox

$$\mathcal{T}' = \{X \sqsubseteq \exists \mathsf{succ}^n.\top \mid X \in \{\mathsf{Node}, \mathsf{Left}, \mathsf{Right}\}, n \geq 0\}$$

satisfies $\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}'$ and thus qualifies as an $\mathcal{EL}$-approximant except for its infinity. However, $\mathcal{EL}$-approximants need to be *finite* and it can be proved that there is no finite $\mathcal{EL}$-TBox with $\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}'$, thus no $\mathcal{EL}$-approximant. If we add $\mathsf{Left} \sqsubseteq \mathsf{Node}$ and $\mathsf{Right} \sqsubseteq \mathsf{Node}$ to $\mathcal{T}$, then an $\mathcal{EL}$-approximant consists of these two CIs plus $\mathsf{Node} \sqsubseteq \exists \mathsf{succ}.\mathsf{Node}$.

Note that it makes sense to approximate classical $\mathcal{ELU}$-TBoxes by *general* $\mathcal{EL}$-TBoxes rather than classical ones. For example, the $\mathcal{EL}$-approximant of

$$\{A \equiv (B_1 \sqcup B_2) \sqcap (B_1' \sqcup B_2')\}$$

consists of the CIs $B_i \sqcap B_j' \sqsubseteq A$, $i, j \in \{1, 2\}$, whereas its approximation as a classical $\mathcal{EL}$-TBox is empty.

Just like uniform $\mathcal{EL}_\Sigma$-interpolants, $\mathcal{EL}$-approximations of classical $\mathcal{ELU}$-TBoxes are useful for answering queries over instance data. In particular, answering instance queries in the presence of classical $\mathcal{ELU}$-TBoxes is coNP-complete regarding data complexity while it is in PTIME for $\mathcal{EL}$-TBoxes and thus it is possible to achieve tractability by replacing a classical $\mathcal{ELU}$-TBox with its $\mathcal{EL}$-approximation. Interestingly, such a replacement is *optimal* in the following sense: for a TBox $\mathcal{T}$, we say that *answering instance queries w.r.t.* $\mathcal{T}$ *is in* PTIME if for every instance query $C(a)$, there is a polytime algorithm that, given an ABox $\mathcal{A}$, decides whether $\mathcal{A}, \mathcal{T} \models C(a)$ (see (Lutz and Wolter 2012) for more information on this *non-uniform view* of data complexity); then, the $\mathcal{EL}$-approximation of an $\mathcal{ELU}$-TBox $\mathcal{T}$ is *the most specific TBox* among all general $\mathcal{ELU}$-TBoxes $\mathcal{T}'$ that are entailed by $\mathcal{T}$ such that answering instance queries w.r.t. $\mathcal{T}'$ is in PTIME. By the results in (Lutz and Wolter 2012), the same is true for the more general conjunctive queries.

Theorem 7. *Let $\mathcal{T}$ be a (general) $\mathcal{ELU}$-TBox and $\mathcal{T}'$ the $\mathcal{EL}$-approximant of $\mathcal{T}$. If $\mathcal{T}''$ is a (general) $\mathcal{ELU}$-TBox with $\mathcal{T} \models \mathcal{T}''$ and answering instance queries w.r.t. $\mathcal{T}''$ is in* PTIME*, then $\mathcal{T}' \models \mathcal{T}''$ (unless* PTIME=coNP*).*

To give a semantic characterization of $\mathcal{EL}$-approximants, we need the product operation on interpretations, as used in an $\mathcal{EL}$ context in (Lutz, Piro, and Wolter 2011). Given a family of interpretations $(\mathcal{I}_i)_{i \in I}$, the *product* $\mathcal{I} = \prod_{i \in I} \mathcal{I}_i$ is defined as follows: the domain $\Delta^\mathcal{I}$ consists of all functions $f : I \to \bigcup_{i \in I} \Delta^{\mathcal{I}_i}$ with $f(i) \in \Delta^{\mathcal{I}_i}$ for all $i \in I$ and all symbols $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$ are interpreted as follows:

- $f \in A^\mathcal{I}$ iff $f(i) \in A^{\mathcal{I}_i}$, for all $i \in I$
- $(f, g) \in r^\mathcal{I}$ iff $(f(i), g(i)) \in r^{\mathcal{I}_i}$, for all $i \in I$.

The connection between $\mathcal{EL}$-concepts and products follows.

Lemma 8. *For all $\mathcal{EL}$-concepts C, families of interpretations $(\mathcal{I}_i)_{i \in I}$, and $f \in \Delta^{\Pi_{i \in I} \mathcal{I}_i}$, we have $f \in C^{\Pi_{i \in I} \mathcal{I}_i}$ iff $f(i) \in C^{\mathcal{I}_i}$ for all $i \in I$.*

Now for the semantic characterization of approximants. For a class $\mathfrak{C}$ of interpretations, we use $\mathsf{cl}_\Pi(\mathfrak{C})$ to denote the *closure under products* of $\mathfrak{C}$, i.e., the class of all interpretations $\mathcal{I}$ such that $\mathcal{I} = \prod_{i \in I} \mathcal{I}_i$ for some (potentially infinite) family of interpretations $(\mathcal{I}_i)_{i \in I}$ contained in $\mathfrak{C}$. We use $\mathfrak{C}_{\restriction \mathsf{finout}}$ to denote the restriction of $\mathfrak{C}$ to interpretations of finite outdegree. It was shown in (Lutz, Piro, and Wolter 2011) that for each $\mathcal{EL}$-TBox $\mathcal{T}$, $\mathsf{mod}(\mathcal{T})$ is closed under products.

Theorem 9. *Let $\mathcal{T}$ be an FO-TBox and $\mathcal{T}'$ an $\mathcal{EL}$-TBox with $\mathcal{T} \models \mathcal{T}'$. Then $\mathcal{T}'$ is an $\mathcal{EL}$-approximant of $\mathcal{T}$ iff* $\mathsf{mod}(\mathcal{T}')_{\restriction \mathsf{finout}} = \mathsf{cl}_\approx(\mathsf{cl}_\Pi(\mathsf{mod}(\mathcal{T})))_{\restriction \mathsf{finout}}$.

The central ingredient to the proof of Theorem 9 is the following result.

Lemma 10. *Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox. The following are equivalent for any interpretation $\mathcal{I}$ of finite outdegree:*

1. *$\mathcal{I} \models C \sqsubseteq D$, for all $\mathcal{EL}$-CIs $C \sqsubseteq D$ with $\mathcal{T} \models C \sqsubseteq D$;*
2. *$\mathcal{I} \in \mathsf{cl}_\approx(\mathsf{cl}_\Pi(\mathsf{mod}(\mathcal{T})))$.*

Proof. We prove that Point 2 implies Point 1 and defer the other direction, which is more involved, to the long version. Assume that $\mathcal{I} \in \mathsf{cl}_\approx(\mathsf{cl}_\Pi(\mathsf{mod}(\mathcal{T})))$, but $d \in C^\mathcal{I} \setminus D^\mathcal{I}$ for some $d \in \Delta^\mathcal{I}$ and $\mathcal{EL}$-CI $C \sqsubseteq D$ with $\mathcal{T} \models C \sqsubseteq D$. Take a family $(\mathcal{I}_i)_{i \in I}$ of models of $\mathcal{T}$ and an $f \in \Delta^{\prod_{i \in I} \mathcal{I}_i}$ such that $(\mathcal{I}, d) \approx (\prod_{i \in I} \mathcal{I}_i, f)$. By Lemma 4, $f \in C^{\prod_{i \in I} \mathcal{I}_i} \setminus D^{\prod_{i \in I} \mathcal{I}_i}$ and by Lemma 8, there is an $i \in I$ with $f(i) \in C^{\mathcal{I}_i} \setminus D^{\mathcal{I}_i}$ in contradiction to $\mathcal{I}_i$ being a model of $\mathcal{T}$. ❏

To show that Theorem 9 is a consequence of Lemma 10, one uses the well-known fact that whenever $\mathcal{T}' \not\models C \sqsubseteq D$ for an $\mathcal{EL}$-TBox $\mathcal{T}'$, then there is an $\mathcal{I} \in \mathsf{mod}(\mathcal{T}')_{\restriction \mathsf{finout}}$ with $C^\mathcal{I} \setminus D^\mathcal{I} \neq \emptyset$. It remains open whether Theorem 9 holds without the restriction to finite outdegree.

5 $\mathcal{EL}$ automata

We introduce $\mathcal{EL}$ automata as a novel automaton model and establish some central properties. Like the automata introduced in (Janin and Walukiewicz 1995; Wilke 2001), an $\mathcal{EL}$ automaton $\mathcal{A}$ runs directly on interpretations. As we will see, $\mathcal{EL}$ automata share many crucial properties with $\mathcal{EL}$-TBoxes. Yet, they are strictly more expressive and we will prove that, in a sense to be made precise later, uniform $\mathcal{EL}_\Sigma$-interpolants of $\mathcal{EL}$-TBoxes and $\mathcal{EL}$-approximants of classical $\mathcal{ELU}$-TBoxes can *always* be represented by a (finite!) $\mathcal{EL}$ automaton. As illustrated by the examples in Sections 3 and 4, this is not the case for $\mathcal{EL}$-TBoxes.

Definition 11. An *$\mathcal{EL}$ automaton (EA)* is a tuple $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$, where Q is a finite set of *bottom up states*, P is a finite set of *top down states*, $\Sigma_N \subseteq \mathsf{N_C}$ is the finite *node alphabet*, $\Sigma_E \subseteq \mathsf{N_R}$ is the finite *edge alphabet*, and δ is a set of transitions of the following form:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\mathsf{true} & \to & q & p & \to & p_1 \\
A & \to & q & p & \to & \langle r \rangle p_1 \\
q_1 \wedge \cdots \wedge q_n & \to & q & p & \to & A \\
\langle r \rangle q_1 & \to & q & p & \to & \mathsf{false} \\
q & \to & p & & &
\end{array}$$

where q, $q_1, \ldots, q_n$ range over Q, p, p_1 range over P, A ranges over Σ_N, and r ranges over Σ_E.

The separation of states into bottom up states and top down states is crucial for attaining some relevant properties of $\mathcal{EL}$ automata, as discussed in more detail below.

Definition 12. Let $\mathcal{I}$ be an interpretation and $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ an EA. A *run* of $\mathcal{A}$ on $\mathcal{I}$ is a map $\rho : \Delta^\mathcal{I} \to 2^{Q \cup P}$ such that for all $d \in \Delta^\mathcal{I}$, we have:

1. if $\mathsf{true} \to q \in \delta$, then $q \in \rho(d)$;
2. if $A \to q \in \delta$ and $d \in A^\mathcal{I}$, then $q \in \rho(d)$;
3. if $q_1, \ldots, q_n \in \rho(d)$ and $q_1 \wedge \cdots \wedge q_n \to q \in \delta$, then $q \in \rho(d)$;

4. if $(d,e) \in r^{\mathcal{I}}$, $q_1 \in \rho(e)$, and $\langle r\rangle q_1 \to q \in \delta$, then $q \in \rho(d)$;
5. if $q \in \rho(d)$ and $q \to p \in \delta$, then $p \in \rho(d)$;
6. if $p \in \rho(d)$ and $p \to p_1 \in \delta$, then $p_1 \in \rho(d)$;
7. if $p \in \rho(d)$ and $p \to \langle r\rangle p_1 \in \delta$, then there is an $(d,e) \in r^{\mathcal{I}}$ with $p_1 \in \rho(e)$;
8. if $p \in \rho(d)$ and $p \to A \in \delta$, then $d \in A^{\mathcal{I}}$;
9. if $p \to \mathsf{false} \in \delta$, then $p \notin \rho(d)$.

We use $L(\mathcal{A})$ to denote the language accepted by $\mathcal{A}$, i.e., the set of interpretations $\mathcal{I}$ such that there is a run of $\mathcal{A}$ on $\mathcal{I}$.

Note that EAs run on interpretations that interpret all symbols in $\mathsf{N_C} \cup \mathsf{N_R}$, not just those in $\Sigma_N \cup \Sigma_E$. However, the transition relation can only use symbols from the latter set. Thus, the alphabets of an EA play the same role as a signature of an $\mathcal{EL}$-TBox.

As an example, consider the following EA, which accepts precisely those interpretations that satisfy the (non-$\mathcal{EL}$) CI $A_1 \sqcap \exists r^*.A_2 \sqsubseteq B$, where r^* is interpreted as the transitive and reflexive closure of r (a property not expressible by an $\mathcal{EL}$-TBox):

$$
\begin{array}{rclrcl}
Q & = & \{q_{A_1}, q_{A_2}, q_\wedge\} & P & = & \{p_B\} \\
\Sigma_N & = & \{A, B\} & \Sigma_E & = & \{r, s\} \\
\delta & = & \{ \quad A_1 \to q_{A_1} & A_2 & \to & q_{A_2} \\
 & & \langle r\rangle q_{A_2} \to q_{A_2} & q_{A_1} \wedge q_{A_2} & \to & q_\wedge \\
 & & q_\wedge \to p_B & p_B & \to & B \quad \}.
\end{array}
$$

The above example shows that EAs are more expressive than $\mathcal{EL}$-TBoxes. Conversely, every $\mathcal{EL}$-TBox is equivalent to some EA. We say that an EA $\mathcal{A}$ is *equivalent* to a TBox $\mathcal{T}$ if $L(\mathcal{A}) = \mathsf{mod}(\mathcal{T})$.

Proposition 13. *Every $\mathcal{EL}$-TBox $\mathcal{T}$ can be converted in polynomial time into an equivalent EA $\mathcal{A}_\mathcal{T}$.*

To prove Proposition 13, let $\mathcal{T}$ be a TBox, $\mathsf{sub}(\mathcal{T})$ the subconcepts of (concepts that occur in) $\mathcal{T}$, and define

$$\mathcal{A}_\mathcal{T} = (Q, P, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_R}, \delta)$$

where

- $Q = \{q_C \mid C \in \mathsf{sub}(\mathcal{T})\}$, $P = \{p_C \mid C \in \mathsf{sub}(\mathcal{T})\}$;
- δ consists of the following transitions:
  - $\mathsf{true} \to q_\top$ if $\top \in \mathsf{sub}(\mathcal{T})$;
  - $A \to q_A$ and $q_A \to p_A$ for all $A \in \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}$;
  - $q_C \wedge q_D \to q_{C\sqcap D}$;
  - $\langle r\rangle q_C \to q_{\exists r.C}$ and $q_{\exists r.C} \to \langle r\rangle p_C$ for all $\exists r.C \in \mathsf{sub}(\mathcal{T})$;
  - $q_C \to q_D$ for all $C, D \in \mathsf{sub}(\mathcal{T})$ with $\mathcal{T} \models C \sqsubseteq D$;
  - $p_A \to A$ for all $A \in \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}$;
  - $p_{\exists r.C} \to \langle r\rangle p_C$ for all $\exists r.C \in \mathsf{sub}(\mathcal{T})$;
  - $p_C \to p_D$ for all $C, D \in \mathsf{sub}(\mathcal{T})$ with $\mathcal{T} \models C \sqsubseteq D$;
  - $p_\bot \to \mathsf{false}$ if $\bot \in \mathsf{sub}(\mathcal{T})$.

Note that $\mathcal{A}_\mathcal{T}$ can be constructed in polynomial time since $\mathcal{EL}$-subsumptions $\mathcal{T} \models C \sqsubseteq D$ can be decided in PTIME (Baader, Brandt, and Lutz 2005).

Lemma 14. $L(\mathcal{A}_\mathcal{T}) = \mathsf{mod}(\mathcal{T})$.

Proof.(sketch) "$\supseteq$". Let $\mathcal{I} \in \mathsf{mod}(\mathcal{T})$. Define $\rho : \Delta^\mathcal{I} \to 2^{P\cup Q}$ by setting for $d \in \Delta^\mathcal{I}$

$$\rho(d) = \{q_C, p_C \mid d \in C^\mathcal{I}, C \in \mathsf{sub}(\mathcal{T})\}$$

One can show that ρ is a run of $\mathcal{A}_\mathcal{T}$ on $\mathcal{I}$. Hence $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$ as required.

"$\subseteq$". Let $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$. It can be proved by induction on the structure of C that for all $\mathcal{EL}$-concepts $C \in \mathsf{sub}(\mathcal{T})$, all runs ρ and all $d \in \Delta^\mathcal{I}$, we have

1. $d \in C^\mathcal{I}$ implies $q_C \in \rho(d)$;
2. $p_C \in \rho(d)$ implies $d \in C^\mathcal{I}$.

Since $\mathcal{A}_\mathcal{T}$ has transitions $q_C \to q_D$ and $q_D \to p_D$ for each $C \sqsubseteq D \in \mathcal{T}$, it follows that $\mathcal{I}$ satisfies $\mathcal{T}$. ❑

We now analyze some basic properties of $\mathcal{EL}$ automata, in particular the complexity of deciding emptiness and containment as well as closure properties of languages accepted by a single EA and the class of languages accepted by EAs in general.

Theorem 15. *For EAs, emptiness can be decided in* PTIME *and containment is* EXPTIME*-complete.*

Proof. (sketch) The PTIME upper bound for EA emptiness is proved by a straightforward reduction to unsatisfiability of $\mathcal{EL}$ concepts w.r.t. general TBoxes (recall that, in this paper, $\mathcal{EL}$ includes $\bot$), which is in PTIME (Baader, Brandt, and Lutz 2005); essentially, states are translated into concept names and each transition gives rise to one CI. The lower bound for containment is a consequence of the fact that deciding $\mathcal{EL}$-conservative extensions is EXPTIME-hard (Lutz and Wolter 2010) and, by Theorem 28 proved later on, can be reduced in polynomial time to containment of EAs. The upper bound is obtained by a polytime translation of EAs into alternating parity tree automata in the style of Wilke for which containment is known to be in EXPTIME (Wilke 2001). ❑

We now turn to closure properties. An interpretation $\mathcal{I}$ is the *disjoint union* of a (potentially infinite) family of interpretations $(\mathcal{I}_i)_{i\in I}$ with pairwise disjoint domains if $\Delta^\mathcal{I} = \bigcup_{i\in I} \Delta^{\mathcal{I}_i}$ and $X^\mathcal{I} = \bigcup_{i\in I} X^{\mathcal{I}_i}$ for all $X \in \mathsf{N_C} \cup \mathsf{N_R}$. For a class $\mathfrak{C}$ of interpretations, we use $\mathsf{cl}_\Delta(\mathfrak{C})$ to denote the *closure under disjoint unions* of $\mathfrak{C}$, i.e., the class of all interpretations $\mathcal{I}$ such that $\mathcal{I}$ is the disjoint union of a family of interpretations $(\mathcal{I}_i)_{i\in I}$ contained in $\mathfrak{C}$. It was shown in (Lutz, Piro, and Wolter 2011) that an FO-TBox $\mathcal{T}$ is equivalent to an $\mathcal{EL}$-TBox if and only if $\mathsf{mod}(\mathcal{T})$ is closed under global equisimulations, products, and disjoint unions. We show that EAs enjoy the same closure properties as $\mathcal{EL}$-TBoxes.

Lemma 16. *For every EA $\mathcal{A}$, $L(\mathcal{A})$ is closed under global equisimulations, products, and disjoint unions.*

Proof. $\mathsf{cl}_\Delta(L(\mathcal{A})) \subseteq L(\mathcal{A})$. Let $\mathcal{I}$ be the disjoint union of $\mathcal{I}_i \in L(\mathcal{A})$, $i \in I$, and let ρ_i be a run of $\mathcal{A}$ on $\mathcal{I}_i$. Then $\rho = \bigcup_{i\in I} \rho_i$ is a run of $\mathcal{A}$ on $\mathcal{I}$. Hence $\mathcal{I} \in L(\mathcal{A})$.

$\mathsf{cl}_\Pi(L(\mathcal{A})) \subseteq L(\mathcal{A})$. Let $\mathcal{I}$ be the product of $\mathcal{I}_i \in L(\mathcal{A})$, $i \in I$, and let ρ_i be a run of $\mathcal{A}$ on $\mathcal{I}_i$. Define $\rho : \prod_{i\in I} \Delta^{\mathcal{I}_i} \to 2^{Q\cup P}$ by setting for $f \in \prod_{i\in I} \Delta^{\mathcal{I}_i}$: $\rho(f) = \bigcap_{i\in I} \rho(f(i))$ One can show that ρ is a run of $\mathcal{A}$ on $\mathcal{I}$. Hence $\mathcal{I} \in L(\mathcal{A})$.

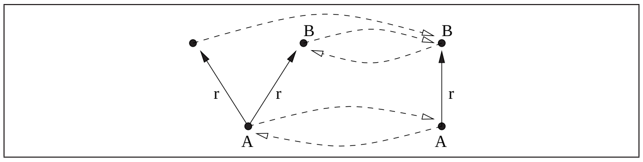

Figure 1: $\mathcal{I}_1$ and $\mathcal{I}_2$ with mutual simulations

$\mathsf{cl}_\approx(L(\mathcal{A})) \subseteq L(\mathcal{A})$. Let $\mathcal{I} \in \mathsf{cl}_\approx(L(\mathcal{A}))$. Take for every $d \in \Delta^\mathcal{I}$ an $\mathcal{I}_d \in L(\mathcal{A})$ and $f_d \in \Delta^{\mathcal{I}_d}$ such that $(\mathcal{I}, d) \approx (\mathcal{I}_d, f_d)$. Let $\mathcal{J}$ be the disjoint union of the $\mathcal{I}_d$, $d \in \Delta^\mathcal{I}$. Then $\mathcal{J} \in L(\mathcal{A})$, by Point 1. Let ρ_1 be a run of $\mathcal{A}$ on $\mathcal{J}$. Define ρ_2 by setting for $d \in \Delta^{\mathcal{I}_2}$:

$$\rho_2(d) = (Q \cap \bigcap_{(\mathcal{I},d) \leq (\mathcal{J},d')} \rho_1(d')) \cup (P \cap \bigcup_{(\mathcal{I},d) \geq (\mathcal{J},d')} \rho_1(d'))$$

One can show that ρ_2 is a run of $\mathcal{A}$ on $\mathcal{I}$. ❑

We conjecture that the operations of Lemma 16 characterize languages $L(\mathcal{A})$ of EAs $\mathcal{A}$ within MSO (monadic second-order logic). An MSO-TBox is defined in the same way as an FO-TBox with the exception that we admit second-order quantifiers for sets.

Conjecture 17. *For any MSO-TBox $\mathcal{T}$, there exists an EA $\mathcal{A}$ with $\mathsf{mod}(\mathcal{T}) = L(\mathcal{A})$ iff $\mathsf{mod}(\mathcal{T})$ is closed under global equisimulations, products, and disjoint unions.*

Note that closure under global equisimulations fails when we give up the separation of states into bottom-up and top-down states. To see this, consider the following non-EA:

$$\begin{array}{rclrcl} Q & = & \{q, q'\} & P & = & \{p_B\} \\ \Sigma_N & = & \{A, B\} & \Sigma_E & = & \{r\} \\ \delta & = & \multicolumn{4}{l}{\{A \rightarrow \langle r \rangle q,\ B \rightarrow q',\ q \wedge q' \rightarrow \bot\ \}.} \end{array}$$

With the natural definition of runs, this EA would accept the interpretation $\mathcal{I}_1$ on the left-hand side of Figure 1, but not the interpretation $\mathcal{I}_2$ on the right-hand side. However, as indicated by the dashed edges, $\mathcal{I}_2 \in \mathsf{cl}_\approx(\{\mathcal{I}_1\})$.

In passing, we also summarize some closure properties of the class of languages that can be accepted with the new automaton model.

Observation 18. *The class of languages accepted by EAs is closed under intersection. It is not closed under complementation and union.*

6 From $\mathcal{EL}$ automata to $\mathcal{EL}$-TBoxes

We start with making precise what it means for an $\mathcal{EL}$ automaton to represent a uniform interpolant or an $\mathcal{EL}$-approximant. For an EA $\mathcal{A}$, we write $\mathcal{A} \models C \sqsubseteq D$ if every $\mathcal{I} \in L(\mathcal{A})$ satisfies $C \sqsubseteq D$. This allows us to speak about $\mathcal{EL}_\Sigma$-entailment and $\mathcal{EL}_\Sigma$-inseparability between EAs and TBoxes. For example, an EA $\mathcal{A}$ and a TBox $\mathcal{T}$ are $\mathcal{EL}_\Sigma$-inseparable, in symbols $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$, if $\mathcal{A} \models C \sqsubseteq D$ iff $\mathcal{T} \models C \sqsubseteq D$ for all $\mathcal{EL}_\Sigma$ inclusions $C \sqsubseteq D$. We say that $\mathcal{A}$

- *represents the uniform $\mathcal{EL}_\Sigma$-interpolant of an $\mathcal{EL}$-TBox $\mathcal{T}$* if $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$ and $\Sigma = \Sigma_N \cup \Sigma_E$ is the signature of $\mathcal{A}$ (see the note after Definition 1);
- *represents the $\mathcal{EL}$-approximant of a classical $\mathcal{ELU}$-TBox $\mathcal{T}$* if $\mathcal{A} \equiv^{\mathcal{EL}} \mathcal{T}$ (see the note after Definition 6).

A main aim of this paper is to give decision procedures for the existence of uniform interpolants and $\mathcal{EL}$-approximants (represented as a TBox). The approach is to construct an EA $\mathcal{A}$ that represents the interpolant/approximant, and then to decide for a suitable signature Σ whether there exists an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ such that $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$. In this section, we show that the latter can be done in ExpTime. In fact, we will later construct the relevant EAs $\mathcal{A}$ such that the 'suitable signature' Σ is simply the signature of $\mathcal{A}$. Our aim is thus to prove the following.

Theorem 19. *Given an EA $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$, it can be decided in* ExpTime *whether there is an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ such that $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$, where $\Sigma = \Sigma_N \cup \Sigma_E$.*

To prove Theorem 19, we proceed in two steps: first, we characterize the non-existence of an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ with $\mathcal{A} \equiv_\Sigma^{\mathcal{EL}} \mathcal{T}$ in terms of the existence of certain interpretations, and then we show that the latter can be checked in ExpTime using alternating parity tree automata (APTAs) in the form defined by Wilke (Wilke 2001). The technical tools developed in this section can be regarded as '$\mathcal{EL}$-analogues' of the tools developed for deciding the existence of uniform $\mathcal{ALC}$-interpolants in (Lutz and Wolter 2011). All proofs are deferred to the long version of this paper.

Let $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be an EA and $\Sigma = \Sigma_N \cup \Sigma_E$. First observe that it is trivial to find an *infinite* TBox that is $\mathcal{EL}_\Sigma$-inseparable from $\mathcal{A}$, namely the set

$$\mathcal{T}_\Sigma(\mathcal{A}) = \{C \sqsubseteq D \mid \mathcal{A} \models C \sqsubseteq D,\ C, D\ \mathcal{EL}_\Sigma\text{-concepts}\}.$$

Candidates for finite TBoxes with the same property are provided by the following subsets of $\mathcal{T}_\Sigma(\mathcal{A})$. The *role depth* $\mathsf{rd}(C)$ of a concept C is the nesting depth of existential restrictions in C. For every $m \geq 0$, we can fix a *finite* set $\mathcal{EL}_f^m(\Sigma)$ of $\mathcal{EL}_\Sigma$-concepts D with $\mathsf{rd}(D) \leq m$ such that every $\mathcal{EL}_\Sigma$-concept C with $\mathsf{rd}(C) \leq m$ is equivalent to some $D \in \mathcal{EL}_f^m(\Sigma)$. Set

$$\mathcal{T}_\Sigma^m(\mathcal{A}) = \{C \sqsubseteq D \mid \mathcal{A} \models C \sqsubseteq D \text{ and } C, D \in \mathcal{EL}_f^m(\Sigma)\}.$$

Obviously, the following are equivalent:

(a) no (finite!) $\mathcal{EL}_\Sigma$-TBox is $\mathcal{EL}_\Sigma$-inseparable from $\mathcal{A}$;

(b) $\mathcal{T}_\Sigma^m(\mathcal{A})$ is not $\mathcal{EL}_\Sigma$-inseparable from $\mathcal{A}$, for all $m \geq 0$;

(c) for all $m \geq 0$ there is a $k > m$ s.t. $\mathcal{T}_\Sigma^m(\mathcal{A}) \not\models \mathcal{T}_\Sigma^k(\mathcal{A})$.

We now present the announced characterization of (a). An interpretation $\mathcal{I}$ is a *tree-interpretation* if $(\Delta^\mathcal{I}, \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I})$ is a (possibly infinite) tree and $r^\mathcal{I} \cap s^\mathcal{I} = \emptyset$ for any two distinct $r, s \in \mathsf{N_R}$. By $\rho^\mathcal{I}$ we denote the root of $\mathcal{I}$. Note that every $\mathcal{EL}$-TBox $\mathcal{T}$ is determined by tree interpretations: if $\mathcal{T} \not\models C \sqsubseteq D$, then there exists a tree interpretation $\mathcal{I}$ that is a model of $\mathcal{T}$ such that $\rho^\mathcal{I} \in C^\mathcal{I} \setminus D^\mathcal{I}$. For $m \geq 0$, we use $\mathcal{I}^{\leq m}$ to denote the restriction of $\mathcal{I}$ to those elements of $\Delta^\mathcal{I}$ that can be reached from $\rho^\mathcal{I}$ in at most m steps in the graph $(\Delta^\mathcal{I}, \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I})$. For any $d \in \Delta^\mathcal{I}$, $\mathcal{I}(d)$ denotes the restriction of $\mathcal{I}$ to those elements of $\Delta^\mathcal{I}$ that can be reached from d in the graph $(\Delta^\mathcal{I}, \bigcup_{r \in \mathsf{N_R}} r^\mathcal{I})$.

The proof of the following result consists of an adaptation of the proof of Theorem 9 in (Lutz and Wolter 2011) and additionally requires the use and analysis of "m-equisimulations", which capture the expressive power of $\mathcal{EL}$-concept of role-depth $\leq m$.

Theorem 20. *Let $\mathcal{A}$ be an EA. For all $m > 0$, the following conditions are equivalent:*

1. *There exists $k > m$ such that $\mathcal{T}^m_\Sigma(\mathcal{A}) \not\models \mathcal{T}^k_\Sigma(\mathcal{A})$;*
2. *there exist two tree interpretations, $\mathcal{I}_1$ and $\mathcal{I}_2$, of finite outdegree such that*
   - $\mathcal{I}_1^{\leq m} = \mathcal{I}_2^{\leq m}$;
   - $\mathcal{I}_1 \in L(\mathcal{A})$;
   - $\mathcal{I}_2 \notin L(\mathcal{A})$;
   - *$\mathcal{I}_2(d) \in L(\mathcal{A})$, for all d such that $(\rho^{\mathcal{I}_2}, d) \in r^{\mathcal{I}_2}$ for some r.*

To provide the announced characterization of (a), we prove that rather than testing Point 2 of Theorem 20 for all m, it suffices to consider a single number m. Technically, this is achieved by a pumping argument as in (Lutz and Wolter 2011). Since we are concerned with $\mathcal{EL}$ rather than with $\mathcal{ALC}$, we obtain a single exponential bound on m rather than a double exponential one. The proof makes intensive use of what we call *canonical pre-runs* of EAs, the automaton counterpart of the well-known canonical models of an $\mathcal{EL}$-TBox (Lutz, Toman, and Wolter 2009).

Theorem 21. *Let $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be an EA and $\Sigma = \Sigma_N \cup \Sigma_E$. There does not exist an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ with $\mathcal{A} \equiv^{\mathcal{EL}}_\Sigma \mathcal{T}$ iff Point 2 of Theorem 20 holds with $m = 2^{2|Q\cup P|} + 1$.*

We now turn to the second step in the proof of Theorem 19 and show that the characterization provided by Theorem 21 leads to an EXPTIME decision procedure with an APTA emptiness check at its basis. We refer to the long version of this paper for a precise definition of APTAs. Like EAs, APTAs run directly on interpretations (actually, pointed interpretations). The proof of the following result is an easy adaptation of the proof of the corresponding theorem in (Lutz and Wolter 2011).

Theorem 22. *Let $\mathcal{A} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be an EA, $\Sigma = \Sigma_N \cup \Sigma_E$, and $m \geq 0$. Then there is an APTA $\mathcal{A}_{\mathcal{A},\Sigma,m}$ with state set Q and node and edge alphabets Σ'_N and Σ'_E such that $L(\mathcal{A}_{\mathcal{A},\Sigma,m}) \neq \emptyset$ iff Point 2 of Theorem 20 is satisfied. Moreover, $|Q| \in \mathcal{O}(n + \log^2 m)$ and $|\Sigma'_N|, |\Sigma'_E| \in \mathcal{O}(n + \log m)$, where $n = |Q \cup P|$.*

The size of $\mathcal{A}_{\mathcal{T},\Sigma,m}$ is polynomial in $|Q\cup P|$ and logarithmic in m. By Theorem 21, we can use $m = 2^{2|Q\cup P|}+1$, and thus the size of $\mathcal{A}_{\mathcal{T},\Sigma,m}$ is polynomial in $|Q\cup P|$. As emptiness of APTAs can be decided in EXPTIME, we obtain an EXPTIME decision procedure for (a).

The above algorithm decides for a given EA $\mathcal{A}$ whether there is an $\mathcal{EL}_\Sigma$-TBox that has the same $\mathcal{EL}_\Sigma$-CIs as consequences as $\mathcal{A}$, with Σ the signature of $\mathcal{A}$. It is interesting to note (and will be useful in the subsequent section) that this is equivalent to the existence of an $\mathcal{EL}_\Sigma$-TBox $\mathcal{T}$ that is *equivalent to* $\mathcal{A}$ in the sense that $L(\mathcal{A}) = \mathsf{mod}(\mathcal{T})$. This is a consequence of Proposition 13 and the following, proved in the appendix.

Proposition 23. *For all EAs $\mathcal{A}_1, \mathcal{A}_2$ over the same alphabets Σ_N and Σ_E, we have $\mathcal{T}_\Sigma(\mathcal{A}_2) \subseteq \mathcal{T}_\Sigma(\mathcal{A}_1)$ iff $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$ when $\Sigma = \Sigma_N \cup \Sigma_E$.*

7 Automata for Uniform Interpolants and Conservative Extensions

We prove that deciding the existence of uniform $\mathcal{EL}_\Sigma$-interpolants is EXPTIME-complete. We also show that, using EAs and bypassing the machinery in Section 6, conservative extensions can be decided in EXPTIME. A matching lower bound is known from (Lutz and Wolter 2010). We first show that for every $\mathcal{EL}$-TBox $\mathcal{T}$ and signature Σ, there is an EA that represents the uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$. In fact, the EA is even stronger than required at the beginning of Section 6 as it accepts precisely the 'right' class of models rather than only have the 'right' $\mathcal{EL}$-consequences.

Theorem 24. *Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox and $\Sigma \subseteq \mathsf{sig}(\mathcal{T})$ a signature. Then one can construct in polynomial time an EA $\mathcal{A}_{\mathcal{T},\Sigma} = (Q, P, \Sigma \cap \mathsf{N_C}, \Sigma \cap \mathsf{N_R}, \delta)$ with $|Q \cup P| \in \mathcal{O}(|\mathcal{T}|)$ such that $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}))$.*

More specifically, the automaton $\mathcal{A}_{\mathcal{T},\Sigma}$ for Theorem 24 can be constructed as follows. Let $\mathcal{A}_\mathcal{T} = (Q, P, \Sigma_N, \Sigma_E, \delta)$ be the automaton from the proof of Proposition 13. We have $\Sigma \subseteq \Sigma_N \cup \Sigma_E$. The Σ-*restriction* of $\mathcal{A}_\mathcal{T}$ is the automaton $(Q, P, \Sigma \cap \Sigma_N, \Sigma \cap \Sigma_E, \delta')$ where δ' is obtained from δ by dropping all transitions $A \to q$ and $p \to A$ with $A \notin \Sigma$ and all transitions $\langle r \rangle q_1 \to q$ and $p \to \langle r \rangle p_1$ with $r \notin \Sigma$. Call the resulting automaton $\mathcal{A}_{\mathcal{T},\Sigma}$. By the following lemma, it is as required for Theorem 24.

Lemma 25. $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}))$.

Proof.(sketch) "$\supseteq$". By Lemma 16, it is sufficient to show that $\mathsf{mod}(\mathcal{T}) \subseteq L(\mathcal{A}_{\mathcal{T},\Sigma})$. This can be done as in the proof of Lemma 14.

"$\subseteq$". Let $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T},\Sigma})$ and ρ be a run of $\mathcal{A}_{\mathcal{T},\Sigma}$ on $\mathcal{I}$. We construct a model $\mathcal{J}$ of $\mathcal{T}$ such that for all $d \in \Delta^\mathcal{I}$, there is an $e \in \Delta^\mathcal{J}$ with $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, e)$. Fix, for each $D = \exists r.C \in \mathsf{sub}(\mathcal{T})$ and each $d \in \Delta^\mathcal{I}$ with $q_D \in \rho(d)$ a least tree model $\mathcal{J}_{D,d}$ of D and $\mathcal{T}$, i.e., $\mathcal{J}_{D,d}$ is a tree model of $\mathcal{T}$ with root $d \in D^{\mathcal{J}_{D,d}}$ and for all models $\mathcal{J}$ of $\mathcal{T}$ and $e \in D^\mathcal{J}$, we have $(\mathcal{J}_{D,d}, d) \leq (\mathcal{J}, e)$. Assume w.l.o.g. that the domains of all chosen models are pairwise disjoint, and that each model $\mathcal{J}_{D,d}$ shares with $\mathcal{I}$ only the domain element d. Let Γ be the set of all models $\mathcal{J}_{D,d}$ chosen. Now set

$$\begin{aligned}
\Delta^\mathcal{J} &= \Delta^\mathcal{I} \cup \bigcup_{\mathcal{J}_{D,d}\in\Gamma} \Delta^{\mathcal{J}_{D,d}} \\
A^\mathcal{J} &= \{d \in \Delta^\mathcal{I} \mid q_A \in \rho(d)\} \cup \bigcup_{\mathcal{J}_{D,d}\in\Gamma} A^{\mathcal{J}_{D,d}} \\
r^\mathcal{J} &= r^\mathcal{I} \cup \bigcup_{\mathcal{J}_{D,d}\in\Gamma} r^{\mathcal{J}_{D,d}}
\end{aligned}$$

In the long version, we carefully analyze $\mathcal{J}$ to show that it is a model of $\mathcal{T}$ and, as intended, $(\mathcal{I}, d) \approx_\Sigma (\mathcal{J}, e)$ for $e = d$. ❑

We are now able to deliver our promise and prove the "⇒" direction of Point 1 of Theorem 5 without the restriction to interpretations of finite outdegree. To this end, assume that $\mathcal{T}$ Σ-entails $\mathcal{T}'$ and let $\mathcal{I}$ be a model of $\mathcal{T}$. By definition, $\mathcal{T}_\Sigma(\mathcal{A}_{\mathcal{T}',\Sigma}) \subseteq \mathcal{T}_\Sigma(\mathcal{A}_{\mathcal{T},\Sigma})$. Hence, by Proposition 23, $L(\mathcal{A}_{\mathcal{T},\Sigma}) \subseteq L(\mathcal{A}_{\mathcal{T}',\Sigma})$. Summing up, we thus have

$$\mathcal{I} \in \mathsf{mod}(\mathcal{T}) \subseteq L(\mathcal{A}_{\mathcal{T},\Sigma}) \subseteq L(\mathcal{A}_{\mathcal{T}',\Sigma}) \subseteq \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}')$$

and are done. We come to the main theorem of this section.

Theorem 26. *Given an $\mathcal{EL}$-TBox $\mathcal{T}$ and signature Σ, it can be decided in* ExpTime *whether there is an $\mathcal{EL}$-TBox that is the uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$.*

Proof. By Theorem 24, $\mathcal{T}$ can be converted in polynomial time into an EA $\mathcal{A}_{\mathcal{T},\Sigma}$ such that $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}))$. It follows by Lemma 4 that $\mathcal{T}$ and $\mathcal{A}_{\mathcal{T},\Sigma}$ are $\mathcal{EL}_\Sigma$-inseparable, i.e., $\mathcal{A}_{\mathcal{T},\Sigma}$ represents the uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$. It remains to apply Theorem 19. ❑

A matching lower bound can be proved by a careful analysis and adaptation of the ExpTime lower bound for deciding $\mathcal{EL}$-conservative extensions given in (Lutz and Wolter 2010).

Theorem 27. *Given an $\mathcal{EL}$-TBox $\mathcal{T}$ and signature Σ, it is* ExpTime*-hard to decide whether there is an $\mathcal{EL}$-TBox that is the uniform $\mathcal{EL}_\Sigma$-interpolant of $\mathcal{T}$.*

We now turn our attention to conservative extensions. It was originally proved in (Lutz and Wolter 2010) that deciding conservative extensions in $\mathcal{EL}$ is ExpTime-complete. We give an alternative proof of the upper bound based on EA containment, which is arguably more transparent than the original proof. Note that our proof completely bypasses the (somewhat intricate) machinery established in Section 6 and only relies on EA containment. The construction actually works for the more general problem of $\mathcal{EL}_\Sigma$-entailment.

Lemma 28. *Let $\mathcal{T}$ and $\mathcal{T}'$ be $\mathcal{EL}$-TBoxes and Σ a signature. Then $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ iff $L(\mathcal{A}_{\mathcal{T},\Sigma}) \subseteq L(\mathcal{A}_{\mathcal{T}',\Sigma})$.*

Proof. (⇒) Suppose $\mathcal{T}$ $\mathcal{EL}_\Sigma$-entails $\mathcal{T}'$ and $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T},\Sigma})$. To prove that $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T}',\Sigma})$, since $L(\mathcal{A}_{\mathcal{T}',\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$ it is sufficient to show $\mathcal{I} \in \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$. Let $d \in \Delta^\mathcal{I}$. Since $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}))$, there is a model $\mathcal{I}_1$ of $\mathcal{T}$ and a $d_1 \in \Delta^{\mathcal{I}_1}$ such that $(\mathcal{I}, d) \approx_\Sigma (\mathcal{I}_1, d_1)$. By Theorem 5, there is a model $\mathcal{I}_2$ of $\mathcal{T}'$ and a $d_2 \in \Delta^{\mathcal{I}_2}$ such that $(\mathcal{I}_1, d_1) \approx_\Sigma (\mathcal{I}_2, d_2)$. By closure under composition of Σ-equisimulations, $(\mathcal{I}, d) \approx_\Sigma (\mathcal{I}_2, d_2)$ as required.

(⇐) Suppose $L(\mathcal{A}_{\mathcal{T},\Sigma}) \subseteq L(\mathcal{A}_{\mathcal{T}',\Sigma})$. By Theorem 5, it suffices to show that $\mathsf{mod}(\mathcal{T}) \subseteq \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$. To this aim, let $\mathcal{I}$ be a model of $\mathcal{T}$. Since $L(\mathcal{A}_{\mathcal{T},\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}))$, $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T},\Sigma})$ and, therefore, $\mathcal{I} \in L(\mathcal{A}_{\mathcal{T}',\Sigma})$. Since $L(\mathcal{A}_{\mathcal{T}',\Sigma}) = \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$, this implies $\mathcal{I} \in \mathsf{cl}^\Sigma_\approx(\mathsf{mod}(\mathcal{T}'))$, as required. ❑

Together with Theorem 19, we obtain the desired result.

Theorem 29. *(Lutz and Wolter 2010) $\mathcal{EL}_\Sigma$-entailment and $\mathcal{EL}$-conservative extensions can be decided in* ExpTime.

8 Automata for Approximation

We prove that the existence of $\mathcal{EL}$-approximants of a classical $\mathcal{ELU}$-TBox can be decided in 2ExpTime using EAs. An ExpTime lower bound is established as well, but the precise complexity remains open.

In analogy to what was done in the previous section, the central observation is that for every classical $\mathcal{ELU}$-TBox $\mathcal{T}$, there is an EA $\mathcal{A}_\mathcal{T}$ of exponential size that represents the $\mathcal{EL}$-approximant of $\mathcal{T}$.

Theorem 30. *Let $\mathcal{T}$ be a classical $\mathcal{ELU}$-TBox. Then one can construct an EA $\mathcal{A}_\mathcal{T} = (Q, P, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_R}, \delta)$ with $|Q \cup P| \in \mathcal{O}(2^{|\mathcal{T}|})$ such that $\mathcal{A}_\mathcal{T} \equiv^{\mathcal{EL}} \mathcal{T}$.*

More specifically, the automaton $\mathcal{A}_\mathcal{T}$ from Theorem 30 can be constructed as follows. Let $\mathcal{T}$ be a classical $\mathcal{ELU}$-TBox, $\mathsf{sub}(\mathcal{T})$ the subconcepts of $\mathcal{T}$, and $\mathsf{dis}(\mathcal{T})$ the set of all disjunctions of concepts from $\mathsf{sub}(\mathcal{T})$ that do not contain duplicate disjuncts. Define the EA $\mathcal{A}_\mathcal{T} = (Q, P, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_C}, \mathsf{sig}(\mathcal{T}) \cap \mathsf{N_R}, \delta)$ as follows:

- $Q = \{q_C \mid C \in \mathsf{dis}(\mathcal{T})\}$ and $P = \{p_C \mid C \in \mathsf{dis}(\mathcal{T})\}$;
- δ consists of the following transitions:
  - $\mathsf{true} \to q_\top$;
  - $A \to q_A$ and $q_A \to p_A$ for all $A \in \mathsf{sub}(C) \cap \mathsf{N_C}$;
  - $q_{C_1} \wedge \cdots \wedge q_{C_n} \to q_C$ for all $C_1, \ldots, C_n, C \in \mathsf{dis}(\mathcal{T})$ with $\mathcal{T} \models C_1 \sqcap \cdots \sqcap C_n \sqsubseteq C$;
  - $\langle r \rangle q_C \to q_D$ for all $C, D \in \mathsf{dis}(\mathcal{T})$ and $r \in \Sigma_E$ with $\mathcal{T} \models \exists r.C \sqsubseteq D$;
  - $q_C \to p_C$ for all $C \in \mathsf{dis}(\mathcal{T})$;
  - $p_C \to A$ for all $C \in \mathsf{dis}(\mathcal{T})$ and $A \in \mathsf{sub}(\mathcal{T}) \cap \mathsf{N_C}$ such that $\mathcal{T} \models C \sqsubseteq A$;
  - $p_C \to \langle r \rangle p_D$ for all $C, D \in \mathsf{dis}(\mathcal{T})$ and $r \in \Sigma_E$ such that $\mathcal{T} \models C \sqsubseteq \exists r.D$.

Using completeness w.r.t. interpretations of finite outdegree, Theorem 30 follows from the following lemma.

Lemma 31.

1. *If $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$, then $\mathcal{I} \models C \sqsubseteq D$ for all $\mathcal{EL}$-CIs $C \sqsubseteq D$ with $\mathcal{T} \models C \sqsubseteq D$.*
2. *If $\mathcal{I}$ has finite outdegree and $\mathcal{I} \models C \sqsubseteq D$ for all $\mathcal{EL}$-CIs with $\mathcal{T} \models C \sqsubseteq D$, then $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$.*

Proof. The proof of (1.) is given in the long version. For (2.), by Lemma 10, is is sufficient to show $\mathsf{cl}_\approx(\mathsf{cl}_\Pi(\mathsf{mod}(\mathcal{T}))) \subseteq L(\mathcal{A}_\mathcal{T})$. This follows from Lemma 16 if $\mathsf{mod}(\mathcal{T}) \subseteq L(\mathcal{A}_\mathcal{T})$. To show the inclusion let $\mathcal{I} \in \mathsf{mod}(\mathcal{T})$. Define a mapping $\rho : \Delta^\mathcal{I} \to 2^{P \cup Q}$ by setting $\rho(d) = \{q_C, p_C \mid d \in C^\mathcal{I}, C \in \mathsf{dis}(\mathcal{T})\}$. One can show that ρ is a run of $\mathcal{A}_\mathcal{T}$ on $\mathcal{I}$; hence $\mathcal{I} \in L(\mathcal{A}_\mathcal{T})$. ❑

Point 2 of Lemma 31 fails without the restriction to finite outdegree. To see this, consider the example TBox $\mathcal{T}$ about nodes in a tree given in Section 4. Let $\mathcal{I}$ be an interpretation that has a root node d_0 satisfying Node, Left, and Right and having outgoing succ-chains of unbounded finite length (but no infinite such chain!). $\mathcal{I}$ satisfies all $\mathcal{EL}$-CIs that follow from $\mathcal{T}$, but $\mathcal{I} \notin L(\mathcal{A}_\mathcal{T})$. We come to the main theorem of this section.

Theorem 32. *Given a classical $\mathcal{ELU}$-TBox $\mathcal{T}$, it can be decided in* 2EXPTIME *whether there is an $\mathcal{EL}$-TBox that is the $\mathcal{EL}$-approximant of $\mathcal{T}$.*

Proof. Construct the automaton $\mathcal{A}_\mathcal{T}$ from the proof of Theorem 30 and apply the APTA-based decision procedure underlying Theorem 19. Despite the double exponential number of transitions of $\mathcal{A}_\mathcal{T}$, one can achieve a 2EXPTIME-procedure by using on-the-fly versions of the construction of $\mathcal{A}_\mathcal{T}$ and of the APTA $\mathcal{A}_{\mathcal{A},\Sigma,m}$ from Theorem 22 when checking emptiness of the latter. ❑

We do not know whether the bound established in Theorem 32 is tight. However, we have the following.

Theorem 33. *Given a classical $\mathcal{ELU}$-TBox $\mathcal{T}$, it is* EXPTIME*-hard to decide whether there is an $\mathcal{EL}$-TBox that is the $\mathcal{EL}$-approximant of $\mathcal{T}$.*

Proof. The proof is a modification of the EXPTIME hardness proof in (Haase and Lutz 2008) for subsumption in classical $\mathcal{ELU}$-TBoxes. Denote by $\mathcal{EL}^\neg$ the extension of $\mathcal{EL}$ with negation. Since $\mathcal{EL}^\neg$ has the same expressive power as $\mathcal{ALC}$, it is EXPTIME-complete to decide whether a classical $\mathcal{EL}^\neg$-TBox is satisfiable. We reduce this problem to deciding whether there is an $\mathcal{EL}$-approximant of an $\mathcal{ELU}$-TBox.

Let $\mathcal{T}$ be a classical $\mathcal{EL}^\neg$-TBox. We may assume w.l.o.g. that $\mathcal{T}$ contains only one role name r and is of the form

$$\{A_1 \equiv C_1, \dots, A_n \equiv C_n\},$$

where each C_i is of the form $\top$, P, $\neg B$, $\exists r.B$ or $B_1 \sqcap B_2$ with P not occurring on the left-hand side of any CI in $\mathcal{T}$ (a 'primitive' concept name in $\mathcal{T}$) and B, B_1, B_2 occurring on a left-hand side ('defined' concept names in $\mathcal{T}$).

We convert $\mathcal{T}$ into a classical $\mathcal{ELU}$-TBox $\mathcal{T}'$ such that $\mathcal{T}$ is satisfiable iff there is no $\mathcal{EL}$-approximant of $\mathcal{T}'$. Introduce new concept names $\overline{A}_i$, $1 \leq i \leq n$, which intuitively represent $\neg A_i$. Let D be an abbreviation for

$$\bigsqcap_{1 \leq i \leq n} (A_i \sqcup \overline{A}_i).$$

We also use new concept names E_1, E_2, F and M and a new role name s. To obtain $\mathcal{T}'$, first replace in $\mathcal{T}$

- every $A_i \equiv \top$ with $A_i \equiv D \sqcup M$;
- every $A_i \equiv P$ with $A_i \equiv (P \sqcap D) \sqcup M$;
- every $A_i \equiv \neg A_j$ with $A_i \equiv (\overline{A}_j \sqcap D) \sqcup M$;
- every $A_i \equiv B_1 \sqcap B_2$ with $A_i \equiv (B_1 \sqcap B_2 \sqcap D) \sqcup M$;
- every $A_i \equiv \exists r.B$ with $A_i \equiv (D \sqcap \exists r.(B \sqcap D)) \sqcup M$

and then add

$$\begin{aligned} M &\equiv \exists r.M \sqcup \bigsqcup_{1 \leq i \leq n} (\overline{A_i} \sqcap A_i) \\ E_i &\sqsubseteq (D \sqcap \exists s.(D \sqcap (E_1 \sqcup E_2))), \text{ for } i = 1, 2 \\ F &\equiv (E_1 \sqcup E_2) \sqcap M. \end{aligned}$$

Intuitively, M plays the role of a marker that is set whenever there is an 'inconsistency' in the sense that a concept name A_i and its 'negation' $\overline{A}_i$ are both true at the same element. By the second last CI, we have $\mathcal{T} \models E_i \sqsubseteq \exists s^n.\top$ for $i \in \{1, 2\}$ which results in $\mathcal{T}'$ not being $\mathcal{EL}$-approximable when $\mathcal{T}$ is satisfiable. If $\mathcal{T}$ is not satisfiable, then the inconsistency marker M is true at the relevant elements in every model, which means that the above $\mathcal{EL}$-consequences can be approximated by $E_i \sqsubseteq \exists s.F$. The overall $\mathcal{EL}$-approximation of $\mathcal{T}'$ contains additional concept inclusions, as analyzed in more detail in the appendix of the long version. ❑

9 Future Work

Two interesting research directions are to exploit the techniques introduced in this paper for the actual computation of interpolants and approximants, and to extend the decision procedure for the existence of $\mathcal{EL}$-aproximants from classical $\mathcal{ELU}$-TBoxes to more expressive DLs and TBox formalisms.

Regarding the first point, we conjecture that EAs that represent uniform interpolants/approximants can be exploited to compute small interpolants/approximants, if they exist. A refinement of the proof of Proposition 23 given in the long version should be a promising starting point.

Regarding the second point, note that a naive adaptation of our decision procedure to TBoxes formulated in expressive DLs such as $\mathcal{ALC}$ does not work. For example, the $\mathcal{ALC}$-TBox $\mathcal{T} = \{A \sqsubseteq \forall r.B\}$ does not have an $\mathcal{EL}$-approximant. To see this, note that $\mathcal{T}$ has

$$A \sqcap \exists r.C \sqsubseteq \exists r.(B \sqcap C)$$

as an $\mathcal{EL}$-consequence, for any $\mathcal{EL}$-concept C. In particular, we thus have as a consequence the $\mathcal{EL}$-inclusion

$$A \sqcap \exists r.\exists s^n.E \sqsubseteq \exists r.(B \sqcap \exists s^m.E)$$

iff $n = m$, where E and s are fresh concept and role names. Semantically, this means that, for every rs^n-path in an interpretation $\mathcal{I}$ that starts at an element $d \in A^\mathcal{I}$, there must exist a (potentially different) rs^n-path that starts at d and satisfies B at the second element. It is standard to show that such languages, which involve 'unbounded counting', are not recognizable by an APTA (with parity/Rabin acceptance condition) and, therefore, not by an EA either.

Interestingly, a similar effect can be observed already for general $\mathcal{ELU}$-TBoxes. As an example, consider

$$\begin{aligned} \mathcal{T}' = \{A \sqcap \exists r.\overline{B} &\sqsubseteq \exists r.(B \sqcap (S \sqcup S')) \\ (S \sqcup S') &\sqsubseteq \exists s.(S \sqcup S') \\ D &\sqsubseteq B \sqcup \overline{B}\ \} \end{aligned}$$

$\mathcal{T}'$ does not have an $\mathcal{EL}$-approximant, for similar reasons as the above $\mathcal{ALC}$-TBox $\mathcal{T}$. As a hint, note that the $\mathcal{EL}$-inclusion

$$A \sqcap \exists r.(D \sqcap \exists s^n.E) \sqsubseteq \exists r.(B \sqcap \exists s^m.E)$$

is a consequence of $\mathcal{T}$ if, and only if, $n = m$. Despite these difficulties, one can well imagine EAs as a central ingredient of a more intricate procedure for checking $\mathcal{EL}$-approximability of general $\mathcal{ELU}$- or even $\mathcal{ALC}$-TBoxes, as a second step after first checking recognizability by APTAs.

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