

# Homogeneous Logical Proportions: Their Uniqueness and Their Role in Similarity-Based Prediction

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## Abstract

Given a 4-tuple of Boolean variables  $(a, b, c, d)$ , logical proportions are modeled by a pair of equivalences relating similarity indicators  $(a \wedge b$  and  $\bar{a} \wedge \bar{b})$ , or dissimilarity indicators  $(a \wedge \bar{b}$  and  $\bar{a} \wedge b)$  pertaining to the pair  $(a, b)$ , to the ones associated with the pair  $(c, d)$ . Logical proportions are homogeneous when they are based on equivalences between indicators of the same kind. There are only 4 such homogeneous proportions, which respectively express that i) “ $a$  differs from  $b$  as  $c$  differs from  $d$ ” (and “ $b$  differs from  $a$  as  $d$  differs from  $c$ ”), ii) “ $a$  differs from  $b$  as  $d$  differs from  $c$ ” (and “ $b$  differs from  $a$  as  $c$  differs from  $d$ ”), iii) “what  $a$  and  $b$  have in common  $c$  and  $d$  have it also”, iv) “what  $a$  and  $b$  have in common neither  $c$  nor  $d$  have it”. We prove that each of these proportions is the unique Boolean formula (up to equivalence) that satisfies groups of remarkable properties including a stability property w.r.t. a specific permutation of the terms of the proportion. The first one (i) is shown to be the only one to satisfy the standard postulates of an analogical proportion. The paper also studies how two analogical proportions can be combined into a new one. We then examine how homogeneous proportions can be used for diverse prediction tasks. We particularly focus on the completion of analogical-like series, and on missing value abduction problems. Finally, the paper compares our approach with other existing works on qualitative prediction based on ideas of betweenness, or of matrix abduction.

## Introduction

Proportions, understood as the identity of relations between two ordered pairs of entities, say  $(A, B)$  and  $(C, D)$ , play a crucial role in the way the human mind perceives the world and tries to make sense of it. Proportions thus involve four terms, which may not be all distinct. In mathematics, a proportion is a statement of equality between the result of operations involving numerical quantities (i.e.,  $A, B, C, D$  are numbers). The geometric proportion amounts to state the equality of two ratios, i.e.,  $A/B = C/D$ , while the arithmetic proportion compares two pairs of numbers in terms of their differences, i.e.,  $A - B = C - D$ . In these equalities, which emphasize the *symmetric* role of the pairs  $(A, B)$

and  $(C, D)$ , geometric or arithmetic ratios have an implicit comparative flavor, and the proportions express the invariance of the ratios. Note that by cross-product for geometric proportion, or by addition for the arithmetic one, the two proportions are respectively equivalent to  $AD = BC$  and to  $A + D = B + C$ , which makes clear that  $B$  and  $C$ , or  $A$  and  $D$ , can be *permuted* without changing the validity of the proportion. Moreover, mathematical proportions are at the basis of reasoning procedures that enable us to “*extrapolate*” the fourth value knowing three of the four quantities. Indeed, assuming that  $D$  is unknown, one can deduce  $D = C \times B/A$  in the first case, which corresponds to the well-known “rule of three”, or  $D = C + (B - A)$  in the second case. Besides, so-called *continuous* proportions where  $B = C$  are directly related to the idea of averaging, since taking  $B = C$  as the unknown respectively yields the geometric mean  $(AD)^{1/2}$  and the arithmetic mean  $(A + D)/2$ .

Already in Ancient Greek time, numerical proportions were used at the conceptual level for discussing philosophical matters. For instance, in the Book 5 about Justice of his *Nicomachean Ethics*, Aristotle makes explicit reference to geometric proportions when discussing what is “fair”. Since Aristotle’s time, analogical reasoning has received a lot of attention from researchers in many areas, and in particular from scholars in philosophy, anthropology, cognitive psychology and linguistics (see, e.g., (Hesse 1959; Durrenberger and Morrison 1977; Gick and Holyoak 1980; Gentner 1983; Gentner, Holyoak, and Kokinov 2001; Holyoak and Thagard 1989; French 2002; Lepage, Migeot, and Guillerme 2009)), including artificial intelligence more recently (Helman 1988). However, strangely enough, it seems that there has been no attempt at providing some *logical* model of analogical proportions up to two noticeable exceptions, which have been however fully ignored by the mainstream literature. The first exception is a proposal by the psychologist Jean Piaget (1953) (pp. 35–37), where the following definition of a so-called *logical proportion* is given: 4 propositions  $A, B, C$ , and  $D$  make a logical proportion if the two following conditions hold  $A \wedge D = B \wedge C$  and  $A \vee D = B \vee C$ . This logical proportion turns out, as we shall see, to be one among the possible (equivalent) definitions of an *analogical* proportion, usually denoted  $A : B :: C : D$ , which reads “ $A$  is to  $B$  as  $C$  is to  $D$ ”. The second exception is provided by Sheldon Klein (1982), a computer scientist with a strong

background in anthropology and linguistics, who introduced a so-called ATO operator (where ATO stands for “Appositional Transformation Operator”). This Boolean-like operator, which is based on the logical equivalence connective, amounts to compute the 4th argument of an analogical proportion between Boolean vectors (describing the items in terms of binary features) by applying  $D = C \equiv (A \equiv B)$  componentwise. However, strictly speaking, this calculation does not always fit with the correct definition of an analogical proportion, as we shall see.

Motivated by Klein’s proposal, a logical definition of the analogical proportion expressing that “ $A$  differs from  $B$  as  $C$  differs from  $D$  and  $B$  differs from  $A$  as  $D$  differs from  $C$ ” has been justified in (Miclet and Prade 2009). More recently, the authors of this paper, before rediscovering (Piaget 1953), have introduced the idea of *logical proportions* defined as 2 joint equivalences between indicators pertaining to the similarity or the dissimilarity of  $A$  and  $B$ , and of  $C$  and  $D$  respectively (Prade and Richard 2010c). More precisely, the conjunction of two positive or two negative atoms, by modeling their common truth or common falsity, expresses similarity, while the conjunction of a positive atom and a negative atom expresses dissimilarity. The analogical proportion then appears as a particular logical proportion, still especially remarkable, in company of two other companion proportions obtainable through permutations. Among a rather large number of logical proportions (Prade and Richard 2010c; 2010a), a subset of 15 logical proportions acknowledging that  $(A, A)$  and  $(A, A)$  should make a proportion (“full identity”), has been singled out.

In this paper, we provide new steps in this study, first by showing that a subset of 4 logical proportions, called *homogeneous* are the only ones to satisfy symmetry together with a property of independency with respect to encoding. These 4 proportions are the analogical proportion and its two companions, together with a new one violating “full identity”. This latter proportion expresses that “what  $A$  and  $B$  have in common, neither  $C$  nor  $D$  have it, and conversely what  $C$  and  $D$  have in common, neither  $A$  nor  $B$  have it”. We shall show that it is the only logical proportion that is invariant under any permutation of its four components, while the three other proportions are characterized by distinct permutation properties. Secondly, we exploit the extrapolation power of logical proportions for discussing the missing value abduction problem, which amounts to complete empty cells in a table on the basis of (a usually small number of) complete examples.

The paper is organized as follows. After a background gathering the previous results needed for the study, we identify homogeneous logical proportions, and characterize them in different ways, by laying bare their remarkable properties, and emphasizing their semantics. We then show how the patterns of homogeneous proportions are useful for completing empty cells in tables, using both IQ test (including Raven’s test) and commonsense prediction problems for illustrating the extrapolation and interpolation power of these proportions. The approach is compared to the few existing approaches attacking the same plausible prediction problem.

## Background on logical proportions

We first introduce the similarity and dissimilarity indicators used for defining logical proportions, from which we identify the 4 homogeneous proportions that can be defined.

**Similarity and dissimilarity indicators** Generally speaking, the comparison of two items  $A$  and  $B$  relies on the representation of these items. For instance, if an item is represented as a subset of a referential of binary features  $X$  (Tversky 1977), a precise meaning can be given to common features and specificities, then to similarities and dissimilarities by simply using set operators. Nevertheless, since we are not looking for any global measure, we adopt a logical setting that we explain now. Let  $\varphi$  be a property, which can be seen as a predicate:  $\varphi(A)$  may be true (in that case  $\neg\varphi(A)$  is false), or false. When comparing two items  $A$  and  $B$  w.r.t.  $\varphi$ , it makes sense to consider  $A$  and  $B$  *similar* (w.r.t.  $\varphi$ ):

- when  $\varphi(A) \wedge \varphi(B)$  is true or
- when  $\neg\varphi(A) \wedge \neg\varphi(B)$  is true.

In the remaining cases:

- when  $\neg\varphi(A) \wedge \varphi(B)$  is true or
- when  $\varphi(A) \wedge \neg\varphi(B)$  is true,

we can consider  $A$  and  $B$  as *dissimilar* w.r.t. property  $\varphi$ .  $\varphi(A)$  and  $\varphi(B)$  being ground formulas, they can be considered as Boolean variables, denoted  $a$  and  $b$  by abstracting w.r.t.  $\varphi$ . Then  $A, B, C, D$  can be viewed as represented respectively by vectors  $(a_1, \dots, a_n), (b_1, \dots, b_n), (c_1, \dots, c_n), (d_1, \dots, d_n)$ , where  $a_i = \varphi_i(A), b_i = \varphi_i(B), c_i = \varphi_i(C), d_i = \varphi_i(D)$  are instances of a set of Boolean variables encoding the truth value of  $n$  properties  $\varphi_i$  applicable to  $A, B, C, D$ . If the conjunction  $a \wedge b$  is true, the property is satisfied by both items  $A$  and  $B$ , while the property is satisfied by neither  $A$  nor  $B$  if  $\bar{a} \wedge \bar{b}$ <sup>1</sup> is true. This leads us to call such a conjunction of Boolean literals an *indicator*, and for a given pair of Boolean variables  $(a, b)$ , we have 4 distinct indicators:

- $a \wedge b$  and  $\bar{a} \wedge \bar{b}$  that we call *similarity indicators*,
- $a \wedge \bar{b}$  and  $\bar{a} \wedge b$  that we call *dissimilarity indicators*.

In a logical proportion, 4 items are involved:  $A, B, C$ , and  $D$ . The simplest way for expressing a comparison between the pair  $(A, B)$  and the pair  $(C, D)$  appears to consider an equivalence between 2 indicators, like for instance  $a \wedge \bar{b} \equiv c \wedge d$ . For the remaining of the paper, we stick to the lower case notation when  $a, b, c$  and  $d$  denote Boolean variables.

**Analogical and other logical proportions** Following the introductory discussion, it makes sense to encode an analogical proportion with the conjunction

$$(a \wedge \bar{b} \equiv c \wedge \bar{d}) \wedge (\bar{a} \wedge b \equiv \bar{c} \wedge d)$$

as it is the logical counterpart of “ $a$  differs from  $b$  as  $c$  differs from  $d$ ”, and conversely. As a consequence, it is legitimate to consider all the conjunctions of 2 equivalences between indicators: such a conjunction is called a *logical proportion* (Prade and Richard 2010c; 2010a). More formally, let  $I_{(a,b)}$

<sup>1</sup>The overline denotes Boolean negation.

and  $I'_{(a,b)}$ <sup>2</sup> (resp.  $I_{(c,d)}$  and  $I'_{(c,d)}$ ) denote 2 indicators for  $(a, b)$  (resp.  $(c, d)$ ).

**Definition 1** A logical proportion  $T(a, b, c, d)$  is the conjunction of 2 distinct equivalences between indicators of the form

$$(I_{(a,b)} \equiv I_{(c,d)}) \wedge (I'_{(a,b)} \equiv I'_{(c,d)})$$

Since we have to choose 2 distinct equivalences among  $4 \times 4 = 16$  possible ones for defining a logical proportion, we have  $\binom{16}{2} = 120$  such proportions and it has been shown that they are all semantically distinct. Consequently, if two proportions are semantically equivalent, they have the same expression as a conjunction of two equivalences between indicators (up to the symmetry of the conjunction). This formal definition goes beyond what may be expected from the informal idea of “logical proportion”, since equivalences may be put between things that are not homogeneous, i.e., mixing similarity and dissimilarity indicators in various ways. An example of such an “heterogenous” proportion is

$$((a \wedge \bar{b}) \equiv (c \wedge d)) \wedge ((a \wedge b) \equiv (\bar{c} \wedge \bar{d})).$$

**The 4 homogeneous logical proportions** When we add the constraint of considering fully homogeneous equivalences only, i.e., considering logical proportions that involve only dissimilarity, or only similarity indicators, 4 logical proportions remain which are listed below with their name:

- *analogy*:  $A(a, b, c, d)$ , defined by

$$((a \wedge \bar{b}) \equiv (c \wedge \bar{d})) \wedge ((\bar{a} \wedge b) \equiv (\bar{c} \wedge d))$$

- *reverse analogy*:  $R(a, b, c, d)$ , defined by

$$((a \wedge \bar{b}) \equiv (\bar{c} \wedge d)) \wedge ((\bar{a} \wedge b) \equiv (c \wedge \bar{d}))$$

- *paralogy*:  $P(a, b, c, d)$ , defined by

$$((a \wedge b) \equiv (c \wedge d)) \wedge ((\bar{a} \wedge \bar{b}) \equiv (\bar{c} \wedge \bar{d}))$$

- *inverse paralogy*:  $I(a, b, c, d)$ , defined by

$$((a \wedge b) \equiv (\bar{c} \wedge \bar{d})) \wedge ((\bar{a} \wedge \bar{b}) \equiv (c \wedge d))$$

Reverse analogy expresses that “ $a$  differs from  $b$  as  $d$  differs from  $c$ ”, and conversely; paralogy expresses that “what  $a$  and  $b$  have in common,  $c$  and  $d$  have it also”. The truth tables of these homogeneous proportions are recalled in Table 1, where we only show the lines leading to truth value 1. For them, we observe that only 6 lines among  $2^4 = 16$  lead to truth value 1. It can be proved that this is a general property of any of the 120 logical proportions. When we realize that, among all the  $\binom{16}{6} = 8008$  Boolean formulas involving 4 variables and having exactly 6 lines leading to true, we have 120 logical proportions, it makes these proportions all the more singular. The following proposition, easily deducible from the definition, establishes a link between analogy, reverse analogy and paralogy (while inverse paralogy  $I$  is not related to the 3 others through a simple permutation):

<sup>2</sup>Note that  $I_{(a,b)}$  (or  $I'_{(a,b)}$ ) refers to one element in the set  $\{a \wedge b, \bar{a} \wedge b, a \wedge \bar{b}, \bar{a} \wedge \bar{b}\}$ , and should not be considered as a functional symbol:  $I_{(a,b)}$  and  $I_{(c,d)}$  may be indicators of two different kinds. Still, we use this notation for the sake of simplicity.

Table 1: Analogy, Reverse analogy, Paralogy, Inverse Paralogy truth tables

A				R				
0	0	0	0		0	0	0	0
1	1	1	1		1	1	1	1
0	0	1	1		0	0	1	1
1	1	0	0		1	1	0	0
0	1	0	1		0	1	1	0
1	0	1	0		1	0	0	1
P				I				
0	0	0	0		1	1	0	0
1	1	1	1		0	0	1	1
1	0	0	1		1	0	0	1
0	1	1	0		0	1	1	0
0	1	0	1		0	1	0	1
1	0	1	0		1	0	1	0

**Proposition 1**  $A(a, b, c, d) \leftrightarrow R(a, b, d, c)$  and  $A(a, b, c, d) \leftrightarrow P(a, d, c, b)$ .

It is not the aim of this paper to go deeper in a global investigation of the whole set of logical proportions: we focus on the 4 homogeneous proportions  $A, R, P, I$  and we show that they stand out of the logical proportions.

### Semantics of homogeneous proportions and their uniqueness

It is interesting to take a closer look at the truth tables of the four homogeneous proportions. First, one can observe in Table 1, that 8 possible valuations for  $(a, b, c, d)$  never appear among the patterns that make  $A, R, P$ , or  $I$  true: these 8 valuations are of the form  $xxxy, xxyx, xyxx, yxxx$  with  $x \neq y$  and  $(x, y) \in \{0, 1\}^2$ . As can be seen, it corresponds to situations where  $a = b$  and  $c \neq d$ , or  $a \neq b$  and  $c = d$ , i.e., similarity holds between the components of one of the pairs, and dissimilarity holds in the other pair. Moreover, the truth table of each of the four homogeneous proportions, is built in the same manner:

- 1) two lines of the table correspond to the characteristic pattern of the proportion; namely the two lines where one of the two equivalences in its definition holds true under the form  $1 \equiv 1$  (rather than  $0 \equiv 0$ ). Thus,
  - $A$  is characterized by the pattern  $xyxy$  (corresponding to valuations  $1010$  and  $0101$ ), i.e. we have the same difference between  $a$  and  $b$  as between  $c$  and  $d$ ;
  - $R$  is characterized by the pattern  $yxyx$  (corresponding to valuations  $1001$  and  $0110$ ), i.e. the differences between  $a$  and  $b$  and between  $c$  and  $d$  are in opposite directions;
  - $P$  is characterized by the pattern  $xxxx$  (corresponding to valuations  $1111$  and  $0000$ ), i.e. what  $a$  and  $b$  have in common,  $c$  and  $d$  have it also;
  - $I$  is characterized by the pattern  $xyyy$  (corresponding to valuations  $1100$  and  $0011$ ), i.e. what  $a$  and  $b$  have in common,  $c$  and  $d$  do not have it, and conversely. Thus, the six lines of the truth table of  $A$  that makes it true are induced by the characteristic patterns of  $A$ ,

$P$ , and  $I$ <sup>3</sup>, the six valuations that makes  $P$  true are induced by the characteristic patterns of  $P$ ,  $A$ , and  $R$ , and so on for  $R$  and  $I$ .

- 2) the four other lines of the truth table of an homogeneous proportion  $T$  are generated by the characteristic patterns of the two other proportions that are not opposed to  $T$  (in the sense that  $A$  and  $R$  are opposed, as well as  $P$  and  $I$ ). For these four lines, the proportion holds true since its expression reduces to  $(0 \equiv 0) \wedge (0 \equiv 0)$ .

As can be seen, starting from analogical proportion, we have extended the well known notion of numerical proportion to the idea of logical proportions, and identify four homogeneous proportions. We now investigate what could be expected from homogeneous proportions, which would be the logical counterpart of properties observed for numerical proportions. We start with the property of independency with respect to the encoding, which is implicitly at work in the above analysis in terms of patterns, where we did not distinguish between the values 1 or 0 for  $x$ , and for  $y \neq x$ .

**Code independency** Just as a numerical proportion  $\frac{a}{b} = \frac{c}{d}$  holds independently of the base used for encoding numbers, it seems natural to expect that the logical proportions should be independent of the way we encode items in terms of the truth or the falsity of properties. For instance  $a \wedge \bar{b}$  represents what is specific to  $a$  w.r.t.  $b$ , without any consideration about the way we represent the truth and the falsity. As a consequence, the formula defining that proportion should be valid when we switch 0 to 1 and 1 to 0 in the coding of a valuation. The formal expression of this requirement is:

$$T(a, b, c, d) \rightarrow T(\bar{a}, \bar{b}, \bar{c}, \bar{d})$$

and we call this property “code independency”. Another reason to look for the above property is to consider that the negation is the counterpart for logical proportions of the inverse (resp. opposite) in geometric (resp. arithmetic) proportions, where we have  $\frac{a}{b} = \frac{c}{d}$  implies  $\frac{1}{b} = \frac{1}{d}$  and  $a - b = c - d$  implies  $-a - (-b) = -c - (-d)$ <sup>4</sup>. The “code independency” property appears to be a very restrictive property since we have the following result:

**Proposition 2** *There are exactly 8 proportions satisfying the code independency property including the 4 homogeneous proportions  $A, R, P, I$ .*

*Proof:* Since both  $T(a, b, c, d)$  and  $T(\bar{a}, \bar{b}, \bar{c}, \bar{d})$  are logical proportions (in the sense of Definition 1), code independency tells us that they have the same truth table so the 2 proportions should be identical (up to a permutation of the 2 equivalences). This exactly means that the second (resp.

<sup>3</sup>The measure of analogical dissimilarity introduced in (Miclet, Bayouhd, and Delhay 2008) is 0 for the valuations corresponding to the characteristic patterns of  $A, P$ , and  $I$ , maximal for the valuations corresponding to the characteristic patterns of  $R$ , and takes the same intermediary value for the 8 valuations characterized by one of the patterns  $xxxy, xxyx, xyxx, or yxxx$ .

<sup>4</sup>For arithmetic proportions between numbers in  $[0, 1]$ , and the complementation to 1 as a pseudo inverse, we have: if  $a - b = c - d$  then  $(1 - a) - (1 - b) = (1 - c) - (1 - d)$ .

first) equivalence is obtained from the first (resp. second) one by negating all the variables. Since we have  $4 \times 4$  equalities between indicators, we can build exactly  $16/2 = 8$  proportions satisfying code independency property: each time we choose an equivalence, we use it and its negated form to build up a suitable proportion.  $\square$

**Symmetry** In the numerical case, due to the symmetry of the = operator, when  $\frac{a}{b} = \frac{c}{d}$  holds, then  $\frac{c}{d} = \frac{a}{b}$  holds as well. Its counterpart for logical proportions is formally expressed via the following “symmetry property”:

$$T(a, b, c, d) \rightarrow T(c, d, a, b)$$

A very remarkable result is then:

**Proposition 3**  *$A, R, P, I$  are the only logical proportions to satisfy code independency and symmetry.*

*Proof:* Obviously,  $A, R, P, I$  satisfy both properties. The fact that the 4 other proportions satisfying code independency are not symmetric can be easily checked.  $\square$

It means that, despite the fact we have many options for defining a logical proportion in terms of equivalences, only 4 of them really fit with the natural concept of proportion.

**Transitivity** Another property that we might expect for a proportion is the transitivity coming from the equality:  $\frac{a}{b} = \frac{c}{d}$  and  $\frac{c}{d} = \frac{e}{f}$  entails  $\frac{a}{b} = \frac{e}{f}$ . For a logical proportion  $T$ , this is formally translated into:

$$T(a, b, c, d) \wedge T(c, d, e, f) \rightarrow T(a, b, e, f)$$

We have the following result (easy to check on truth tables):

**Proposition 4**  *$A$  and  $P$  are transitive,  $R$  and  $I$  are not transitive. Moreover,*

$$\begin{aligned} R(a, b, c, d) \wedge R(c, d, e, f) &\rightarrow A(a, b, e, f) \\ I(a, b, c, d) \wedge I(c, d, e, f) &\rightarrow P(a, b, e, f) \end{aligned}$$

Thus two reverse analogies in cascade make an analogy. The last property fits with the intuition that paralogy expresses a form of logical parallelism, while inverse paralogy expresses a form of logical orthogonality (taking two times orthogonal directions in a two dimension space builds a parallel).

**Permutations** In the numerical setting, there is a famous property known as “the permutation of extremes and means property”, which in fact covers a pair of properties:

- means’s permutation: if  $\frac{a}{b} = \frac{c}{d}$  holds then  $\frac{a}{c} = \frac{b}{d}$  holds
- extremes’s permutation: if  $\frac{a}{b} = \frac{c}{d}$  holds then  $\frac{d}{b} = \frac{c}{a}$  holds

Investigating the same idea in the logical setting, we may look for the logical proportions satisfying a permutation of the means for instance, i.e., a property such as:

$$T(a, b, c, d) \rightarrow T(a, c, b, d)$$

Since we have 4 variables involved, we have a set of 24 permutations, but we are only interested here in permutations exchanging the place of two elements, the so-called transpositions. There are exactly 6 such permutations, and we denote them with obvious notations by  $p12, p13, p14, p23, p24, p34$ . Thus,  $p23$  is the permutation of the means and  $p14$  is the permutation of the extremes. The property above is just the stability of a proportion  $T$  w.r.t.  $p23$ . We have the following results:

**Proposition 5** • *A and I are the only logical proportions satisfying symmetry and being stable for permutation p23, i.e. the permutation of the means. The same result holds replacing p23 by p14 (permutation of extremes).*

- *P and I are the only logical proportions satisfying symmetry and being stable for permutation p12. The same result holds replacing p12 by p34.*
- *R and I are the only logical proportions satisfying symmetry and being stable for permutation p24. The same result holds replacing p13 by p24.*

Moreover, the following amazing result holds:

**Proposition 6** *I is the only logical proportion stable for each of the 6 transpositions.*

*Proof:* It is easy to check that these permutations induce a partition of the set of valuations into 5 classes, each of them being closed for these 6 permutations:

- the class {0000} and the class {1111}
- the class {0111, 1011, 1101, 1110}
- the class {1000, 0100, 0010, 0001}
- the class {0101, 1100, 0011, 1010, 1001, 0110}

Taking into account that a logical proportion is true for only 6 valuations, we only have 3 options: a proportion valid for {0000}, {1111} and {0111, 1011, 1101, 1110}, or for {0000}, {1111} and {1000, 0100, 0010, 0001}, or for {0101, 1100, 0011, 1010, 1001, 0110}. It appears that the latter class is just the truth table of inverse paralogy. Now let us consider an equivalence between indicators

$$l_1 \wedge l_2 \equiv l_3 \wedge l_4$$

validated by the class {0111, 1011, 1101, 1110}. The 2 last valid valuations show that the truth value of  $l_3 \wedge l_4$  does not change when we exchange 0 and 1. There are only 2 indicators for  $(c, d)$  satisfying this property:  $c \wedge d$  and  $\bar{c} \wedge \bar{d}$ . The same reasoning applies considering the 2 first valuations showing that an equivalence between indicators satisfying this class of 4 valuations should be of the type  $a \wedge b \equiv c \wedge d$  or  $a \wedge b \equiv \bar{c} \wedge \bar{d}$ , which obviously are not satisfied by the 4 valuations {0111, 1011, 1101, 1110}. In fact, we have proved that the class {0111, 1011, 1101, 1110} cannot validate an equivalence between indicators, and thus not a logical proportion (which requires two such equivalences). The same reasoning applies to {1000, 0100, 0010, 0001}, achieving the proof.  $\square$

**Trivial proportions** For numerical proportions, we obviously have  $\frac{a}{b} = \frac{a}{b}$  and  $\frac{a}{a} = \frac{b}{b}$ , whose translation for logical proportions means:

$$T(a, b, a, b) \text{ and } T(a, a, b, b) \text{ are true.}$$

We have the following result:

**Proposition 7** *A is the unique proportion satisfying  $T(a, b, a, b)$  and p23 (and thus also  $T(a, a, b, b)$ ).*

*P is the unique proportion satisfying  $T(a, b, a, b)$  and p34 (and thus also  $T(a, b, b, a)$ ).*

*R is the unique proportion satisfying  $T(a, a, b, b)$  and p24 (and thus also  $T(a, b, b, a)$ ).*

*Proof:* Similar to the proof of Proposition 6. Let us consider the first statement for instance.  $T(a, b, a, b)$  implies that  $T(0, 0, 0, 0), T(1, 1, 1, 1), T(1, 0, 1, 0)$  and  $T(0, 1, 0, 1)$  hold. Adding the fact that  $T$  is stable for the permutation of the mean p23, we get that  $T(1, 1, 0, 0)$  and  $T(0, 0, 1, 1)$  hold as well, leading to the truth table of  $A$ .  $\square$

**Inverse and negation** Proposition 2 has established a first link between the 4 homogeneous proportions  $A, R, P, I$  and the negation operator, namely code interdependency:

$$T(a, b, c, d) \rightarrow T(\bar{a}, \bar{b}, \bar{c}, \bar{d}).$$

When we deal with the arithmetic or geometric proportions between real numbers, we respectively have the opposite of a number  $a$ , namely  $-a$ , and the multiplicative inverse, namely  $\frac{1}{a}$  (when  $a \neq 0$ ), as inverse. Then, we have the following trivial numerical proportions:

$$a - b = (-b) - (-a) \quad (1); \quad \frac{a}{b} = \frac{\frac{1}{b}}{\frac{1}{a}} \quad (2)$$

whose Boolean counterpart is  $T(a, b, \bar{b}, \bar{a})$ . These concerns lead to the following result (just checking the definitions):

**Proposition 8** *Among the homogeneous proportions (i.e.  $A, R, P, I$ ),*

*A and I are the only ones satisfying  $T(a, b, \bar{b}, \bar{a})$ ,*

*R and I are the only ones satisfying  $T(a, b, \bar{a}, \bar{b})$ ,*

*P and I are the only ones satisfying  $T(a, \bar{a}, b, \bar{b})$ .*

The four homogeneous proportions  $A, R, P, I$  can be related through permutations and negation as follows (trivial proof):

**Proposition 9**

$$A(a, b, c, d) \leftrightarrow R(a, b, \bar{c}, \bar{d}),$$

$$A(a, b, c, d) \leftrightarrow P(a, \bar{b}, c, \bar{d}),$$

$$A(a, b, c, d) \leftrightarrow I(a, \bar{b}, \bar{c}, d).$$

Only the analogical proportion  $A$  sticks to the properties of arithmetic or geometric numerical proportions. Indeed inverse paralogy  $I$  misses one basic requirement, namely full identity, i.e.  $T(a, a, a, a)$ . We have seen that among all the logical proportions, the homogeneous ones are made singular by the remarkable properties they enjoy, while being closely related in different ways. Then it should not come as a surprise that  $A, R, P, I$  play an important role in diverse prediction activities as we shall see in the next sections.

## Completing an analogical pattern

As previously mentioned, the notion of proportion is closely related to the idea of extrapolation, i.e. to guess/compute a new value on the ground of existing values. In other words, if for some reason, it is believed or known that a proportion holds between 4 binary items, 3 of them being known, then one may look for the value of the 4th one. For geometric proportions, it leads to solve the equation  $\frac{a}{b} = \frac{c}{x}$  when the last item is missing. This may be considered as a simple abduction principle: knowing that  $d$  should be in proportion with  $a, b$  and  $c$ , we can take  $d$  as being the solution of the equation  $\frac{a}{b} = \frac{c}{x}$  (i.e.  $\frac{b \cdot c}{a}$ ) where  $x$  is the unknown. This is nothing but the “rule of three”. The counterpart of such an equation solving process can be easily adapted to homogeneous logical proportions as we shall see in the next section.

## Logical proportions and equation solving

In the context of logical proportions, the equation solving problem can be stated as follows. Given a logical proportion  $T$  and a valuation  $v$  such that  $v(a), v(b), v(c)$  are known, does it exist a Boolean value  $x$  such that  $v(T(a, b, c, d)) = 1$  when  $v(d) = x$ , and in that case, is this value unique? For the sake of simplicity, a propositional variable  $a$  is denoted as its truth value  $v(a)$ , and we use the equational notation  $T(a, b, c, x) = 1$ , where  $x \in \{0, 1\}$  is unknown. First of all, it is easy to see that there are always cases where the equation has no solution. Indeed, the triple  $a, b, c$  may take  $2^3 = 8$  values, while any proportion  $T$  is true only for 6 distinct valuations, leaving at least 2 cases with no solution. For instance, when we deal with analogy  $A$ , the equations  $A(1, 0, 0, x)$  and  $A(0, 1, 1, x)$  have no solution. When considering the homogeneous proportions  $A, R, P, I$ , we have the following result (already shown in (Prade and Richard 2010c) except for  $I$ ):

### Proposition 10

*The analogical equation  $A(a, b, c, x)$  is solvable iff  $(a \equiv b) \vee (a \equiv c)$  holds.*

*The reverse analogical equation  $R(a, b, c, x)$  is solvable iff  $(b \equiv a) \vee (b \equiv c)$  holds.*

*The paralogical equation  $P(a, b, c, x)$  is solvable iff  $(c \equiv b) \vee (c \equiv a)$  holds.*

*In each of the three above cases, when it exists, the unique solution is given by  $x = c \equiv (a \equiv b)$ , i.e.  $x = a \equiv b \equiv c$ .*

*The inverse paralogical equation  $I(a, b, c, x)$  is solvable iff  $(a \not\equiv b) \vee (b \not\equiv c)$  holds. In that case, the unique solution is  $x = c \not\equiv (a \not\equiv b)$ .*

As we can see, the first 3 homogeneous proportions  $A, R, P$  behave similarly. Still, their conditions of equation solvability differ. Moreover, it can be checked that at least 2 of these proportions are always simultaneously solvable. Besides, when they are solvable, there is a common expression that yields the solution. This again points out a close relationship between  $A, R$ , and  $P$ . This contrasts with proportion  $I$  which in some sense behaves in an opposite manner.

This simple equation-solving process allows us to complete a sequence of 3 Boolean values, but we can do much more than that, just by extending the notion of proportion from  $\mathbb{B}$  to Boolean vectors in  $\mathbb{B}^n$  as follows:

$$T(\vec{a}, \vec{b}, \vec{c}, \vec{d}) \text{ iff } \forall i \in [1, n], T(a_i, b_i, c_i, d_i).$$

The solving process is still effective: instead of getting one Boolean value, we get a Boolean vector, by solving equations componentwise. If we consider a Boolean vector as the *bitmap* description of a picture (thus considered at the pixel level), solving an equation  $A(\vec{a}, \vec{b}, \vec{c}, \vec{x})$  is just about trying to complete a sequence of 3 pictures as it is often the case in IQ tests. In fact, a noticeable part of the IQ tests are based on providing incomplete analogical proportions (see, e.g., (French 2002)): the 3 first items  $a, b, c$  are given and the 4th item  $d$  has to be chosen among several plausible options. When the items are pictures, our method applies, possibly leading to a solution as it is the case for Figure 1. This tech-

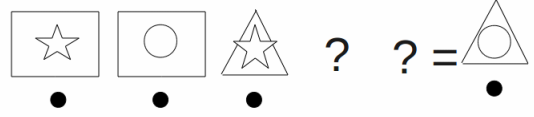


Figure 1: IQ test: Graphical analogy

nique<sup>5</sup> automatically builds triangles, circles, and more generally any geometric figures without having any knowledge of what a triangle or a circle is, or of any geometric concept, just by considering the objects at a pixel level, as recently pointed out in (Prade and Richard 2011).

This agrees with a *logical* encoding of items  $A, B, C$  described respectively, in the example of Figure 1 by vectors  $(1, 0, 1, 0, 1)$ ,  $(1, 0, 0, 1, 1)$ ,  $(0, 1, 1, 0, 1)$ , where the vector components refer respectively to the presence (or not) of a square, of a triangle, of a star, of a circle, of a black point. By the componentwise solving of the analogical proportion equations expressing that  $A(a_i, b_i, c_i, x_i)$  holds true for  $i = 1, 5$ , we easily get  $X = (0, 1, 0, 1, 1)$ , which corresponds to the result exhibited in Figure 1. Note that  $X$  is directly computed with this method, not chosen among a set of more or less “distant” potential solutions. It is not difficult to build sequences of 4 pictures, where the display of squares, triangles, stars, circles and black dots is different, and where the fourth picture would be obtained via one of the three other homogeneous proportions  $R, P$ , or  $I$ .

However, such a basic process, based on a straightforward extension of logical proportions to a multiple-dimensional setting, is not powerful enough to complete more sophisticated sequences of pictures or proportions like “ $abc$  is to  $abd$  what  $ijk$  is to ?””, where a vectorial representation is not suitable. We need a more general approach.

## Extensions of the analogical proportion patterns

The above example suggests that it is advisable to extend the notion of analogical proportion beyond Boolean lattices and to take into account the underlying structure if any. We may do that in two manners:

i) Let us start from a typical example to understand the matter: let us suppose we have to complete the sequence  $1 : 2 :: 7 : ?$  in an analogical way. In that case, we are naturally led to try to apply to the third item 7 the function which may have been applied to 1 to get 2. This is equivalent to take as granted an analogical pattern such as

$$A(\vec{x}, f(\vec{x}), \vec{y}, f(\vec{y})) \quad (3)$$

where  $f$  is a function from  $\mathcal{X}$  to  $\mathcal{X}'$ , including  $x$  and  $y$  in its domain  $\mathcal{X}$ , and  $f(x)$  and  $f(y)$  in its co-domain  $\mathcal{X}'$ . Having  $\mathcal{X} \neq \mathcal{X}'$  gives the freedom to handle items from different types in the same analogical proportion. In fact, a purely

<sup>5</sup>It is clear that the analogical proportion-based technique applied at the pixel level would fail to work unless all the geometric shapes (squares, triangles, stars, circles) use exactly the same pixels in all cases.

presence of	x	y	f
x	1	0	0
f(x)	1	0	1
y	0	1	0
?	?	?	?

Figure 2: A Boolean representation of  $A(\vec{x}, f(\vec{x}), \vec{y}, ?)$

Boolean vector interpretation as in Figure 2 (namely  $x$  is represented by  $\vec{x} = (1, 0, 0)$ ,  $y$  by  $\vec{y} = (0, 1, 0)$ , and  $f(x)$  by  $f(\vec{x}) = (1, 0, 1)$ ) exhibits the fact that we get the vector  $(0, 1, 1)$  that represents  $f(y)$  as solution of the analogical equation, componentwise. This shows the full agreement of the logical view with the pattern (3). Thus, this pattern provides a Boolean representation of the informal analogical proportion pattern  $x : f(x) :: y : f(y)$ . ii) Another option is to consider existing composition laws and to compose 2 analogical proportions to build up a new one. For instance, a simple result can be proved within our Boolean framework:

**Proposition 11**

$$A(a, b, c, d) \rightarrow A(a \vee e, b \vee e, c \vee e, d \vee e)$$

$$A(a, b, c, d) \rightarrow A(a \wedge e, b \wedge e, c \wedge e, d \wedge e)$$

These properties express a restrictive form of compatibility of the analogical proportion with the 2 internal laws of the Boolean lattice. Namely, the analogical proportion  $A(a, b, c, d)$  is composed with the trivial proportion  $A(e, e, e, e)$  using  $\wedge$  or  $\vee$ , and we get a new analogical proportion. A general form of such a rule would be:

$$A(a, b, c, d) \wedge A(a', b', c', d') \rightarrow A(a.a', b.b', c.c', d.d')$$

where  $.$  denotes an internal law on a Boolean lattice. In fact, as soon as we have an algebraic structure over the underlying universe  $\mathcal{X}$ , the notion of analogical proportion can be defined and the inductive principle, based on the equation solving process, can be generalized. We refer to (Stroppa and Yvon 2005; 2006) for a detailed investigation of such an idea. When we have an internal composition law, which is associative, it is possible to introduce the notion of factorization of an element of the underlying set. Starting from this factorization, the authors provide a definition for analogical proportion which is “factor-wise” in some sense. This definition applies to lattices since they are equipped with commutative and associative laws. However, the definition of analogical proportions that they obtain in the case of Boolean lattice structure turns out to be much less restrictive than ours since then  $A(a, b, c, d)$  holds true not only for the 6 valuations (shown in Table 1) characterizing the analogical proportion, but also for the valuations 0111, 1011, 1101, 1110.<sup>6</sup> We summarize in Table 2 the generic analogical patterns that we have defined. The letters  $a, b, c, d$  now represent items of a general universe, which is not necessarily a Boolean structure,  $f$  denotes a function, and  $.$  a binary operator (we omit the symbol  $A$ ).

<sup>6</sup>In fact, the definition given in (Stroppa and Yvon 2006) amounts, for distributive lattices, to defining  $A(a, b, c, d)$  only by the equivalence  $(a \vee b) \equiv (c \vee d)$ , which holds true for 10 distinct valuations over 16.

$a b a b$ or $a a b b$	(1) basic patterns
$a f(a) b f(b)$	(2) functional extension
$\vec{a} \vec{b} \vec{c} \vec{d}$	(3) vector extension
$a b c d \mid \rightarrow a.a' b.b' c.c' d.d'$ $a' b' c' d' \mid$	(4) compositional extension

Table 2: Generic analogical patterns

**Copycat example** If instead of dealing with a Boolean lattice, we extend analogical proportions to the set of strings, we are able to deal with analogical proportion puzzles like the following one taken from the work (known as the *Copycat project*) of (Hofstadter and Mitchell 1995),

$$\text{if } abc - - > abd \text{ then } ijk - - > ?$$

In that case, we proceed as follows:

- $ab ab ij ij$  (pattern 1)
- $c succ(c) k succ(k)$  (pattern 2) where  $succ$  is the successor function on the Latin alphabet
- $ab.c ab.succ(c) ij.k ij.succ(k)$  (pattern 4) where the  $.$  operator is the string concatenation

Then we get  $?$  as  $ijl$ . Let us note that  $abc abd ijk ijk$ , sometimes suggested since  $c$  does not appear in  $ijk$  is simply not an option if we admit that we have to follow the pattern of an analogical proportion: indeed,  $a b c c$  is not an analogical pattern when  $a \neq b$ .

**Analogical proportions in Raven’s tests**

A mine of examples is coming from the IQ tests literature. As in the previous case, some IQ tests are based on sequences of letters or words (or sentences) to be completed. Nevertheless, in order to avoid the bias of a cultural background, a lot of IQ tests are picture-based instead of vocabulary-based. There is a set of well-known IQ tests, the so-called Raven’s Progressive Matrices (Raven 2000), which are picture-based, and are considered as a reference for measuring the reasoning component of “the general intelligence”. Moreover, recently (Lovett, Forbus, and Usher 2010) have investigated a computational model for solving Ravens Progressive Matrices. This model combines qualitative spatial representations with analogical comparison via structure-mapping (Gentner 1983). In the following, we suggest that the Boolean approach can be also used for solving such a test (see (Prade and Richard 2011) for another example).

Each test is constituted with a 3x3 matrix  $pic[i, j]$  of pictures where the last picture  $pic[3, 3]$  is missing and has to be chosen among a panel of 8 candidate pictures. An example is given in Figure and its solution in Figure 4. We assume that the Raven matrices can be understood in the following way, with respect to rows and columns:

$$\forall i \in [1, 2], \exists f \text{ such that } pic[i, 3] = f(pic[i, 1], pic[i, 2])$$

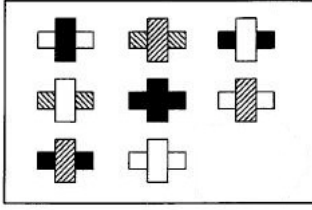


Figure 3: Raven test



Figure 4: Raven test: the solution

$$\forall j \in [1, 2], \exists g \text{ such that } pic[3, j] = g(pic[1, j], pic[2, j])$$

The two complete rows (resp. columns) are supposed to help to discover  $f$  (resp.  $g$ ) and to predict the missing picture  $pic[3, 3]$  as  $f(pic[3, 1], pic[3, 2])$  (resp.  $g(pic[1, 3], pic[2, 3])$ ).

Obviously, these tests do not fit the standard equation solving scheme, but they follow an extended one. Indeed, the extended analogical scheme (pattern (2)) has to be applied for telling us that  $A((a, b), f(a, b), (c, d), f(c, d))$  holds for lines and  $A((a, b), g(a, b), (c, d), g(c, d))$  for columns, i.e.

$$A((pic[1, 1], pic[1, 2]), pic[1, 3], (pic[2, 1], pic[2, 2]), pic[2, 3])$$

$$A((pic[1, 1], pic[2, 1]), pic[3, 1], (pic[1, 2], pic[2, 2]), pic[3, 2])$$

Thus, in that case, we have to consider a pair of cells  $(pic[i, 1], pic[i, 2])$  as the first element of an analogical proportion, and then the pair  $((pic[i, 1], pic[i, 2]), pic[i, 3])$  provides the 2 first element  $a$  and  $b$  of the analogical proportion we are considering. In terms of coding, in the example of Figure 3, we may consider the pictures as represented by a pair (or vector)  $(hr, vr)$  with one horizontal rectangle  $hr$  and a vertical one  $vr$ , each of these rectangles having one color among *Black, White, Grey*, we have then the following obvious encoding of the matrix in Table 3. It leads to the following analogical patterns (using the traditional notation):

$$(WB, GG) : BW :: (GW, BB) : WG \text{ (1st and 2nd rows)}$$

$$(WB, GG) : BW :: (BG, WW) : ?i?ii \text{ (1st and 3rd rows)}$$

where  $BW = f(WB, GG)$  and  $WG = f(GW, BB)$ .

$$(WB, GW) : BG :: (GG, BB) : WW \text{ (1st and 2nd columns)}$$

$$(WB, GW) : BG :: (BW, WG) : ?i?ii \text{ (1st and 3rd columns)}$$

where  $BG = f(WB, GW)$  and  $WW = g(GG, BB)$ ,

	1	2	3
1	WB	GG	BW
2	GW	BB	WG
3	BG	WW	?i?ii

Table 3: A coding of the Raven matrix example

or if we prefer, since analogical proportions holds componentwise, we have the following proportions

- for the horizontal bars:

$$(W, G) : B :: (G, B) : W \text{ (horizontal analysis)}$$

$$(W, G) : B :: (B, W) : ?i \text{ (horizontal analysis)}$$

$$(W, G) : B :: (G, B) : W \text{ (vertical analysis)}$$

$$(W, G) : B :: (B, W) : ?i \text{ (vertical analysis)}$$

- for the vertical bars:

$$(B, G) : W :: (W, B) : G \text{ (horizontal analysis)}$$

$$(B, G) : W :: (G, W) : ?ii \text{ (horizontal analysis)}$$

$$(B, W) : G :: (G, B) : W \text{ (vertical analysis)}$$

$$(B, W) : G :: (W, G) : ?ii \text{ (vertical analysis)}$$

One can observe that the item  $(B, W)$  appears only in the analogical proportions with question marks for horizontal bars, while the items  $(G, W)$  and  $(W, G)$  appear only in the analogical proportions with question marks for vertical bars. Analogical proportions coming from both horizontal or vertical analysis are insufficient for concluding here. However, we can consider the Raven matrix provides a set of analogical associations without any distinction between those ones coming from the horizontal bars and those ones coming from vertical bars. In other words, we now relax the componentwise reading by considering that what applies to horizontal bars, may apply to vertical bars, and vice-versa. With this viewpoint, it appears that the pair  $(B, W)$  and the pair  $(W, G)$  are respectively associated to  $G$  (vertical association for vertical bar) and  $B$  (horizontal association for horizontal bar), which encodes the expected solution  $GB$  (as pictured in Figure 4). Note that  $(G, W)$  cannot help predicting ?ii.

## Matrix abduction

The problem of completing a matrix where some values are missing is not new and there are diverse techniques to deal with this issue, generally linked to the characteristics of the problem:

- we can deal with large matrices or small matrices: in the case of large matrices, statistical approaches are applicable;
- the values to be computed could be real, symbolic or discrete;

- the meaning of a cell  $(i, j)$  may be the value of an attribute  $j$  for the item  $i$ , or may express a relation between the item  $i$  and the item  $j$  (for instance the distance between  $i$  and  $j$ , or still in a preference matrix, it may express how  $i$  is preferred to / is more important than  $j$ , etc.).

Whatever the technique, the main question is to know if the extra knowledge that we may have about the problem, and the available data carry sufficient information for an accurate reconstruction of the missing cells. This is not always the case, especially when we have very few available data. In the following, we focus on a particular case, called “matrix abduction problem”, using (Abraham, Gabbay, and Schild 2009)’s terminology. It consists in guessing plausible values for cells having empty information in a matrix where each line corresponds to a situation described according to different binary features (so each column corresponds to a particular feature). Such a problem may be encountered in daily



	<i>P</i>	<i>C</i>	<i>I</i>	<i>R</i>	<i>D</i>	<i>S</i>
<i>screen1</i>	0	1	0	1	0	1
<i>screen2</i>	0	0	1	1	0	1
<i>screen3</i>	0	0	0	0	1	?
<i>screen4</i>	1	1	0	0	1	1

Table 4: The screen example

life, where you want to compare incompletely described situations (e.g., the characteristics of objects to be sold on the web), or for judging a new situation on the basis of other (incompletely described) situations. According to (Abraham, Gabbay, and Schild 2009) such problems were already encountered by Talmudic commentators for determining what behavior is the most appropriate in a given situation!

Let us consider the screen example used by (Abraham, Gabbay, and Schild 2009), where computer screens are described by 6 characteristic features: *P* is for price over 450, *C* for self collection, *I* for screen bigger than 24 inch, *R* for reaction time below 4ms, *D* for dot size less than 0.275, and *S* for stereophonic; 1 means “yes” and 0 means “no”. We have 3 screens (screen 1, screen 2 and screen 4) whose characteristics are known and screen 3 where the truth value of the attribute *S* is missing (see Table 4). Various approaches may be thought of for attacking such a common sense problem. We present how homogeneous proportions may be useful for solving it, together with a brief mention of two other recent approaches, which are handling it somewhat differently. In each approach, however, it is admitted that the problem has not always a solution, i.e. there are cases where the problem of providing a plausible Boolean value for some unknown feature of a considered situation, will remain undecided.

A general idea, explicitly or implicitly, common to all approaches is that replacing an unknown value ? by either 1 or 0 should result in the least possible perturbation of the matrix. This idea may be implemented differently. In the proposed approach we suggest to enforce an homogeneous proportion *T* that already holds for completely informed features. In (Abraham, Gabbay, and Schild 2009) the idea is to choose the value that, when it is added, least perturbs the partial ordering that what was existing between the column vectors of the matrix. In (Schockaert and Prade 2011), the idea is rather to respect betweenness and parallelism relations that hold in conceptual spaces.

**Homogeneous proportion approach** Assume we have a Boolean vector describing incompletely a situation with respect to a set of  $n + 1$  considered features, say  $v = (v_1, \dots, v_n, x_{n+1})$ , where for simplicity we assume that only  $x_{n+1}$  is unknown. For trying to make a plausible guess of the value of  $x_{n+1}$ , we have a collection (which may be rather small) of completely informed examples  $e^i = (e_1^i, \dots, e_n^i, e_{n+1}^i)$  for  $i = 1, n$ . Then one may have at least three strategies:

i) comparing  $v$  to each  $e^i$  separately, and using a  $k$ -nearest

neighbors approach; this extends the idea that the proportion  $T(e, e, e, v)$  should hold true and has  $v = e$  as a solution.

ii) looking for pairs  $e^i, e^j$  such that  $T(e_h^i, v_h, v_h, e_h^j)$  makes a continuous homogeneous proportion *T* for a maximal number of features  $h$ ; it thus implements some idea of having  $v_h$  between  $e_h^i$  and  $e_h^j$ ; observe however, that in the Boolean case, this would force to have the trivial situations  $T(1, 1, 1, 1)$  or  $T(0, 0, 0, 0)$  on a maximal number of features, and to tolerate some “approximate” patterns  $T(1, 1, 1, 0)$ ,  $T(0, 1, 1, 1)$ ,  $T(0, 0, 0, 1)$ , or  $T(1, 0, 0, 0)$ , while rejecting patterns  $T(0, 1, 1, 0)$  and  $T(1, 0, 0, 1)$ , in agreement with footnote 3.

iii) looking for triples  $e^i, e^j, e^k$  such that  $T(e_h^i, e_h^j, e_h^k, v_h)$  makes an homogeneous proportion *T* for a maximal number of features  $h$ .

In cases ii) or iii), the principle amounts to say that if an homogeneous proportion – we restrict to homogeneous proportions due to their symmetry and encoding independency properties – holds for a number of features as great as possible among features  $h$  such that  $1 \leq h \leq n$ , it should still hold for feature  $n+1$ , which provides an equation for finding  $x_{n+1}$  if solvable. If there are several triples that are equally good in terms of numbers of features for which the proportion holds, but lead to different predictions, one may then consider that there is no acceptable plausible value for  $x_{n+1}$ .

**Discussing the screen example** Abraham, Gabbay, and Schild (2009) compare the two graphs induced by the partial ordering between the columns of the matrix when the missing value is replaced respectively by 1 and 0. It appears that the graph associated with 1 is superior to the one associated with 0, because it is connected, while the other one is not. This corresponds to an instance of a more general graph comparison procedure (not recalled here) and it leads here to promote 1 as the value we are looking for. Let us examine now the application of the three strategies described above.

The application of the first strategy yields 1 considering that screen 3 is already identical to screen 4 on 3 features.

Using the second strategy, we observe that screen 3 is only in “between” screen 2 and screen 4 in the sense described above, leading again to 1 as a solution.

Using the third strategy that should involve 4 distinct items, we can observe that the analogical proportion *screen 1* : *screen 2* :: *screen 4* : *screen 3* component-wise for features *C*, *R*, and *D* (while it fails on *P* and *I*). Again we get 1 as a solution for ensuring an analogical proportion (namely  $A(1, 1, 1, 1)$ ) on *S*. Observe also that whatever the order in which the screens are considered, an homogeneous proportion holds for features *C*, *R*, *D*, and *S*. For instance, if we consider the permutation (*screen 4* : *screen 2* :: *screen 1* : *screen 3*), these four features satisfy a paralogical, as well as a reverse analogical proportion. Since there are only 4 screens in this example, we have no option to consider other triples of screens that together with the incompletely described screen will have the same homogeneous proportion satisfied on possibly more features. Note that when *T* is an homogeneous proportion, the equation  $T(1, 1, 1, x)$  may have only  $x = 1$  as a solution (for proportions *A*, *R*, *P*). Considering other triples (if available)

may lead to other equations having 0 as a solution. A prediction based on the triple making an homogeneous proportion with the incompletely described item on a maximal number of features, should be preferred. In case of ties on this maximal number of features between concurrent triples leading to opposite predictions, no prediction can be given.

The screen example is clearly a toy example. Abraham, Gabbay, and Schild (2009) also discuss another very similar examples with four cameras, which can be handled in the same way, as well as several other small examples mainly originated from jurisprudence rules in Jewish, Islamic and Indian legal reasoning. These latter examples where the meaning of the rows and columns may vary, could be also handled using homogeneous proportions, sometimes working with columns rather than lines when the matrix has two lines only (or even with the cells themselves if the matrix has only 4 cells). However, in such examples, the matrix appears to go beyond the simple description of items in terms of relevant features, and one may wonder if the matrix abduction problem remains exactly of the same kind, for instance, when the matrix associates causes and effects, which is not the case in the screen or in the camera example.

It is worth noticing that in their approach, the above-cited authors emphasize that the use of 0 and 1 in the Boolean coding in their matrices is not just a matter of convention. They write commenting on a matrix that would pertain to hurricanes that “To turn the data into 0–1 data we need to decide on a cut-off point. Say for winds we choose 150 miles per hour. We have two choices for the wind column. Do we take 1 to mean over 150 miles per hour or do we take it to be 1 = under 150 miles per hour? The reader might think it is a matter of notation but it is not! We need to assume that all the column features pull in the same direction. In the hurricane case the direction we can take is the capacity for damage. In the LCD screen and camera case it is performance.” Such an hypothesis clearly agrees with their intuition of basing their approach on the partial ordering existing between the columns. In our approach, when we constraint our proportions to allocate to 0 and 1 the same role (code independency property), we get that  $A, R, P, I$  are the only logical proportions to satisfy code independency and symmetry. In fact, considering  $A, R, P, I$  together has the advantage of making the conclusion obtained independent of the ordering in which the tuples are considered in a triple, as explained now.

**Why using the  $A, R, P, I$  proportions** Observe that a tuple of 4 Boolean values, corresponding to the values of the same feature for 4 items, may correspond to  $2^4 = 16$  valuations. 8 of them have an odd number of 0’s (and an odd number of 1’s), e.g.,  $(0, 0, 0, 1)$  or  $(1, 1, 0, 1)$ , and are thus “irregular” in the sense that all values are equal except one. The 8 remaining valuations have an even number of 0’s and 1’s. They are exactly the valuations encountered in the truth tables of the homogeneous proportions, as can be checked on Table 1. In fact,  $A, R, P, I$  are the only code independent proportions which are true for 6 of these 8 valuations (since a logical proportion is true for only 6 valuations among 16). Moreover, one can notice that these 8 valuations correspond to 4 basic intuitively meaningful sit-

uations: i)  $(1, 1, 1, 1)$  and  $(0, 0, 0, 0)$  to complete similarity; ii)  $(1, 0, 1, 0)$  and  $(0, 1, 0, 1)$  to changes made in the same direction for the two first components and the two last ones; iii)  $(1, 0, 0, 1)$  and  $(0, 1, 1, 0)$  to changes made in opposite directions; iv)  $(1, 1, 0, 0)$  and  $(1, 1, 0, 0)$  to partial similarity of the two first components and of the two last ones, thus referring to two different contexts.

It is also worth noticing that when we consider 4 Boolean vectors, which are such as for any component (i.e., feature) we have a pattern with an even number of 0’s and 1’s, the following property holds: If a maximum number of 3 of the 4 above types of even patterns are present in the 4 Boolean vectors componentwise, then there exists at least one homogeneous proportion that holds for *all* components. Note that none of the homogeneous proportions can cover the 4 types of even patterns. Moreover, when we change the ordering in which the vectors are considered, there will always exist an homogeneous proportion that holds for *all* components (provided that 3 or less of the 4 above types of even patterns are present). Finally, the homogeneous proportion covers the component where one value is missing for one of the vectors, which means that the proportion equation is then solvable, its solution will remain the same when changing the ordering of the vectors, and solving the new corresponding proportion equation. Thus in a given matrix, looking for a triple of vectors that maximizes the number of features where the same homogeneous proportion holds true (or is solvable) together with the vector having the missing value, is a way of taking advantage of the regularities existing in the matrix to predict the missing values.

Clearly in such problems, the prediction is only “plausible”, whatever the method. The proposed approach makes sense, beyond its properties and the principles that are at work in it, may be also somewhat indirectly justified by the fact that the approach applies as well to IQ tests, as exemplified in this paper, and can be rather successfully scaled up into a learning device (at least for homogeneous proportions  $A$  and  $P$  (see (Prade, Richard, and Yao 2012) for preliminary results). In practice, we may also use different methods for matrix abduction and to “validate” a prediction only if all methods agree.

## Concluding remarks

This paper has singled out four homogeneous logical proportions (including the analogical proportion) which satisfy properties that are similar to the ones enjoyed by arithmetical or geometric proportions. Each of these proportions have distinctive properties that have been laid bare. Their respective roles is still to be investigated in greater details. Still the underlying logical view of analogical proportions has been shown to be able to handle sophisticated IQ tests (but we might consider the application of the other homogeneous proportions as well). Homogeneous logical proportions also provide a general setting for proposing solutions to the commonsense matrix abduction problem. A detailed comparison of the different recently proposed approaches to this problem, whose applicability conditions somewhat differ, is still to be done. Other lines of future research include the extension of the four homogeneous logical proportions to

multiple-valued settings in order to handle non-Boolean features (see (Prade and Richard 2010b) for the homogeneous proportions  $A, R, P$ ). It also appears that what is at work in geometrical IQ proportional analogies might be also transposed to other types of scenes such as topographic maps, see (O'Donoghue, Bohan, and Keane 2006) on this issue.

## References

- Abraham, M.; Gabbay, D. M.; and Schild, U. J. 2009. Analysis of the talmudic argumentum a fortiori inference rule (kal vachomer) using matrix abduction. *Studia Logica* 92(3):281–364.
- Durrenberger, P., and Morrison, J. W. 1977. A theory of analogy. *J. of Anthropological Research* 33(4):372–387.
- French, R. M. 2002. The computational modeling of analogy-making. *Trends in Cognitive Sciences* 6(5):200 – 205.
- Gentner, D.; Holyoak, K. J.; and Kokinov, B. N. 2001. *The Analogical Mind: Perspectives from Cognitive Science, Cognitive Science, and Philosophy*. Cambridge, MA: MIT Press.
- Gentner, D. 1983. Structure-mapping: A theoretical framework for analogy. *Cognitive Science* 7(2):155–170.
- Gick, N., and Holyoak, K. J. 1980. Analogical problem solving. In *Cognitive Psychology*, volume 12. 306–335.
- Helman, D. H., ed. 1988. *Analogical Reasoning: Perspectives of Artificial Intelligence*. Cognitive Science, and Philosophy. Dordrecht: Kluwer.
- Hesse, M. 1959. On defining analogy. *Proceedings of the Aristotelian Society* 60:79–100.
- Hofstadter, D., and Mitchell, M. 1995. The Copycat project: A model of mental fluidity and analogy-making. In Hofstadter, D., and The Fluid Analogies Research Group., eds., *Fluid Concepts and Creative Analogies: Computer Models of the Fundamental Mechanisms of Thought*, 205–267. New York, NY: Basic Books, Inc.
- Holyoak, K. J., and Thagard, P. 1989. Analogical mapping by constraint satisfaction. *Cognitive Science* 13:295–355.
- Klein, S. 1982. Culture, mysticism & social structure and the calculation of behavior. In *Proc. 5th Europ. Conf. in Artificial Intelligence (ECAI'82), Paris Orsay*, 141–146.
- Lepage, Y.; Migeot, J.; and Guillermin, E. 2009. A measure of the number of true analogies between chunks in Japanese. In Vetulani, Z., and Uszkoreit, H., eds., *Human Language Technology. Challenges of the Information Society, Third Language and Technology Conference, LTC 2007, Poznan, Poland, October 5-7, 2007, Revised Selected Papers*, volume 5603 of LNCS, 154–164. Springer.
- Lovett, A.; Forbus, K.; and Usher, J. 2010. A structure-mapping model of Raven's progressive matrices. In *Proc. of the 32nd Annual Conference of the Cognitive Science Society, Portland, OR*.
- Miclet, L., and Prade, H. 2009. Handling analogical proportions in classical logic and fuzzy logics settings. In *Proc. 10th Eur. Conf. on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'09), Verona*, 638–650. Springer, LNCS 5590.
- Miclet, L.; Bayouh, S.; and Delhay, A. 2008. Analogical dissimilarity: definition, algorithms and two experiments in machine learning. *JAIR*, 32 793–824.
- O'Donoghue, D. P.; Bohan, A. J.; and Keane, M. 2006. Seeing things: Inventive reasoning with geometric analogies and topographic maps. *New Generation Comput.* 24:267–288.
- Piaget, J. 1953. *Logic and Psychology*. Manchester Univ. Press.
- Prade, H., and Richard, G. 2010a. Logical proportions - typology and roadmap. In Hüllermeier, E.; Kruse, R.; and Hoffmann, F., eds., *Computational Intelligence for Knowledge-Based Systems Design: Proc. 13th Inter. Conf. on Information Processing and Management of Uncertainty (IPMU'10), Dortmund, June 28 - July 2*, volume 6178 of LNCS, 757–767. Springer.
- Prade, H., and Richard, G. 2010b. Multiple-valued logic interpretations of analogical, reverse analogical, and paralogical proportions. In *Proc. 40th IEEE Int. Symp. on Multiple-Valued Logic (ISMVL'10), Barcelona*, 258–263.
- Prade, H., and Richard, G. 2010c. Reasoning with logical proportions. In *Proc. 12th Inter. Conf. on Principles of Knowledge Representation and Reasoning, KR 2010, Toronto, Ontario, Canada, May 9-13, 2010 (F. Z. Lin, U. Sattler, M. Truszczynski, eds.)*, 545–555. AAAI Press.
- Prade, H., and Richard, G. 2011. Analogy-making for solving IQ tests: A logical view. In *Proc. 19th Inter. Conf. on Case-Based Reasoning, Greenwich, London, 12-15 Sept.*, volume 6880 of LNAI, 241–250. Springer.
- Prade, H.; Richard, G.; and Yao, B. 2012. Enforcing regularity by means of analogy-related proportions—a new approach to classification. *International Journal of Computer Information Systems and Industrial Management Applications* 4:648–658.
- Raven, J. 2000. The Raven's progressive matrices: Change and stability over culture and time. *Cognitive Psychology* 41(1):1 – 48.
- Schockaert, S., and Prade, H. 2011. Qualitative reasoning about incomplete categorization rules based on interpolation and extrapolation in conceptual spaces. In Benferhat, S., and Grant, J., eds., *Proc. 5th Inter. Conf. on Scalable Uncertainty Management (SUM'11), Dayton, OH, Oct.10-13*, volume 6929 of LNCS, 303–316. Springer.
- Stroppa, N., and Yvon, F. 2005. Analogical learning and formal proportions: Definitions and methodological issues. ENST Paris report.
- Stroppa, N., and Yvon, F. 2006. Du quatrième de proportion comme principe inductif : une proposition et son application à l'apprentissage de la morphologie. *Traitement Automatique des Langues* 47(2):1–27.
- Tversky, A. 1977. Features of similarity. In *Psychological Review*, number 84. American Psychological Association. 327–352.