Characterizing the Codimension of Zero Singularities for Time-Delay Systems: A Link with Vandermonde and Birkhoff Incidence Matrices

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Abstract The analysis of time-delay systems mainly relies on detecting and understanding the spectral values bifurcations when crossing the imaginary axis. This paper deals with the zero singularity, essentially when the zero spectral value is multiple. The simplest case in such a configuration is characterized by an algebraic multiplicity two and a geometric multiplicity one, known as the Bogdanov-Takens singularity. Moreover, in some cases the codimension of the zero spectral value exceeds the number of the coupled scalar-differential equations. Nevertheless, to the best of the author's knowledge, the bounds of such a multiplicity have not been deeply investigated in the literature. It is worth mentioning that the knowledge of such an information is crucial for nonlinear analysis purposes since the dimension of the projected state on the center manifold is none other than the sum of the dimensions of the generalized eigenspaces associated with spectral values with zero real parts. Motivated by a control-oriented problems, this paper provides an answer to this question for time-delay systems, taking into account the parameters' algebraic constraints that may occur in applications. We emphasize the link between such a problem and the incidence matrices associated with the Birkhoff interpolation problem. In this context, symbolic algorithms for LU-factorization for functional confluent Vandermonde as well as some classes of bivariate functional Birkhoff matrices are also proposed.

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Contents

1 Introduction

[1]

Matrices arising from a wide range of problems in mathematics and engineering typically display a characteristic structure. Exploiting such a structure is the means to the design of efficient algorithms, see for instance [3]. This study is a crossroad between the investigation of a class of such structured matrices originally involved in *Multivariate Interpolation Problems* (namely, the well-known *Birkhoff Interpolation Problem*) and the estimation of the upper bound for the codimension of *spectral values* of linear *Time-Delay Systems* (TDS) (which are the zeros of some characteristic entire function called *characteristic quasipolynomial*). The aim of this paper is three fold: firstly, it emphasizes the link between the above two quoted issues. Secondly, it shows that the codimension of the zero spectral value of a given TDS is characterized by some algebraic properties of an appropriate *functional Birkhoff Matrix.* Finally, it shows the effectiveness of the proposed constructive approach by exploring the generic settings as well as investigating some specific but significant sparsity patterns. In both cases, symbolic algorithms for LU-factorization are established for some novel classes of Birkhoff matrices. It is worth mentioning that such an attempt can be exploited for further classes of Birkhoff matrices and should be of interest in some linear algebra problems involving structured matrices as well as in applications including polynomial interpolation.

The class of systems considered throughout this paper is infinite-dimensional with N discrete (constant) delays. Let $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ denote the state-vector, then the system reads as follows

[1] Some of the results proposed in this work have been presented in The 13th European Control Conference, June 24-27, 2014, Strasbourg, France [1] and The 21st International Symposium on Mathematical Theory of Networks and Systems July 7-11, 2014, Groningen, The Netherlands [2]

$$\dot{z} = \sum_{k=0}^{N} A_k z(t - \tau_k), \tag{1}$$

under appropriate initial conditions belonging to the Banach space of continuous functions $\mathcal{C}([-\tau_N, 0], \mathbb{R}^n)$. Here $\tau_k,\ k = 1 \dots N$ are strictly increasing positive constant delays such that $\tau_0 = 0$ and $\tau = (\tau_1, \dots, \tau_N)$, and $A_k \in \mathcal{M}_n(\mathbb{R})$ for $k = 0 \dots N$. It is well known that the exponential stability of the solutions of (1) is derived by the location of the spectrum χ, where χ designates the set of roots of the *characteristic function* [4, ?]. Notice that such a function, denoted in the sequel $\Delta(\lambda, \tau)$, is transcendental in the Laplace variable λ. Such a spectrum χ can be split as $\chi = \chi_+ \cup \chi_0 \cup \chi_-$ where $\chi_+ = \{\lambda \in \mathbb{C},\ \Delta(\lambda, \tau) = 0,\ \Re(\lambda) > 0\}$, $\chi_- = \{\lambda \in \mathbb{C},\ \Delta(\lambda, \tau) = 0,\ \Re(\lambda) < 0\}$ and $\chi_0 = \{\lambda \in \mathbb{C},\ \Delta(\lambda, \tau) = 0,\ \Re(\lambda) = 0\}$. More precisely, the characteristic function of system (1) $\Delta :\ \mathbb{C} \times {\mathbb{R}^N}_+ \to \mathbb{C}$ which reads as

$$\Delta(\lambda, \tau) = \det \left(\lambda\, I - A_0 - \sum_{k=1}^{N} A_k\, e^{-\tau_k \lambda} \right), \tag{2}$$

One can prove that the quasipolynomial function (2) admits an infinite number of zeros, see for instance the references [5, 6, 7]. The study of zeros of an entire function [7] of the form (2) plays a crucial role in the analysis of asymptotic stability of the zero solution of some given system (1). Indeed, the zero solution is asymptotically stable if all the zeros of (2) are in the open left-half complex plane [8]. Accordingly to this observation, the parameter space which is spanned by the coefficients of the polynomials P_i, can be split into domains, each of them with a constant number of right half-plane characteristic roots (which is nothing but the so-called D-decomposition, see for instance [8] and references therein). These domains are separated by a boundary corresponding to the case when at least one characteristic root belongs to the spectrum. In the stability analysis of TDS, we are particularly interested by the stability domains (all characteristic roots with strictly negative real parts) as well as their boundary. Moreover, under appropriate algebraic restrictions, a given root associated to such a boundary may have high codimension. In this paper, we are concerned with the codimension of the zero spectral value. The typical example for non-simple zero spectral value is the Bogdanov-Takens singularity which is characterized by an algebraic multiplicity two and a geometric multiplicity one. Cases with higher order multiplicities of the zero spectral value are known to us as generalized Bogdanov-Takens singularities. Those types of configurations are not only theoretical and are involved in concrete applications. Indeed, the Bogdanov-Takens singularity is identified in [9] where the case of two coupled scalar delay equations modeling a physiological control problem is studied. In [10] and [11] this type of singularity is also encountered in the study of coupled axial-torsional vibrations of some oilwell rotary drilling system. Moreover, the paper [12] is devoted to the analysis of such type of singularities where codimension two and three are studied, and the associated center manifolds are explicitly computed. It is commonly accepted that the time-delay induces desynchronizing and/or destabilizing effects on the dynamics. However, new theoretical developments in control of finite-dimensional dynamical systems suggest the use of delays in the control laws for stabilization purposes. For instance, [13, 14] are concerned by the stabilization of the inverted pendulum by delayed control laws and provide concrete situations where the codimension of the zero spectral value exceeds the number of the coupled scalar equations modeling the inverted pendulum on cart. In [13], the authors prove that the delayed proportional-derivative (PD) controller stabilizes the inverted pendulum by identifying a codimension three singularity for a system of two coupled delay equations. In [14], the same singularity is characterized by using a particular delay block configuration. It is shown that two delay blocks offset a PD delayed controller.

Although the algebraic multiplicity of each spectral value of a time-delay system is finite (a direct consequence of Rouché lemma, see [15]), to the best of the author's knowledge, the computation of *the upper bound of the codimension of the zero spectral value* did not receive a complete characterization especially in the case when the physical parameters of a given time-delay model are subject to algebraic constraints. Namely, if the root at the origin is invariant with respect to the delay parameters, however, its multiplicity is strongly dependent on the existing links between the delays and the other parameters of the system.

Furthermore, the knowledge of the codimension of such a spectral value as well as the number of purely imaginary spectral values counting their multiplicities are valuable information. For instance, such an information allows to estimate the number of unstable roots; $\mathbf{card}(\chi_+)$ where $\chi_+ = \{\lambda \in \mathbb{C},\ \Delta(\lambda, \tau) = 0,\ \Re(\lambda) > 0\}$ for a given time-delay system (1). Actually, the main theorem from [16, p. 223], which is reported in the Appendix, emphasizes the link between $\mathbf{card}(\chi_+)$ and $\mathbf{card}(\chi_0)$ (Multiplicity is taken into account). In the light of the quoted result and its potential consequences on designing new approaches for the characterization of the linear stability analysis of time-delay systems, the need of for greater emphasis on the study of the zero singularity and, more generally, the imaginary roots becomes obvious. Finally, it is worth mentioning that such a study is also interesting from a nonlinear analysis viewpoint, which gives another motivation for the present investigation. When the unstable spectrum is an empty set or equivalently $\chi = \chi_- \cup \chi_0$, a complete knowledge of the imaginary roots as well as their multiplicities becomes crucial predominatingly when the center manifold and the normal forms theory are involved for deriving an accurate local qualitative description of the studied dynamical system, see [17]. In particular, in this case, when the zero spectral value is the only spectral value with zero real part; $\chi_0 = \{0\}$, then the center manifold dimension is none other than the codimension of the generalized Bogdanov-Takens singularity [18, 17, 19].

In the context mentioned above, as a first estimation on the bound for the codimension of the zero spectral value for quasipolynomial functions, we emphasize the Pólya-Szegö bound [15, pp. 144], which will be denoted $\sharp_{PS}$ in the sequel. The proof of such a result is based on the Rouché lemma and

Cauchy's argument principle, see also [20, pp. 198] for further insights on the distribution of zeros of quasipolynomial functions. Recall that $\sharp_{PS}$ bound is nothing but the degree of the considered quasipolynomial function, see also [21] for a modern formulation of the mentioned result. Originally, the Pólya-Szegö result, which is reported in the Appendix, gives a bound for the number of roots of a quasipolynomial function in a horizontal strip $\alpha \leq \mathcal{I}(z) \leq \beta$. Setting $\alpha = \beta = 0$ provides a bound for the number of real spectral values, which is a natural bound for the multiplicity of the zero spectral value. It will be stressed in the sequel that the Pólya-Szegö bound is a sharp bound for the zero spectral value multiplicity in the case of quasipolynomial functions consisting in the so-called *complete* or *dense* polynomials. Nevertheless, it is obvious that the Pólya-Szegö bound remains unchanged when certain coefficients $c_{i,j}$ vanish without affecting the degree of the quasipolynomial function. Such a simple remark allows us claiming that Pólya-Szegö bound does not take into account the algebraic constraints on the parameters. However, such constraints appear naturally in applications. In fact, models issued from applications often consist in *lacunary* ou *sparse* structures [22], illustrative examples will be given in the next section concerned by motivations. Moreover, when one needs conditions insuring a given multiplicity bounded by $\sharp_{PS}$, then computations of the successive differentiations of the quasipolynomial have to be made.

In the sequel, among others, we emphasize a systematic approach allowing to a sharper bound for the zero spectral value multiplicity. Indeed, the proposed approach does not only take into account the algebraic constraints on the coefficients $c_{i,j}$ but it also gives appropriate conditions guaranteeing any admissible multiplicity. Furthermore, the symbolic approach we adopt in this study underlines the connection between the codimension of the zero singularity problem and *incidence matrices* of the so-called *Confluent Vandermonde Matrix* as well as the *Birkhoff Matrix*, see for instance [23,24,25,26]. To the best of the author's knowledge, the first time the Vandermonde matrix appears in a control problem is reported in [27, p. 121], where the controllability of a finite dimensional dynamical system is guaranteed by the invertibility of such a matrix, see also [28,29]. Next, in the context of time-delay systems, the use of the standard Vandermonde matrix properties was proposed by [30,8] when controlling one chain of integrators by delay blocks. Here we further exploit the algebraic properties of such matrices into a different context.

The remaining paper is organized as follows. Section 2 presents some prerequisites and the problem statement. It is concluded by a more focused description of the paper objectives and contributions. Section 3 exhibits some motivating examples including some illustrations of the limitations of the Pólya-Szegö bound $\sharp_{PS}$. Next, the main results are proposed and proved in Section 4 and Section 5. Namely, Section 4 provides some symbolic algorithms for the LU-factorisation associated to some classes of *functional Birkhoff Matrices.* Then, the results of the later section are exploited in Section 5 which provides an adaptive bound for the multiplicity of the zero spectral value for quasipolynomial functions. Various illustrative examples and control-oriented discussions are presented in Section 6. Finally, some concluding remarks end the paper.

2 Prerequisites and statement of the problem

Let us start by setting a new parametrization for the quasipolynomial function (2) characterizing the time-delay system (1) and defining some useful notations adopted through the paper.

Some straightforward computations give the following formal expression of the quasipolynomial function (2)

$$\Delta(\lambda,\sigma) = P_0(\lambda) + \sum_{M^k \in S_{N,n}} P_{M^k}(\lambda)\, e^{\sigma_{M^k}\,\lambda}, \tag{3}$$

where $\sigma_{M^k} = -M^k\,\tau^T$ and $S_{N,n}$ is the set of all the possible row vectors $M^k = (M_1^k,\,\ldots,\,M_N^k)$ belonging to $\mathbb{N}^N$ such that $1 \leq M_1^k + \ldots + M_N^k \leq n$. Furthermore, by running the index from 1 to the cardinality $\tilde{N}_{N,n} \triangleq \mathbf{card}(S_{N,n})$ the quasipolynomial (3) is written in the following compact form

$$\Delta(\lambda,\sigma) = P_0(\lambda) + \sum_{k=1}^{\tilde{N}_{N,n}} P_k(\lambda)\, e^{\sigma_k\,\lambda}. \tag{4}$$

For instance,

$$\begin{aligned} S_{3,2} = \{&(1,0,0),\ (0,1,0),\ (0,0,1),\ (2,0,0),\ (1,1,0),\ (1,0,1),\ (0,2,0),\\ &(0,1,1),\ (0,0,2)\}\,, \end{aligned}$$

is ordered first by increasing sums ($\sum_{i=1}^N M_i^k$) then by lexicographical order. In this case one has:

$$M^2 = (0,1,0) \quad \text{and} \quad \tilde{N}_{3,2} = 9.$$

A generical property of retarded systems (1) allow considering P_0 as a monic polynomial of degree n in λ and the polynomials P_{M^k} satisfying $\deg(P_{M^k}) = n - \sum_{s=1}^N M_s^k \leq (n-1)\ \forall M^k \in S_{N,n}$. In the sequel, $P_0(\lambda)$ will be called the *delay-free polynomial* and the quasipolynomial function $\sum_{k=1}^{\tilde{N}_{N,n}} P_{M^k}(\lambda)\, e^{\sigma_k\,\lambda}$ will be called the *transcendental part of the quasipolynomial.*

Next, let define $a_{j,k}$ as the coefficient of the monomial λ^k for the polynomial P_{M^j}, $1 \leq j \leq \tilde{N}_{N,n}$, and denote $P_{M^0} = P_0$. Thus, $a_{0,n} = 1$ and $a_{j,k} = 0\ \forall k \geq d_j = n - \sum_{s=1}^N M_s^j$, here $d_j - 1$ is nothing but the degree of P_{M^j}. Furthermore, the following notations are adopted: $a_0 = (a_{0,0},\, a_{0,1},\, \ldots,\, a_{0,n-1})^T$ is the vector of the coefficients of the delay-free polynomial and $a_j = (a_{j,0},\, a_{j,1},\, \ldots,\, a_{j,d_j-1})^T$ is the vector of the coefficients of the polynomial associated to the auxiliary delay σ_j for $1 \leq j \leq \tilde{N}_{N,n}$. Next, set the delay auxiliary vector $\sigma = (\sigma_1,\, \sigma_2,\, \ldots,\, \sigma_{\tilde{N}_{N,n}})$ and $a = (a_1/\, a_2/\, \ldots /\, a_{\tilde{N}_{N,n}})^T$ where

$$(x/y) = (x_1, \ldots, x_{d_x}, y_1, \ldots, y_{d_y})^T$$

for $x = (x_1, \ldots, x_{d_x})^T$ and $y = (y_1, \ldots, y_{d_y})^T$. Let us denote by $\Delta^{(k)}(\lambda, \sigma)$ the k-th derivative of $\Delta(\lambda, \sigma)$ with respect to the variable λ. We say that zero is an eigenvalue of algebraic multiplicity/codimension $m \geq 1$ for (1) if $\Delta(0, \sigma^*) = \Delta^{(k)}(0, \sigma^*) = 0$ for all $k = 1, \ldots, m-1$ and $\Delta^{(m)}(0, \sigma^*) \neq 0$. We assume also in what follows that $\sigma_k \neq \sigma_{k'}$ for any $k \neq k'$ where $k, k' \in S_{N,n}$. Indeed, if for some value of the delay vector τ there exists some $k \neq k'$ such that $\sigma_k = \sigma_{k'}$, then the number of auxiliary delays and the number of polynomials is reduced by considering a new family of polynomials $\tilde{P}$ satisfying $\tilde{P}_{M^k} = P_{M^k} + P_{M^{k'}}$. In the sequel, D_q will designate *the degree* of the transcendental part of the quasipolynomial. [2]

Now, to characterize the structure of a given quasipolynomial function one needs to introduce a vector $\mathcal{V}$, which will be called in the sequel *incidence vector*, reproducing the data on the vanishing components of the vector "a" defined above. Thus, $\mathcal{V}$ is a sparsity patterns indicator for the transcendental part of the quasipolynomial. To do so, we introduce the symbol "*star*" ($\star$) to indicate the vanishing of a given coefficient of the transcendental part of the quasipolynomial.

To illustrate the notion of "incidence vector" as well as the symbol "*star*", let consider the following quasipolynomial function:

$$\Delta(\lambda, \sigma) = P_0(\lambda) + (a_{1,0,0,0} + a_{1,0,0,1}\lambda)\ e^{\sigma_{1,0,0}\lambda} + a_{0,1,0,2}\lambda^2\, e^{\sigma_{0,1,0}\lambda} + a_{0,0,1,1}\lambda\, e^{\sigma_{0,0,1}\lambda}. \tag{5}$$

According to the above considerations, P_0 is a polynomial with $\deg(P_0) = n \geq 3$. The transcendental part of (5), is characterized by the following incidence vector

$$\mathcal{V} = (x_1, x_1, \star, \star, x_2, \star, x_3). \tag{6}$$

Namely, the first two components of $\mathcal{V}$ indicate that P_{M^1} is a *complete* polynomial of degree 1, the three components $\star, \star, x_2$, indicate that $a_{0,1,0,0} = a_{0,1,0,1} = 0$ and P_{M^2} is a *lacunary* polynomial of degree 2 and finally, the last components $\star, x_3$ say that $a_{0,0,1,0}$ and P_{M^3} is lacunary polynomial of degree 1.

Let us better formalize the description of the shape of a given incidence vector. As a matter of fact, the above definition, may help in the sequel, for describing the organisation of the distribution of symbol $\star$ in the components of the incidence vector $\mathcal{V}$.

Definition 21. *The symbol $\star$ appearing in a given incidence vector $\mathcal{V}$ is classified as follows:*

- *If the symbol $\star$ (or a sequence of the symbol $\star$) starts a sequence of an interpolating point x_k in the incidence vector $\mathcal{V}$ then we call it a* starter star.

[2] The sum of the degrees of the polynomials involved in the quasi-polynomial plus the number of polynomials involved minus one is called the degree of a given quasi-polynomial. Further discussions on such a notion can be found in [21].

- *If the symbol $\star$ (or a sequence of the symbol $\star$) is strictly included in a sequence of an interpolating point x_k (the symbol $\star$ is not located at the border of a given sequence x_k) in the incidence vector $\mathcal{V}$ it is called an* intermediate star*.*
- *The length of a sequence of "stars" is the number of the repetition of symbol $\star$ without interruption.*

We recognize that the notion of incidence vectors we introduced above is closely inspired from the notion of the *incidence matrices*. Such matrices are known to be involved in defining the structure of the well known *Birkhoff matrices*.

Initially, Birkhoff and Vandermonde matrices are derived from the problem of polynomial interpolation of some unknown function g, this can be presented in a general way by describing the interpolation conditions in terms of *incidence matrices*, see for instance [31]. For given integers $n \geq 1$ and $r \geq 0$, the matrix

$$\mathcal{E} = \begin{pmatrix} e_{1,0} & \cdots & e_{1,r} \\ \vdots & & \vdots \\ e_{n,0} & \cdots & e_{n,r} \end{pmatrix}$$

is called an incidence matrix if $e_{i,j} \in \{0,1\}$ for every i and j. Such a matrix contains the data providing the known information about the function g. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $x_1 < \ldots < x_n$, the problem of determining a polynomial $\hat{P} \in \mathbb{R}[x]$ with degree less or equal to ι ($\iota + 1 = \sum_{1 \leq i \leq n,\ 1 \leq j \leq r} e_{i,j}$) that interpolates g at $(x, \mathcal{E})$, i.e. which satisfies the conditions:

$$\hat{P}^{(j)}(x_i) = g^{(j)}(x_i),$$

is known as the *Birkhoff interpolation problem.* Recall that $e_{i,j} = 1$ when $g^{(j)}(x_i)$ is known, otherwise $e_{i,j} = 0$. Furthermore, an incidence matrix $\mathcal{E}$ is said to be *poised* if such a polynomial $\hat{P}$ is unique. This amounts to saying that, if, $n = \sum_{i=1}^{n} \sum_{j=1}^{r} e_{i,j}$ then the coefficients of the interpolating polynomial $\hat{P}$ are solutions of a linear square system with associated square matrix Υ that we call *Birkhoff matrix* in the sequel. This matrix is parametrized in x and is shaped by $\mathcal{E}$. It turns out that the incidence matrix $\mathcal{E}$ is poised if, and only if, the Birkhoff matrix Υ is non singular for all x such that $x_1 < \ldots < x_n$. The characterization of poised incidence matrices is solved for interpolation problems of low degrees. As a matter of fact, the problem is still unsolved for any degree greater than six, see for instance [26,32].

Remark 1. *Unlike Hermite interpolation problem, for which the knowledge of the value of a given order derivative of the interpolating polynomial at a given interpolating point impose the values of all the lower orders derivatives of the interpolating polynomial at that point, the Birkhoff interpolation problem release such a restriction. Thereby justifying the qualification of "lacunary" to describe the Birkhoff interpolation problem.*

The analogy between the introduced incidence vector $\mathcal{V}$ for time-delay system analysis purposes and the incidence matrix $\mathcal{E}$ characterizing multivariate interpolation problems can be interpreted as follows. Each $k-$th row of $\mathcal{E}$ indicate the distribution of the symbol $\star$ in the sequence of x_k corresponding to $\mathcal{V}$. Namely, in that row of $\mathcal{E}$, each "1" corresponds to an x_k of $\mathcal{V}$ and each "0" (situated at the left of at least a "1") corresponds to a symbol $\star$ in $\mathcal{V}$.

Proposition 21. *There exists a one to one mapping between $\mathcal{V}$ and $\mathcal{E}$.*

Now, to illustrate the analogy between the two concepts of incidence vector/matrix, let us consider the reduced example from [32] where the incidence matrix $\mathcal{E}$ is given by

$$\mathcal{E} = \begin{pmatrix} 1\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \end{pmatrix}. \tag{6'}$$

The first row of $\mathcal{E}$ indicates that $g(x_1)$ and $g'(x_1)$ are known. In terms of time-delay systems purposes, this reproduces the first two components of $\mathcal{V}$, namely, x_1, x_1. The second row of $\mathcal{E}$ says that only $g''(x_2)$ is known, which in terms of $\mathcal{V}$, reproduces the components $\star, \star, x_2$. Finally, the third row says that only $g'(x_3)$ is known, which reproduces the last two components $\star, x_3$ of $\mathcal{V}$. This, shows that the incidence matrix (6') and the incidence vector (6) reproduce exactly the same information. For the sake of saving space, it is more appropriate to consider $\mathcal{V}$ in the sequel. Recall that one associates to (6') (or equivalently to (6)) the following Birkhoff matrix

$$\Upsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 1 \\ x_1^2 & 2x_1 & 2 & 2x_3 \\ x_1^3 & 3x_1^2 & 6x_2 & 3x_3^2 \end{pmatrix}.$$

Accordingly, the interpolation problem is solvable if, and only if,

$$12\, x_3\, x_2 + 6\, x_1^2 - 12\, x_2\, x_1 - 6\, x_3^2$$

does not vanish for all values of x such that $x_1 < x_2 < x_3$.

In the spirit of the definition of functional confluent Vandermonde matrices introduced in [28], we provide a definition for functional Birkhoff matrices.

Definition 22. *The square* functional Birkhoff matrix Υ *is associated to a sufficiently regular function ϖ and an incidence matrix $\mathcal{E}$ (or equivalently an incidence vector $\mathcal{V}$) and is defined by:*

$$\Upsilon = [\Upsilon^1\, \Upsilon^2\, \ldots\, \Upsilon^M] \in \mathcal{M}_\delta(\mathbb{R}) \tag{7}$$

where

$$\Upsilon^i = [\kappa^{(k_{i_1})}(x_i)\; \kappa^{(k_{i_2})}(x_i)\; \ldots\; \kappa^{(k_{i_{d_i}})}(x_i)] \tag{8}$$

such that $k_{i_l} \geq 0$ for all $(i,l) \in \{1,\ldots,M\} \times \{1,\ldots,d_i\}$ and $\sum_{i=1}^M d_i = \delta$ where

$$\kappa(x_i) = \varpi(x_i)[1\, \ldots\, x_i^{\delta-1}]^T, \quad \textit{for} \quad 1 \leq i \leq M. \tag{9}$$

Remark 2. *In the sequel, for the time-delay systems analysis purposes, we shall be concerned with square functional Birkhoff matrices such that* $\varpi(x_i) = x_i^s$*, where* s *is a given positive integer. Furthermore, in terms of the quasipolynomial function* (4)*, the degree of the delay-free polynomial* P_0 *is fixed thanks to the explicit choice of the function* $\varpi(x_i) = x_i^s$*. More precisely,* $n \triangleq s-1$*. Analogously to the Birkhoff interpolation problem, the non degeneracy of such functional Birkhoff matrices will be a fondamental assumption for investigating the codimension of the zero spectral values for time-delay systems.*

Remark 3. *When* $s = 0$*, the matrix* Υ *is nothing but the standard Birkhoff matrix and thus* $\varpi(x_i) = 1$*. If, in addition,* $\mathcal{V}$ *does not contain "stars" then we recover the confluent Vandermonde matrix [28]. The particular case* $d_i = 1$ *for* $i = 1 \ldots M$ *corresponds to the standard Vandermonde matrix and in this case* $M = \delta$ *since* Υ *is assumed to be a square matrix.*

The explicit development of numeric/symbolic algorithms for LU factorization and inversion of the confluent Vandermonde and Birkhoff matrices [29, 33,34,35] is still an attracting topic due to their specific structure and their implications in various applications, see for instance [27,36] and references therein. Furthermore, in our opinion, such an objective is still challenging when reduced complexity algorithms are needed to factorize such matrices. For instance, the reader is referred to [29] where a numerical recipe is derived for computing the inversion of the confluent Vandermonde matrix with a computational complexity $O(n^2)$, where n corresponds to the matrix dimension. However, as emphasized in [37], deriving explicit fully analytical formulae for such factorizations is of great help in order to perform such efficient algorithms.

It is worth mentioning that one of the contributions of this paper is to propose an explicit recursive formula for the LU-factorization for several configurations of the functional Birkhoff matrix defined by (7)-(9). The proposed formulas are in the spirit of the symbolic expressions established in [35] for the standard Vandermonde case. To the best of the author's knowledge, such an explicit formulas seems to be unavailable in the open literature, see e. g., [26,35]. In fact, the historical note in [35] emphasizes that rather the extensive numerical literature on practical solutions to Vandermonde systems fails to reveal the explicit factorization formula for the LU-factorization as well as to the symbolic inversion of the standard Vandermonde matrix.

The functional Birkhoff matrix configurations we consider are: the first one, no "stars" in the incidence vector $\mathcal{V}$, that is the functional confluent Vandermonde matrix. Next, the second configuration is when we deal with *starter "stars"*. Finally, an LU-factorization in the case of *successive intermediate "stars"* is established. We claim that, the characterization we present through the paper yields some new possibilities to get formulae in cases combining the two configurations (starter/intermediate "stars"), but, this needs careful inspection of the implicated polynomials and then by adapting the proposed formula to the specific incidence vector $\mathcal{V}$. Since the formulas depend explicitly on the choice of the specific incidence matrix $\mathcal{E}$, then it makes no sense that one goes further in defining some deeper classification. Furthermore, as a byproduct of the approach, we first propose a different proof for the Pólya-Szegö bound $\sharp_{PS}$ of the origin multiplicity deduced from proposition 72 (presented in the appendix), then, we shall establish a sharper bound for such a multiplicity under the non degeneracy condition of an appropriate Birkhoff matrix.

To summarize, the contribution of the present paper is threefold:

1. In the general case, the Birkhoff interpolation problem may or may not have a unique solution. To the best of the author's knowledge, no general pattern for its determinant is known, and thus no general formula for the interpolating polynomial (when it exists) is known. Moreover, it seems that the problem is still unsolved [26, 32] since such a formula depends directly on the chosen incidence matrix among a multitude of configurations. As an attempt, we propose an explicit recursive formula for the LU-factorization of the functional confluent Vandermonde matrix as well as some classes of the functional Birkhoff matrix.
2. We introduce incidence matrices for describing quasipolynomial functions. Then, we identify the existing link between the multiplicity of the zero singularity of time-delay systems (even in the presence of coupling delays) and an appropriate functional Birkhoff matrix, as defined in Definition 22.
3. Finally, in the generic case (all the polynomials $P_{M_k\,k\geq 0}$ are complete), the Pólya-Szegö bound $\sharp_{PS}$ is completely recovered using an alternative Vandermonde-based method. Moreover, when at least one of the polynomials is lacunary [22] (contains a "star" or a sequence of successive "stars"), then under the non degeneracy of an appropriate functional Birkhoff matrix, we establish a bound for the multiplicity of the zero singularity which is sharper than the Pólya-Szegö bound $\sharp_{PS}$.

In order to increase the readability of the paper, the following notations are adopted. Let ξ stand for the real vector composed from x_i counting their repetition d_i through columns of Υ, that is

$$\xi = (\underbrace{x_1, \ldots, x_1}_{d_1}, \ldots, \underbrace{x_M, \ldots, x_M}_{d_M}).$$

For instance, one has $\xi_1 = x_1$ and $\xi_{d_1+d_2+1} = \xi_{d_1+d_2+d_3} = x_3$. Accordingly and setting $d_0 = 0$, without any loss of generality, we have: $\xi_k = \xi_{d_0+\ldots+d_r+\alpha} = \xi_{\sum_{l=0}^{\varrho(k)-1} d_l + \varkappa(k)}$, where $0 \leq r \leq M-1$ and $\alpha \leq d_{r+1}$; here $\varrho(k)$ denotes the index of the component of x associated with ξ_k, that is $x_{\varrho(k)} = \xi_k$ and by $\varkappa(k)$ the order of ξ_k in the sequence of ξ composed only by $x_{\varrho(k)}$. Obviously, $\varrho(k) = r+1$ and $\varkappa(k) = \alpha$.

3 Motivating examples and further observations

3.1 A Vector Disease Model

Consider first a simple scalar differential equation with one delay representing some biological model discussed by Cooke in [38] describing the dynamics of some disease. Namely, the infected host population $x(t)$ is governed by:

$$\dot{x}(t) + a_0\, x(t) + a_1\, x(t-\tau) - a_1\, x(t-\tau)\, x(t) = 0, \tag{10}$$

where $a_1 > 0$ designates the contact rate between infected and uninfected populations and it is assumed that the infection of the host recovery proceeds exponentially at a rate $-a_0 > 0$; see also [39] for more insights on the modeling and stability results. The linearized system is given by

$$\dot{x}(t) + a_0\, x(t) + a_1\, x(t-\tau) = 0, \tag{11}$$

with $(a_0, a_1, \tau) \in \mathbb{R}^2 \times \mathbb{R}_+^*$, then the associated characteristic (transcendental) function Δ becomes

$$\Delta(\lambda, \tau) = \lambda + a_0 + a_1\, e^{-\lambda\tau}, \tag{12}$$

for which, the corresponding incidence vector is $\mathcal{V} = (x_1)$.

Zero is a spectral value for (11) if, and only if, Δ vanishes at zero which is equivalent to $a_0 + a_1 = 0$. Computations of the first derivatives of (12) with respect to λ give, using the notation "$\frac{\partial .}{\partial \lambda} = .'$",

$$\begin{aligned} \Delta'(\lambda, \tau) &= 1 - \tau\, a_1\, e^{-\lambda\tau}, \\ \Delta''(\lambda, \tau) &= \tau^2\, a_1\, e^{-\lambda\tau}. \end{aligned} \tag{13}$$

The vanishing of the two first derivatives allows us to conclude that the codimension of the zero spectral value is at most two (Bogdanov-Takens singularity) since the algebraic multiplicity two is insured by $\tau = 1/a_1,\ a_0 = -a_1$ and $\Delta''(0) \neq 0$ $(\tau \in \mathbb{R}_+^*)$.
This provides the simplest example showing that the codimension of the zero spectral value can exceed the number of scalar equations defining a given system. Moreover, this emphasizes that the codimension of the zero singularity is less than the number of the free parameters involved in the associated (quasipolynomial) characteristic function. In this case, the number of free parameters is three and the upper bound of the codimension is two, which is exactly the Pólya-Szegö bound $\sharp_{PS}$.

3.2 Furuta Pendulum

Consider now the rotary Furuta inverted pendulum, which consists of a driven arm that rotates in the horizontal plane and a pendulum attached to that arm which is free to rotate in the vertical plane, see figure 1. This device has two rotational degrees of freedom and only one actuator and is thus an underactuated system. Balancing the pendulum in the vertical unstable equilibrium position requires continuous correction by a control mechanism, see [40]. We focus now on the use of multiple delayed proportional gain as suggested by [14] in controlling the inverted pendulum on cart. Using the Lagrange formalism and adopting the *Quanser* rotary experiment settings for the physical parameter values [41], we can easily show that the dynamics of the rotary

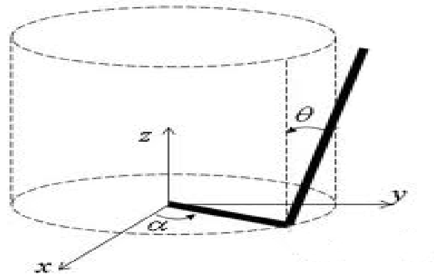

Fig. 1 Inverted Pendulum on a cart

Furuta inverted pendulum in figure 1 is governed by the following system of differential equations:

$$\begin{cases} \ddot{\alpha} = -\dfrac{27692}{3}\,\dfrac{6\,\sin(\theta)\dot{\theta}^2 + 265\,\cos(\theta)\sin(\theta) - 1806T}{-90601 + 39008\,\cos^2(\theta)}, \\ \ddot{\theta} = \dfrac{53}{6}\,\dfrac{4416\,\cos(\theta)\sin(\theta)\dot{\theta}^2 - 1329216\,\cos(\theta)T + 453005\,\sin(\theta)}{-90601 + 39008\,\cos^2(\theta)}, \end{cases} \tag{14}$$

where T is the control law acting on the motor torque. Define

$$\begin{aligned} T(t) =& \beta_{1,0}\theta(t) + \gamma_{1,0}\alpha(t) + \beta_{1,1}\theta(t-\tau_1) + \beta_{1,2}\theta(t-\tau_2) \\ &+ \gamma_{1,1}\alpha(t-\tau_1) + \gamma_{1,2}\alpha(t-\tau_2). \end{aligned} \tag{15}$$

Recall that it is always possible to normalize one of the delays by a simple scaling of time. Thus, without any loss of generality, $\tau_1 = 1$. With the later remark, when T is defined by (15), the linearization around the origin associated with system (14) is given by

$$\dot{x} = A_0\,x + A_1\,x(t-1) + A_2\,x(t-\tau_2), \tag{16}$$

where $A_k = (a_{k,i,j})_{(i,j)\in\{1,\dots,4\}\times\{1,\dots,4\}} \in \mathcal{M}_4(\mathbb{R})$ for $k = 0,\,1,\,2$

$$A_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a_{0,3,1} & a_{0,3,2} & 0 & 0 \\ a_{0,4,1} & a_{0,4,2} & 0 & 0 \end{pmatrix},\ A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{1,3,1} & a_{1,3,2} & 0 & 0 \\ a_{1,4,1} & a_{1,4,2} & 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a_{2,3,1} & a_{2,3,2} & 0 & 0 \\ a_{2,4,1} & a_{2,4,2} & 0 & 0 \end{pmatrix}.$$

where

$$
\begin{aligned}
a_{0,3,1} &= -\frac{16670584}{51593}\,\gamma_{1,0},\; a_{0,3,2} = -\frac{16670584}{51593}\,\beta_{1,0} + \frac{7338380}{154779},\\
a_{0,4,1} &= \frac{11741408}{51593}\,\gamma_{1,0},\; a_{0,4,2} = -\frac{24009265}{309558} + \frac{11741408}{51593}\,\beta_{1,0},\\
a_{1,3,1} &= -\frac{16670584}{51593}\,\gamma_{1,1},\; a_{1,3,2} = -\frac{16670584}{51593}\,\beta_{1,1},\; a_{1,4,1} = \frac{11741408}{51593}\,\gamma_{1,1},\\
a_{1,4,2} &= \frac{11741408}{51593}\,\beta_{1,1},\; a_{2,3,1} = -\frac{16670584}{51593}\,\gamma_{1,2},\; a_{2,3,2} = -\frac{16670584}{51593}\,\beta_{1,2},\\
a_{2,4,1} &= \frac{11741408}{51593}\,\gamma_{1,2},\; a_{2,4,2} = \frac{11741408}{51593}\,\beta_{1,2}
\end{aligned}
$$

System (16) is characterized by the quasipolynomial function

$$
\begin{aligned}
\Delta(\lambda,\tau) =& \lambda^4 + \frac{2208852380}{154779}\gamma_{1,0} + \left(\frac{24009265}{309558} - \frac{11741408}{51593}\beta_{1,0} + \frac{16670584}{51593}\gamma_{1,0}\right)\lambda^2\\
&+ \left((\frac{16670584}{51593}\gamma_{1,1} - \frac{11741408}{51593}\beta_{1,1})\lambda^2 + \frac{2208852380}{154779}\gamma_{1,1}\right)\mathrm{e}^{-\lambda}\\
&+ \left((-\frac{11741408}{51593}\beta_{1,2} + \frac{16670584}{51593}\gamma_{1,2})\lambda^2 + \frac{2208852380}{154779}\gamma_{1,2}\right)\mathrm{e}^{-\lambda\tau_2},
\end{aligned}
\tag{17}
$$

for which is associated the incidence vector $\mathcal{V} = (x_1, \star, x_1, x_2, \star, x_2)$. A zero singularity of codimension five is insured by:

$$
\begin{aligned}
\beta_{1,0} &= \frac{1710742793353}{4582037506368}, \beta_{1,1} = \frac{51593}{2935352}, \beta_{1,2} = -\frac{257965}{10689112},\; \tau_2 = \frac{72}{265},\\
\gamma_{1,0} &= \frac{1654329545}{542135726972}, \gamma_{1,1} = \frac{29777931810}{26158048826399},\; \gamma_{1,2} = -\frac{438397329425}{104632195305596}.
\end{aligned}
\tag{18}
$$

Also, by this example, it is easy to see that, under the delay effect, the codimension of the zero singularity exceeds the dimension of the uncontrolled system (which is free of delays). Moreover, by using the Pólya-Szegö result, one has $\sharp_{PS} = D - 1 = 10$ which is far from the effective sharp bound computationally established. The bound $\sharp_{PS}$ loses its effective value due to the existence of algebraic constraints relating some parameters (with the notations of Proposition 72, $c_{2,1} = c_{2,2} = c_{2,3} = 0$. Such algebraic constraints are not taken into account by the Pólya-Segö approach).

3.3 Further insights on the Pólya-Segö bound

As shown in the examples above, $\sharp_{PS}$ bound for the zero root multiplicity of a given quasipolynomial function is still represents a generic bound. To be more precise, consider for instance the following three quasipolynomial functions:

$$\begin{cases} \Delta_1 = & \lambda^3 + a_{0,2}\lambda^2 + a_{0,1}\lambda + a_{0,0} + \left(a_{1,2}\lambda^2 + a_{1,1}\lambda + a_{1,0}\right) e^{-\lambda\tau_1} \\ & + \left(a_{2,2}\lambda^2 + a_{2,1}\lambda + a_{2,0}\right) e^{-\lambda\tau_2}, \\ \Delta_2 = & \lambda^3 + a_{0,2}\lambda^2 + a_{0,1}\lambda + a_{0,0} + \left(a_{1,2}\lambda^2 + a_{1,1}\lambda + a_{1,0}\right) e^{-\lambda\tau_1} \\ & + \left(a_{2,2}\lambda^2 + a_{2,0}\right) e^{-\lambda\tau_2}, \\ \Delta_3 = & \lambda^3 + a_{1,2}\lambda^2 e^{-\lambda\tau_1} + a_{2,2}\lambda^2 e^{-\lambda\tau_2}. \end{cases} \tag{19}$$

It is easy to see that all of them reduce to polynomials of degree three when the delays vanish $\tau_1 = \tau_2 = 0$. The quasipolynomial function Δ_1 characterizes the generic dynamical system consisting in three coupled differential equation with two discrete delays. One easily observes that Δ_1 has complete polynomials with the corresponding incidence vector $\mathcal{V}_1 = (x_1, x_1, x_1, x_2, x_2, x_2)$, which is not the case for the quasipolynomials Δ_2 (characterized by $\mathcal{V}_2 = (x_1, x_1, x_1, x_2, \star, x_2)$) and Δ_3 (characterized by $\mathcal{V}_3 = (\star, \star, x_1, \star, \star, x_2)$).

Indeed, Δ_2 has the so-called intermediate "star" since the polynomial associated with the second delay τ_2 is sparse; $a_{2,1} = 0$. However, Δ_3 has two connected sequences of starter "stars", since $a_{2,0} = a_{2,1} = 0$ and $a_{1,0} = a_{1,1} = 0$. Moreover, $\sharp_{PS}(\Delta_1) = \sharp_{PS}(\Delta_2) = \sharp_{PS}(\Delta_3)$ because the degree of such quasipolynomials are equal.

Nevertheless, intuitively, the multiplicity of the zero root of Δ_3 is less or equal to the multiplicity of such a root for Δ_1. In addition, the above observation stresses the fact that the zero multiplicity depends on the number of the free parameters as well as on the particular structure of the system rather than on the degree of the quasipolynomial which is a generic bound of the number of free parameters. The next sections provide the main results of the paper. Section 4 provides some new LU developments for classes of functional Birkhoff matrices. In Section 5 , we first recover the Pólya-Segö bound by an effective computational approach, then we establish a sharper bound for the multiplicity of the zero spectral value taking into account the above observation.

4 LU-factirization for some classes of functional Birkhoff matrices

In all generality, the Birkhoff interpolation problem and the "poised"-ness of its incidence matrices are yet open problems [26]. In some reduced cases (two variables), related to our class of systems, we give the explicit LU-factorization of Birkhoff matrices. To the best of the author's knowledge, such formulae seem to be new and then it yields some new possibilities for tackling the Birkhoff interpolation problem.

In this section we intentionally separate the two configurations: the first one, is *the regular case*, that is all the polynomials of the delayed part of the studied quasipolynomial are complete. However, the second configuration occurring when the incidence vector $\mathcal{V}$ contains at least one star.

4.1 On functional confluent Vandermonde matrices

It is well known that Vandermonde and confluent Vandermonde matrices V can be factorized into a lower triangular matrix L and an upper triangular matrix U where $V = LU$, see for instance [42,43]. In what follows, we show that the same applies for the functional confluent Vandermonde matrix (7)-(9) by establishing explicit formulas for L and U where $\Upsilon = LU$. The factorization is *unique* if no row or column interchanges are made and if it is specified that the diagonal elements of L are unitary. The following theorem concerning (7)-(9) with $s = n+1$ will be used in the sequel, but the same arguments can be easily adapted for any positive integer s.

Theorem 41. *Given the functional confluent Vandermonde matrix* (7)-(9) *with incidence vector $\mathcal{V}$ wanting "stars" , the unique LU-factorization with unitary diagonal elements $L_{i,i} = 1$ is given by the formulae:*

$$\begin{cases} L_{i,1} = x_1^{i-1} \quad \textit{for} \quad 1 \le i \le \delta, \\ U_{1,j} = \Upsilon_{1,j} \quad \textit{for} \quad 1 \le j \le \delta, \\ L_{i,j} = L_{i-1,j-1} + L_{i-1,j}\,\xi_j \quad \textit{for} \quad 2 \le j \le i, \\ U_{i,j} = (\varkappa(j)-1)\,U_{i-1,j-1} + U_{i-1,j}\,\left(x_{\varrho(j)} - \xi_{i-1}\right) \quad \textit{for} \quad 2 \le i \le j. \end{cases} \tag{20}$$

Proof (Proof of Theorem 41) Define a matrix Ω by:

$$\Omega_{i,j} = \sum_{k=1}^{D} L_{i,k} U_{k,j} = \sum_{k=1}^{i} L_{i,k} U_{k,j} \quad 1 \le i,j \le \delta. \tag{21}$$

In what follows, we prove that $\Omega_{i,j} = \Upsilon_{i,j}$ $\forall (i,j)$ $1 \le i,j \le \delta$. The proof is a total 2D recurrence-based that can be summarized as follows:

- Initialization by proving:
  - $\Omega_{i,j} = \Upsilon_{i,j}$ for $i = 1$ and $1 \le j \le \delta$ that is $\Upsilon_{1,j}$.
  - $\Omega_{i,j} = \Upsilon_{i,j}$ for $j = 1$ and $1 \le i \le \delta$ that is $\Upsilon_{i,1}$.
- Assuming $\Omega_{i,j} = \Upsilon_{i,j}$ holds for any $1 \le i \le i_0 - 1$ and $1 \le j \le j_0 - 1$ and proving:
  - $\Omega_{i_0,j_0-1} = \Upsilon_{i_0,j_0-1}$.
  - $\Omega_{i_0-1,j_0} = \Upsilon_{i_0-1,j_0}$.
  - $\Omega_{i_0,j_0} = \Upsilon_{i_0,j_0}$.
- Conclude that $\Omega_{i,j} = \Upsilon_{i,j}$ for $1 \le i \le \delta$ and $1 \le j \le \delta$

Since L is a lower triangular matrix with a unitary diagonal and U is an upper triangular, using (20) one proves $\Omega_{1,j} = U_{1,j} \equiv \Upsilon_{1,j}$ and $\Omega_{i,1} = L_{i,1}U_{1,1} \equiv \Upsilon_{i,1}$ for any $1 \le j \le \delta$ and any $1 \le i \le \delta$. Hence, the initialization assumption holds. Assume now that $\Omega_{i,j} = \Upsilon_{i,j}$ is satisfied for any $1 \le i \le i_0 - 1$ and $1 \le j \le j_0 - 1$. According to (20), one gets:

$$\begin{cases} L_{i_0,k} = L_{i_0-1,k-1} + L_{i_0-1,k}\,\xi_k, \\ U_{k,j_0-1} = (\varkappa(j_0-1)-1)\,U_{k-1,j_0-2} + U_{k-1,j_0-1}\,\left(x_{\varrho(j_0-1)} - \xi_{k-1}\right), \end{cases}$$

then

$$\begin{aligned} L_{i_0,k}U_{k,j_0-1} =& (\varkappa(j_0-1)-1)\, L_{i_0-1,k-1}U_{k-1,j_0-2} + x_{\varrho(j_0-1)}\, L_{i_0-1,k-1}U_{k-1,j_0-1} \\ & -\xi_{k-1}\, L_{i_0-1,k-1}U_{k-1,j_0-1} + \xi_k\, L_{i_0-1,k}U_{k,j_0-1}. \end{aligned}$$

Thus,

$$\begin{aligned} \Omega_{i_0,j_0-1} &= x_{\varrho(j_0-1)} \sum_{k=1}^{i_0} L_{i_0-1,k}U_{k,j_0-1} + \sum_{k=1}^{i_0} (\varkappa(j_0-1)-1)\, L_{i_0-1,k}U_{k,j_0-2} \\ &= x_{\varrho(j_0-1)}\, \Upsilon_{i_0-1,j_0-1} + (\varkappa(j_0-1)-1)\, \Upsilon_{i_0-1,j_0-2} \triangleq \Upsilon_{i_0,j_0-1}. \end{aligned}$$

The same argument gives

$$\begin{cases} L_{i_0-1,k} = L_{i_0-2,k-1} + L_{i_0-2,k}\, \xi_k, \\ U_{k,j_0} = (\varkappa(j_0)-1)\, U_{k-1,j_0-1} + U_{k-1,j_0}\, \left(x_{\varrho(j_0)} - \xi_{k-1}\right), \end{cases}$$

then

$$\begin{aligned} L_{i_0-1,k}U_{k,j_0} =& x_{\varrho(j_0)}\, L_{i_0-2,k-1}U_{k-1,j_0} + (\varkappa(j_0)-1)\, L_{i_0-2,k-1}U_{k-1,j_0-1} \\ & -\xi_{k-1}\, L_{i_0-2,k-1}U_{k-1,j_0} + \xi_k\, L_{i_0-2,k}U_{k,j_0}. \end{aligned}$$

Thus,

$$\begin{aligned} \Omega_{i_0-1,j_0} &= x_{\varrho(j_0)} \sum_{k=1}^{i_0} L_{i_0-2,k}U_{k,j_0} + \sum_{k=1}^{i_0} (\varkappa(j_0)-1)\, L_{i_0-2,k}U_{k,j_0-1} \\ &= x_{\varrho(j_0)}\, \Upsilon_{i_0-2,j_0} + (\varkappa(j_0)-1)\, \Upsilon_{i_0-2,j_0-1} \triangleq \Upsilon_{i_0-1,j_0}. \end{aligned}$$

By using again (20) one obtains:

$$\begin{cases} L_{i_0,k} = L_{i_0-1,k-1} + L_{i_0-1,k}\, \xi_k, \\ U_{k,j_0} = (\varkappa(j_0)-1)\, U_{k-1,j_0-1} + U_{k-1,j_0}\, \left(x_{\varrho(j_0)} - \xi_{k-1}\right), \end{cases}$$

leading to:

$$\begin{aligned} L_{i_0,k}U_{k,j_0} =& x_{\varrho(j_0)}\, L_{i_0-1,k-1}U_{k-1,j_0} + (\varkappa(j_0)-1)\, L_{i_0-1,k-1}U_{k-1,j_0-1} \\ & +\xi_k\, L_{i_0-1,k}U_{k,j_0} - \xi_{k-1}\, L_{i_0-1,k-1}U_{k-1,j_0}. \end{aligned}$$

Hence, we have:

$$\Omega_{i_0,j_0} = x_{\varrho(j_0)}\, \Upsilon_{i_0-1,j_0} + (\varkappa(j_0)-1)\, \Upsilon_{i_0-1,j_0-1} \triangleq \Upsilon_{i_0,j_0},$$

which ends the proof.

The explicit computation of the determinant of the functional confluent Vandermonde matrix Υ follows directly from (20):

Corollary 42. *The determinant of the functional confluent Vandermonde matrix Υ is given by:*

$$det(\Upsilon) = \prod_{j=1}^{\delta} (U_{j,j}),$$

where $U_{j,j}$ for $1 \le j \le \delta$ are defined by:

$$\begin{cases} U_{1,1} = x_1^{n+1}, \\ U_{j,j} = U_{j-1,j} \left(x_{\varrho(j)} - \xi_{j-1}\right) \quad \text{if } j > 1 \text{ and } \varkappa(j) = 1, \\ U_{j,j} = (\varkappa(j) - 1)\, U_{j-1,j-1} \quad \text{otherwise.} \end{cases}$$

Proof Obviously, the determinant of Υ is equal to the determinant of U since $\det(L) = 1$ (triangular with unitary diagonal). The second equation from (20) gives: $U_{1,1} = x_1^{n+1}$. Substituting $i = j$ in the last equation from (20), and consider $j \equiv 1 \mod (d_0 + \ldots + d_r)$ where $1 \le r \le M$ that is $\varkappa(j) = 1$ then $U_{j,j} = U_{j-1,j} \left(x_{\varrho(j)} - \xi_{j-1}\right)$ otherwise $x_{\varrho(j)} = \xi_{j-1}$ which ends the proof.

Corollary 43. *The diagonal elements of the matrix U associated with the functional confluent Vandermonde matrix Υ are obtained as follows:*

$$\begin{cases} U_{1,1} = x_1^{n+1}, \\ U_{j,j} = x_{k+1}^{n+1} \prod_{l=1}^{k} (x_{k+1} - x_l)^{d_l} \quad \text{if } j = 1 + d_k \text{ for } 1 \le k \le M-1, \\ U_{j,j} = (j - 1 - d_k)\, U_{j-1,j-1} \quad \text{if } d_k + 1 < j \le d_{k+1} \text{ for } 1 \le k \le M-1, \end{cases}$$

Moreover, the functional confluent Vandermonde matrix Υ is non degenerate if, and only if, $\forall\, 1 \le i \ne j \le \delta$ we have $x_i \ne 0$ and $x_i \ne x_j$.

Proof Using (20) one easily identifies the induction:

$$\begin{cases} U_{1+d_l,1+d_l} = (\varkappa(1+d_l) - 1)\, U_{d_l,d_l} + U_{d_l,1+d_l} \left(x_{\varrho(1+d_l)} - \xi_{d_l}\right) \\ \qquad\qquad = U_{d_l,1+d_l} \left(x_{l+1} - \xi_{d_l}\right), \\ U_{d_l,1+d_l} = U_{d_l-1,1+d_l} \left(x_{l+1} - \xi_{d_l-1}\right), \\ \quad \vdots \\ U_{2,1+d_l} = U_{1,1+d_l} \left(x_{l+1} - \xi_1\right), \\ U_{1,1+d_l} = x_{l+1}^{n+1}. \end{cases}$$

This ends the proof.

4.2 Non degeneracy domain for two classes of 2D-functional Birkhoff matrices: An LU-factorization

Polynomials in nature (e.g. from applications and modeling) are not necessarily generic. They often have some additional structure which we would like to take into account showing what it reflects in the multiplicity bound for the zero spectral value for time-delay systems.

In this section, we consider functional Birkhoff matrices with incidence vector $\mathcal{V}$ containing "stars" . Two configurations are investigated. The first one consists in a sequence of starter "stars" and the second one, involves a sequence of intermediate "stars" in a sequence of x_i, $i > 1$. We claim that if the particular incidence matrix under study contains "stars" in the two configurations (starter/intermediate) one can benefit from the understanding of each situation separately owing the following results.

In what follows, we present an attempt to extend the results of the previous section to the case of Birkhoff matrix (7)-(9) but with one sequence of x_1 wanting "stars" and a second x_2 containing "stars" . Explicit formulas for LU-factorization will be given in two subclasses. Such developments can be easily adapted in the study of 2D-Birkhoff interpolation problem.

We consider (7)-(9) with $\varpi(x) = x^s$ where $s = n + 1$, but the same algorithms can be easily exploited for any functional Birkhoff matrix with the same incidence matrix and a sufficiently regular function ϖ, in particular, for any integer $s \geq 0$. Moreover, such a restriction to the two variables case $x = (x_1, x_2)$ may be extended to higher number of variables.

In terms of zeros of quasipolynomial functions, this amounts to say that all the illustrations we provide are focused on the two-delay case.

4.2.1 Starter stars: Polynomial LU-factorization

In all generality, a functional Birkhoff matrix (7)-(9) with incidence vector

$$\mathcal{V} = (\underbrace{x_1, \ldots, x_1}_{d_1}, \underbrace{\star, \ldots, \star}_{d_*}, \underbrace{x_2, \ldots, x_2}_{d_2})$$

admits an $LU-$factorization where the associated L and U matrices are with rational coefficients in the variables x_1 and x_2. This claim is illustrated by the following simple example.

Example 41. *Let consider the functional Birkhoff matrix $\Upsilon_{\mathcal{E}}$ associated with the incidence vector $\mathcal{V} = (x_1, x_1, \star, x_2, x_2)$, thus, $n = 4$:*

$$\Upsilon = \begin{bmatrix} {x_1}^4 & 4\,{x_1}^3 & 4\,{x_2}^3 & 12\,{x_2}^2 \\ {x_1}^5 & 5\,{x_1}^4 & 5\,{x_2}^4 & 20\,{x_2}^3 \\ {x_1}^6 & 6\,{x_1}^5 & 6\,{x_2}^5 & 30\,{x_2}^4 \\ {x_1}^7 & 7\,{x_1}^6 & 7\,{x_2}^6 & 42\,{x_2}^5 \end{bmatrix}$$

for which one can computes the LU factorization that gives:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 \\ {x_1}^2 & 2\,x_1 & 1 & 0 \\ {x_1}^3 & 3\,{x_1}^2 & \frac{7\,{x_2}^2+7\,x_1x_2-8\,{x_1}^2}{2\,(3\,x_2-2\,x_1)} & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} {x_1}^4 & 4\,{x_1}^3 & 4\,{x_2}^3 & 12\,{x_2}^2 \\ 0 & {x_1}^4 & {x_2}^3\,(5\,x_2-4\,x_1) & 4\,{x_2}^2\,(5\,x_2-3\,x_1) \\ 0 & 0 & 2\,{x_2}^3\,(3\,x_2-2\,x_1)\,(-x_1+x_2) & 2\,{x_2}^2\left(15\,{x_2}^2+6\,{x_1}^2-20\,x_1x_2\right) \\ 0 & 0 & 0 & \frac{{x_2}^3(-x_1+x_2)\left(10\,{x_1}^2-28\,x_1x_2+21\,{x_2}^2\right)}{3\,x_2-2\,x_1} \end{bmatrix}.$$

Even the coefficients of L and U are rational functions in $x = (x_1, x_2)$, the determinant of $\Upsilon_{\mathcal{E}}$ still have polynomial expression in x as expected. For instance, in the considered example, the denominator of $U_{4,4}$ will be canceled by a factor from $U_{3,3}$.

Nevertheless, there exists a unique configuration in which L and U conserve their polynomial structure (as in the regular case), which occurs when $d_2 = 1$ independently from d_1 and d^*. The following theorem provides an explicit $LU-$factorization for a functional Birkhoff matrix in such a special case:

Theorem 44. *Given the functional Birkhoff matrix* (7)-(9) *with incidence vector*

$$\mathcal{V} = (\underbrace{x_1, \ldots, x_1}_{d_1}, \underbrace{\star, \ldots, \star}_{d_*}, x_2) \tag{22}$$

the unique LU-factorization with unitary diagonal elements $L_{i,i} = 1$ is given by the formulae:

$$\begin{cases} L_{i,1} = x_1^{i-1} \quad \textit{for} \quad 1 \le i \le d_1 + 1, \\ U_{1,j} = \Upsilon_{1,j} \quad \textit{for} \quad 1 \le j \le d_1 + 1, \\ L_{i,j} = L_{i-1,j-1} + L_{i-1,j}\,\xi_j \quad \textit{for} \quad 2 \le j \le i \le d_1 + 1, \\ U_{i,j} = (\varkappa(j) - 1)\,U_{i-1,j-1} + U_{i-1,j}\,\left(x_{\varrho(j)} - \xi_{i-1}\right) \quad \textit{for} \quad 2 \le i \le j \le d_1, \\ U_{i,d_1+1} = \Upsilon_{i,j} - (i-1)\displaystyle\int_0^{x_1} U_{i-1,d_1+1}(y, x_2)dy, \quad \textit{for} \quad 2 \le i \le d_1 + 1. \end{cases} \tag{23}$$

where $\xi = (\underbrace{x_1, \ldots, x_1}_{d_1}, x_2)$.

The proofs of Theorem 44 is provided in the appendix.

Remark 4. *The proposed formulas given in Theorem 44 can be easily extended to incidence matrices:*

$$\mathcal{V} = (\underbrace{x_1, \ldots, x_1}_{d_1}, \ldots, \underbrace{x_{n-1}, \ldots, x_{n-1}}_{d_{n-1}}, \underbrace{\star, \ldots, \star}_{d_*}, x_n), \tag{24}$$

allowing to investigate multiple zero spectral values for the n-delays case.

As a direct consequence of the above Theorem a nondegeneracy condition is given in the following corollary:

Corollary 45. *Let x_1 and x_2 be two distinct nonzero real numbers. The Birkhoff matrix Υ defined by (7)-(9) with incidence vector $\mathcal{V} = (\underbrace{x_1, \ldots, x_1}_{d_1}, \underbrace{\star, \ldots, \star}_{d_*}, x_2)$ is invertible if, and only if, $U_{d_1+1,d_1+1} \neq 0$.*

4.2.2 Intermediate stars: Polynomial LU-factorization

Similarly to the starting "stars" case, a nondegenerate functional Birkhoff matrix (7)-(9) with incidence vector

$$\mathcal{V} = (\underbrace{x_1, \ldots, x_1}_{d_1}, \underbrace{x_2, \ldots, x_2}_{d_2^-}, \underbrace{\star, \ldots, \star}_{d_*}, \underbrace{x_2, \ldots, x_2}_{d_2^+})$$

admits an $LU-$factorization where the associated L and U matrices are with rational coefficients in the variables x_1 and x_2. Here, we provide an illustrative simple example.

Example 42. *Let consider the functional Birkhoff matrix $\Upsilon_{\mathcal{E}}$ associated with the incidence vector $\mathcal{V} = (x_1, x_2, \star, x_2, x_2)$ and $n = 4$:*

$$\Upsilon = \begin{bmatrix} {x_1}^5 & {x_2}^5 & 20\,{x_2}^3 & 60\,{x_2}^2 \\ {x_1}^6 & {x_2}^6 & 30\,{x_2}^4 & 120\,{x_2}^3 \\ {x_1}^7 & {x_2}^7 & 42\,{x_2}^5 & 210\,{x_2}^4 \\ {x_1}^8 & {x_2}^8 & 56\,{x_2}^6 & 336\,{x_2}^5 \end{bmatrix}.$$

The corresponding quasipolynomial function is for which one can computes the LU factorization that gives:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_1 & 1 & 0 & 0 \\ {x_1}^2 & x_2 + x_1 & 1 & 0 \\ {x_1}^3 & {x_2}^2 + x_1 x_2 + {x_1}^2 & \frac{13{x_2}^2 - 5x_1x_2 - 5{x_1}^2}{6\,x_2 - 5\,x_1} & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} {x_1}^5 & {x_2}^5 & 20\,{x_2}^3 & 60\,{x_2}^2 \\ 0 & {x_2}^6 - x_1{x_2}^5 & 30\,{x_2}^4 - 20\,x_1{x_2}^3 & 120\,{x_2}^3 - 60\,x_1{x_2}^2 \\ 0 & 0 & 12\,{x_2}^5 - 10\,x_1{x_2}^4 & 90\,{x_2}^4 - 60\,x_1{x_2}^3 \\ 0 & 0 & 0 & 6\,\frac{{x_2}^4\left(21\,{x_2}^2 - 35\,x_1x_2 + 15\,{x_1}^2\right)}{6\,x_2 - 5\,x_1} \end{pmatrix}.$$

Even the coefficients of L and U are rational functions in $x = (x_1, x_2)$, the determinant of $\Upsilon_{\mathcal{E}}$ still have polynomial expression in x as expected. For instance, in the considered example, the denominator of $U_{4,4}$ will be canceled by a factor from $U_{3,3}$.

The unique configuration in which L and U conserve their polynomial structure (as in the regular case as well the stating "stars" case with $d_2 = 1$), which occurs when $d_2^+ = 1$. The following theorem provides an explicit $LU-$factorization for a functional Birkhoff matrix in such a special case:

Theorem 46. *Given the functional Birkhoff matrix* (7)-(9) *with incidence vector*

$$\mathcal{V} = (\underbrace{x_1, \dots, x_1}_{d_1}, \underbrace{x_2, \dots, x_2}_{d_2^-}, \underbrace{\star, \dots, \star}_{d_*}, x_2) \tag{25}$$

the unique LU-factorization with unitary diagonal elements $L_{i,i} = 1$ is given by the formulae:

$$L_{i,1} = x_1^{i-1} \quad \textit{for} \quad 1 \le i \le d_1 + d_2^- + 1, \tag{26}$$

$$U_{1,j} = \Upsilon_{1,j} \quad \textit{for} \quad 1 \le j \le d_1 + d_2^- + 1, \tag{27}$$

$$L_{i,j} = L_{i-1,j-1} + L_{i-1,j}\, \xi_j \quad \textit{for} \quad 2 \le j \le i \le d_1 + d_2^- + 1, \tag{28}$$

$$U_{i,j} = (\varkappa(j) - 1)\, U_{i-1,j-1} + U_{i-1,j} \left(x_{\varrho(j)} - \xi_{i-1}\right) \quad \textit{for} \quad 2 \le i \le j \le d_1 + d_2^-, \tag{29}$$

$$U_{i,j} = \Upsilon_{i,j} - (i-1) \int_0^{x_1} U_{i-1,j}(y, x_2)\, dy \quad \textit{for} \quad j = d_1 + d_2^- + 1 \textit{ and } 2 \le i \le d_1 + 1, \tag{30}$$

$$U_{i,j} = (j + d^* - (i-1)) \int_0^{x_2} U_{i-1,j}(x_1, y)\, dy \textit{ for } j = d_1 + d_2^- + 1 \textit{ and } d_1 + 2 \le i \le j, \tag{31}$$

where $\xi = (\underbrace{x_1, \dots, x_1}_{d_1}, \underbrace{x_2, \dots, x_2}_{d_2^- + 1})$.

Remark 5. *The proposed formulas given in theorem 46 can be easily extended to incidence matrices:*

$$\mathcal{V} = (\underbrace{x_1, \dots, x_1}_{d_1}, \dots, \underbrace{x_{n-1}, \dots, x_{n-1}}_{d_{n-1}}, \underbrace{x_n, \dots, x_n}_{d_n^-}, \underbrace{\star, \dots, \star}_{d_*}, x_n) \tag{32}$$

As a direct consequence of the above Theorem as well as the auxiliary Lemmas 3-5 presented in the appendix, one can compute analytically the determinant of the considered Birkhoff matrix and then easily deduce its non-degeneracy domain. The above Corollary is in the same spirit of Corollary 42 for the functional confluent Vandermonde matrices.

Corollary 47. *Let x_1 and x_2 be two distinct nonzero real numbers. The determinant of the functional Birkhoff matrix Υ defined by* (7)-(9) *with incidence vector*

$$\mathcal{V} = (\underbrace{x_1, \dots, x_1}_{d_1}, \underbrace{x_2, \dots, x_2}_{d_2^-}, \underbrace{\star, \dots, \star}_{d_*}, x_2)$$

is given by:

$$det(\Upsilon) = \prod_{j=1}^{d_2^- + d_1 + 1} (U_{j,j}) \,,$$

where $U_{j,j}$ *for* $1 \leq j \leq d_2^- + d_1 + 1$ *are defined by:*

$$\begin{cases} U_{1,1} = x_1^{n+1}, \\ U_{d_1+1,d_1+1} = x_2^{n+1} \, (x_2 - x_1)^{d_1} \\ U_{d_1+d_2^-+1,d_1+d_2^-+1} = \\ \prod_{\mu=0}^{d^*-1} (d_2^- + d_* - \mu) \sum_{l=0}^{d_1} \binom{d_1}{l} (-1)^l \, x_1^l \underbrace{\int_0^{x_2} \dots \int_0^{x_2}}_{d_2^-} \Upsilon_{d_1+1-l,d_1+d_2^-+1}(x_1, y) \underbrace{dy \dots dy}_{d_2^-}, \\ U_{j,j} = (\varkappa(j) - 1) \, U_{j-1,j-1} \quad \text{otherwise.} \end{cases}$$

Moreover, the functional Birkhoff matrix Υ *is invertible if, and only if,* $U_{d_1+d_2^-+1,d_1+d_2^-+1} \neq 0$ *.*

We emphasize that the results obtained above can be easily adapted for computing the LU-factorization for any functional Birkhoff matrix (with the same $\mathcal{V}$) with different sufficiently regular function ϖ.

5 Codimension of zero Singularities of TDS

This section includes the main contributions of the paper. In its first part, we give an adaptive bound for the zero spectral value taking into account the system structure. The proof of this proposition is constructive and as such underlines and exploit the existing links between the multiplicity of the zero singularity and Birkhoff matrices. Additionally, it gives the values of the system parameters guaranteeing an admissible multiplicity for the zero spectral value. The second part is devoted to recover the Pólya-Szegö generic bound. In this framework, the provided explicit expressions for the LU-factorization of the functional confluent Vandermonde matrices. Finally, the third part, entitled "On beyond of the Pólya-Segö Bound", concerns some classes of functional Birkhoff matrices.

In the light of the above results on LU-factorization of the considered classes of functional Birkhoff matrices, we are now able to establish a sharper bound for the zero multiplicity under the assumption of nondegeneracy of appropriate functional Birkhoff matrices. Indeed, the following result applies even when the delay associated polynomials are sparse.

Proposition 51. *The following assertions hold:*

i) The multiplicity of the zero root for the generic quasipolynomial function (4) *cannot be larger than* $\sharp_{PS} = D + \tilde{N}_{N,n}$*, where* D *is the sum of degrees of the polynomials involved in the quasipolynomial and* $\tilde{N}_{N,n}+1$ *is the number* *of the associated polynomials. Moreover, such a bound is reached if, and only if, the parameters of* (4) *satisfy simultaneously:*

$$a_{0,k} = -\sum_{i\in S_{N,n}} \left(a_{i,k} + \sum_{l=0}^{k-1} \frac{a_{i,l}{\sigma_i}^{k-l}}{(k-l)!} \right), \quad \textit{for} \quad 0 \le k \le \sharp_{PS} - 1. \tag{33}$$

ii Consider a quasipolynomial function (4) *containing at least one incomplete polynomial for which we associate an incidence vector* $\mathcal{V}_{\tilde{\mathcal{E}}}$ *(which is nothing other than* $\mathcal{V}_{\mathcal{E}}$ *given by* (34) *where a component associated with a vanishing coefficient is indicated by a "star").*
When the associated functional Birkhoff matrix $\Upsilon_{\tilde{\mathcal{E}}}$ *is nonsingular, then the multiplicity of the zero root for the quasipolynomial function* (4) *cannot be larger than "n" plus the number of nonzero coefficients of the polynomial family* $(P_{M^k})_{M^k\in S_{N,n}}$.

Remark 6. *In the generic case, the Pólya-Szegö bound* $\sharp_{PS}$ *is completely recovered by the first assertion of 51. But, its advantage consists in providing the parameter values insuring any admissible multiplicity fo the zero singularity. The proof of Proposition 51 provides a constructive linear algebra alternative for identifying such a bound.*

Remark 7. *Obviously, the number of non-zero coefficients of a given quasipolynomial function is bounded by its degree plus its number of polynomials. Thus, the bound elaborated in Proposition 51 ii) is sharper than* $\sharp_{PS}$*, even in the generic case, that is all the parameters of the quasipolynomial are left free, these two bounds are equal. Indeed, in the generic case, that is when the number of the left free parameters is maximal, the Pólya-Szegö bound* $\sharp_{PS} = D + \tilde{N}_{N,n} = n + D_q + \tilde{N}_{N,n}$ *which is nothing else than n plus the number of parameters of the polynomial family* $(P_{M^k})_{M^k\in S_{N,n}}$.

Remark 8. *When the matrix* $\Upsilon_{\tilde{\mathcal{E}}}$ *is singular, one keeps the generic Pólya-Szegö bound* $\sharp_{PS}$.

Remark 9. *The above proposition can be interpreted as follows. Under the hypothesis:*

$$\Delta(i\omega) = 0 \Rightarrow \omega = 0 \tag{H}$$

(that is, all the imaginary roots are located at the origin), the dimension of the projected state on the center manifold associated with zero singularity for equation (4) *is less or equal to its number of nonzero coefficients minus one. Indeed, under* (H)*, the codimension of the zero spectral value is identically equal to the dimension of the state on the center manifold since, in general, the dimension of the state on the center manifold is none other than the sum of the dimensions of the generalized eigenspaces associated with the spectral values having a zero real part.*

Since we are dealing only with the values of $\Delta_k(0)$, we suggest to translate the problem into the parameter space (the space of the coefficients of the P_i). This is more appropriate and consider a parametrization by σ. In the appendix we introduce a lemma that allows to establish an m-set of multivariate algebraic functions (polynomials) vanishing at zero when the multiplicity of the zero root of the transcendental equation $\Delta(\lambda, \tau) = 0$ is equal to m.

Proof (Proof of Proposition 51:) The condition (33) follows directly from Lemma 1 (see Appendix). Hereafter, we recover the bound $\sharp_{PS}$ by explicit use of functional Vandermonde matrices. Then, assuming that some coefficients of the quasipolynomial vanish without affecting its degree, we show that a sharper bound can be related to the number of nonzero parameters rather than the degree.

i) More precisely, we shall consider the variety associated with the vanishing of the polynomials ∇_k(defined in Lemma 1 in the appendix) , that is $\nabla_0(0) = \ldots = \nabla_{m-1}(0) = 0$ and $\nabla_m(0) \neq 0$ and we aim to find the maximal m (codimension of the zero singularity).
Let us exhibit the first elements from the family ∇_k

$$\begin{cases} \nabla_0(0) = 0 \Leftrightarrow \displaystyle\sum_{s=0}^{\tilde{N}_{N,n}} a_{s,0} = 0, \\ \nabla_1(0) = 0 \Leftrightarrow \displaystyle\sum_{s=0}^{\tilde{N}_{N,n}} a_{s,1} + \sum_{s=1}^{\tilde{N}_{N,n}} a_{s,0}\,\sigma_s = 0, \\ \nabla_2(0) = 0 \Leftrightarrow 2! \displaystyle\sum_{s=0}^{\tilde{N}_{N,n}} a_{s,2} + 2! \sum_{s=1}^{\tilde{N}_{N,n}} a_{s,1}\,\sigma_s + \sum_{s=1}^{N} a_{s,0}\,\sigma_s^2 = 0. \end{cases}$$

If we consider $a_{i,j}$ and σ_k as variables, the derived algebraic system is nonlinear and solving it in all its generality (without attributing values for n and N) becomes a very difficult task. In fact, even the use of Gröbner basis methods [44] seems to be very challenging since the set of variables depends on N and n. However, considering $a_{i,j}$ as variables and σ_k as parameters helps in simplifying the problem to a linear one, as seen in (33). Consider the ideal I_1 generated by polynomials $< \nabla_0(0), \nabla_1(0), \ldots, \nabla_{n-1}(0) >$.
As it can be seen from (33) and Lemma 1 (see appendix), the variety V_1 associated with the ideal I_1 has the following linear representation $a_0 = \underline{\Upsilon}\, a$ such that $\underline{\Upsilon} \in \mathcal{M}_{n, D_q + \tilde{N}_{N,n}}(\mathbb{R}[\sigma])$ where D_q is the degree of $\sum_{k=1}^{\tilde{N}_{N,n}} P_{M_k}$ and $D_q = D - n$ (D the degree of the quasipolynomial (4)). Somehow, in this variety there are no restrictions on the components of a if a_0 is left free. Since $a_{0,k} = 0$ for all $k > n$, the remaining equations consist of an algebraic system only in a and parametrized by σ. Consider now the ideal denoted I_2 and generated by the $D_q + \tilde{N}_{N,n}$ polynomials defined by $I_2 =< \nabla_{n+1}(0), \nabla_{n+2}(0), \ldots, \nabla_{D+\tilde{N}_{N,n}}(0) >$. It can be observed that the variety V_2 associated with I_2 can be written as $\bar{\Upsilon}\, a = 0$ which is nothing but a homogeneous linear system with $\bar{\Upsilon} \in \mathcal{M}_{D_q + \tilde{N}_{N,n}}(\mathbb{R}[\sigma])$. More precisely, $\bar{\Upsilon}$ is a functional confluent Vandermonde matrix (7)-(9) with $x = \sigma$, $s = n$,

$M = \tilde{N}_{N,n}$ and $\delta = D_q + \tilde{N}_{N,n}$ which is associated with some incidence vector:

$$\mathcal{V} = (\underbrace{\sigma_{M^1}, \ldots, \sigma_{M^1}}_{n-\sum_{s=1}^{N} M_s^1}, \underbrace{\sigma_{M^2}, \ldots, \sigma_{M^2}}_{n-\sum_{s=1}^{N} M_s^2}, \ldots, \sigma_{M^{\tilde{N}_{N,n}}}, \ldots, \sigma_{M^{\tilde{N}_{N,n}}}). \tag{34}$$

Now, using Corollaries 42 and 43 and the assumption that σ_i are distinct non zero auxiliary delays, we can conclude that the determinant of $\bar{\Upsilon}$ cannot vanish. Thus the only solution for this subsystem is the zero solution, that is, $a = 0$.

Finally, consider the polynomial defined by $\nabla_n(0)$, Lemma 1 states that (see appendix)

$$\nabla_n(0) = 0 \Leftrightarrow 1 = -\sum_{i=1}^{\tilde{N}_{N,n}} \sum_{s=0}^{n-1} \frac{a_{i,s}{\sigma_i}^{n-s}}{(n-s)!}$$

Now, substituting the unique solution of V_2 into the last equality leads to an incompatibility result. In conclusion, the maximal codimension of the zero singularity is less or equal to $D_q + \tilde{N}_{N,n} + n$ which is exactly the Pólya-Szegö bound $\sharp_{PS} = \underbrace{D_q + (n+1)}_{D+\tilde{N}_{N,n}}$ proving i).

ii) She same arguments apply when z coefficients from the polynomial family $(P_{M^k})_{M^k \in S_{N,n}}$ vanish without affecting the degree of the quasipolynomial, then $a^T \in \mathbb{R}^{D_q+\tilde{N}_{N,n}-z}$ and thus the matrix $\bar{\Upsilon}$ of i) becomes $\Upsilon_{\tilde{\mathcal{E}}} \in \mathcal{M}_{D_q+\tilde{N}_{N,n}-z}(\mathbb{R}[\sigma])$. Thus, the invertibility of the later matrix allows to: the maximal codimension of the zero singularity is less or equal to $D_q + \tilde{N}_{N,n} - z + n < \sharp_{PS}$. Which ends the proof.

Remark 10. *It is noteworthy that the codimension of the zero singularity may decrease if the vector parameter a_0 is not left free. Indeed, if some parameter component $a_{0,k}$ is fixed for $0 \leq k \leq n-1$, then the variety associated to the first ideal I_1 may impose additional restrictions on the vector parameter a.*

6 Illustrative examples: An effective approach vs Pólya-Szegö Bound

A natural consequence of Proposition 51 is to explore the situation when the codimension of zero singularity reaches its upper bound. Starting the section by a generic example, we show the convenience of the proposed approach even in the case of cross-talk between the delays. Then the obtained symbolic results are applied to identify an effective sharp bound in the case of concrete physical system (with constraints on the coefficients). Namely, the stabilization of an inverted pendulum on cart via a multi-delayed feedback. Next, the LU-factorizations are illustrated in the two configurations starter "stars"/intermediate "stars" and then interpreted in terms of the codimension of the zero singularity. This section is ended by a control oriented discussion.

6.1 Two scalar equations with two delays: An inverted pendulum on cart with delayed feedback

We associate to the general planar time-delay system with two positive delays $\tau_1 \neq \tau_2$ the quasipolynomial function:

$$\begin{aligned}\Delta(\lambda,\sigma) =&\lambda^2 + a_{0,0,1}\lambda + a_{0,0,0} + (a_{1,0,0} + a_{1,0,1}\lambda)\,\mathrm{e}^{\lambda\,\sigma_{1,0}} + (a_{0,1,0} + a_{0,1,1}\lambda)\,\mathrm{e}^{\lambda\,\sigma_{0,1}}\\ &+ a_{2,0,0}\mathrm{e}^{\lambda\,\sigma_{2,0}} + a_{1,1,0}\mathrm{e}^{\lambda\,\sigma_{1,1}} + a_{0,2,0}\mathrm{e}^{\lambda\,\sigma_{0,2}}.\end{aligned} \tag{35}$$

Generically, the multiplicity of the zero singularity is bounded by $\sharp_{PS} = 9$. However, in what follows, we present two configurations where such a bound cannot be reached. The first, corresponds to the case when $\sigma_i = \sigma_j$ for $i \neq j$ and the second, when some components of the coefficient vector $a = (a_{1,0,0}, a_{1,0,1}, a_{0,1,0}, a_{0,1,1}, a_{2,0,0}, a_{1,1,0}, a_{0,2,0})^T$ vanish.

Formula (33) allows us to explicitly computing the confluent Vandermonde matrices $\underline{\Upsilon}$ and $\bar{\Upsilon}$ and the expression of $\nabla_2(0)$ from the proof of Proposition 51 such that $\underline{\Upsilon}\,a = a_0$, $\nabla_2(0) = 0$ and $\bar{\Upsilon}\,a = 0$ where $a_0 = (a_{0,0,0}, a_{0,0,1})^T$:

$$\underline{\Upsilon} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ \sigma_{1,0} & 1 & \sigma_{0,1} & 1 & \sigma_{2,0} & \sigma_{1,1} & \sigma_{0,2} \end{bmatrix},$$

$$\nabla_2(0) - 2 = \begin{bmatrix} \sigma_{1,0}{}^2 & 2\,\sigma_{1,0} & \sigma_{0,1}{}^2 & 2\,\sigma_{0,1} & \sigma_{2,0}{}^2 & \sigma_{1,1}{}^2 & \sigma_{0,2}{}^2 \end{bmatrix} a$$

$$\bar{\Upsilon} = \begin{bmatrix} \sigma_{1,0}{}^3 & 3\,\sigma_{1,0}{}^2 & \sigma_{0,1}{}^3 & 3\,\sigma_{0,1}{}^2 & \sigma_{2,0}{}^3 & \sigma_{1,1}{}^3 & \sigma_{0,2}{}^3 \\ \sigma_{1,0}{}^4 & 4\,\sigma_{1,0}{}^3 & \sigma_{0,1}{}^4 & 4\,\sigma_{0,1}{}^3 & \sigma_{2,0}{}^4 & \sigma_{1,1}{}^4 & \sigma_{0,2}{}^4 \\ \sigma_{1,0}{}^5 & 5\,\sigma_{1,0}{}^4 & \sigma_{0,1}{}^5 & 5\,\sigma_{0,1}{}^4 & \sigma_{2,0}{}^5 & \sigma_{1,1}{}^5 & \sigma_{0,2}{}^5 \\ \sigma_{1,0}{}^6 & 6\,\sigma_{1,0}{}^5 & \sigma_{0,1}{}^6 & 6\,\sigma_{0,1}{}^5 & \sigma_{2,0}{}^6 & \sigma_{1,1}{}^6 & \sigma_{0,2}{}^6 \\ \sigma_{1,0}{}^7 & 7\,\sigma_{1,0}{}^6 & \sigma_{0,1}{}^7 & 7\,\sigma_{0,1}{}^6 & \sigma_{2,0}{}^7 & \sigma_{1,1}{}^7 & \sigma_{0,2}{}^7 \\ \sigma_{1,0}{}^8 & 8\,\sigma_{1,0}{}^7 & \sigma_{0,1}{}^8 & 8\,\sigma_{0,1}{}^7 & \sigma_{2,0}{}^8 & \sigma_{1,1}{}^8 & \sigma_{0,2}{}^8 \\ \sigma_{1,0}{}^9 & 9\,\sigma_{1,0}{}^8 & \sigma_{0,1}{}^9 & 9\,\sigma_{0,1}{}^8 & \sigma_{2,0}{}^9 & \sigma_{1,1}{}^9 & \sigma_{0,2}{}^9 \end{bmatrix}.$$

As shown in the proof of Proposition 51, $\bar{\Upsilon}$ is a singular matrix when $\sigma_i = \sigma_j$ for $i \neq j$. For instance, when $\sigma_{2,0} = \sigma_{0,1}$ that is $2\tau_1 = \tau_2$, then the bound of multiplicity of the zero singularity decrease since the polynomials $P_{2,0}$ and $P_{0,1}$ will be collected $\tilde{P}_{0,1} = P_{0,1} + P_{2,0}$.

Consider now a system of two coupled equations with two delays modeling a friction free inverted pendulum on cart. The adopted model is studied in [45, 13, 46, 14] and in the sequel we keep the same notations. In the dimensionless form, the dynamics of the inverted pendulum on a cart in figure 2 is governed by the following second-order differential equation:

$$\left(1 - \frac{3\epsilon}{4}\cos^2(\theta)\right)\ddot{\theta} + \frac{3\epsilon}{8}\dot{\theta}^2\sin(2\theta) - \sin(\theta) + U\,\cos(\theta) = 0, \tag{36}$$

where $\epsilon = m/(m+M)$, M the mass of the cart and m the mass of the pendulum and D represents the control law that is the horizontal driving force. A

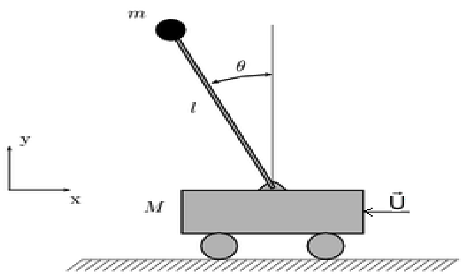

Fig. 2 Inverted Pendulum on a cart

generalized Bogdanov-Takens singularity with codimension three is identified in [13] by using $U = a\,\theta(t-\tau) + b\,\dot\theta(t-\tau)$. Motivated by the technological constraints, it is suggested in [14, 47] to avoid the use of the derivative gain that requires the estimation of the angular velocity that can induce harmful errors for real-time simulations and propose a multi-delayed-proportional controller $U = a_{1,0}\,\theta(t-\tau_1) + a_{2,0}\,\theta(t-\tau_2)$, this choice is argued by the accessibility of the delayed state by some simpler sensor. By this last controller choice and by setting $\epsilon = \frac{3}{4}$, the associated quasipolynomial function Δ becomes:

$$\Delta(\lambda,\tau) = \lambda^2 - \frac{16}{7} + \frac{16\,a_{1,0}}{7}\,\mathrm{e}^{-\lambda\,\tau_1} + \frac{16\,a_{2,0}}{7}\,\mathrm{e}^{-\lambda\,\tau_2}. \tag{37}$$

Thus, the associated incidence vector is $\mathcal{V} = (x_1, x_2)$. A zero singularity with codimension three is identified in [14], see Figure 3 for the map of local bifurcations in the $(a_{1,0}, a_{2,0})$ plan.

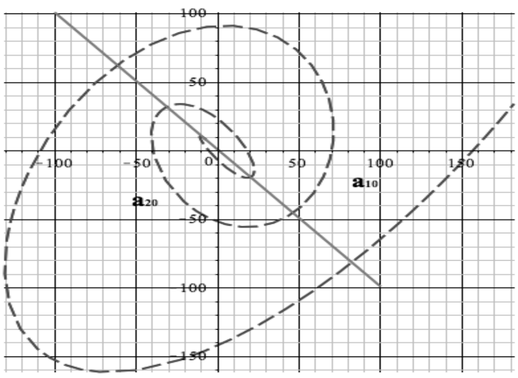

Fig. 3 Bifurcations curves of (37) in the gains $(a_{1,0}, a_{2,0})$ plan (solid red=Pitchfork singularity i.e. the zero singularity, discontinuous blue=Hopf singularity) for the fixed value $\tau_2 = \frac{7}{8}$.

Moreover, it is shown that the upper bound of the codimension for the zero singularity for (36) is three (can be easily checked by (33)) and this configuration is obtained when the gains and delays satisfy simultaneously:

$$a_{1,0} = -\frac{7}{-7+8\,\tau_1},\ a_{2,0} = \frac{8{\tau_1}^2}{-7+8\,{\tau_1}^2},\ \tau_2 = \frac{7}{8\,\tau_1}.$$

However, using Pólya-Szegö result, one has $\sharp_{PS} = D-1 = (3+2+2)-1 = 6$ exceeding the effective bound which is three. This is a further justification for the algebraic constraints on the parameters imposed by the physical model, for instance the sparsity pattern of the delay-free polynomial, namely, the vanishing of $a_{0,1}$.

6.2 The nondegeneracy of a 2D-functional Birkhoff matrix: incidence vector with starter stars

As an illustration of the result given in Corollary 45, consider the functional Birkhoff matrix Υ characterized by the incidence vector $\mathcal{V} = (x_1, x_1, x_1, x_1, x_1, \star, \star, x_2)$. Thus, one has

$$\Upsilon = \begin{bmatrix} {x_1}^8 & 8\,{x_1}^7 & 56\,{x_1}^6 & 336\,{x_1}^5 & 1680\,{x_1}^4 & 56\,{x_2}^6 \\ {x_1}^9 & 9\,{x_1}^8 & 72\,{x_1}^7 & 504\,{x_1}^6 & 3024\,{x_1}^5 & 72\,{x_2}^7 \\ {x_1}^{10} & 10\,{x_1}^9 & 90\,{x_1}^8 & 720\,{x_1}^7 & 5040\,{x_1}^6 & 90\,{x_2}^8 \\ {x_1}^{11} & 11\,{x_1}^{10} & 110\,{x_1}^9 & 990\,{x_1}^8 & 7920\,{x_1}^7 & 110\,{x_2}^9 \\ {x_1}^{12} & 12\,{x_1}^{11} & 132\,{x_1}^{10} & 1320\,{x_1}^9 & 11880\,{x_1}^8 & 132\,{x_2}^{10} \\ {x_1}^{13} & 13\,{x_1}^{12} & 156\,{x_1}^{11} & 1716\,{x_1}^{10} & 17160\,{x_1}^9 & 156\,{x_2}^{11} \end{bmatrix}. \tag{38}$$

Under the assumptions $x_1 x_2 \neq 0$ and $x_1 \neq x_2$, the matrix Υ is a non singular matrix if, and only if, the bivariate polynomial $39\,{x_2}^2 - 48\,x_2 x_1 + 14\,{x_1}^2 \neq 0$, see Figure 4. Consider, the corresponding quasipolynomial function

$$\Delta(\lambda,\sigma) = \lambda^7 + \sum_{k=0}^{6} a_{0,k}\,\lambda^k + e^{\sigma_{1,0}\lambda} \sum_{k=0}^{4} a_{1,0,k}\,\lambda^k + a_{0,1,2}\lambda^2\,e^{\sigma_{0,1}\lambda}. \tag{39}$$

In terms of time-delay systems analysis purpose, the result above asserts that if the auxiliary non zero distinct delays $\sigma_{1,0}$ and $\sigma_{0,1}$ satisfy $39\,{\sigma_{0,1}}^2 - 48\,\sigma_{0,1}\sigma_{1,0} + 14\,{\sigma_{1,0}}^2 \neq 0$, then, the codimension of the zero singularity is bounded by 13. Furthermore, such a multiplicity bound is reached if, and only if, the parameter vectors a and a_0 satisfy equality (33) for $k = 0, \ldots, 12$. Notice that, in this configuration, the Pólya-Segö bound $\sharp_{PS} = 15$.

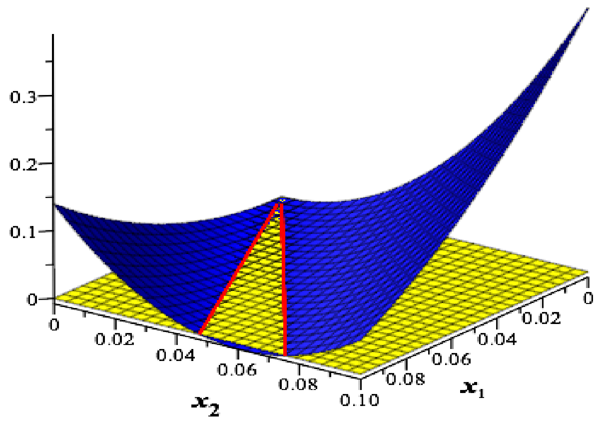

Fig. 4 In blue, the 3D plot of the surface defined by $39\,{x_2}^2 - 48\,x_2x_1 + 14\,{x_1}^2$. The red curves are associated to the degeneracy domain (in the (x_1, x_2) plane) of the matrix Υ.

6.3 The nondegeneracy of a 2D-functional Birkhoff matrix: incidence vector with intermediate stars

As an illustration of the result given in Corollary 47, consider the functional Birkhoff matrix Υ characterized by the incidence vector $\mathcal{V} = (x_1, x_1, x_1, x_1, x_2, \star, \star, x_2)$. Thus, one has

$$\Upsilon = \begin{bmatrix} {\sigma_{1,0}}^8 & 8\,{\sigma_{1,0}}^7 & 56\,{\sigma_{1,0}}^6 & 336\,{\sigma_{1,0}}^5 & {\sigma_{0,1}}^8 & 336\,{\sigma_{0,1}}^5 \\ {\sigma_{1,0}}^9 & 9\,{\sigma_{1,0}}^8 & 72\,{\sigma_{1,0}}^7 & 504\,{\sigma_{1,0}}^6 & {\sigma_{0,1}}^9 & 504\,{\sigma_{0,1}}^6 \\ {\sigma_{1,0}}^{10} & 10\,{\sigma_{1,0}}^9 & 90\,{\sigma_{1,0}}^8 & 720\,{\sigma_{1,0}}^7 & {\sigma_{0,1}}^{10} & 720\,{\sigma_{0,1}}^7 \\ {\sigma_{1,0}}^{11} & 11\,{\sigma_{1,0}}^{10} & 110\,{\sigma_{1,0}}^9 & 990\,{\sigma_{1,0}}^8 & {\sigma_{0,1}}^{11} & 990\,{\sigma_{0,1}}^8 \\ {\sigma_{1,0}}^{12} & 12\,{\sigma_{1,0}}^{11} & 132\,{\sigma_{1,0}}^{10} & 1320\,{\sigma_{1,0}}^9 & {\sigma_{0,1}}^{12} & 1320\,{\sigma_{0,1}}^9 \\ {\sigma_{1,0}}^{13} & 13\,{\sigma_{1,0}}^{12} & 156\,{\sigma_{1,0}}^{11} & 1716\,{\sigma_{1,0}}^{10} & {\sigma_{0,1}}^{13} & 1716\,{\sigma_{0,1}}^{10} \end{bmatrix}.$$

Under the assumptions $x_1x_2 \neq 0$ and $x_1 \neq x_2$, the matrix Υ is a non singular matrix if, and only if, the bivariate polynomial $33\,{x_2}^2 - 44\,x_2x_1 + 14\,{x_1}^2 \neq 0$, see Figure 5.

Now, consider the corresponding quasipolynomial function

$$\Delta(\lambda, \sigma) = \lambda^7 + \sum_{k=0}^{6} a_{0,k}\,\lambda^k + e^{\sigma_{1,0}\lambda}\,\sum_{k=0}^{4} a_{1,0,k}\,\lambda^k + \left(a_{0,1,0} + +a_{0,1,3}\lambda^3\right)\,e^{\sigma_{0,1}\lambda}. \tag{40}$$

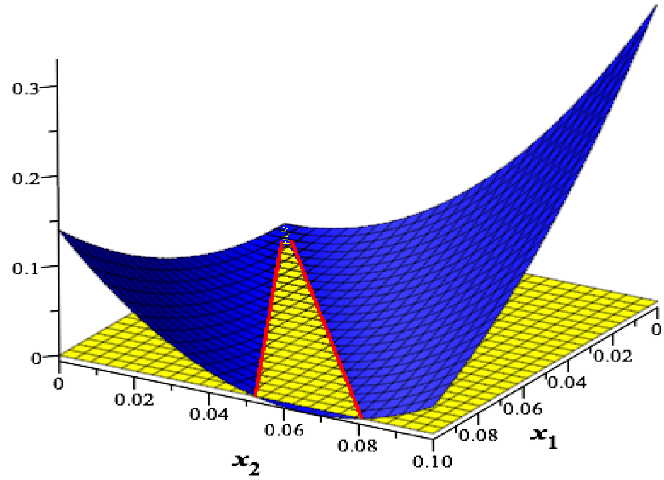

Fig. 5 In blue, the 3D plot of the surface defined by $33\,{x_2}^2 - 44\,x_2 x_1 + 14\,{x_1}^2$. The red curves are associated to the degeneracy of the matrix Υ.

The result above asserts that if the auxiliary non zero distinct delays $\sigma_{1,0}$ and $\sigma_{0,1}$ satisfy $33\,{\sigma_{0,1}}^2 - 44\,\sigma_{0,1}\sigma_{1,0} + 14\,{\sigma_{1,0}}^2 \neq 0$, then, the codimension of the zero singularity is bounded by 14. Furthermore, such a multiplicity bound is reached if, and only if, the parameter vectors a and a_0 satisfy equality (33) for $k = 0, \ldots, 13$. Notice that, in this configuration, the Pólya-Segö bound $\sharp_{PS} = 16$.

6.4 Controlling Generalized Bogdanov-Takens Singularity

Commonly, the generalized Bogdanov-Takens singularity (typically a chain of integrators) represents an undesired configuration when dealing with stability problems. Our control idea is based on it and it can be summarized as follow: first, we introduce sufficiently many delayed proportional controllers with free gains [30,48]. Next we identify the appropriate parameters (delays and gains) values allowing to reach the configuration of a spectrum consisting of stable spectrum and a multiple-zero singularity. Hence, we develop the spectral projection in the finite dimensional subspace (the generalized eigenspace) associated with the zero singularity. This suggests the computation of the center manifold and the normal form of the equations governing the dynamics on it for the parameters-perturbed system; which are known to be powerful tools for the local qualitative study of the dynamics. In other words, this reduces our problem to the control of a finite dimensional dynamical system. It is well known that the stability of the obtained finite dimensional projected dynamics means the stability of the original time-delay system, see [18]. Moreover, the matrix associated with the linear part of the finite dimensional projection of the parameters-perturbed system (associated with the generalized Bogdanov-Takens singularity) is nothing but a companion matrix (depending on the parameters-perturbation). It is always possible to design an appropriate perturbation that makes this matrix Hurwitz, [30, 48]. This approach proved its efficiency particularly in suppressing undesired dynamics of mechanical systems. In [45] it is shown that the only use of a proportional controller is not sufficient in stabilizing the inverted pendulum and it is proved that an additional delay in the control signal is necessary for a successful stabilization. The described approach is emphasized in some recent contribution of J. Sieber & B. Krauskopf [13] for stabilizing the inverted pendulum by designing a delayed PD controller. Moreover, they established a linearized stability analysis allowing to characterize all the possible local bifurcations additionally to the nonlinear analysis. This analysis involves the center manifold theory and normal forms. The study underlined the existence of a codimension-three triple zero bifurcation. It is also shown that the stabilization of the inverted pendulum in its upright position cannot be achieved by a PD controller when the delay exceeds some critical value τ_c. In [46], the authors investigate some modifications of the delayed PD scheme allowing to extend the range of the permissible delays by introducing an additional parameter. For that, two options were proposed, either to additionally take into account the angular acceleration or to consider an intentional additional delay in the angular position feedback. In [14] the authors introduce a multi-delayed-proportional controller allowing the stabilization of the inverted pendulum without the use of derivative measurements. Usually, the use of PD controller needs the knowledge of the velocity history but in general we are only able to have approximate measurements due to technological constraints. In absence of measurement of the derivative, a classical idea is to use an observer to reconstruct the state, but this task is computationally involved. It is shown in [47] that this type of singularity (triple zero singularity) can be avoided by offsetting the delayed derivative gain by introducing two-delayed-proportional controller. The interest of considering control laws of the form $\sum_{k=1}^{m} \gamma_k \, x(t-\tau_k)$ lies in the simplicity of the controller as well as in its practical implementation facility, suggesting the only use of position sensor. In a similar manner other type of singularities can be avoided. In [49], S.A. Campbell *et. al.* considered a proportional controller to locally maintain the pendulum in the upright position. The authors have shown that when this proportional is delayed and if the time-delay sampling is not too large, the controller still locally stabilizes the system. Among others, using the center manifold theorem and normal forms, they show the loss of stability when the delay exceeds a critical value and a supercritical Andronov-Hopf Bifurcation [19] occurs generating stable limit cycles. Finally, the described approach as well as all the above cited results emphasize the importance of the imaginary roots and motivate investigations making them more understood.

7 Concluding remarks and future work

This paper addressed the problem of characterizing the dimension of the eigenspace associated with a zero singularity for time-delay systems (the explicit conditions guaranteeing a given dimension) as well as an effective sharp bound for such a dimension. The existing links between the codimension of the zero singularity and functional Birkhoff matrices are emphasized. It is shown that the codimension bound of the zero singularity relies on the number of nonzero coefficients rather than the degree of the corresponding quasipolynomial. As a matter of fact, for generic quasipolynomials, a linear algebra alternative proof for the Pólya-Segö bound [15] is proposed. In the case of sparse quasipolynomials, a sharper bound for the codimension of the zero singularity is established under the non degeneracy of an appropriate functional Birkhoff matrix. It is worth noting, that the established LU-developments yields some new possibilities in the study of "poised"- ness of Birkhoff incidence matrices. Finally, we emphasize that the proposed approach can be extended to wider classes of functional equations, for instance, neutral systems and delay-difference equations. In the next step, the effect of the rational dependency of the delays on the codimension bound of the zero singularity will be explored.

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Appendix

In this section, we first summarize the main notations in Table 1. Then, for the sake of self-containment, we report some results selected from the literature. Finally, some useful auxiliary lemmas are presented and proved. The proofs of Theorem 44 and Theorem 46 are provided.

Here, we reporte some useful results from the mentioned literature. The main theorem from [16] emphasizes the link between $\mathbf{card}(\chi_+)$ and $\mathbf{card}(\chi_0)$, both take into account the multiplicity.

Theorem 71 (Hassard, [16], pp. 223). *Consider the quasipolynomial function Δ defined by (4). Let $\rho_1, \ldots, \rho_r$ be the positive roots of $\mathcal{R}(y) = \Re(i^n\, \Delta(i\, y))$, counted by their multiplicities and ordered so that $0 < \rho_1 \leq \ldots \leq \rho_r$. For each $j = 1, \ldots, r$ such that $\Delta(i\, \rho_j) = 0$, assume that the multiplicity of $i\rho_j$ as a zero of $\Delta(\lambda, \tau)$ is the same as the multiplicity of ρ_j as a root of $\mathcal{R}(y)$. Then*

Table 1 Table of the Main Notations

λ	a complexe number.
$\mathcal{R}(\lambda)$	real part of λ.
$\mathcal{I}(\lambda)$	imaginary part of λ.
τ	a delay vector.
σ	an auxiliary delay vector.
z	the vector state of the dynamical system.
n	the number of scalar differential equations defining a system.
N	the number of the delays involved in the dynamical system.
$\Delta(\lambda,\tau)$	a quasipolynomial function.
$P_i(\lambda)$	the quasipolynomial associated polynomials.
$a_{l,l}$	coefficient of λ^l of the l-th polynomial of Δ.
χ	spectrum associated to a given quasipolynomial.
χ_+	the set of instable spectral values.
χ_-	the set of stable spectral values.
χ_0	the set of critical spectral values i.e. crossing imaginary roots.
$\sharp$	cardinality of a set.
g	unknown function to interpolate.
x_i	interpolating points.
$\hat{P}$	interpolating polynomial.
$\mathcal{E}$	incidence matrix.
$\mathcal{V}$	the corresponding incidence vector.
$e_{i,j}$	entries of the incidence matrix.
Υ	a functional Birkhoff matrix corresponding to a sufficiently regular function ϖ.
$\hat{\Upsilon}$	coalescence matrix associeted to Υ.

$\mathbf{card}(\chi_+)$ *is given by the formula:*

$$\mathbf{card}(\chi_+) = \frac{n - \mathbf{card}(\chi_0)}{2} + \frac{(-1)^r}{2}\operatorname{sgn}\mathcal{I}^{(\mu)}(0) + \sum_{j=1}^{r}\operatorname{sgn}\mathcal{I}(\rho_j), \qquad (41)$$

where μ designate the multiplicity of the zero spectral value of $\Delta(\lambda,\tau) = 0$ and $\mathcal{I}(y) = \Im(i^{-n}\Delta(iy))$. Furthermore, $\mathbf{card}(\chi_+)$ *is odd (respectively, even) if $\Delta^{(\mu)}(0) < 0$ ($\Delta^{(\mu)}(0) > 0$). If $\mathcal{R}(y) = 0$ has no positive zeros, set $r = 0$ and omit the summation term in the expression of* $\mathbf{card}(\chi_+)$*. If $\lambda = 0$ is not a root of the characteristic equation, set $\mu = 0$ and interpret $\mathcal{I}^{(0)}(0)$ as $\mathcal{I}(0)$ and $\Delta^{(0)}(0)$ as $\Delta(0)$.*

The following result from [15] gives a valuable information allowing to have a first estimation on the bound for the codimension of the zero spectral value.

Proposition 72 (Pólya-Szegö, [15], pp. 144)**.** *Let $\tau_1, \ldots, \tau_N$ denote real numbers such that*

$$\tau_1 < \tau_2 < \ldots < \tau_N,$$

and $d_1, \ldots, d_N$ positive integers satisfying

$$d_1 \geq 1,\ d_2 \geq 1\ \ldots\ d_N \geq 1, \quad d_1 + d_2 + \ldots + d_N = D + N.$$

Let $f_{i,j}(s)$ stands for the function $f_{i,j}(s) = s^{j-1}\, e^{\tau_i\, s}$, for $1 \leq j \leq d_i$ and $1 \leq i \leq N$.

Let $\sharp$ be the number of zeros of the function

$$f(s) = \sum_{1 \le i \le N, 1 \le j \le d_i} c_{i,j}\, f_{i,j}(s),$$

that are contained in the horizontal strip $\alpha \le \mathcal{I}(z) \le \beta$.
Assuming that

$$\sum_{1 \le k \le d_1} |c_{1,k}| > 0, \; \ldots, \; \sum_{1 \le k \le d_N} |c_{N,k}| > 0,$$

then

$$\frac{(\tau_N - \tau_1)\,(\beta - \alpha)}{2\,\pi} - D + 1 \le \sharp \le \frac{(\tau_N - \tau_1)\,(\beta - \alpha)}{2\,\pi} + D + N - 1.$$

Setting $\alpha = \beta = 0$, the above Proposition allows to $\sharp_{PS} \le D + N - 1$ where D stands for the sum of the degrees of the polynomials involved in the quasipolynomial function f and N designate the associated number of polynomials. This gives a sharp bound in the case of complete polynomials.

In the sequel, we present some useful lemmas as well as the proofs of the claimed theorems.

Lemma 1. *Zero is a root of $\Delta^{(k)}(\lambda)$ for $k \ge 0$ if, and only if, the coefficients of P_{M^j} for $0 \le j \le \tilde{N}_{N,n}$ satisfy the following assertion*

$$a_{0,k} = -\sum_{i \in S_{N,n}} \left[a_{i,k} + \sum_{l=0}^{k-1} \frac{a_{i,l} {\sigma_i}^{k-l}}{(k-l)!} \right]. \tag{A.1}$$

Proof We define the family ∇_k for all $k \ge 0$ by

$$\nabla_k(\lambda) = \sum_{i=0}^{\tilde{N}_{N,n}} \frac{d^k}{d\lambda^k} P_{M^i}(\lambda) + \sum_{j=0}^{k-1} \left(\binom{k}{j} \sum_{i=1}^{\tilde{N}_{N,n}} {\sigma_i}^{k-j} \frac{d^j}{d\lambda^j} P_{M^i}(\lambda) \right), \tag{A.2}$$

here, $M^0 \triangleq 0$ and $\frac{d^0}{d\lambda^0} f(\lambda) \triangleq f(\lambda)$. Obviously, the defined family ∇_k is polynomial since P_i and their derivatives are polynomials. Moreover, zero is a root of $\Delta^{(k)}(\lambda)$ for $k \ge 0$ if, and only if, zero is a root of $\nabla_k(\lambda)$. This can be proved by induction. More precisely, differentiating k times $\Delta(\lambda, \tau)$ the following recursive formula is obtained:

$$\Delta^{(k)}(\lambda) = \sum_{i=0}^{\tilde{N}_{N,n}} \frac{d^k}{d\lambda^k} P_{M^i}(\lambda)\, e^{\sigma_i \lambda} + \sum_{j=0}^{k-1} \left(\binom{k}{j} \sum_{i=1}^{\tilde{N}_{N,n}} {\sigma_i}^{k-j} \frac{d^j}{d\lambda^j} P_{M^i}(\lambda)\, e^{\sigma_i \lambda} \right).$$

Since only the zero root is of interest, we can set $e^{\sigma_i \lambda} = 1$ which define the polynomial functions ∇_k. Moreover, careful inspection of the obtained quantities presented in (A.2) and substituting $\frac{d^k}{d\lambda^k} P_i(0) = k!\, a_{i,k}$ leads to the formula (A.1).

Here, we prove the results given in section 4.2.1, that is, we consider the incidence vector:

$$\mathcal{V} = (\underbrace{x_1, \ldots, x_1}_{d_1}, \underbrace{\star, \ldots, \star}_{d_*}, x_2).$$

The right hand side of the last equality from (23) defining U_{i,d_1+1} for $2 \le i \le d_1 + 1$ can be also written as follows.

Lemma 2. *For $2 \le i \le d_1 + 1$ the following equality is satisfied:*

$$\Upsilon_{i,d_1+1} - (i-1) \int_0^{x_1} U_{i-1,d_1+1}(y, x_2) dy = \sum_{k=0}^{i-1} \binom{i-1}{k} (-1)^{i-1-k} x_1^{i-1-k} \Upsilon_{k+1,d_1+1}.$$

Proof (Proof of Lemma 2) First, one has $U_{2,d_1+1} = \Upsilon_{2,d_1+1} - x_1 \Upsilon_{1,d_1+1} = \Upsilon_{2,d_1+1} - \int_0^{x_1} U_{1,d_1+1}(y, x_2)\, dy$ since $U_{1,d1+1} = \Upsilon_{1,d_1+1}(x_2)$.

Now, let assume that for $2 \le i \le p$ where $p < d_1 + 1$ the following equality is satisfied:

$$\sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{i-1-l}\, x_1^{i-1-l}\, \Upsilon_{l+1,d_1+1} = \Upsilon_{i,d_1+1} - (i-1) \int_0^{x_1} U_{i-1,d_1+1}(y, x_2)\, dy.$$

One has to show that for $i = p + 1$:

$$\sum_{l=0}^{p} \binom{p}{l} (-1)^{p-l}\, x_1^{p-l}\, \Upsilon_{l+1,d_1+1} = \Upsilon_{p+1,d_1+1} - (p) \int_0^{x_1} U_{p,d_1+1}(y, x_2)\, dy.$$

Indeed,

$$\begin{cases} -\displaystyle\int_0^{x_1} p\, U_{p,d_1+1}(y, x_2)\, dy = -\int_0^{x_1} p \sum_{l=0}^{p-1} \binom{p-1}{l} (-1)^{p-1-l}\, s^{p-1-l}\, \Upsilon_{l+1,d_1+1}\, ds, \\ = -\displaystyle\sum_{l=0}^{p-1} \frac{p!}{l!\,(p-l-1)!} (-1)^{p-1-l} \Upsilon_{l+1,d_1+1} \int_0^{x_1} s^{p-1-l} ds, \\ = \displaystyle\sum_{l=0}^{p-1} \binom{p}{l} (-1)^{p-l}\, x_1^{p-l}\, \Upsilon_{l+1,d_1+1}. \end{cases}$$

Proof (Proof of Theorem 44) The only difference between algorithms (23) and (20) lies in definition of the last column of the matrix U. Thus, one has to show that for any $2 \le i \le d_1 + 1$ the following equality holds $\Upsilon_{i,d_1+1} = \sum_{k=1}^{i} L_{i,k} U_{k,d_1+1}$. By definition, one has:

$$\begin{cases} \Upsilon_{2,d_1+1} = \displaystyle\sum_{k=1}^{2} L_{2,k}\, U_{k,d_1+1} \\ \qquad\quad = L_{2,1}\, U_{1,d_1+1} + L_{2,2}\, U_{2,d_1+1} \\ \qquad\quad = x_1\, \Upsilon_{1,d_1+1} + U_{2,d_1+1}. \end{cases} \tag{42}$$

Now, let assume that for $2 \le i \le p$ where $p < d_1 + 1$ the following equality is satisfied:

$$U_{i,d_1+1} = \Upsilon_{i,d_1+1} - (i-1) \int_0^{x_1} U_{i-1,d_1+1}(y, x_2)\, dy,$$

or equivalently, from Lemma 2

$$U_{i,d_1+1} = \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^{i-1-l}\, x_1^{i-1-l}\, \Upsilon_{l+1,d_1+1}.$$

It stills to show that the last equality from (23) holds for U_{p+1,d_1+1} when $p < d_1 + 1$. Indeed, by definition

$$U_{p+1,d_1+1} = \Upsilon_{p+1,d_1+1} - \sum_{k=1}^{p} L_{p+1,k} U_{k,d_1+1}.$$

Moreover, (for same arguments as the ones given in the proof of Lemma 6 presented in the sequel), one has $L_{p+1,k} = \frac{1}{k-1} \frac{\partial L_{p+1,k-1}}{\partial x_1}$. Thus, $L_{p+1,k} = \frac{1}{(k-1)!} \frac{\partial^{k-1} L_{p+1,1}}{\partial x_1^{k-1}} = \frac{1}{(k-1)!} \frac{\partial^{k-1} x_1^p}{\partial x_1^{k-1}} = \frac{p!\, x_1^{p-k+1}}{(p-k+1)!\,(k-1)!}$. So that, one has:

$$L_{p+1,k} = \binom{p}{k-1} x_1^{p-(k-1)}. \tag{43}$$

Now, by definition of U_{p+1,d_1+1} and using (43) as well as the recurrence assumption, we obtain

$$\begin{cases} U_{p+1,d_1+1} = \Upsilon_{p+1,d_1+1} - \sum_{l=1}^{p} L_{p+1,l} U_{l,d_1+1} \\ \qquad = \Upsilon_{p+1,d_1+1} - \sum_{l=1}^{p} \sum_{l=0}^{k-1} \binom{l-1}{l} \binom{p}{l-1} (-1)^{l-l-1}\, x_1^{l-l-1}\, x_1^{p-(l-1)}\, \Upsilon_{l+1,d_1+1} \\ \qquad = \Upsilon_{p+1,d_1+1} - \sum_{l=1}^{p} \sum_{l=0}^{l-1} \binom{l-1}{l} \binom{p}{l-1} (-1)^{l-1-l}\, x_1^{p-l}\, \Upsilon_{l+1,d_1+1} \end{cases}$$

Thus, one has to prove that

$$\sum_{k=0}^{p-1} \binom{p}{k} (-1)^{p-k}\, x_1^{p-k}\, \Upsilon_{k+1,d_1+1} = - \sum_{l=1}^{p} \sum_{l=0}^{l-1} \binom{l-1}{l} \binom{p}{l-1} (-1)^{l-1-l}\, x_1^{p-l}\, \Upsilon_{l+1,d_1+1}. \tag{44}$$

Recall that, the two side expressions of (44) are polynomials in x_1 and x_2. The only quantities depending in x_2 are $(\Upsilon_{k,d_1+1})_{1 \le k \le p}$. Since, $\deg(\Upsilon_{k,d_1+1}) \neq \deg(\Upsilon_{k',d_1+1})$ for $k \neq k'$, it will be enough to we examine the equality of coefficients of the two side expressions in Υ_{m+1,d_1+1} for arbitrarily chosen $0 \le m \le p-1$. So that, let $m = k_0$ for which corresponds $m = l_0$ in the right hand side quantity from (44). Then consider the coefficient of $x_1^{p-m} \Upsilon_{m+1,d_1+1}$ from the two sides of (44) . Now, one easily check that $\sum_{l=m}^{p} \binom{l-1}{m} \binom{p}{l-1} (-1)^{l-m} = (-1)^{p-m} \binom{p}{m}$ is always satisfied, which ends the proof.

In what follow, we propose some lemmas exhibiting some interesting properties of functional Birkhoff matrices. Those will be useful for the analytical proof of Theorem 46.

Lemma 3. *Equation* (30) *is equivalent to:*

$$U_{i,j} = \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^l \, x_1^l \, \Upsilon_{i-l,j} \quad \textit{for} \quad j = d_1 + d_2^- + 1 \textit{ and } 2 \leq i \leq d_1 + 1. \tag{45}$$

Proof (Proof of Lemma 3) The equality (45) follows directly by induction. First, one checks that

$$\Upsilon_{2,d_1+d_2^-+1} = U_{2,d_1+d_2^-+1} + x_1 \, \Upsilon_{1,d_1+d_2^-+1}.$$

Indeed,

$$\begin{cases} \Upsilon_{2,d_1+d_2^-+1} = \displaystyle\sum_{k=1}^{2} L_{2,k} \, U_{k,d_1+d_2^-+1} \\ \qquad\qquad\quad = L_{2,1} \, U_{1,d_1+d_2^-+1} + L_{2,2} \, U_{2,d_1+d_2^-+1} \\ \qquad\qquad\quad = x_1 \, \Upsilon_{1,d_1+d_2^-+1} + U_{2,d_1+d_2^-+1}, \end{cases} \tag{46}$$

since $L_{2,2} = 1$. Now, let assume that

$$U_{i,j} = \sum_{l=0}^{i-1} \binom{i-1}{l} (-1)^l \, x_1^l \, \Upsilon_{i-l,j} \quad \text{for} \quad j = d_1+d_2^-+1 \textit{ and } 2 \leq i \leq p \textit{ and } p < d_1+1, \tag{47}$$

From Equation (30) one has

$$U_{p+1,d_1+d_2^-+1} = \Upsilon_{p+1,d_1+d_2^-+1} - p \int_0^{x_1} U_{p,d_1+d_2^-+1}(y, x_2) dy.$$

Using (47), one has,

$$\begin{cases} U_{p+1,d_1+d_2^-+1} \\ =\Upsilon_{p+1,d_1+d_2^-+1} \\ \quad - p \displaystyle\int_0^{x_1} \left(\Upsilon_{p,d_1+d_2^-+1}(y, x_2) + \sum_{l=1}^{p-1} \binom{p-1}{l} (-1)^l \, y^l \, \Upsilon_{p-l,d_1+d_2^-+1}(y, x_2) \right) dy \\ \quad = \Upsilon_{p+1,d_1+d_2^-+1} - p \, \Upsilon_{p,d_1+d_2^-+1} x_1 + \displaystyle\sum_{l=1}^{p-1} p \binom{p-1}{l} (-1)^l \, \Upsilon_{p-l,d_1+d_2^-+1} \int_0^{x_1} y^l \, dy \\ \quad = \displaystyle\sum_{l=0}^{p} \binom{p}{l} (-1)^l \, x_1^l \, \Upsilon_{p+1-l,d_1+d_2^-+1}. \end{cases}$$

which ends the proof.

Lemma 4.

$$\Upsilon_{i+1,j} = x_2\,\Upsilon_{i,j} + (d_2^- + d^*) \int_0^{x_2} \Upsilon_{i,j}(y)\,dy \quad \textit{for} \quad j = d_1 + d_2^- + 1 \textit{ and } 1 \le i \le d_1 + d_2^-. \tag{48}$$

Proof (Proof of Lemma 4) Let consider the *coalescence* [50] confluent Vandermonde matrice $\hat{\Upsilon}$ which regularize the considered Birkhoff matrice Υ. That is $\hat{\Upsilon}$ is the rectangular matrix associated with the incidence matrix

$$\mathcal{V} = (\underbrace{x_1, \ldots, x_1}_{d_1}, \underbrace{x_2, \ldots, x_2}_{d_2^-}, \underbrace{x_2, \ldots, x_2}_{d_*}, x_2).$$

Here, the "stars" $\star$ in (32) are simply replaced by x_2. Thus, Υ and $\hat{\Upsilon}$ have the same number of rows, but the number of columns of $\hat{\Upsilon}$ exceeds the columns number of Υ by d^*. We point out that $\Upsilon_{i+1,d_1+d_2^-+1}$ is nothing but $\hat{\Upsilon}_{i+1,d_1+d_2^-+1+d^*}$. This means that the term $(d_2^- + d^*) \int_0^{x_2} \Upsilon_{i,j}$ in (48) is exactly $\hat{\Upsilon}_{i+1,d_1+d_2^-+d^*}$. Thus, equality (48) turns to be

$$\bar{\Upsilon}_{i+1,j} = x_2\,\bar{\Upsilon}_{i,j} + \bar{\Upsilon}_{i,j-1} \quad \text{for} \quad j = d_1 + d_2^- + 1 + d^* \textit{ and } 1 \le i \le d_1 + d_2^-.$$

This last equality can be easily proved by using a 2-D recurrence in terms of $\bar{\Upsilon}$ (regular matrix) as in the proof of Theorem 41 to show that it applies even for $d_1 + 2 \le j \le d_1 + d_2^- + 1 + d^*$.

The following Lemma provides an other way defining the components of U given by (30).

Lemma 5. *for all $i = 1, \ldots, d_1$ and $j = d_1 + d_2^- + 1$ the following equality applies*

$$U_{i+1,j^*} = (x_2 - x_1)\,U_{i,j^*} + (d_2^- + d^*) \int_0^{x_2} U_{i,j^*}(y)dy. \tag{49}$$

Proof (Proof of Lemma 5) Let set

$$\mathcal{I}_k = U_{k+1,j^*} + (x_1 - x_2)\,U_{k,j^*} - (d_2^- + d^*) \int_0^{x_2} U_{k,j^*}(y)dy.$$

where $j^* = d_1 + d_2^- + 1 + d^*$ and $1 \le k \le d_1 + 1$.

Substitute equation (45) from lemma 3 in $\mathcal{I}_k$, to obtain

$$\begin{aligned}\mathcal{I}_k =& \sum_{l=0}^{k} \binom{k}{l} (-1)^l x_1^l \Upsilon_{k+1-l,j^*} \\ &- \sum_{l=0}^{k-1} \binom{k-1}{l} (-1)^l x_1^l \left((x_2 - x_1)\Upsilon_{k-l,j^*} + (d_2^- + d^*) \int_0^{x_2} \Upsilon_{k-l,j^*}(y)dy \right).\end{aligned}$$

Using lemma 4, one obtains

$$\begin{aligned}\mathcal{I}_k &= \sum_{l=1}^{k-1} (-1)^l\, x_1^l\, \left[\left(\binom{k}{l} - \binom{k-1}{l} \right) \Upsilon_{k+1-l,j^*} + x_1 \binom{k-1}{l} \Upsilon_{k-l,j^*} \right] \\ &\quad + (-1)^k\, x_1^k\, \Upsilon_{1,j^*} + x_1\, \Upsilon_{k,j^*} \\ &= \sum_{l=1}^{k-1} (-1)^l\, x_1^l\, \left(\binom{k-1}{l-1} \Upsilon_{k+1-l,j^*} + x_1 \binom{k-1}{l} \Upsilon_{k-l,j^*} \right) + (-1)^k\, x_1^k\, \Upsilon_{1,j^*} \\ &\quad + x_1\, \Upsilon_{k,j^*}\end{aligned}$$

which is as expected identically zero, that ends the proof.

The following Lemma provides a differential relation between the coefficients of L matrix.

Lemma 6. *for all* $1 \leq k \leq p$ *the following equality holds*

$$\frac{\partial L_{d_1+p,d_1+k}}{\partial x_2} = k\, L_{d_1+p,d_1+k+1} \tag{50}$$

The following result applies when dealing with $\Upsilon_{i,j}$ and $\Upsilon_{i,j-1}$ are in the same variable block. We emphasize that such a property is inherited by the expressions of L defined in (28).

Proof (Proof of Lemma 6) The proof is 2-D recurrence-based. First, one easily check that for $p = 2$ then $k = 1$

$$L_{d_1+2,d_1+2} = \frac{\partial L_{d_1+2,d_1+1}}{\partial x_2}$$

since by definition of L one has $L_{d_1+2,d_1+1} = L_{d_1+1,d_1} + x_2\, L_{d_1+1,d_1+1} = L_{d_1+1,d_1} + x_2$ and $\frac{\partial L_{d_1+1,d_1}}{\partial x_2} = 0$. When assuming that

$$L_{d_1+p,d_1+2} = \frac{\partial L_{d_1+p,d_1+1}}{\partial x_2},$$

and again, using the definition of L, one obtains,

$$\begin{aligned}L_{d_1+p+1,d_1+2} &= L_{d_1+p,d_1+1} + x_2 L_{d_1+p,d_1+2}, \\ L_{d_1+p+1,d_1+1} &= L_{d_1+p,d_1} + x_2 L_{d_1+p,d_1+1},\end{aligned}$$

which as expected gives:

$$\begin{aligned}\frac{\partial L_{d_1+p+1,d_1+1}}{\partial x_2} &= L_{d_1+p,d_1+1} + x_2 \frac{\partial L_{d_1+p,d_1+1}}{\partial x_2} \\ &= L_{d_1+p,d_1+1} + x_2 L_{d_1+p,d_1+2} = L_{d_1+p+1,d_1+2}.\end{aligned}$$

Let assume that for any $2 < p < d_2^- + 1$ and $k = 1, \ldots, p-1$ one has

$$\frac{\partial L_{d_1+p,d_1+k}}{\partial x_2} = k\, L_{d_1+p,d_1+k+1}.$$

One has to prove the following equalities :

$$\begin{cases} \dfrac{\partial L_{d_1+p+1,d_1+k}}{\partial x_2} = k\,L_{d_1+p+1,d_1+k+1}, \\ \dfrac{\partial L_{d_1+p,d_1+k+1}}{\partial x_2} = (k+1)\,L_{d_1+p,d_1+k+2}, \\ \dfrac{\partial L_{d_1+p+1,d_1+k+1}}{\partial x_2} = (k+1)\,L_{d_1+p+1,d_1+k+2}. \end{cases} \tag{51}$$

Let us consider the first equality of (51), using the definition of L that asserts that

$$\begin{aligned} L_{d_1+p+1,d_1+k+1} &= L_{d_1+p,d_1+k} + x_2 L_{d_1+p,d_1+k+1} \\ L_{d_1+p+1,d_1+k} &= L_{d_1+p,d_1+k-1} + x_2 L_{d_1+p,d_1+k}. \end{aligned}$$

Which gives

$$\begin{aligned} \frac{\partial L_{d_1+p+1,d_1+k}}{\partial x_2} &= \frac{\partial L_{d_1+p,d_1+k-1}}{\partial x_2} + x_2 \frac{\partial L_{d_1+p,d_1+k}}{\partial x_2} + L_{d_1+p,d_1+k} \\ &= (k-1)L_{d_1+p,d_1+k} + k\,L_{d_1+p,d_1+k+1} + L_{d_1+p,d_1+k} \\ &= k\,L_{d_1+p+1,d_1+k+1}. \end{aligned}$$

By the same way, the remaining two equality from (51) are obtained:

$$\begin{aligned} \frac{\partial L_{d_1+p,d_1+k+1}}{\partial x_2} &= \frac{\partial \left(L_{d_1+p-1,d_1+k} + x_2 L_{d_1+p-1,d_1+k+1}\right)}{\partial x_2} \\ &= \frac{\partial L_{d_1+p-1,d_1+k}}{\partial x_2} + x_2 \frac{\partial L_{d_1+p-1,d_1+k+1}}{\partial x_2} + L_{d_1+p-1,d_1+k+1} \\ &= kL_{d_1+p-1,d_1+k+1} + (k+1)\,x_2 L_{d_1+p-1,d_1+k+2} + L_{d_1+p-1,d_1+k+1} \\ &= (k+1)\,L_{d_1+p,d_1+k+2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial L_{d_1+p+1,d_1+k+1}}{\partial x_2} &= \frac{\partial \left(L_{d_1+p,d_1+k} + x_2 L_{d_1+p,d_1+k+1}\right)}{\partial x_2} \\ &= \frac{\partial L_{d_1+p,d_1+k}}{\partial x_2} + x_2 \frac{\partial L_{d_1+p,d_1+k+1}}{\partial x_2} + L_{d_1+p,d_1+k+1} \\ &= kL_{d_1+p,d_1+k+1} + (k+1)\,x_2 L_{d_1+p,d_1+k+2} + L_{d_1+p,d_1+k+1} \\ &= (k+1)\,L_{d_1+p+1,d_1+k+2}. \end{aligned}$$

that ends the proof.

Proof (Proof of Theorem 46) The only change occurring in (26)-(31) compared with (20) is the way in defining the column $d_1 + d_2^- + 1$ of U. Moreover, such a column is only involved in computing the column $d_1 + d_2^- + 1$ of Υ. Thus, it stills to show that the equalities (30) and (31); this will be done by recurrence. Equation (30) follow directly by induction from lemma 3.

Let us focus now on (31) and denote $j^* = d_1 + d_2^- + 1$. First, let us check that

$$U_{d_1+2,j^*} = (d_2^- + d^*) \int_0^{x_2} U_{d_1+1,j^*}(x_1, y)\, dy.$$

From the one side, using lemma 4 one has

$$\begin{aligned} \Upsilon_{d+2,j^*} &= x_2\, \Upsilon_{d+1,j^*} + (d_2^- + d^*) \int_0^{x_2} \Upsilon_{d+1,j^*}(y)\, dy, \\ &= x_2 \sum_{k=1}^{d_1+1} L_{d_1+1,k} U_{k,j^*} + (d_2^- + d^*) \int_0^{x_2} \sum_{k=1}^{d_1+1} L_{d_1+1,k} U_{k,j^*}(y)\, dy. \end{aligned}$$

Since by definition one has $L_{d_1+1,d_1+1} = 1$ and $L_{d_1+1,k} = L_{d_1+1,k}(x_1)$ for $k \in \{1, \ldots, d_1\}$ then

$$\begin{aligned} \Upsilon_{d+2,j^*} =& x_2 \sum_{k=1}^{d_1+1} L_{d_1+1,k} U_{k,j^*} + (d_2^- + d^*) \sum_{k=1}^{d_1} L_{d_1+1,k} \int_0^{x_2} U_{k,j^*}(y) dy \\ &+ (d_2^- + d^*) \int_0^{x_2} U_{d_1+1,j^*}(y) dy. \end{aligned}$$

From the other side and by definition of Υ,

$$\Upsilon_{d_1+2,j^*} = \sum_{k=1}^{d_1+2} L_{d_1+2,k} U_{k,j^*} = U_{d_1+2,j^*} + \sum_{k=1}^{d_1+1} L_{d_1+2,k} U_{k,j^*}.$$

To prove (31) for $i = d_1 + 2$ one has to prove that

$$\sum_{k=1}^{d_1+1} L_{d_1+2,k} U_{k,j^*} = x_2 \sum_{k=1}^{d_1+1} L_{d_1+1,k} U_{k,j^*} + (d_2^- + d^*) \sum_{k=1}^{d_1} L_{d_1+1,k} \int_0^{x_2} U_{k,j^*}(y) dy,$$

or equivalently to prove

$$\sum_{k=1}^{d_1+1} \left(L_{d_1+2,k} - x_2\, L_{d_1+1,k}\right) U_{k,j^*} - (d_2^- + d^*) \sum_{k=1}^{d_1} L_{d_1+1,k} \int_0^{x_2} U_{k,j^*}(y) dy = 0. \tag{52}$$

Using equation (28), one obtain

$$\begin{aligned} &L_{d_1+2,k} - x_2\, L_{d_1+1,k} = L_{d_1+1,k-1} + (x_1 - x_2)\, L_{d_1+1,k}, \quad \text{for } k = 1, \ldots, d_1, \\ &L_{d_1+2,d_1+1} - x_2\, L_{d_1+1,d_1+1} = L_{d_1+1,d_1}. \end{aligned}$$

Thus, the right hand side of (52) becomes

$$\sum_{k=1}^{d_1} L_{d_1+1,k} U_{k+1,j^*} + (x_1 - x_2) \sum_{k=1}^{d_1} L_{d_1+1,k} U_{k,j^*}$$
$$- (d_2^- + d^*) \sum_{k=1}^{d_1} L_{d_1+1,k} \int_0^{x_2} U_{k,j^*}(y) dy =$$
$$\sum_{k=1}^{d_1} L_{d_1+1,k} \left(U_{k+1,j^*} + (x_1 - x_2) U_{k,j^*} - (d_2^- + d^*) \int_0^{x_2} U_{k,j^*}(y) dy \right).$$

Lemma 5 asserts that for all $i = 1, \ldots, d_1$ and $j = d_1 + d_2^- + 1$ one has

$$U_{k+1,j^*} + (x_1 - x_2) U_{k,j^*} - (d_2^- + d^*) \int_0^{x_2} U_{k,j^*}(y) dy = 0 \qquad (53)$$

which implies that (31) applies for $i = d_1 + 2$.

Let assume now that, (31) is satisfied for $i = d_1 + 2, \ldots, d_1 + p$ where $1 < p < d_2^- + d^*$. It stills to prove that (31) is satisfied for $i = d_1 + p + 1$.

By the same argument as for $i = d_1 + 2$, one has

$$\begin{aligned}
\Upsilon_{d_1+p+1,j^*} =& x_2 \Upsilon_{d+p,j^*} + (d_2^- + d^*) \int_0^{x_2} \Upsilon_{d+p,j^*}(y)\, dy, \\
=& x_2 \sum_{k=1}^{d_1+p} L_{d_1+p,k} U_{k,j^*} (d_2^- + d^*) \sum_{k=1}^{d_1+p-1} \int_0^{x_2} L_{d_1+p,k} U_{k,j^*}(y) dy \\
&+ (d_2^- + d^*) \int_0^{x_2} U_{d_1+p,j^*}(y) dy \\
=& x_2 \sum_{k=1}^{d_1+p} L_{d_1+p,k} U_{k,j^*} + (d_2^- + d^*) \sum_{k=1}^{d_1+p-1} \int_0^{x_2} L_{d_1+p,k} U_{k,j^*}(y) dy \\
&+ (p-1) \int_0^{x_2} U_{d_1+p,j^*}(y) dy + (d_2^- + d^* - p + 1) \int_0^{x_2} U_{d_1+p,j^*}(y) dy
\end{aligned}$$

From the other side, we obtain

$$\Upsilon_{d_1+p+1,j^*} = \sum_{k=1}^{d_1+p} L_{d_1+p+1,k} U_{k,j^*} + U_{d_1+p+1,j^*}.$$

Hence, we have to prove that

$$\sum_{k=1}^{d_1+p} L_{d_1+p+1,k} U_{k,j^*} - x_2 \sum_{k=1}^{d_1+p} L_{d_1+p,k} U_{k,j^*} - (d_2^- + d^*) \sum_{k=1}^{d_1+p-1} \int_0^{x_2} L_{d_1+p,k} U_{k,j^*}(y) dy$$
$$= (p-1) \int_0^{x_2} U_{d_1+p,j^*}(y) dy.$$

Now, using the result from Lemma 5, one has to prove that

$$\sum_{k=d_1+2}^{d_1+p} L_{d_1+p+1,k}U_{k,j^*} - x_2 \sum_{k=d_1+2}^{d_1+p} L_{d_1+p,k}U_{k,j^*} \tag{54}$$

$$- (d_2^- + d^*) \sum_{k=d_1+1}^{d_1+p-1} \int_0^{x_2} L_{d_1+p,k}U_{k,j^*}(y)dy \tag{55}$$

$$= (p-1) \int_0^{x_2} U_{d_1+p,j^*}(y)dy. \tag{56}$$

Using equation (28), one obtains

$$L_{d_1+p+1,k} - x_2\, L_{d_1+p,k} = L_{d_1+p,k-1}, \quad \text{for} \quad k = d_1+2, \ldots, d_1+p.$$

Finally, equation (54) becomes

$$\begin{aligned} E =& \sum_{k=1}^{p-1} L_{d_1+p,d_1+k}U_{d_1+k+1,j^*} - (d_2^- + d^*) \sum_{k=1}^{p-1} \int_0^{x_2} L_{d_1+p,d_1+k}U_{d_1+k,j^*}(y)dy \\ & - (p-1) \int_0^{x_2} U_{d_1+p,j^*}(y)dy = 0. \end{aligned} \tag{57}$$

Differentiating E given in (57) with respect to the variable x_2 one obtains

$$\begin{aligned} \frac{\partial E}{\partial x_2} =& \sum_{k=1}^{p-1} \left(\frac{\partial L_{d_1+p,d_1+k}}{\partial x_2} U_{d_1+k+1,j^*} + L_{d_1+p,d_1+k} \frac{\partial U_{d_1+k+1,j^*}}{\partial x_2} \right) \\ & - (d_2^- + d^*) \sum_{k=1}^{p-1} L_{d_1+p,d_1+k}U_{d_1+k,j^*} - (p-1)U_{d_1+p,j^*} \\ =& \sum_{k=1}^{p-1} \frac{\partial L_{d_1+p,d_1+k}}{\partial x_2} U_{d_1+k+1,j^*} \\ & + \sum_{k=1}^{p-1} L_{d_1+p,d_1+k} \left(\frac{\partial U_{d_1+k+1,j^*}}{\partial x_2} - (d_2^- + d^*)U_{d_1+k,j^*} \right) - (p-1)U_{d_1+p,j^*} \end{aligned}$$

By using the recurrence assumption, one obtains,

$$\begin{aligned}\frac{\partial E}{\partial x_2} =&\sum_{k=1}^{p-1} \frac{\partial L_{d_1+p,d_1+k}}{\partial x_2} U_{d_1+k+1,j^*} - (p-1)U_{d_1+p,j^*} \\ &+ \sum_{k=1}^{p-1} L_{d_1+p,d_1+k} \left((d_2^- + d^* - (k-1))U_{d_1+k,j^*} - (d_2^- + d^*)U_{d_1+k,j^*}\right) \\ =&\sum_{k=1}^{p-1} \frac{\partial L_{d_1+p,d_1+k}}{\partial x_2} U_{d_1+k+1,j^*} - \sum_{k=2}^{p-1} (k-1)\, L_{d_1+p,d_1+k}\, U_{d_1+k,j^*} - (p-1)U_{d_1+p,j^*} \\ =&\sum_{k=1}^{p-2} \left(\frac{\partial L_{d_1+p,d_1+k}}{\partial x_2} - k\, L_{d_1+p,d_1+1+k}\right) U_{d_1+1+k,j^*} \\ &+ \left(\frac{\partial L_{d_1+p,d_1+p-1}}{\partial x_2} - (p-1)\right) U_{d_1+p,j^*} \equiv 0,\end{aligned}$$

which is as expected zero since Lemma 6 asserts that each factor is identically zero, that ends the proof.

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