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Abstract: This paper focuses on the study of the behavior of critical roots when a dynamical system is stabilized by a PD-controller, for which the derivative action has been approximated by using two commensurate delays. The use of such an approximation leads to a characteristic quasipolynomial whose coefficients depend explicitly on the delay parameter. The aim of the paper is to address the way the delay parameter may affect the location of the roots of the corresponding characteristic function, and in particular the cases when "small" delays induce instability in the closed-loop systems. Such an analysis is performed by expressing the critical solution of the system as a delay-dependent power series. Illustrative examples complete the presentation.

Keywords: PD-control, Delay-dependent Coefficients, Improperly-Posed Linear Systems.

1. INTRODUCTION

It is well known that, due to its simple implementation and effectiveness for solving real world problems, PID controllers are among the most popular controllers in the industry (O'Dwyer, 2009; da Silva et al., 2020; Astrom and Hagglund, 1995). The implementation of such controllers has been studied for several years by control engineers, and it has been pointed out that the use of a derivative action may induce some "*bad behaviors*". Indeed, there exists a problem with the implementation of such an operator, since it tends to amplify any noise presented on the signal. In this sense, Figure 1 illustrates such a phenomenon: *a very small noise signal is strongly amplified when the derivative action is applied.* A classical solution to such problems is to add a low-pass filter (Michiels, 2022) to the derivative action. Alternatively, another solution to the above problem is to approximate the derivative action by intentionally introducing a single delay in the measured signal (see, for instance (Ramírez et al., 2021; Jin et al., 2018, 2017)). The study of time-delay based controllers is not new, see for example Villafuerte et al. (2012); Villafuerte-Segura et al. (2019); Ochoa-Ortega et al. (2020). Such tools have demonstrated to be both powerful and useful, since, as mentioned in Appeltans et al. (2022) and Méndez-Barrios et al. (2021), there exist cases where neither the pure derivative action nor the low-pass filter implementation stabilize the system, but the time delay approximation does. However, there are some restrictions on its implementations. On the one hand, the approximations using a single time-lag needs the delay τ to be very small in order to correctly reproduce the derivative, but it is not a trivial task to induce such a small delay, thus, one may increase the precision of the approximation by adding more delays to it (as one might expect, increasing the number of delays may also produce a noise amplification effect, since we are obtaining an operator that is *closer* to the ideal derivative action) (Ackleh et al., 2009). On the other hand, the use of this kind of approximation bears a characteristic equation with coefficients that explicitly depend on the delay τ. Such a characteristic equation may have solutions exhibiting singular behavior, and in certain cases, such a solution

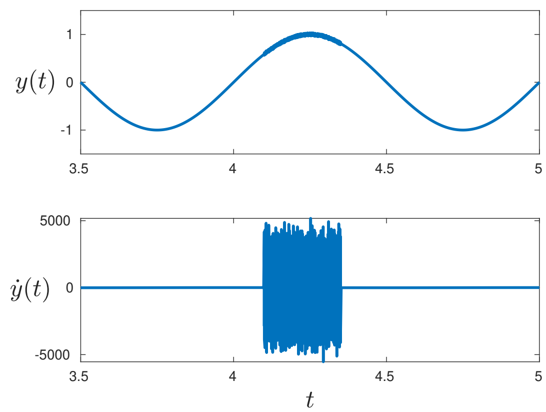

Fig. 1. Derivative effect on signal with small noise.

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can be an unstable one. It is also worth mentioning that the use of delays may induce essential problems due to the sensitivity of the system to such a class of parameter (see Logemann et al. (1996) and Engelborghs et al. (2001)).

In the light of these observations, it is of special importance to characterize the solutions of systems under the action of controllers for which the derivative action has been approximated using time-delays. That, in the simplest case (one delay only), is

$$\dot{y}(t) \approx \frac{y(t) - y(t-\tau)}{\tau} \quad (1)$$

Observe that considering approximation (1) generates a time-delay system with parameters that are also delay-dependent. While there have been several studies of stability of time delay systems (see for instance Niculescu (2001); Sipahi et al. (2011); Michiels and Niculescu (2007)), the analysis of systems with delay-dependent parameters is not common. It is, however, a very common practice to study stability in terms of the system parameters. In particular the $\tau-$decomposition method presented in Lee and Hsu (1969) studies stability in terms of the system's delay, however, it does not consider that the system parameters explicitly depend on such delay.

The case of the approximation (1) of the derivative action has already been investigated on Méndez-Barrios et al. (2021). There, the concept of an improperly-posed closed-loop system is presented and studied by means of the Newton Diagram method. In Sipahi et al. (2006), the authors also consider the approximation of the derivative action, but the presented analysis is based on a geometric approach.

As it has already been mentioned, this methodology allows to reduce the noise amplification in a very effective way, but the derivative may not be precise enough. One might obtain a more precise approximation to the derivative by considering more points. Consider, for example, the three-points approximation, which is obtained via the finite-difference method (see for instance LeVeque (2007))

$$\dot{y}(t) \approx \frac{1}{2\tau}[3y(t) - 4y(t-\tau) + y(t-2\tau)]. \quad (2)$$

Indeed, as we increase the number of points, we get a more precise approximation, as shown in Figure 2.

Despite the benefits of using approximations such as those given by (1) and (2), another problem may arise. In point of fact, there might be cases where a controller with an ideal derivative action stabilizes the system but, for the same set of control gains, the approximation produces an unstable behavior on the system. A system for which this situation arises is known as an improperly-posed system.

The main contribution of this paper is twofold. First, to extend a previous result where a one delay approximation is considered to the case of a two delays approximation. Such a case is of interest since adding more points to the approximation allows to obtain a good approximation to the derivative action without the need of a *very small* delay, which is not easy to implement. Another important contribution of the present note is to study the singular behavior of some roots of the characteristic quasi-polynomial when implementing the derivative action by means of a discretization. In this vein, some properties of the auxiliary

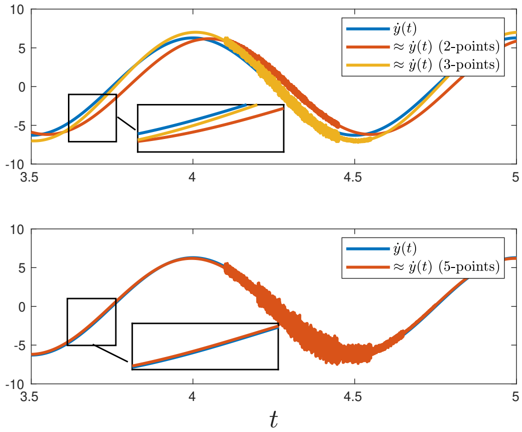

Fig. 2. Approximation of the derivative with different number of points. The pure derivative action is presented with no noise in order to depict the accuracy of each approximation. The five points case was added to the image in order to depict how adding points brings a better approximation to the derivative, but at the same time produce a bigger noise amplification. The zoom presented in both graphs shows how there exist a bigger gap between the derivative and the approximation when less points are considered.

quasi-polynomial, that were not previously shown in the study of simpler cases of the approximation, are presented. To the best of the authors' knowledge, does not exist similar contributions in the open literature devoted to such topics.

The remaining of this article is organized as follows: Section 2 introduces some preliminary results and definitions, as well as the problem formulation. The characterization and analysis of the improperly posed case for the three points approximation of the derivative action is presented in Section 3. In Section 4 some numerical examples are included. Finally, some concluding remarks are presented in Section 5.

Notations: Throughout this paper, the following standard notations are used: $\mathbb{C}$ represents the set of complex numbers. The set of real numbers is denoted by $\mathbb{R}$. Finally, $\deg(f)$ represents the degree of the polynomial f.

2. PRELIMINARY RESULTS AND PROBLEM FORMULATION

Consider the class of *strictly proper* LTI SISO system described by the following transfer function:

$$H_{yu}(s) := \frac{P(s)}{Q(s)}, \quad (3)$$

where Q and P are polynomials in s, with real coefficients:

$$Q(s) := s^n + \sum_{j=0}^{n-1} q_j s^j, \quad P(s) := \sum_{j=0}^{m} p_j s^j, \quad q_j, p_j \in \mathbb{R}, \quad (4)$$

such that $n := \deg(Q) > m := \deg(P)$. The control scheme considered in this paper, is the classical PD-feedback law with the following structure:

$$u(t) = -k_p y(t) - k_d \dot{y}(t), \quad (5)$$

where $(k_p, k_d) \in \mathbb{R}^2 \setminus \{(0,0)\}$. In frequency-domain, the corresponding PD-controller will be denoted by C_0 and given by

$$C_0(s) := -k_p - k_d s. \tag{6}$$

In the sequel, we denote by $Stab(H)$ the set of all stabilizing controllers and assume that this set is nonempty.

Consider now the closed-loop system (3) & (6) with the characteristic function $\Delta : \mathbb{C} \mapsto \mathbb{C}$ given by:

$$\Delta(s) := Q(s) + (k_p + k_d s)P(s). \tag{7}$$

Under the assumption that $C_0 \in Stab(H)$, the interest is to find at least one pair $(k_p, k_d) \in \mathbb{R}^2$ such that (7) has all of its roots located in the left-hand side of the complex plane. As it has already been pointed out in Section 1, due to the different problems associated to the derivative operator, one may consider an approximation for which the frequency-domain controller writes as:

$$C_\tau(s) = -k_p - k_d \frac{3 - 4e^{-\tau s} + e^{-2\tau s}}{2\tau}. \tag{8}$$

In this vein, consider the following definition from Méndez-Barrios et al. (2021):

Definition 2.1. (Improperly-posed system). Consider the LTI SISO system with frequency domain representation given by the transfer function (3). Suppose that C_0 with the form (6) is a stabilizing controller $C_0 \in Stab(H)$ and is replaced by C_τ given in (8). If there exists a sequence of real numbers $(\tau_n)_{n\in\mathbb{N}}$, $\tau_n \to 0^+$ when $n \to \infty$ such that for all $\epsilon > 0$, there exists some positive integer n_u, with $\tau_{n_u} < \epsilon$ and $C_{\tau_{n_u}} \notin Stab(H)$ the controller C_τ is called an improperly-posed controller for "small" delays. In this case, the closed-loop system is improperly posed [1].

2.1 Motivating example

Consider the first-order dynamical system given by the scalar differential equation:

$$\dot{y}(t) = p_0 y(t) + u(t), \tag{9}$$

where $u(t)$ is a classic PD controller given by

$$u(t) = -k_p y(t) - k_d \dot{y}(t).$$

Consider now both the three and two "points" (two and one delay respectively) approximation previously mentioned. With $\tau \to 0$, the closed-loop characteristic function associated to the three points approximation is given as

$$\Delta_1(s;\tau) = (s - p_0) - \left(k_p + \frac{k_d}{2\tau}(3 - 4e^{-\tau s} + e^{-2s\tau})\right),$$

while the one linked to the two points approximation writes as:

$$\Delta_2(s;\tau) = (s - p_0) - \left(k_p + \frac{k_d}{\tau}(1 - e^{-s\tau})\right).$$

Both characteristic functions can be respectively rewritten as follows:

$$s\hat{f}_1(s;\tau) = s\left[(1 - \frac{p_0}{s}) - \frac{k_p}{s} - \frac{k_d}{2s\tau}(3 - 4e^{-\tau s} + e^{-2s\tau})\right],$$

$$s\hat{f}_2(s;\tau) = s\left[(1 - \frac{p_0}{s}) - \frac{k_p}{s} - \frac{k_d}{s\tau}(1 - e^{-s\tau})\right],$$

where $\hat{f}_j(s;\tau) = f_0(s) - f_j(s\tau)$, $j \in \{1,2\}$ with the functions $f_0(s)$ and $f_j(s\tau)$ defined as:

$$f_0(z) = 1 - \frac{p_0 + k_p}{z}$$

and

$$f_1(z) = k_d \frac{(1 - e^{-z})(3 - e^{-z})}{2z}; \quad f_2(z) = k_d \frac{(1 - e^{-z})}{z}.$$

It is easy to observe that s_0 such that $f_0(s_0) = f_j(s_0\tau)$ is a root of Δ_j. Considering that $k_d > 1$ and $p_0 + k_p > 0$ (conditions for the stability when the PD controller stabilizes the system free of delay, that is, $\tau = 0$) there exists a root located in the right-hand plane that persists for any positive value of τ, as depicted on Figure 3.

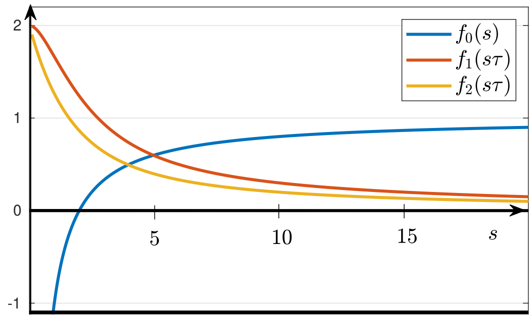

Fig. 3. Plot of $f_0(s)$ and $f_j(s\tau)$.

Remark 2.1. Figure 3 shows that even though the stability conditions for the ideal derivative implementation case are fulfilled, there is always a positive root for both approximation cases. This follows from the fact that there exist a point s_0 where $f_0(s_0) = f_j(s_0\tau)$, meaning that s_0 is a solution of $\Delta_j(s;\tau)$ for $j \in \{1,2\}$. It is worth mentioning that Fig.3 shows only positive values of s.

Observe that, as we increase the number of points used in the approximation, the crossing point between the functions f_0 and $f_{j\tau}$ moves to the right. This observation suggests that, regardless of the number of points considered for the derivative action approximation, the singular root may persist.

The main objective of this note is to characterize the singular unstable root presented whenever a system subjected to a PD controller is improperly-posed. The main tool to achieve such an objective is known as the Newton Diagram method which, as presented in the next section, allows us to find an adequate change of variable, in order to regularize the singular root.

3. ASYMPTOTIC BEHAVIOR ANALYSIS

3.1 Improperly-posed case characterization

As it has been pointed out in the previous sections, the interest of this work is to characterize a singular unstable root. For such a task consider the closed-loop characteristic function given as

$$\Delta(s;\tau) = Q(s) + \left(k_p + k_d\left(\frac{1 - e^{-\tau s}}{\tau}\right)\left(\frac{3 - e^{-\tau s}}{2}\right)\right)P(s). \tag{10}$$

Inspired by the methodology presented in Simmonds and Mann Jr (1998), we seek for a convenient change of variable that allows observing the singularity presented in a certain solution s^* of (10). In this vein, a tool that helps to choose the adequate change of variable is the Newton's Diagram

[1] for "small" delays

(readers may refer to (Baumgärtel, 1985; Walker, 1978; Krantz and Parks, 2002) for a deep explanation of such a tool). First, consider the following expansion of (10) into a power series

$$\Delta(s;\tau) = Q(s) + (k_p + k_d \sum_{j=1}^{\infty} a_j s^j \tau^{j-1}) P(s), \quad (11)$$

where,

$$a_j = -\frac{(-1)^j 2 + (-1)^{j+1} 2^{j-1}}{j!}.$$

To simplify the problem, we may want to consider rewriting (11) as $\tilde{\Delta}(s;\tau) = \tau\Delta(s;\tau)$. The Newton's Polygon of $\tilde{\Delta}(s;\tau)$ can be obtained using the algorithm presented in Martínez-González et al. (2019).

Proposition 3.1. For an improperly-posed system, the characteristic function (11) has a singular solution given by

$$s^*(\tau) = \frac{1}{\tau}\lambda^*(\tau), \quad (12)$$

where $\lambda(\tau)$ is an analytic function, such that $\lambda(0) \neq 0$.

Proof. The Newton's Polygon gives the information to construct a power series for a solution of the form:

$$s(\tau) = \mu_{\epsilon_1}\tau^{\epsilon_1} + \mu_{\epsilon_2}\tau^{\epsilon_2} + \ldots$$

where $\epsilon = -\beta$ is given by the slopes on the Newton's Polygon of the equation for which we are finding roots. Applying the algorithm to (11) we obtain Figure 4.

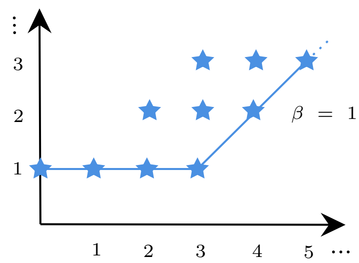

Fig. 4. Newton's polygon of quasi-polynomial (11).

One can observe that there exists a constant slope identified by the value of 1. ∎

Taking into consideration the above proof along with Figure 4, we may consider the change of variable $s = \tau s$ obtaining

$$\tau^{-1}\tilde{\Delta}(\tau s;\tau) = Q(\tau s) + (k_p + k_d \sum_{j=1}^{\infty} a_j s^j \tau^{2j-1}) P(\tau s). \quad (13)$$

Observe that such a change of variable regularizes the problem since $s^*\tau = \lambda^*(\tau)$, which presents no-singular behavior.

Now that the problem has been regularized, consider the following auxiliary quasi-polynomial together with Proposition 3.2

$$p_a(w) := \lim_{\tau\to 0^+} \tilde{\Delta}(w;\tau), \quad (14)$$

which is

$$p_a(w) = w^{n-1}\left[w + \frac{1}{2}(1 - e^{-w})(3 - e^{-w})k_d p_{n-1}\right], \quad (15)$$

where n is the degree of $Q(s)$ and p_{n-1} is the corresponding coefficient to s^{n-1} on P.

Proposition 3.2. For a quasi-polynomial of the form (11) at $\tau \to 0_+$, the singular root s^* is given by

$$s^*(\tau) = \frac{1}{\tau}z^* + \mathcal{O}(1),$$

where z^* is a solution of the auxiliary quasi-polynomial p_a

Remark 3.1. It is easy to observe that $w = 0$ is always a solution of (15).

Remark 3.2. In the case of transfer function having the relative degree equal to one, some simple, but tedious algebraic manipulations allow concluding that the closed-loop system is properly-posed as long as $| k_d p_{n-1} | < 1$. Such a condition was proposed in Appeltans et al. (2022) by using a different argument. It should be noted that such a *sufficient* (inequality) condition is *not necessary* for guaranteeing that the corresponding closed-loop system is properly posed (see, for instance, the discussions in Méndez-Barrios et al. (2021) in the case of a simpler approximation scheme).

4. NUMERICAL EXAMPLES

Through this section, numerical examples are presented to emphasize the methodology proposed in this work.

4.1 Example 1: Second order improperly-posed system

Consider the system described by the transfer function given as

$$H(s) = \frac{-s}{s^2 - \omega_0^2}. \quad (16)$$

We know that the system is improperly-posed for the gains $(k_p, k_d) = (1, 2)$. Following the procedure presented in this work, the auxiliary quasi-polynomial is given by:

$$p_a(w) = 4e^{-w} - e^{-2w} - 3 + w.$$

Rewriting $p_a(w)$ allows to observe that, as remarked before, there exists a root at $\omega = 0$

$$p_a(w) = w + (1 - e^{-w})(e^{-w} - 3).$$

If the system is improperly-posed then $p_a(w)$ must have at least one positive solution. Observe that for $p_a(\infty^+) = \infty$ and, since $w = 0$ is a solution, it suffices to show that for certain values of w, $p_a(w) < 0$. In this case, evaluating the derivative of p_a at $w = 0$ shows that at that point the function is decreasing: $\dot{p}_a(0) = -2$. Finally, solving p_a using the QPmR algorithm (see Vyhlídal and Zítek (2014)) the real root z^* of p_a is given by the value $z^* = 2.7478$, which means that for small delays the characteristic function of the system has a singular root

$$s^*(\tau) = \frac{2.7478}{\tau} + \mathcal{O}(1).$$

Taking for example $\tau = 0.1$, the singular root will be located around $s^*(0.1) \approx 27.4$, which is very close to its real value which is $s^*(0.1) = 28.9$.

Figure 5 depicts the behavior of the system roots as the value of τ decreases. Observe that the rightmost root moves to the right as $\tau \to 0$.

4.2 Example 2: Third order improperly-posed system

Consider the system described by the following transfer function

$$H(s) = \frac{1 - 5s^2}{s^3 + 8s^2 - 13s - 8}. \quad (17)$$

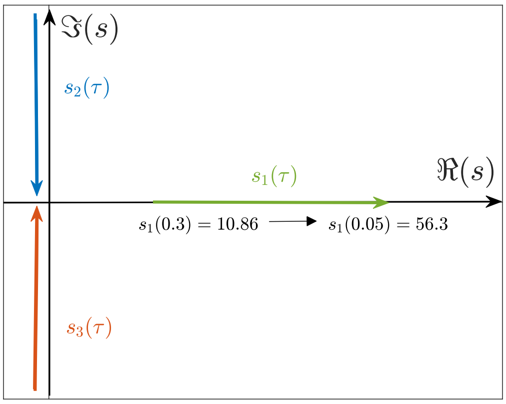

Fig. 5. Roots behavior of system (16) as τ moves from 0.3 to 0.05.

Under the action of a PD controller with parameters $(k_p, k_d) = (4, 1)$ the system in closed-loop is stabilized. However, it is improperly posed. Its corresponding auxiliary quasi-polynomial is given by

$$p_a(w) = -\frac{5}{2}e^{-2w} + 10e^{-w} + w - \frac{15}{2}.$$

Similarly to the last example, rewriting $p_a(w)$ as

$$p_a(w) = w + \frac{1}{2}(1 - e^{-w})(5e^{-w} - 15).$$

Following a similar procedure to the one presented in the previous example, we observe that $p_a(\infty^+) = \infty$, and $\dot{p}_a(0) = -8$. Indicating that the system is improperly-posed. Using the already mentioned QPmR algorithm we find a solution of p_a at $z^* = 7.4944$, meaning that there exists a singular root of the characteristic function given by

$$s^*(\tau) = \frac{7.4944}{\tau} + \mathcal{O}(1).$$

Figure 6 shows the behavior of the rightmost root $s^*(\tau)$ as τ decreases its value. It can be observed that it moves to the right.

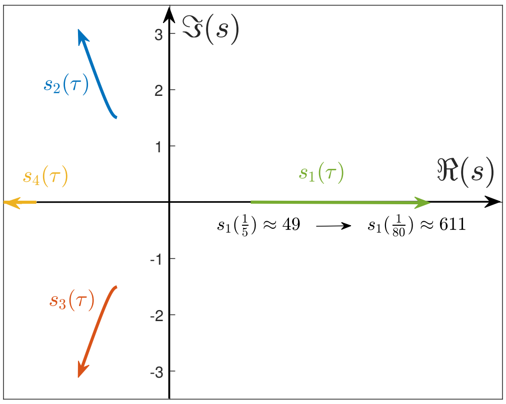

Fig. 6. Rightmost root behavior of system (17) for decreasing τ.

4.3 Example 3: Properly-posed system

The previous examples shown the behavior of the system roots when the system is improperly-posed. However, a system that is improperly-posed for a certain pair (k_p, k_d) may be properly-posed for some other pair. In this vein, for a last example, consider the system described by the following transfer function

$$H(s) = \frac{-2s}{s^2 + 3s + 1}, \tag{18}$$

subjected to the PD controller with gains $(k_p, k_d) = (1, \frac{1}{3})$. It is clear that such gains suffice to stabilize the closed-loop system. Now, considering the two-delays approximation of the derivative action, the auxiliary quasi-polynomial is given by

$$p_a(w) = w - \frac{1}{3}(e^{-w} - 3)(e^{-w} - 1).$$

Similarly to the previous examples, $p_a(w)$ has a solution at the origin. Derivating w.r.t. w we obtain:

$$\frac{dp_a(w)}{dw} = 1 + \frac{2}{3}e^{-2w} - \frac{4}{3}e^{-w},$$

which is positive for all positive w. This means that there are not positive roots, and thus, the singular root is placed on the LHP. Figure 7 shows the behavior of the roots placed more to the right of the system (18).

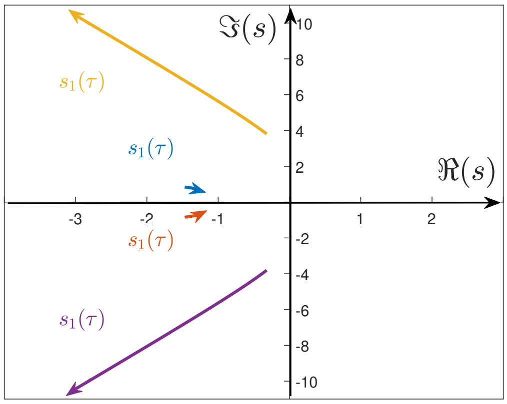

Fig. 7. Roots behavior of properly-posed system (18).

5. CONCLUSIONS

In this note, we studied quasi-polynomials with delay-dependent coefficients. Such dependency comes from an approximation of a derivative action. The main objective was to analyze the case when the quasi-polynomial is improperly-posed. In such a case, there exist unbounded (singular) roots for $\tau \to 0$. We analyze this problem by expressing the singular solution into a power series, and regularizing it by applying an adequate change of variable.

Future work might be focused on finding conditions under which the system is improperly-posed in the general case, that is, for an approximation of the derivative action via n-points discretization, as well as determining the ideal number of delays depending on the application. It is also of interest to study the behavior of the approximation when a higher order derivative is implemented. Indeed, as seen in Sahib (2015) and Saab and Jaafar (2021), some engineering problems require derivative action of second or higher order and, for such actions, the same problems associated to this operator appear. Future research may also consider the analysis of a system with a delayed input subjected to a PD controller for which the derivative action has been approximated by means of time-delays.

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