A Complete Characterization of Minima of the Spectral Abscissa and Rightmost Roots of Second-Order Systems with Input Delay

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A COMPLETE CHARACTERIZATION OF MINIMA OF THE SPECTRAL ABSCISSA AND RIGHTMOST ROOTS OF SECOND-ORDER SYSTEMS WITH INPUT DELAY

WIM MICHIELS*, SILVIU-IULIAN NICULESCU†, AND ISLAM BOUSSAADA‡

Abstract. The numerical minimization of the spectral abscissa function of linear time-invariant time-delay systems, an established approach to compute stabilizing controllers with a fixed structure or dimension, often gives rise to minima characterized by active characteristic roots with multiplicity higher than one. At the same time, recent theoretical results reveal situations where the so-called multiplicity induced dominancy (MID) property holds, i.e., a sufficiently high multiplicity implies that the root is dominant, leading to designs based on directly assigning multiple roots. Using an integrative approach, combining analytical characterizations, computation of characteristic roots and numerical optimization, a complete characterization of the stabilizability of second-order systems with input delays is provided, for both state feedback and delayed output feedback with two terms in the control. The level sets of the minimal achievable spectral abscissa are also characterized. These results shed light on the complex relations between (configurations involving) multiple roots, the property of being dominant roots, and the property of corresponding to (local/global) minimizers of the spectral abscissa function.

Key words. Time-delay systems, stability optimization, spectral abscissa, multiplicity-induced dominancy, stabilizability

AMS subject classifications. 93B25, 93B40, 93B52, 93C23, 93D15

1. Introduction.

In this paper we perform a semi-analytical, semi-computational study of the limitations of feedback delay on the stabilizability of linear time-invariant systems using state and delayed output feedback, motivated by and building on three complementary developments in the area of spectrum based analysis and control.

First, in order to design stabilizing controllers with a prescribed structure or dimension, methods and algorithms have been proposed that rely on the direct minimization of the spectral abscissa as a function of controller parameters, see, e.g., [22, 23, 15, 8]. These methods have been implemented in the software tool `tds_stabil`, corresponding to article [15], and integrated in the software package `TDS-CONTROL` [1]. Even for retarded type models in which system matrices smoothly depend on controller parameter, the spectral abscissa function is typically non-smooth at parameter values for which there are multiple rightmost characteristic root (in the upper half plane), hence, its minimization requires dedicated optimization algorithms. The spectral abscissa function may even fail to be locally Lipschitz continuous due to the presence of multiple defective rightmost characteristic roots, a situation often observed in local minimizers [22]. The precisely configuration of rightmost roots corresponding to minima is however difficult to predict a priori (i.e., without executing the optimization algorithms) as it strongly depends on the model structure and system parameters.

Second, for various classes of systems whose characteristic equation is affine in the controller parameters, situations have been recently characterized in which the multiplicity-induced dominancy (MID) property holds, in the sense that that a suffi-

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ciently high multiplicity of a characteristic roots implies that it must be a rightmost root, see, e.g., [6, 19, 14, 13, 3] and the references therein. This property has inspired analytic design approaches based on directly assigning multiple roots using the available controller parameters [19]. The technical arguments used in the proof of the dominancy, however, do not allow to make direct assertions about local or global minimality of the spectral abscissa function.

Third, recent contributions in eigenvalue perturbation theory concern the characterization of the splitting behavior of multiple characteristic roots as a function of parameter changes [16, 19]. For example, Theorem 2.2 of [16] implies that if there is only one real or one pair of of rightmost characteristic roots with geometric multiplicity larger than two and algebraic multiplicity one, the spectral abscissa will strictly increase whenever some non-degeneracy condition with respect to the adopted parameter variation is satisfied (implying the property of a completely regular splitting). Such results suggest relations between multiple rightmost characteristic roots and local minimizers of the spectral abscissa function.

All these considerations bring us to the following research questions.

1. What are the possible configurations of rightmost characteristic roots in the minima of the spectral abscissa function (counting multiplicity), and how do they depend on the characteristics of the uncontrolled plant, on the controller structure and on feedback delay?
2. Under which conditions do particular characteristic root configurations that can be directly imposed by an appropriate choice of controller parameters, correspond to rightmost roots?
3. If this is the case, under which conditions do they correspond to local or global minimizers of the spectral abscissa function?

These questions turn out not to be have simple answers, even when significantly restricting the class of systems under consideration. To shed light on the complexity of the problems and provide insight in the relations (multiple root, dominant root, minimizer of the spectral abscissa), the article presents a complete characterization of the stabilizability and of the mimima of the spectral abscissa function for the SISO controllable second-order system with input delay τ,

$$\dot{x}(t) = Ax(t) + Bu(t-\tau), \quad x(t) \in \mathbb{R}^2, \tag{1.1}$$

where we assume that the model is already in controller canonical form,

$$A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and we consider two classes of control laws. The first class corresponds to instantaneous state feedback,

$$u(t) = Kx(t), \tag{1.2}$$

with $K = \begin{bmatrix} -k_2 & -k_1 \end{bmatrix}$. What concerns the second class, we will assume that only output $y(t) = Cx(t)$ is available for measurement, with $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and will consider the control law

$$u(t) = -k_1 y(t) - k_2 y(t-\tau). \tag{1.3}$$

With the selected output, PD control corresponds to state feedback, and control law (1.3) might be motivated by using a finite-difference approximation of the output derivative. Note that control law (1.3) is an example of a so-called propertional-retarded controller (see, e.g., [21]).

The stabilizability and minima of the spectral abscissa of the closed-loop system will be characterized as a function of the *three plant parameters* (a_1, a_2, τ). For the plant description, we will use, besides the couple (a_1, a_2), the eigenvalues of matrix A,

$$\lambda_{1,2} = \begin{cases} -a_1 \pm \sqrt{a_1^2 - 4a_2}, & a_1^2 - 4a_2 \geq 0, \\ -a_1 \pm \sqrt{4a_2 - a_1^2}\, \imath, & a_1^2 - 4a_2 < 0, \end{cases}$$

where for λ_1 the plus-sign is selected, as well as the pair $(\lambda_{\text{av}}, \lambda_{\text{diff}})$, defined through

$$\lambda_{\text{av}} = \frac{\lambda_1 + \lambda_2}{2}, \quad \lambda_{\text{diff}} = \frac{\lambda_1 - \lambda_2}{2}.$$

At various places, we will make connections with the literature for special instances of the plant parameters.

From a methodological point of view, computing stabilizability regions in a plant parameter space is more challenging than computing stability regions in (any) parameter space. In the former case, one namely needs to determine plant parameters for which the global minimum of the spectral abscissa in function of the controller parameters is negative. In the latter case, one is looking for parameters (of the closed-loop system) for which the spectral abscissa itself is negative. To tackle the stabilizability analysis, we used a mixed analytic, computational approach as in [18, 11, 2]. More specifically, we rely on numerical optimization and eigenvalue computations (using the method of [15]) to characterize characteristic root configurations, along with numerical continuation, supplemented by analytic/symbolic computations.

The structure of the paper is as follows. In Section 2 we characterize the stabilizability of (1.1) as well as the level sets of the minimal spectral abscissa, using state feedback (1.2). In Section 3 we address delayed output feedback (1.3). In Section 4 we briefly comment on imposing maximal multiplicity of rightmost roots in case both gains and delay are available as controller parameters, and in Section 5 we present some concluding remarks. Finally, in the appendix we elaborate on the methodology to compute the boundary between stablizable and non-stabilizable systems in the relevant plant parameter space.

2. State feedback. With control law (1.2), the characteristic function of the closed-loop system is given by

$$H(\lambda;\ k_1, k_2, a_1, a_2, \tau) = \lambda^2 + (a_1 + k_1 e^{-\lambda\tau})\lambda + a_2 + k_2 e^{-\lambda\tau}. \tag{2.1}$$

Note that in the adopted notation for functions, a dot comma separates arguments and parameters (optional, to stress the dependence). The closed loop system is exponentially stable if and only if the spectral abscissa α is negative,

$$\alpha(k_1, k_2;\ a_1, a_2, \tau) := \sup_{\lambda \in \mathbb{C}} \left\{ \Re(\lambda) : H(\lambda;\ k_1, k_2, a_1, a_2, \tau) = 0 \right\}.$$

A characteristic root whose real part is equal to the spectral abscissa is called an *active* characteristic root. The closed loop system is exponentially stabilizable if and only if

$$\left(\inf_{(k_1,k_2) \in \mathbb{R}^2} \alpha(k_1, k_2;\ a_1, a_2, \tau) \right) < 0.$$

Based on the concept of convex directions of quasi-polynomials, the following important property of the spectral abscissa can be proven.

PROPOSITION 2.1. *The function*

$$(k_1, k_2) \mapsto \alpha(k_1, k_2;\ a_1, a_2, \tau),$$

corresponding to (2.1), is quasi-convex.

Proof. Let c be such that the sublevel set $\mathcal{R}_c := \{(k_1, k_2):\ \alpha(k_1, k_2) \leq c\}$ is not empty. In what follows, we show that this set is convex. The substitution $\lambda \leftarrow \lambda - c$ makes clear that $\mathcal{R}_c$ is the stability region of quasi-polynomial

$$p(K_1, K_2) := \lambda^2 + (a_1 + 2c)\lambda + (a_2 + c^2) + (K_1\lambda + K_2)\, e^{-\lambda\tau}, \tag{2.2}$$

where

$$K_1 = k_1 e^{-c\tau},\ K_2 = (k_2 + k_1 c)e^{-c\tau}. \tag{2.3}$$

Assume now that (2.2) is stable for $(K_1, K_2) = \left(K_1^{(i)}, K_2^{(i)}\right)$, $i \in \{1, 2\}$. From [12, Lemma] it follows that any convex combination of these gains is stabilizing if

$$\frac{d \arg(g(\imath\omega))}{d\omega} \leq -\frac{\tau}{2} + \left|\frac{\sin(2\arg(g(\imath\omega))) + \omega\tau}{2\omega}\right|, \tag{2.4}$$

whenever the derivative is well defined, with g the "direction",

$$g(\lambda) := \left(\left(K_1^{(2)} - K_1^{(1)}\right)\lambda + \left(K_2^{(2)} - K_2^{(1)}\right)\right) e^{-\lambda\tau}.$$

From [10, Proposition 6] it follows that condition (2.4) is always satisfied for a direction of this form. Hence, the set of stabilizing (K_1, K_2)-values for (2.2) is convex. The proof is completed by noting that the linear transformation (2.3) preserves convexity. □

Hence, a convenient way to check stabilizability consists of numerically minimize the spectral abscissa as a function of the controller parameters using the continuous pole placement method [17], as in [18], or using the method of [15], which relies on non-smooth optimization. By Proposition 2.1, a computed strict local minimizer is a global minimizer. To keep the presentation of the stabilizability analysis simple, we will first assume that $\tau = 1$. In Section 2.3 we will remove that restriction.

2.1. Stabilizability for $\tau = 1$. The plant parameters under consideration are (a_1, a_2). For $c \in \mathbb{R}$ an important role will be given to the set

$$S_c = \left\{(a_1, a_2) \in \mathbb{R}^2 :\ \inf_{(k_1, k2) \in \mathbb{R}^2} \alpha(k_1, k_2;\ a_1, a_2) = c\right\}.$$

Note that stabilizable and unstabilizable plant in the (a_1, a_2)-parameter space are separated by the set $\mathcal{S}_0$.

Consider the line in the (a_1, a_2)-plane,

$$a_2 = -2a_1 - 2, \tag{2.5}$$

uniquely determined by imposing a triple characteristic root at zero, i.e.,

$$H(0;\ k_1, k_2, a_1, a_2) = H'(0) = H''(0) = 0, \tag{2.6}$$

and the curve

$$(a_1,a_2)=\begin{cases}\left(-\frac{\omega(2\omega-\sin(2\omega))}{\omega^2-(\sin(\omega))^2},\frac{\omega^2\left(\omega^2+(\sin(\omega))^2\right)}{\omega^2-(\sin(\omega))^2}\right), & \omega>0,\\ (-4,6), & \omega=0,\end{cases} \tag{2.7}$$

uniquely determined by imposing a pair of imaginary axis roots with multiplicity two,

$$H(\imath\omega)=H'(\imath\omega)=0, \tag{2.8}$$

followed by elimination of k_1, k_2. Note that the curve and line intersect at $(a_1,a_2)=(-4,6)$, which can be directly obtained by imposing $H(0)=H'(0)=H''(0)=H'''(0)=0$.

In [18] it has been shown that the set $\mathcal{S}_0$ for the considered state feedback satisfies

$$\mathcal{S}_0=\left\{(a_1,a_2)\in\mathbb{R}^2:\ \text{either (2.5) holds for } a_1\geq -4,\ \text{or (2.7) holds for some } \omega\geq 0\right\}.$$

Furthermore, if $(a_1,a_2)\in\mathcal{S}_0$ and (2.5) holds for $a_1\geq -4$ (what we will refer to as lying on the *line segment* of $\mathcal{S}_0$), the (global) minimum of the spectral abscissa is characterized by a zero root with multiplicity three. If $(a_1,a_2)\in\mathcal{S}_0$ and (2.7) holds (what we will refer to as lying on the *curved segment* of $\mathcal{S}_0$), then the minimum of the spectral abscissa is characterized by a pair of imaginary axis roots $\pm\imath\omega$ of multiplicity two. As a consequence, the multiple roots induced by (2.6) and (2.8) are then active, i.e., the dominant roots. In Figure 2.1, we display the set $\mathcal{S}_0$. Note that if (2.5) holds with $a_1<-4$, a zero root with multiplicity three can be imposed, but since the system is not stabilizable, this triple root is not dominant.

We now characterize the set S_c, with $c\neq 0$. By substituting $\mu=\lambda-c$, the characteristic equation of the closed-loop system becomes

$$\mu^2+(a_1+2c)\mu+(a_2+a_1c+c^2)+k_1e^{-c}\mu e^{-\mu}+(k_2+k_1c)e^{-c}e^{-\mu}=0.$$

Note that the controller dependent gains, k_1e^{-c} and $(k_2+k_1c)e^{-c}$, can still be chosen independently of each other. Hence, it holds that

$$(a_1,a_2)\in\mathcal{S}_c\ \Leftrightarrow\ \left(a_1+2c,a_2+a_1c+c^2\right)\in\mathcal{S}_0, \tag{2.9}$$

inducing

$$\mathcal{S}_c=\left\{\left(a_1-2c,a_2-a_1c+c^2\right):\ (a_1,a_2)\in\mathcal{S}_0\right\}. \tag{2.10}$$

In Figure 2.2 we display the set $\mathcal{S}_c$ for different values of c. Inherited from $\mathcal{S}_0$, the line and curved segment correspond to a triple rightmost root at c, and to a pair of rightmost complex conjugate roots with real part c and multiplicity two, respectively, as a characteristic of the minimum of the spectral abscissa function.

At the transition between the line and curved segment of $\mathcal{S}_c$, the global optimum of the spectral abscissa function is characterized by a real rightmost root c with multiplicity four. From (2.9) this situation occurs if and only if

$$(a_1+2c,a_2+a_1c+c^2)=(-4,6).$$

Eliminating c leads us to the quadratic function

$$a_2=\frac{1}{4}a_1^2+2. \tag{2.11}$$

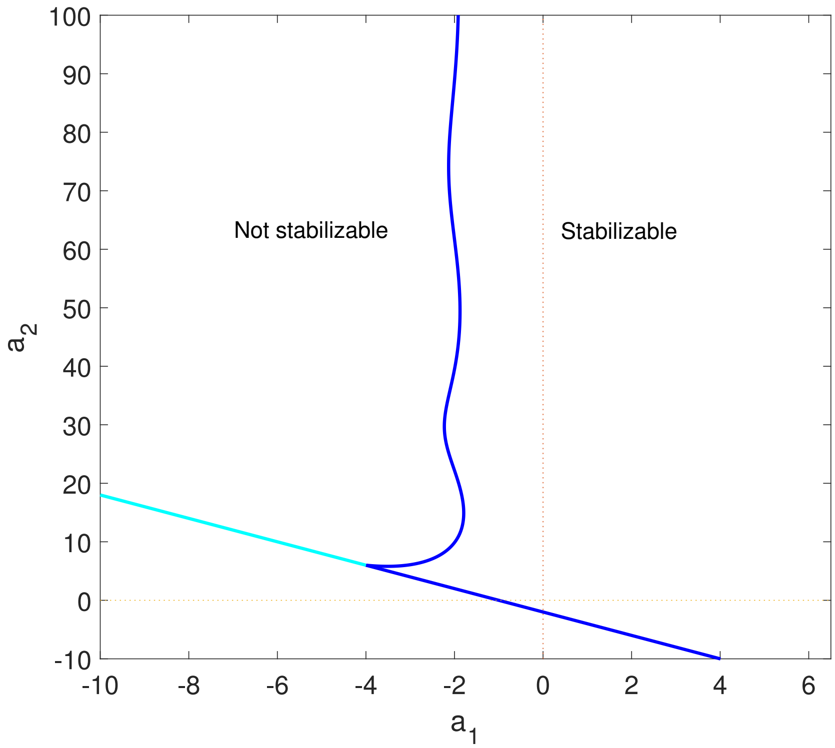

FIG. 2.1. *The regions in the (a_1, a_2)-parameter space for which the plant is stabilizable is bounded by the set $\mathcal{S}_0$, indicated in blue. On its line segment, the global minimum of the spectral abscissa is determined by a real root at zero with multiplicity three, on the curved segment by a complex pair of imaginary axis roots with multiplicity two. For (a_1, a_2) on the cyan line segment, an assignment of a triple root at zero is possible, yet this root is not dominant.*

From (2.5) and (2.10), the line segment of $\mathcal{S}_c$ is described by

$$a_2 = -(2+c)a_1 - (2+4c+c^2), \quad a_1 \geq -4-2c, \tag{2.12}$$

whose slope in the (a_1, a_2)-plane, $-(2+c)$, reveals the following property, visualized also in Figure 2.2.

PROPOSITION 2.2. *The line segment of $\mathcal{S}_c$, described by (2.12), is tangent to (2.11) at its end point ($a_1 = -4-2c$).*

2.2. Multiple root assignment for $\tau = 1$. From Figure 2.2, it is clear that the curve (2.11) takes the role of separatrix. If $\underline{a_2 > \frac{1}{4}a_1^2 + 2}$, which can also be expressed as $|\Im(\lambda_i)| > \sqrt{2},\ i = 1, 2$, it is possible to assign a complex conjugate pair with multiplicity two. These multiple roots are rightmost roots and correspond to a minimum of the spectral abscissa function. It is however not possible to assign a triple real root.

If $\underline{a_2 < \frac{1}{2}a_1^4 + 2}$, or equivalently $|\Im(\lambda_i)| < \sqrt{2},\ i = 1, 2$, it is not possible to assign a complex conjugate pair with multiplicity two. However, there are two distinct possibilities to assign a triple real root, as visualized in Figure 2.3. Let $(\hat{a}_1, \hat{a}_2)$ be fixed plant parameters satisfying $\hat{a}_2 < \frac{1}{2}\hat{a}_1^2 + 4$. Invoking Proposition 2.2, we require

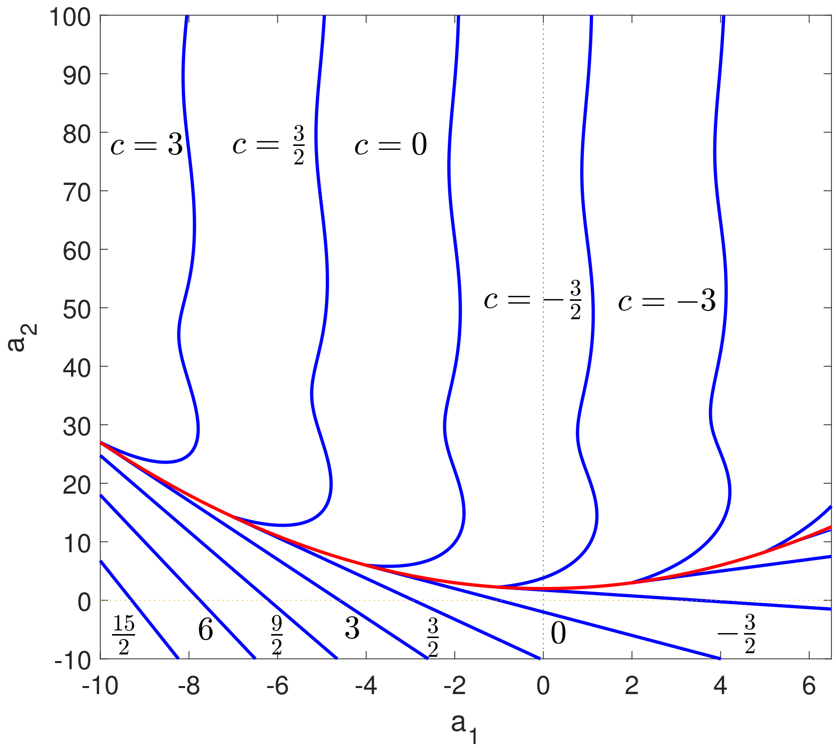

FIG. 2.2. *The set $\mathcal{S}_c$, for various values of c. The value of c corresponds to the minimal value of the spectral abscissa as a function of controller parameter (k_1, k_2). The red curve corresponds to the parabola (2.11).*

that a line, tangent to curve (2.11), passes through $(\hat{a}_1, \hat{a}_2)$, which leads us to the equation

$$(6 + 4c + c^2 - \hat{a}_2) = (-2 - c)(-4 - 2c - \hat{a}_1),$$

see Figure 2.3. The solutions for c are

$$c_{\pm} = -2 - \frac{\hat{a}_1}{2} \pm \sqrt{\frac{\hat{a}_1^2}{4} + 2 - \hat{a}_2},$$

which can also be expressed as

$$c_{\pm} = -2 + \lambda_{\text{av}} \pm \sqrt{2 + \lambda_{\text{diff}}^2}. \tag{2.13}$$

When assigning a triple root at $c = c_+$ the spectral abscissa is minimal and the root is dominant, since $(\hat{a}_1, \hat{a}_2) \in \mathcal{S}_{c_+}$. When assigning a triple root at c_-, it cannot be dominant, because the latter would be in contradiction with the minimal spectral abscissa $c_+ > c_-$.

Finally, if $a_2 = \frac{1}{4}a_1^2 + 2$, a real root with multiplicity four can be uniquely assigned. It is dominant and minimizes the spectral abscissa.

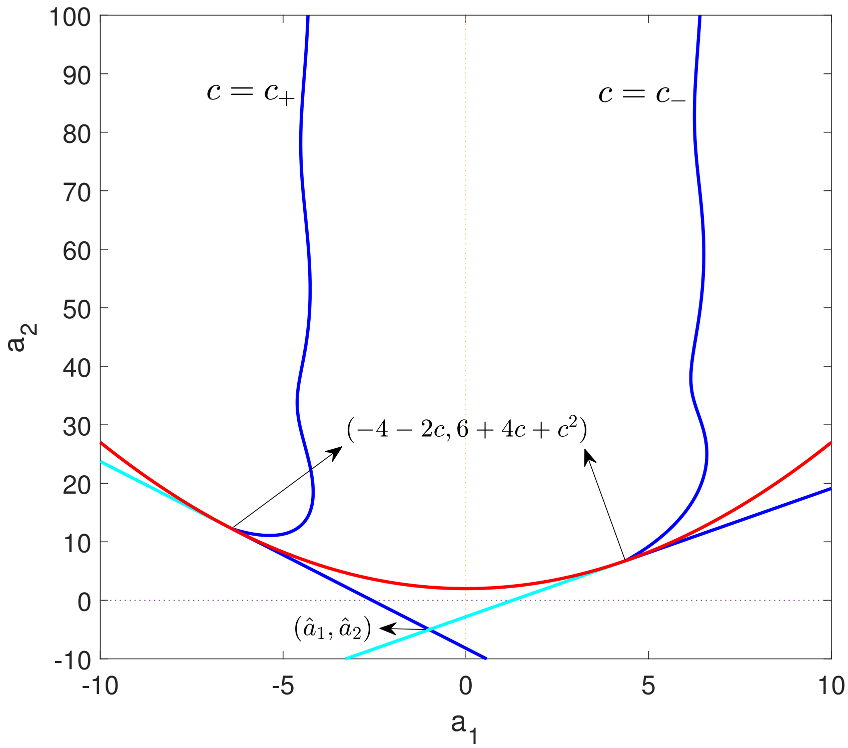

FIG. 2.3. *For plant parameters $(\hat{a}_1, \hat{a}_2)$, there are two possibilities to assign a triple real root. Assigning it to c_+ induces dominance and a minimum of the spectral abscissa function, Assigning it to c_- it is not dominant.*

2.3. Stabilizability and multiple root assignment for $\tau \neq 1$. Making the substitution $\lambda\tau = \nu$ in (2.1), which corresponds to a scaling of time, the characteristic equation becomes

$$\nu^2 + (a_1\tau + k_1\tau e^{-\nu})\nu + a_2\tau^2 + k_2\tau^2 e^{-\nu} = 0.$$

Once again, the controller dependent gains, $k_1\tau$ and $k_2\tau^2$, can be freely assigned in the stabilizability analysis. Hence, the results obtained in Sections 2.1-2.2 remain valid provided that the substitution

$$(a_1, a_2, c) \leftarrow \left(a_1\tau, a_2\tau^2, c\tau\right) \tag{2.14}$$

is made. In particular, we can conclude the following.

- If $|\Im(\lambda_i)| > \frac{\sqrt{2}}{\tau}$, $i = 1, 2$, the minimum of the spectral abscissa is characterized by a pair of complex conjugate roots of multiplicity two. Assigning a triple real root is not possible.
- If $|\Im(\lambda_i)| < \frac{\sqrt{2}}{\tau}$, $i = 1, 2$, it is possible to assign a triple real root at

$$c_\pm = -\frac{2}{\tau} + \lambda_{\text{av}} \pm \sqrt{\frac{2}{\tau^2} + \lambda_{\text{diff}}^2}.$$

The triple root at c_+ is dominant and corresponds to a minimum of the spectral abscissa. Assigning a complex conjugate pair with multiplicity two is not possible.

- If $|\Im(\lambda_i)| = \frac{\sqrt{2}}{\tau}$, $i = 1, 2$, a real root with multiplicity four can be assigned. It corresponds to the minimum of the spectral abscissa.

REMARK 2.3. *The degree of the quasipolynomial H is equal to 4. For a deeper discussion on such a notion, we refer to [13] and the references therein. In our case, it should be noted that this leads to a maximal multiplicity equal to 4 for a real root and equal to 2 for (strictly) complex-conjugate characteristic roots. In other words, the conclusions drawn above allow covering all these "limit" cases.*

REMARK 2.4. *Recently, it has been shown that a quasipolynomial satisfying the MID property admits a representation in terms of the* Kummer confluent hypergeometric functions*, see for instance [13, 3]. This connection leads to a further link with the well known* Padé approximation *as already pointed out in [4]. As a matter of fact, if the closed-loop quasipolynomial H introduced in* (2.1) *admits a root c_0 with multiplicity four*[1]*, then it is said to satisfy the* generic multiplicity induced multiplicity *(or GMID for short) and such a root is necessarily real and for every $\lambda \in \mathbb{C}$ one has:*

$$H(\lambda;\ k_1, k_2, a_1, a_2, \tau) = \tau^2\ (\lambda - c_0)^4 \int_0^1 t(1-t)^2 e^{(c_0-\lambda)\,\tau\, t} dt. \tag{2.15}$$

The last integral representation of the quasipolynomial allows to write H in terms of the Kummer function, as for every $\lambda \in \mathbb{C}$ one has:

$$H(\lambda;\ k_1, k_2, a_1, a_2, \tau) = \frac{\tau^2\ (\lambda - c_0)^4}{12}\, \Phi(2, 5, \tau\,(c_0 - \lambda)) \tag{2.16}$$

where the Kummer function $\Phi(a, b, \cdot) : \mathbb{C} \to \mathbb{C}$ is the entire function defined for $\lambda \in \mathbb{C}$ by the series

$$\Phi(a, b, \lambda) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{\lambda^k}{k!}, \tag{2.17}$$

where we recall that, for $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$, $(\alpha)_k$ is the Pochhammer symbol, see for instance [7, 9, 20]. So that, on beyond of the dominant root c_0, the closed-loop quasipolynomial H shares its remaining roots with $\Phi(2, 5, \tau\,(c_0 - \lambda))$.

Furthermore, as established in [5], if $|\Im(\lambda_i)| < \frac{\sqrt{2}}{\tau}$ and $\lambda = c_+$ is the triple H defined above, then there exists an $A \in \mathbb{R}$ such that

$$H(\lambda;\ k_1, k_2, a_1, a_2, \tau) = \tau(\lambda - c_+)^3 \int_0^1 (1-t)(1-At)e^{-t\tau(\lambda-\lambda_0)} dt.$$

In terms of Kummer functions, this gives precisely:

$$\begin{aligned} H(\lambda;\ k_1, k_2, a_1, a_2, \tau) = \tau\,(\lambda - c_+)^3\ [2\,(1 + \tau\, c_+)\,\Phi\,(1,\, 3,\, \tau\,(c_+ - \lambda)) \\ - (1 + 2\,\tau\, c_+)\,\Phi\,(1,\, 4,\, \tau\,(c_+ - \lambda))]\,. \end{aligned} \tag{2.18}$$

Thanks to this remarkable connection, as has been established in [3], one may exhaustively locate the spectrum of quasipolynomials satisfying the MID property by exploiting standard results on such special functions, see for instance [20, Chapter 13].

[1]In this case, as mentioned in the previous remark, four corresponds to the degree of the quasipolynomial

3. Delayed output feedback. In this section, we consider control law (1.3). In what follows, we use the same notations as in the previous section for spectral abscissa, characteristic function etc. Without explicit reference, they apply to closed-loop system (1.1) and (1.3). The characteristic function of this closed-loop system is given by

$$H(\lambda;\ k_1, k_, a_1, a_2, \tau) = \lambda^2 + a_1\lambda + a_2 + k_1 e^{-\lambda\tau} + k_2 e^{-2\lambda\tau}. \tag{3.1}$$

3.1. Stabilizability and rightmost roots configuration for $\tau = 1$. We start with two theoretical results on the optimization problem of the spectral abscissa function. The first says that for given plant parameters (a_1, a_2), the spectral abscissa function is unbounded in all directions in the controller parameter space.

PROPOSITION 3.1. *Let $(k_1, k_2) = k(\kappa_1, \kappa_2)$, with $k > 0$ a free parameter and (κ_1, κ_2) a given direction in the controller parameter space. Then, there exists a (root) function $\mathbb{R}_+ \ni k \mapsto R(k)$ satisfying*

$$H(R(k);\ k\kappa_1, k\kappa_2, a_1, a_2) = 0, \quad \lim_{k\to\infty} \left| \frac{R(k) - \zeta(k)}{\zeta(k)} \right| = 0,$$

where

$$\zeta(k) = 2W_0\left(\sqrt{-\frac{k\kappa_1}{4}}\right) \tag{3.2}$$

if $\kappa_1 \neq 0$, and

$$\zeta(k) = W_0\left(\sqrt{-k\kappa_2}\right) \tag{3.3}$$

otherwise, with W_0 the principal branch of the Lambert W function. It follows that

$$\lim_{k\to+\infty} \alpha(k\kappa_1, k\kappa_2) = +\infty.$$

Proof. We first address the case where $\kappa_1 \neq 0$. The starting point is the characteristic function

$$H(\lambda;\ k\kappa_1, k\kappa_2, a_1.a_2) = \lambda^2 + a_1\lambda + a_2 + k\kappa_1 e^{-\lambda} + k\kappa_2 e^{-2\lambda},$$

where we perform the following change of variable from λ to ν,

$$\lambda = 2W_0\left(\sqrt{-\frac{k\kappa_1}{4}}\right)\left(1 + \frac{\nu}{L(k)}\right), \tag{3.4}$$

with

$$L(k) := \log\left(-\frac{4}{k\kappa_1}\left(W_0\left(\sqrt{-\frac{k\kappa_1}{4}}\right)\right)^2\right).$$

When omitting the argument of W_0 to simplify the notations, we get from (3.4):

$$e^{-\lambda} = \left(e^{-2W_0}\right)^{1+\frac{\nu}{L(k)}}.$$

From the definition of the Lambert W function we have $W_0 e^{W_0} = \sqrt{-\frac{k\kappa_1}{4}}$, leading to

$$e^{-2W_0} = -\frac{4W_0^2}{k\kappa_1}$$

and

$$e^{-\lambda} = -\frac{4W_0^2}{k\kappa_1}\left(-\frac{4W_0^2}{k\kappa_1}\right)^{\overline{\log\left(-\frac{4W_0^2}{k\kappa_1}\right)}^{\nu}} = -\frac{4W_0^2}{k\kappa_1}\, e^{\nu}, \qquad e^{-2\lambda} = \frac{16W_0^4}{(k\kappa_1)^2}\, e^{2\nu}. \tag{3.5}$$

Substituting (3.4)-(3.5) in the characteristic function, and dividing the latter by $4W_0^2$ yields

$$h(\nu;\ k) := \left(1 + \frac{\nu}{L(k)}\right)^2 + \frac{a_1}{2W_0}\left(1 + \frac{\nu}{L(k)}\right) + \frac{a_2}{4W_0^2} - e^{\nu} + \frac{4W_0^2\kappa_2}{k\kappa_1^2}e^{2\nu}.$$

Since

$$\lim_{k\to\infty}\left|W_0\left(\sqrt{-\frac{k\kappa_1}{4}}\right)\right| = \infty, \quad \lim_{k\to\infty}|L(k)| = \infty, \quad \lim_{k\to\infty}\left|\frac{W_0^2}{k}\right| = 0,$$

function $h(\nu;\ k)$ uniformly converges on compact sets of the complex plane to the function $\hat{h}(\nu) := 1 - e^{-\nu}$ as $k \to \infty$, which has a zero in the origin. From Rouché's theorem, it then follows that there is a threshold $\underline{k}$ and a root function $k \mapsto \underline{\nu}(k)$ satisfying $h(\underline{\nu}(k);\ k) = 0$ for all $k \geq \underline{k}$ and $\lim_{k\to\infty} \underline{\nu}(k) = 0$. Combining this result with (3.4), we can now set

$$R(k) := 2W_0\left(\sqrt{-\frac{k\kappa_1}{4}}\right)\left(1 + \frac{\underline{\nu}(k)}{L(k)}\right)$$

for $k \geq \underline{k}$, with the property

$$\lim_{k\to\infty}\left|\frac{R(k) - \zeta(k)}{\zeta(k)}\right| = \lim_{k\to\infty}\left|\frac{\underline{\nu}(k)}{L(k)}\right| = 0.$$

The proof for $\kappa_1 = 0,\ \kappa_2 \neq 0$ follows the same arguments. For this, notice that likewise (3.2) exactly solves $\lambda^2 + k\kappa_1 e^{-\lambda} = 0$, (3.3) solves $\lambda^2 + k\kappa_2 e^{-2\lambda} = 0$. □

The second theoretical result concerns the minimum number of active characteristic roots corresponding to minima of the spectral abscissa function.

PROPOSITION 3.2. *If the spectral abscissa of closed-loop system (1.1) and (1.3) has a local minimum, then there are more than two active characteristic roots (counting multiplicity).*

Proof. The proof is by contradiction. We namely show that two characteristic roots (ν_1, ν_2) for gains (k_1^*, k_2^*), either both real or forming a complex conjugate pair, can always be shifted to the left in the complex plane by an appropriate change of the controller gains. Since such changes of the gains can be chosen arbitrarily small, as we shall see, the spectral abscissa can always be reduced if there are at most two active characteristic roots, which contradicts the assumption of a minimum.

In what follows, we distinguish between three cases, based on different, but complementary assumptions on (ν_1, ν_2).

Case 1: there does not exist an $\ell \in \mathbb{Z}$ such that $\nu_2 - \nu_1 = \ell 2\pi\imath$. The presence of roots $(\nu_1 - \epsilon, \nu_2 - \epsilon)$, with $\epsilon \in \mathbb{R}$ a parameter, leads to the conditions

$$\begin{bmatrix} e^{-\nu_1+\epsilon} & e^{-2\nu_1+2\epsilon} \\ e^{-\nu_2+\epsilon} & e^{-2\nu_2+2\epsilon} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = - \begin{bmatrix} (\nu_1-\epsilon)^2 + a_1(\nu_1-\epsilon) + a_2 \\ (\nu_2-\epsilon)^2 + a_1(\nu_2-\epsilon) + a_2 \end{bmatrix}. \tag{3.6}$$

We have

$$\det\left(\begin{bmatrix} e^{-\nu_1} & e^{-2\nu_1} \\ e^{-\nu_2} & e^{-2\nu_2} \end{bmatrix}\right) = e^{-\nu_1-2\nu_2}\left(1 - e^{\nu_2-\nu_1}\right) \neq 0.$$

Invoking the implicit function theorem, Equations (3.6) uniquely define a continuous function $\epsilon \mapsto (k_1(\epsilon), k_2(\epsilon))$ around $\epsilon = 0$, such that $(k_1(0), k_2(0)) = (k_1^*, k_2^*)$. Hence, both roots can be shifted to the left by an infinitesimal change of (k_1, k_2).

Case 2: $\nu_1 = \nu_2$. A double root at $\nu_1 - \epsilon$ satisfies

$$\begin{bmatrix} e^{-\nu_1+\epsilon} & e^{-2\nu_1+2\epsilon} \\ -e^{-\nu_1+\epsilon} & -2e^{-2\nu_1+2\epsilon} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = - \begin{bmatrix} (\nu_1-\epsilon)^2 + a_1(\nu_1-\epsilon) + a_2 \\ 2(\nu_1-\epsilon) + a_1 \end{bmatrix}. \tag{3.7}$$

Since we now have

$$\det\left(\begin{bmatrix} e^{-\nu_1} & e^{-2\nu_1} \\ -e^{-\nu_1} & -2e^{-2\nu_1} \end{bmatrix}\right) = -e^{-3\nu_1} \neq 0,$$

we can proceed as in Case 1.

Case 3: $\nu_2 - \nu_1 = \ell 2\pi\imath$, for some $\ell \in \mathbb{Z}$. Without loosing generality, we can set $\nu_1 = \bar{\nu}_2 = \alpha + \ell\pi\imath$ with $\alpha \in \mathbb{R}$. Substitution this expression in the characteristic equation and separating real and imaginary part yield

$$2\alpha + a_1 = 0, \quad \alpha^2 - (\ell\pi)^2 + a_1\alpha + k_1 e^{-\alpha}(-1)^\ell + k_2 e^{-2\alpha} = 0. \tag{3.8}$$

Thus, such roots must satisfy $\alpha = -\frac{a_1}{2}$. Furthermore, there is a *continuum* of corresponding gain values, lying on a line in the (k_1, k_2)-plane with normal vector $\left((-1)^\ell, e^{-\alpha}\right)$. Obviously, (k_1^*, k_2^*) lies on this line.

We now consider a perturbation of the nominal gain values in the direction of the normal vector: $(k_1(\epsilon), k_2(\epsilon)) = (k_1^*, k_2^*) + \epsilon\left((-1)^\ell, e^{-\alpha}\right)$, with $\epsilon \in \mathbb{R}$. As ν_1 is assumed a simple root for $\epsilon = 0$, there exists a corresponding continuously differentiable root function $\epsilon \mapsto \hat{\lambda}(\epsilon)$, defined around $\epsilon = 0$ and satisfying $\hat{\lambda}(0) = \nu_1$. We get

$$\begin{aligned} \hat{\lambda}'(0) &= -\frac{(-1)^\ell e^{-\nu_1} + e^{-\alpha}e^{-2\nu_1}}{2\nu_1 + a_1 - k_1^* e^{-\nu_1} - 2k_2^* e^{-2\nu_1}} \\ &= \frac{e^{-\alpha} + e^{-3\alpha}}{k_1^*(-1)^\ell e^{-\alpha} + 2k_2^* e^{-2\alpha} - 2\ell\pi\imath}, \end{aligned}$$

where we used the first equation of (3.8). Hence, we have

$$\text{sign}\, \Re\left(\hat{\lambda}'(0)\right) = \text{sign}\left(k_1^*(-1)^\ell e^{-\alpha} + 2k_2^* e^{-2\alpha}\right).$$

Consequently, if $k_1^*(-1)^\ell e^{-\alpha} + 2k_2^* e^{-2\alpha} > 0$ (< 0), then the roots (ν_1, ν_2) are simultaneously shifted to the left by taking a perturbation with small $\epsilon < 0$ (> 0).

On the line defined by the second equation of (3.8), there is only one point where $k_1(-1)^\ell e^{-\alpha} + 2k_2 e^{-2\alpha} = 0$. If this is the case for $(k_1, k_2) = (k_1^*, k_2^*)$, then a shift to the left of characteristic roots can be induced by first shifting (k_1, k_2) in the direction of the line (keeping the pair of characteristic roots (ν_1, ν_2) invariant), followed by a perturbation of the gain values in the (opposite) direction of its normal. Again, the overall change of the gains can be made arbitrarily small. □

In extensive numerical experiments with the algorithm of [15] for minimizing the spectral abscissa as a function of (k_1, k_2), three configurations of active characteristic roots corresponding to the minimum were encountered, depending on plant parameters (a_1, a_2). We refer to the appendix for an instance of the three configurations, which all involve more than two active characteristic roots, consistently with Proposition 3.2.

The *first configuration* corresponds to a real root with multiplicity three. If this happens for $(a_1, a_2) \in \mathcal{S}_0$, i.e., on the stabilizability boundary, the conditions $H(0) = H'(0) = H''(0) = 0$ are satisfied, from which once can directly compute

$$a_2 = -\frac{3}{2}a_1 - 1. \tag{3.9}$$

The *second configuration* corresponds to a pair of complex conjugate roots with multiplicity two. If this happen for $(a_1, a_2) \in \mathcal{S}_0$, we get the conditions

$$H(\imath\omega) = H'(\imath\omega) = 0.$$

Considering $\omega \geq 0$ as a parameter, we obtain from these conditions

$$(a_1, a_2) = \begin{cases} \left(-\frac{4\omega(\sin(\omega)^2)}{2\omega - \sin(2\omega)}, \frac{\omega\left(\sin(\omega) - \omega(\cos(\omega))^3 + \omega^2\sin(\omega)\right)}{(\cos(\omega))^3 - \cos(\omega) + \omega\sin(\omega)}\right), & \omega \neq k\pi,\ k \in \mathbb{Z}_+, \\ \left(-3, \frac{7}{2}\right), & \omega = 0. \end{cases} \tag{3.10}$$

In contrast to (2.7), holding for state feedback, there are infinitely many branches. Note that the first branch, defined for $\omega \in [0,\ \pi)$, starts from a point on the line (3.9). In what follows, we call the branch defined for $\omega \in ((k-1)\pi, k\pi)$, with $k \geq 2$, the k-the branch.

REMARK 3.3. *The presence of characteristic roots $\pm\imath\omega,\ \omega > 0$, can be expressed in the form*

$$\begin{bmatrix} e^{-\imath\omega} & e^{-2\imath\omega} \\ e^{\imath\omega} & e^{2\imath\omega} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = -\begin{bmatrix} -\omega^2 + a_1\imath\omega + a_2 \\ -\omega^2 - a_1\imath\omega + a_2 \end{bmatrix}. \tag{3.11}$$

The matrix of this linear system is singular if and only if $\sin(\omega) = 0$. *Hence, assigning roots to $\pm\imath k\pi,\ k \geq 1$ is in general not possible. This is reflected in the exclusion of $\omega = k\pi$ in (3.10) and, by extension, in the presence of infinitely many branches. The fact that arbitrary assignment of two roots is not possible using control law (1.3), unlike using state feedback, is also at the origin of the distinction between Case 1 and Case 3 in the proof of Proposition 3.2.*

Furthermore, also a *third situation* occurs for some (fixed) values of (a_1, a_2). In the process of minimizing the spectral abscissa, one encounters in an intermediate iteration a real root and a complex pair with (approximately) the same real part. These roots are gradually shifted to the left in the complex plane in subsequent iterations until a minimum is reached, without *any* interaction with other characteristic roots or changing the configuration of the three rightmost roots. Mathematically, this situation can be described as follows. A real root α and a complex pair $\alpha \pm \imath\omega$ satisfy the three equations

$$H(\alpha) = 0,\ \Re\left(H(\alpha + \imath\omega)\right) = 0,\ \Im\left(H(\alpha + \imath\omega)\right) = 0,$$

which can be compactly written as $F(X,\ \alpha) = 0$ with $X = (k_1, k_2, \omega)$. The implicit function theorem says that these equations allow to locally define X as a function of α if the Jacobian of F with respect to X, denoted by J_F, is nonsingular. Hence, it is then possible to further reduce α under the non-degeneracy condition $\frac{\partial F}{\partial \alpha} \neq 0$ that is satisfied. A minimum must thus be characterized by the additional condition $\det J_F(X,\ \alpha) = 0$. If this third situation occurs for values of (a_1, a_2) on the stabilizability boundary $\mathcal{S}_0$, we can set $\alpha = 0$. Freeing parameters (a_1, a_2) yields

$$\left\{ \begin{array}{l} H(0;\ k_1, k_2, a_1, a_2) = 0 \\ \Re\left(H(\imath\omega;\ k_1, k_2, a_1, a_2)\right) = 0 \\ \Im\left(H(\imath\omega;\ k_1, k_2, a_1, a_2)\right) = 0 \\ \det J_F(X,\ 0) = 0, \end{array} \right. \tag{3.12}$$

where

$$\begin{aligned} \det J_F(X,\ 0) = (-1 - \cos(\omega) + 2(\cos(\omega))^2)a_1 + (1 - \cos(\omega))k_1 \\ + (-2 + 2\cos(\omega))k_2 - 2\omega\sin(2\omega) + 2\omega\sin(\omega). \end{aligned} \tag{3.13}$$

Note that relations (3.12), consisting of four equations in five unknowns $(k_1, k_2, a_1, a_2, \omega)$, define curves parametrized by ω. Elimination of (k_1, k_2) leads us to an explicit expression for (a_1, a_2),

$$\left\{ \begin{array}{lcl} a_1 & = & \frac{\omega\,(\cos(\omega)+1)\,\left(4\cos(\omega)^2 - 6\,\cos(\omega) + \omega\,\sin(\omega) + 2\right)}{(\cos(\omega)-1)\,(2\,\omega - \sin(2\,\omega) - \sin(\omega) + \omega\,\cos(\omega))}, \\ a_2 & = & \frac{-\omega^2\,(3\,\omega - \sin(3\,\omega))}{2\,(\cos(\omega)-1)\,(2\,\omega - \sin(2\,\omega) - \sin(\omega) + \omega\,\cos(\omega))}, \qquad \omega \neq k2\pi,\ k \in \mathbb{Z}, \end{array} \right. \tag{3.14}$$

and $(a_1, a_2) = \left(-\frac{11}{3}, \frac{9}{2}\right)$ for $\omega = 0$. Again there are infinitely many branches, which we number in the same way as the branches of (3.10). Note that the first one, defined for $\omega \in [0,\ 2\pi)$, also emanates from the line (3.9).

Finally, in Figures 3.1-3.2 we plot the line (3.9) and the curves (3.10) and (3.14). By using a bigger line width, we indicate how the stabilizability boundary $\mathcal{S}_0$ is composed from segments of these curves. More information on determining the relevant segments can be found in the appendix.

Remark 3.4. *A similar reasoning as in Remark 3.3 can be made regarding (3.14), but a distinction between two cases is necessary. If* $\omega = k2\pi,\ k \geq 1$*, the system matrix of (3.11) is singular and the right-hand side is not in its column space. Hence, there is no solution, as is always the case for (3.10). If* $\omega = (2k-1)\pi,\ k \geq 1$*, linear system (3.7) is still solvable because, with* (a_1, a_2) *given by (3.14), the right-hand side lies in the column space of the (singular) system matrix. These observations explain why the parameter range for each branch is an interval of length* 2π*, instead of length* π*.*

Next to the rightmost roots configuration inducing (3.14), another particular feature for control law (1.3) is that multiple strict local minima of the spectral abscissa may coexist (implying that a result similar to Proposition 2.1 does not exist for control law (1.3)), with the property that along the stabilizability boundary, the global minimum may switch between two branches of local minima. This situation is illustrated in Figure 3.3, where we display the spectral abscissa as a function of controller parameters (k_1, k_2), for values of (a_1, a_2) corresponding to point A in Figure 3.1. There are two minima equal to zero, both of them characterized by a pair of rightmost characteristic roots with multiplicity two. The latter is expected as point A lies at the intersection of two branches of (3.10).

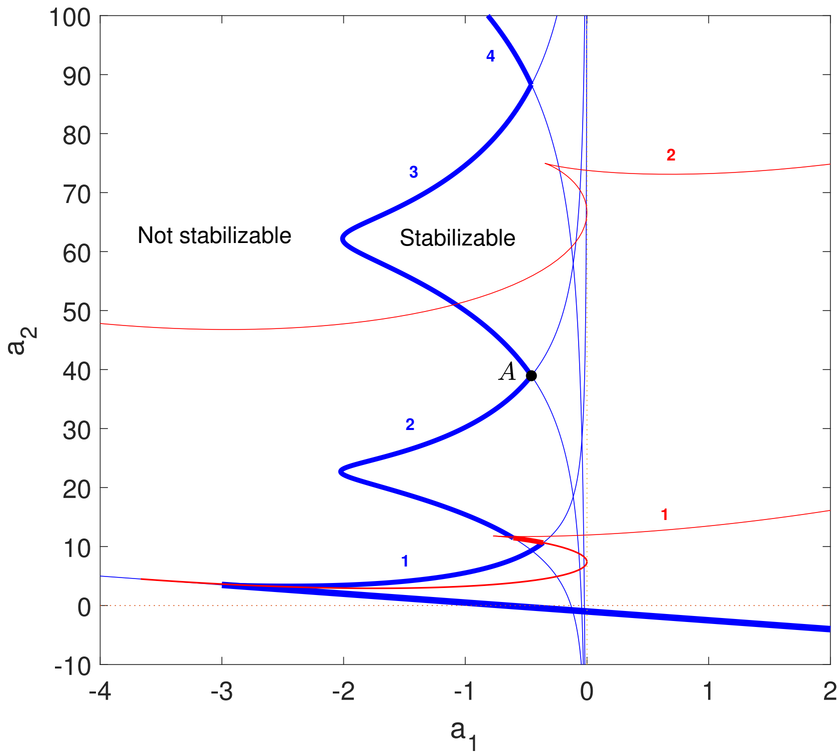

FIG. 3.1. *In the plant parameter space (a_1, a_2), the line (3.9) and the curves (3.10) are displayed in a blue color, the curves (3.14) in a red color, with indication of the branch numbers. The stabilizability boundary $\mathcal{S}_0$ is displayed in bold.*

3.2. Characterization of the sets $\mathcal{S}_c$ for $\tau = 1$. The set $\mathcal{S}_c$, with $c \neq 0$, can be obtained from $\mathcal{S}_0$ by a transformation, see (2.9)-(2.10). For various values of c it is shown in Figure 3.4.

The point $(a_1, a_2) = \left(-3, \frac{7}{2}\right)$, connecting the segments of $\mathcal{S}_0$ described by (3.9) and the first branch of (3.10), is transformed via the relation $(a_1 + 2c, a_2 + a_1 c + c^2) = \left(-3, \frac{7}{2}\right)$. Eliminating c leads us to

$$a_2 = \frac{1}{4} a_1^2 + \frac{5}{4}. \tag{3.15}$$

Note that (3.9) is tangential to (3.15) at $a_1 = -3$, which is the direct analog of Proposition 2.2.

Similarly, the transformation of the two points which connect the segments of $\mathcal{S}_0$ described by (3.10) and (3.14), see Figure 3.2, leads to the quadratic functions

$$a_2 = \frac{1}{4} a_1^2 + \gamma_j, \; j = 1, 2, \tag{3.16}$$

with

$$\gamma_1 = 10.6166, \; \gamma_2 = 11.3640. \tag{3.17}$$

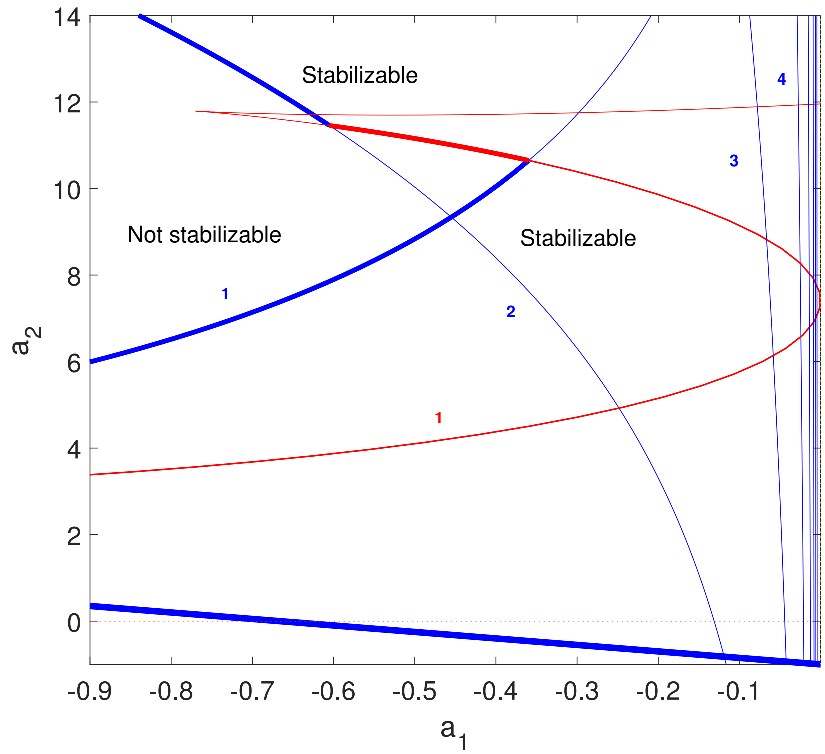

FIG. 3.2. *Zoom of Figure 3.1, highlighting the contribution of (3.14) to the stabilizability boundary $\mathcal{S}_0$.*

Functions (3.15)-(3.16) are also visualized in Figure 3.4. It is important to mention that conditions (3.15) and (3.16) can be alternatively expressed in terms of the eigenvalues of A as

$$|\Im(\lambda_i)| = \sqrt{\frac{5}{4}}$$

and

$$|\Im(\lambda_i)| = \sqrt{\gamma_1},\ |\Im(\lambda_i)| = \sqrt{\gamma_2}, \quad i = 1, 2,$$

respectively.

3.3. Stabilizability and multiple root assignment for $\tau \neq 1$. The situation where the tuple (a_1, a_2, τ) is considered as plant parameters can be addressed by the transformation $\lambda \leftarrow \tau\lambda$, leading to the substitution (2.14) in the characterizations for $\tau = 1$.

We can conclude the following about the global minima of the spectral abscissa function.

- If $|\Im(\lambda_i)| \in \left(\frac{\sqrt{5}}{2\tau},\ \frac{\sqrt{\gamma_1}}{\tau}\right)$ or $|\Im(\lambda_i)| > \frac{\sqrt{\gamma_2}}{\tau}$, with γ_1 and γ_2 given by (3.17) and $i \in \{1, 2\}$, then the minimum of the spectral abscissa is characterized by a pair of complex conjugate rightmost roots of multiplicity two.

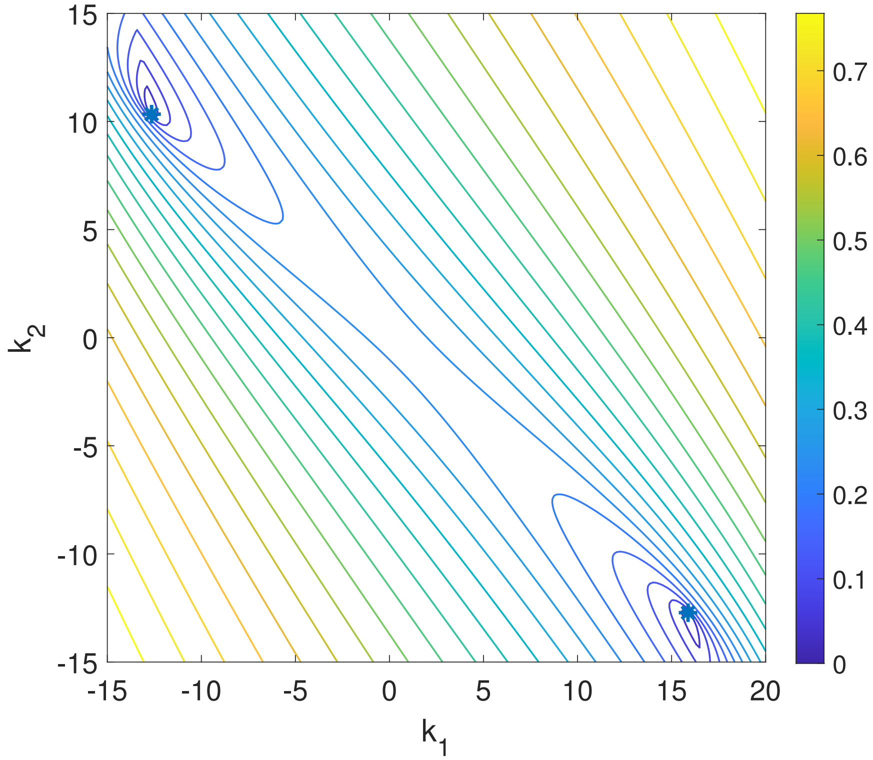

FIG. 3.3. *Contour plot of the spectral abscissa function for* $(a_1, a_2) = (-0.45667, 38.956)$ *as a function of controller parameters* (k_1, k_2)*, corresponding to point A, indicated in Figure 3.1. The minimizers are indicated with an asterisk.*

- If $|\Im(\lambda_i)| < \frac{\sqrt{5}}{2\tau}$, the minimum of the spectral abscissa is characterized by a real rightmost root of multiplicity three.
- If $|\Im(\lambda_i)| \in \left(\frac{\sqrt{\gamma_1}}{\tau}, \ \frac{\sqrt{\gamma_2}}{\tau}\right)$, the minimum is characterized by a real root and a complex conjugate pair of roots with the same real part c, such that (after transforming the delay to one), also condition $\det J_F(X, \ c) = 0$ is satisfied (see the discussion preceding Equation (3.12)) .
- For $|\Im(\lambda_i)| = \frac{\sqrt{\gamma_1}}{\tau}$, $|\Im(\lambda_i)| = \frac{\sqrt{\gamma_2}}{\tau}$ and at the intersections of different branches of (3.10), multiple global minima coexist. For $|\Im(\lambda_i)| = \frac{\sqrt{5}}{2\tau}$, the minimum of the spectral abscissa is uniquely determined by a real rightmost root of multiplicity four.

Regarding the direct assignment of multiple roots, we can conclude the following.

- If $\Im(\lambda_i) \leq \frac{\sqrt{5}}{2\tau}, \ i \in \{1, 2\}$, it is possible to assign a triple real root at

$$c_{\pm} = -\frac{3}{2\tau} + \lambda_{\text{av}} \pm \sqrt{\frac{5}{4\tau^2} + \lambda_{\text{diff}}^2}. \tag{3.18}$$

  The triple root at c_+ is dominant and corresponds to a minimum of the spectral abscissa. If $\Im(\lambda) > \frac{\sqrt{5}}{2\tau}$, assigning a triple real root is not possible. The derivation of (3.18) is based on a geometric argument and follows the

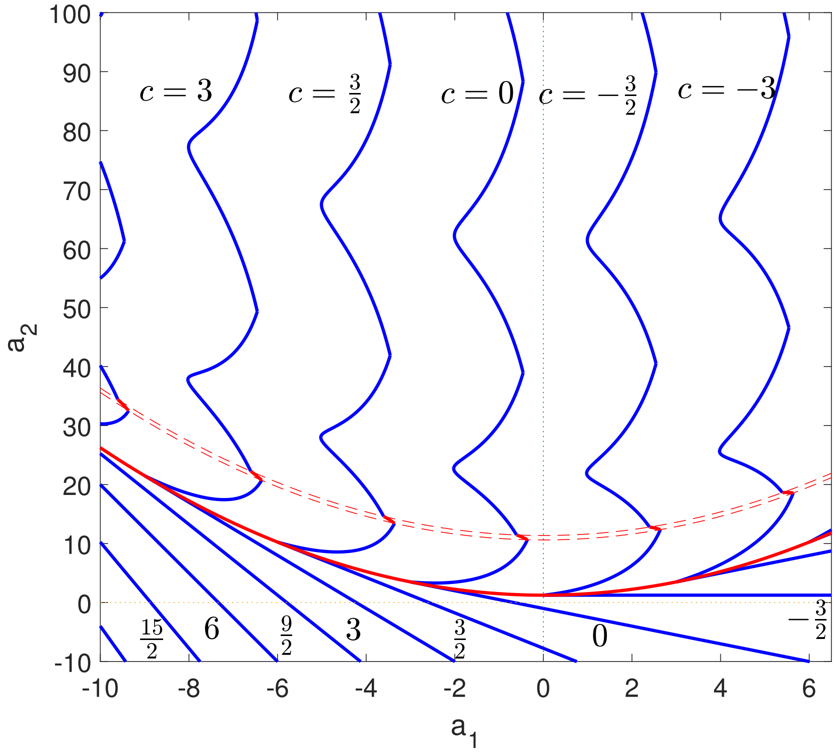

FIG. 3.4. *The set $\mathcal{S}_c$, for various values of c. The value of c corresponds to the minimal value of the spectral abscissa as a function of controller parameter (k_1, k_2). The full red curve corresponds to the parabola (3.15), the dashed red curves to (3.16).*

same steps as the derivation of (2.13).

- For any $(a_1, a_2, \tau) \in \mathbb{R}^2 \times \mathbb{R}_+$, there are an infinite but countable number of possibilities of assigning a complex conjugate pair of multiplicity two. This follows from the infinite number of branches of (3.10), which have the a_2-axis as vertical asymptote (see Figure 3.2) and which are shifted to the line $a_1 = -2c$ by the transformation shifting $\mathcal{S}_0$ to $\mathcal{S}_c$. If $|\Im(\lambda_i)| \in \left(\frac{\sqrt{5}}{2\tau},\ \frac{\sqrt{\gamma_1}}{\tau}\right)$ or $|\Im(\lambda_i)| > \frac{\sqrt{\gamma_2}}{\tau}$, then one of these possibilities corresponds to a minimum of the spectral abscissa.
- Assigning a real characteristic root and a complex conjugate pair of roots with the same real part c leads to a system of under-determined equations. With the additional condition $\det J_F(X,\ c) = 0$, the system becomes fully determined. As is clear from Figure 3.1, whether there are solutions depends on the values of (a_1, a_2, τ). For

$$|\Im(\lambda_i)| \in \left(\frac{\sqrt{\gamma_1}}{\tau},\ \frac{\sqrt{\gamma_2}}{\tau}\right),$$

$i \in \{1, 2\}$, there is a unique solution that corresponds to a minimum of the spectral abscissa function.

Finally, let us illustrate the infinitely many possibilities to assign a complex conjugate pair of characteristic roots with multiplicity two, by considering plant parameters $(a_1, a_2) = (0, 0)$ and $\tau = 1$, i.e., a double integrator with unit input delay. By eliminating (k_1, k_2, α) from the equations

$$\Re\left(H(\alpha + \imath\omega)\right) = \Im\left(H(\alpha + \imath\omega)\right) = \Re\left(H'(\alpha + \imath\omega)\right) = \Im\left(H'(\alpha + \imath\omega)\right) = 0,$$

which characterize such a pair $\alpha + \imath\omega$, we find that the possible values of ω are the zeros of function

$$f(\omega) := \sin(\omega) - \frac{\omega^2 \cos(\omega) + \cos(\omega) - \cos(\omega)^3 - 2\,\omega \cos(\omega)^2 \sin(\omega)}{\omega^3}, \qquad (3.19)$$

while the values of α can be obtained from

$$\alpha = -\frac{\omega\,(1 - \cos(2\omega))}{2\omega - \sin(2\omega)}.$$

It is clear that function (3.19) has infinitely many positive zeros, $\{\omega_n\}_{n\geq 1}$, satisfying $\lim_{n\to\infty} \omega_n = \infty$, while the corresponding values of α, $\{\alpha_n\}_{n\geq 1}$, satisfy $\lim_{n\to\infty} \alpha_n = 0$, in agreement with Figure 3.2.

4. Note on the delay as controller parameter and maximum root multiplicity. If the delay $\tau \in \mathbb{R}_+$ is not a plant parameters but can be freely used as controller parameters, then achieving stability is possible for any $(a_1, a_2) \in \mathbb{R}^2$, both in the setting of Section 2 and of Section 3. In the former case, it is sufficient to take $\tau = 0$ and consider a stabilizing state feedback law. In the latter case, one may start from a stabilizing PD controller and replace the output derivative by a finite difference approximation of sufficient precision, by taking τ sufficiently small.

The discussion on both Figure 2.2 and Figure 3.4 made clear that the maximal encountered multiplicity of a rightmost root at the minimum of the spectral abscissa is four, corresponding to curves (2.11) and (3.15), respectively. We now characterize plant parameters (a_1, a_2), for which this maximum multiplicity is achievable by choosing (k_1, k_2, τ), starting with the state feedback setting from Section 2.

Recall first that if $\tau \neq 1$, we have to make the substitution (2.14), also in the labels of the figures generated in the previous sections. This is done in Figure 4.1, which displays the set $\mathcal{S}_c$ for two values of c (similarly to Figure 2.2). Next, we consider plant parameters $(\hat{a}_1, \hat{a}_2)$. If delay τ is varied, the normalized parameters $(\hat{a}_1\tau, \hat{a}_2\tau^2)$ move on a parabola, indicated by the green dashed curve. A minimum spectral abscissa, characterized by a real root with multiplicity four, is possible if $\tau > 0$ can be chosen such that the normalized plant parameters can be moved to a point on the parabola $(a_2\tau)^2 = \frac{1}{4}(a_1\tau)^2 + 2$, indicated in red in Figure 4.1. This leads to the condition

$$(\hat{a}_2\tau)^2 = \frac{(\hat{a}_1\tau)^2}{4} + 2 \Leftrightarrow \hat{a}_2 - \frac{\hat{a}_1^2}{4} = \frac{2}{\tau^2}.$$

As the delay must be non-negative, the condition for achieving maximum multiplicity using (k_1, k_2, τ) as controller parameters is

$$\hat{a}_2 - \frac{\hat{a}_1^2}{4} > 0 \;\Leftrightarrow\; |\Im(\lambda_i)| > 0,\; i = 1, 2.$$

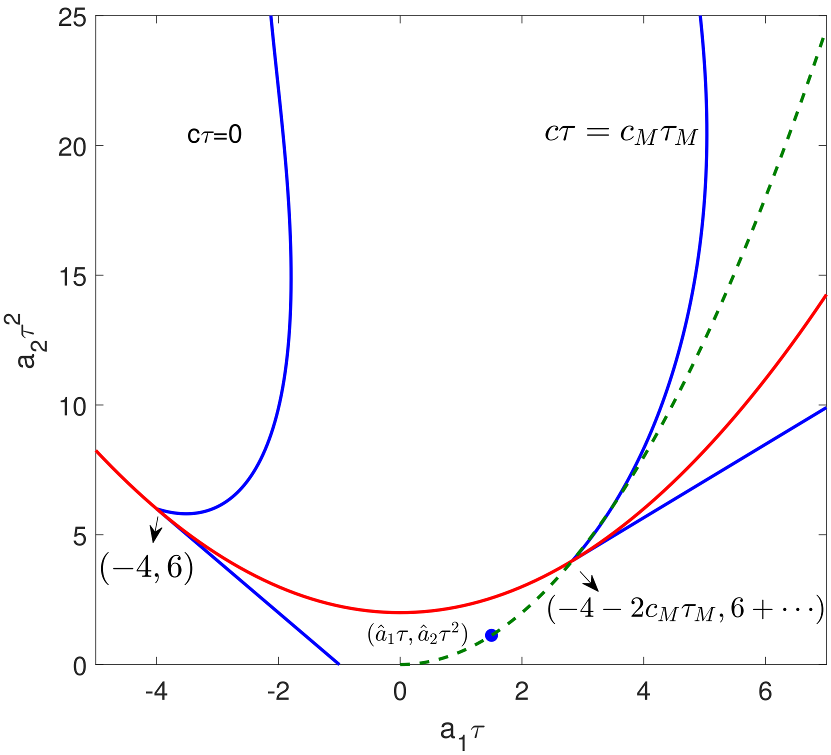

FIG. 4.1. *The set $\mathcal{S}_c$, for two values of c. The value of c corresponds to the minimal value of the spectral abscissa as a function of controller parameter (k_1, k_2).*

The corresponding delay, τ_M, is given by

$$\tau_M = \frac{\sqrt{2}}{|\Im(\lambda_i)|},\ i \in \{1, 2\}, \tag{4.1}$$

and the corresponding minimum value of the spectral abscissa, c_M, satisfies (see Figure 4.1)

$$-4 - 2c_M\tau_M = \hat{a}_1\tau_M,$$

from which we get

$$c_M = \lambda_{\text{av}} - \sqrt{2}\,|\Im(\lambda_i)|,\ i \in \{1, 2\}. \tag{4.2}$$

Expressions (4.1) and (4.2) are consistent with [6, Theorem 4.1]. Note that if $\Im(\lambda) < 0$, a triple real root can be assigned for any τ, see the conclusions of Section 2.

Finally, for the setting in Section 3, the derivation is completely analogous, leading to the same condition $|\Im(\lambda_i)| > 0$ for the assignability of a root of maximum multiplicity four, and to the expressions

$$\tau_M = \frac{\sqrt{5}}{2|\Im(\lambda_i)|},\quad c_M = \lambda_{\text{av}} - \frac{4}{\sqrt{5}}\,|\Im(\lambda_i)|,\quad i \in \{1, 2\}.$$

5. Conclusions. The global minima of the spectral abscissa function corresponding to (2.1) and (3.1), with gains (k_1, k_2) as the optimized variables, were characterized as a function of plant parameters (a_1, a_2) (or, equivalently, $(\lambda_{\text{av}},\ \lambda_{\text{diff}})$) and input delay τ. For these problems, an answer to the questions stated in the introduction was provided in Section 2.3 and Section 3.3. Further considerations are the following.

- In the optima of the spectral abscissa, not all degrees are freedom in the controller parameter space need to be related to maximizing multiplicity, as illustrated with the third situation for control law (1.3) that gives rise to branches (3.14). In such a situation, the optimal parameters cannot be determined by direct assignment of the characteristic roots configuration (as the latter leads to an under-determined system of equations).
- Every encountered characteristic roots configuration, except for the pair of complex roots with multiplicity two in case of state feedback, may or may not correspond to dominant roots, depending on the plant parameter.
- Directly imposing a particular characteristic roots configuration may result in a fully determined system of equations having *infinitely* many isolated solutions, as illustrated at the end of Section 3.3.
- Switches between several segments of the stabilizability boundary for (1.3) and of level sets of the spectral abscissa function illustrate that multiple local and global minima of the spectral abscissa may coexist. If a considered roots configuration does not correspond to a global minimum, it may correspond to non-dominant roots, or to a local minimum that is not global.

Finally, it should be stressed that the surprisingly rich characterizations and conclusions are *all* deduced from *only one curve*, namely the one in bold in Figure 2.1 for state feedback and the one in bold in Figure 3.1 for output feedback. For example, level sets of the spectral abscissa and characterizations with (a_1, a_2, τ) as free parameters then follow from an affine transformation and scaling of the plant parameters, respectively. Proposition 2.2, rooted in the mathematical description of the curve in Figure 2.1, induces the parabola (2.11), acting as a separatrix in the classification at the end of Section 2, while similar conclusions can be made for output feedback in Section 3. Also, the analysis in Section 4 relies on the aforementioned scaling. Hence, conceptually the computed curves in Figures 2.1 and 3.1, with extensive validation (see the appendix), can be considered as analogs of mathematical axioms on which a theory is built.

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Appendix A. Spectral abscissa minimization .

We briefly discuss the spectral abscissa minimization of closed-loop system (1.1) and (1.3) with $\tau = 1$, in relation to determining the sets $\mathcal{S}_c$, with $c \in \mathbb{R}$. For this we used the software `tds_stabil`, related to article [15], which relies on its turn on computing rightmost characteristic roots. As mentioned in Section 3.1, qualitatively three different configurations of active characteristic roots appeared in the experiments, depending on the values of plant parameters (a_1, a_2). For every configuration, we give an example accompanied by the output of the optimization process.

1. For plant parameters $(a_1, a_2) = (-0.5, 0.7)$ the software provides optimal

gains

$$(k_1, k_2) = (-0.544872, -0.110982)$$

and corresponding rightmost characteristic roots (formatted as program output)

$$\begin{array}{l} -4.673625603391569e-01+6.186681269977292e-03\imath \\ -4.673625603391569e-01-6.186681269977292e-03\imath \\ -4.673625700900494e-01+0.000000000000000e+00\imath \end{array}$$

which are in fact perturbations of a triple real root with geometric multiplicity one. Due to the high local sensitivity of defective multiple roots [16], they cannot be very accurately generated from eigenvalue computations (e.g., if the computational errors behaved as errors on the system data of order of magnitude ϵ, the error on perturbations of a triple root with geometric multiplicity three would generically be of order $\epsilon^{\frac{1}{3}}$). In addition, in this situation the spectral abscissa is not differentiable, even not locally Lipschitz continuous in the minimum, challenging the optimization algorithm.
To confirm the assertion about the presence of a triple root, we can however do a post-processing step. From the output of the optimization process, we generate initial values for Newton's method, applied to the system of nonlinear equations $H(\alpha) = H'(\alpha) = H''(\alpha) = 0$ in variables (k_1, k_2, c), which is in general well conditioned. The Newton iterations converge to a solution, given by

$$\alpha = -0.467376,\ k_1 = -0.544870,\ k_2 = -0.110982.$$

2. Plant parameter $(a_1, a_2) = (-1.580, 22.5)$ lead to the optimal gains $(k_1, k_2) = (7.660857, 0.4447815)$ and rightmost characteristic roots

$$\begin{array}{l} -2.168120934607645e-01-4.539125356401833e+00\imath \\ -2.168120934607645e-01+4.539125356401833e+00\imath \\ -2.166962038272997e-01-4.745666832168020e+00\imath \\ -2.166962038272997e-01+4.745666832168020e+00\imath \end{array}$$

which are perturbations of a pair with multiplicity two. As a confirmation of this assertion, solving directly the nonlinear equations

$$\Re H(\alpha+\imath\omega)=0,\ \Im H(\alpha+\imath\omega)=0,\ \Re H'(\alpha+\imath\omega)=0,\ \Im H'(\alpha+\imath\omega)=0$$

in variables (k_1, k_2, c, ω), with starting values generated from the results of the optimization procedure, yields

$$\alpha = -0.220251,\ \omega = 4.642891,\ k_1 = 7.621287,\ k_2 = 0.442270.$$

3. For $(a_1, a_2) = (-0.45, 11.25)$, we get optimized gains $(k_1, k_1) = (-5.450615, -5.108210)$ and corresponding rightmost characteristic roots

$$\begin{array}{l} -4.386724345485593e-02+0.000000000000000e+00\imath \\ -4.386724373548123e-02-3.634288150609573e+00\imath \\ -4.386724373548123e-02+3.634288150609573e+00\imath . \end{array}$$

Furthermore, in the computed optimum we have $\sigma_{\min}(J_F(X,\ c)) = 3.794513\,10^{-8}$, with $\sigma_{\min}(\cdot)$ denoting the smallest singular value, and $\left\|\frac{\partial F}{\partial c}\right\|_2 = 16.335444$.

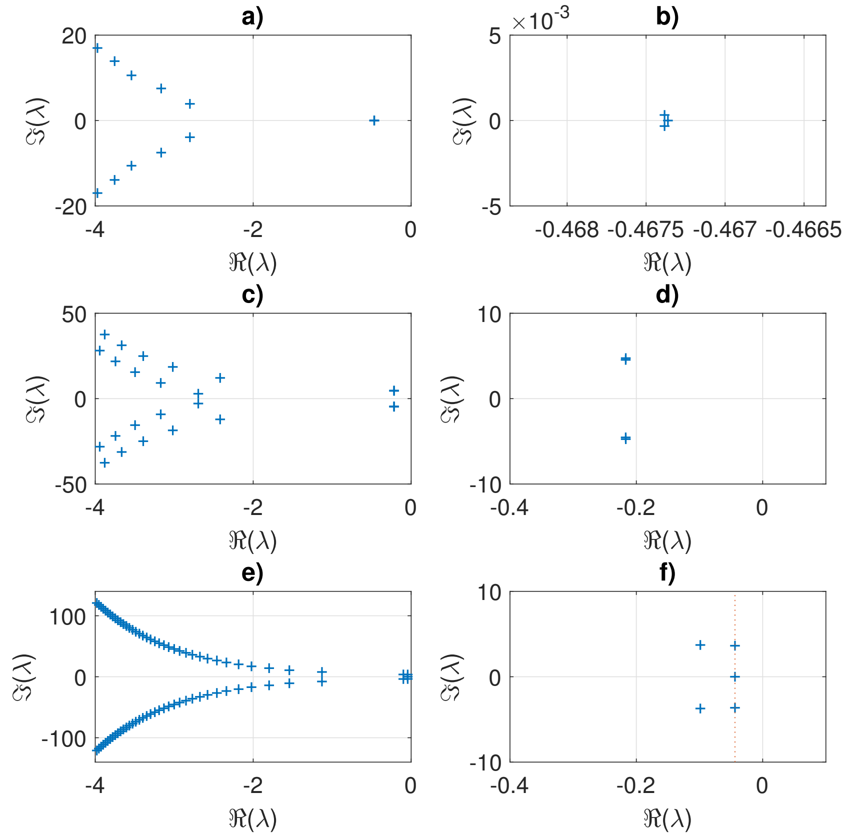

FIG. A.1. *Rightmost characteristic roots corresponding to the minimum of the spectral abscissa function, for* $\tau = 1$*, and* $(a_1, a_2) = (-0.5, 0.7)$ *(top row),* $(a_1, a_2) = (-1.580, 22.5)$ *(middle row) and* $(a_1, a_2) = (-0.45, 11.25)$ *(bottom row). Each right pane is obtained from zooming in on the corresponding left pane.*

In Figure A.1 we plot the rightmost characteristic roots in the computed (global) optima of the spectral abscissa function, for the these examples.

The spectral abscissa function of (1.1) and (1.3), as a function of the controller gains, is in general neither convex nor quasi-convex, as illustrated with the two local optima in Figure 3.3. For this reason. the software `tds_stabil` uses by default five randomly generated starting values for the optimization algorithm and selects the best solution (this number can be changed and initial conditions can also be supplied by the user). Since the sets $\mathcal{S}_c$ are defined in terms of global optima of the spectral abscissa function, additional measures have been taken to ascertain reliability of the derived characterizations, resulting in the following procedure.

Based on the spectral abscissa optimization for (a_1, a_2)-values on a course grid in the region of interest shown in Figure 3.1 (further increasing a_2 and a_1 leads to stationary behavior), followed by analyzing the configuration of the active characteristic roots in the minima (as for the above examples), it was hypothesized that the stabilizability boundary $\mathcal{S}_0$ consists of a concatenation of the specific segments from line (3.9) and curves (3.10) and (3.14), highlighted in Figures 3.1-3.2. Subsequently, this candidate stabilizability boundary was discretized with a very fine resolution. For each corresponding point (a_1, a_2), the optimization algorithm was run and, in addition, the spectral abscissa was computed on a fine grid in a square region in the (k_1, k_2)-space of size 40 by 40 around the minimizer, to confirm zero is the global minimum. The restriction to a compact set is supported by Proposition 3.1. Note that such an a-posteriori brute force validation can be restricted to the set $\mathcal{S}_0$ only, since sets $\mathcal{S}_c$, with $c \neq 0$, are induced by $\mathcal{S}_0$ via the relation (2.10).

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