Multiplicity-induced-dominancy for delay systems: Comprehensive examples in the scalar neutral case

Amina Benarab, Islam Boussaada, Silviu-Iulian Niculescu, Karim Trabelsi

Multiplicity-Induced-Dominancy for Delay Systems: Comprehensive Examples in the Scalar Neutral Case

Amina Benarab[a,b], Islam Boussaada[a,b,*], Silviu-Iulian Niculescu[a], Karim L. Trabelsi[b]

[a] Université Paris-Saclay, CNRS, CentraleSupélec, Inria, Laboratoire des signaux et systèmes, 91190, Gif-sur-Yvette, France
[b] Institut Polytechnique des Sciences Avancées (IPSA), 63 boulevard de Brandebourg, 94200 Ivry-sur-Seine, France.

Abstract

This article focuses on the characterization of a particular spectral property called *Multiplicity-induced-dominancy* applying for linear dynamical systems described by delay-differential equations. More precisely, we characterize the property in the scalar neutral case with respect to the system parameters. Particular attention is paid to the so-called over-order multiplicities corresponding to *real double* and *triple* characteristic roots.

Keywords: Functional differential equations, Neutral equations, Control theory, Control design, Stability theory, Stabilization of dynamical systems, partial pole placement.

1. INTRODUCTION

Dynamical systems with delays provide useful models in a wide range of scientific and technological domains such as biology, chemistry, economics, physics, or engineering, where the presence of the delays is inherent to propagation phenomena, such as of material, energy, or information, with a finite propagation speed. Due to their numerous applications, these kinds of systems have been the subject of much attention by researchers in several fields, in particular since the 1950s and 1960s. More precisely, and to the best of the authors' knowledge, modeling *propagation* and *transport* phenomena by delay-differential algebraic equations dates back to the 50s; see, for instance, a few examples in [1], [2], [3], [4] and the references therein.

On the one hand, various electrical and fluid dynamical systems initially described by partial differential equations (PDEs) of hyperbolic type with mixed initial, and derivative boundary conditions in feedback interconnection, can be integrated along the characteristics to arrive at a set of delay differential-algebraic equations (DDAEs), i.e. coupled delay-differential equations and -difference equations in continuous-time (see, e.g. [5], [6]). For a more comprehensive introduction to the subject including a long list of references we refer to Răsvan [7]. On the other hand, the presence of a delay in the input-output channels in the case of proper dynamical systems may lead to DDAEs for the closed-loop schemes (see, e.g., [4], [8]), whose stability may be sensitive to the delay parameter as shown in [9].

An interesting property, entitled *multiplicity-induced-dominancy* (MID in short), which corresponds to conditions on the system's parameters for which a multiple root defines the spectral abscissa[1] of the corresponding quasipolynomial. The MID property applying for the spectrum of the linear delay-differential equations (DDEs), was recently introduced in [10] and proved in [11, 12] in the *generic* MID case (GMID for short), which corresponds to roots whose multiplicity is equal with the degree of the corresponding

*Corresponding author
Email addresses: amina.benarab@centralesupelec.fr (Amina Benarab), islam.boussaada@centralesupelec.fr (Islam Boussaada), silviu.niculescu@centralesupelec.fr (Silviu-Iulian Niculescu), karim.trabelsi@ipsa.fr (Karim L. Trabelsi)

[1] The spectral abscissa is defined by the real part of the rightmost root of the spectrum of the corresponding characteristic function; see, for instance, [4] and the references therein for a deeper discussion of the notion.

quasipolynomial defines necessarily the spectral abscissa in both retarded and neutral cases. The GMID is shown by using an appropriate integral factorization of the corresponding quasipolynomial which appears to be nothing but a Kummer hypergeometric function (see also [13]). Such a multiplicity is called *generic*, and it is always larger than the degree of the delay-free polynomial. By exploiting different properties of Whittaker (confluent hypergeometric) functions, its extension to neutral DDEs can be found in [12]. For an overview of existing methods for characterizing multiple characteristic roots we refer to [14].

As discussed in the references above, MID triggers interesting perspectives in the control area by providing a new methodology based on the so-called *partial pole placement*; see, for instance, some examples and discussions in [15] on human balancing and [16] on vibration control. In our opinion, the said method is easy to implement, and gives an explicit tuning rule for a prescribed decay rate of the solutions of the closed-loop system. Finally, it should be mentioned that this method further exploits the idea of using the delay as a *control parameter* (see also [17] for an overview on existing results).

To the best of the authors' knowledge, excepting some sufficient conditions proposed by [18], there are no explicit proofs of the MID property holding in the non-generic case, and this paper offers new insights for a better understanding of the property. With the remarks above, the aim of this paper is twofold. First, a full characterization of the MID property in the case of scalar neutral systems in both generic and non-generic cases is carried out. Indeed, if the generic case corresponds to the triple characteristic root located on the real axis (see, e.g., [11], [12]), the non-generic case (double root) corresponds to the so-called *over-order (intermediate) multiplicity*[2] (IMID). Surprisingly, although the stability of the scalar neutral DDE was fully addressed in the open literature and complete characterization of the stability regions in the parameter space exists (see, for instance, [19, 8, 1, 2, 3], however, the link between the multiplicity of the real roots and the corresponding spectral abscissa was not explicitly characterized. However, it should be mentioned that, in characterizing the stability charts in the scalar neutral DDE case, Wright [20] observed that the characteristic function can have three real roots but there is no an explicit discussion regarding double and/or triple real characteristic roots.

Second, in the non-generic case (that is, double and triple real roots), the analysis exhibits the advantages and the limitations of the MID with respect to the corresponding "free" parameter and reinforces the idea that the delay, seen as a *control parameter*, can be beneficial in closed-loop.

The remaining of the paper is organized as follows. Some preliminary results as well as a motivating example are presented in Section 2. Section 3 includes the main results as well as various discussions on the over-order multiplicities (double and triple characteristic real roots). Some remarks conclude the paper.

Throughout this paper, the following notations are used: $\mathbb{R}(\mathbb{R}_+)$ and $\mathbb{C}$ denote the sets of real (positive) numbers and the set of complex numbers, respectively. For a complex number λ, $\Re(\lambda)$ ($\Im(\lambda)$) denote its real (imaginary) part. Finally, for a (quasi)polynomial $P(\cdot)$, $\deg(P)$ denotes its degree.

2. PREREQUISITES

In the study of linear time-invariant (LTI) dynamical delay systems, we deal with transfer functions involving quasipolynomials, which are defined hereafter.

Definition 2.1. *A quasipolynomial is a particular entire function $\Delta : \mathbb{C} \times \mathbb{R}_+^k \mapsto \mathbb{C}$ which may be written as follows*

$$\Delta(s;\tau_1,\cdots,\tau_i) = \sum_{i=0}^{k} P_i(s)\,\mathrm{e}^{-\tau_i s}, \tag{1}$$

where k is a positive integer, τ_i ($i = 0..k$) are pairwise distinct non-negative real numbers and P_i ($i = 0..k$) are polynomials of degree $d_i \geq 0$. The degree D of the quasipolynomial Δ is equal to the sum of the degrees of the involved polynomials P_i plus the number of delays, i.e.,

$$D = k + \sum_{i=0}^{k} d_i.$$

[2]multiplicity larger than the degree of the corresponding polynomial in the delay-free case and smaller than the degree of the quasipolynomial

An important result in the open literature, known as *Polya-Szegö bound*, shows that there exists an explicit link between the degree of a quasipolynomial and the number of its roots in horizontal strips of the complex plane $\mathbb{C}$.

Proposition 2.1. *[21, Problem 206.2, page 144 and page 347]. Let Δ be a quasipolynomial of degree D as in* (1), *and α, $\beta \in \mathbb{R}$ satisfy $\alpha \leq \beta$. If M is the number of roots of Δ contained in the set $\{s \in \mathbb{C} \,|\, \alpha \leq \Im(s) \leq \beta\}$ counting multiplicities, then*

$$\frac{(\tau_k - \tau_0)\,(\beta - \alpha)}{2\pi} - D \leq M \leq \frac{(\tau_k - \tau_0)\,(\beta - \alpha)}{2\pi} + D.$$

Furthermore, for a given root $s_0 \in \mathbb{C}$ of a quasipolynomial Δ, one obtains the following link between the multiplicity of s_0 and the degree of Δ.

Corollary 2.1. *Let Δ be a quasipolynomial of degree D. Then, any root $s_0 \in \mathbb{C}$ of Δ exhibits a multiplicity at most equal to D.*

Remark 2.1. *Corollary 2.1 is obtained immediately by letting $\alpha = \beta = \Im(s_0)$ in Proposition 2.1. Notice also that Polya-Szegö bound has been recovered in [22] using a constructive approach based on* functional Birkhoff matrices*. Furthermore, if some coefficients of the polynomials P_i defined in* (1) *vanish, then a sharper bound for the multiplicity is provided in [22].*

In what follows, we give a precise definition of the *dominant root*.

Definition 2.2. *A spectral eigenvalue (root) s_0 is said to be a dominant (respectively, strictly dominant) root of Δ, if the following inequality holds $\Re(\tilde{s}) \leq \Re(s_0)$ (respectively, $\Re(\tilde{s}) < \Re(s_0)$) for any $\tilde{s} \in \mathbb{C} \backslash \{s_0\}$, a distinct eigenvalue (root) of Δ.*

2.1. Motivating example: Feedback stabilization for a scalar conservation law with PI boundary control

Consider the problem of stabilization of solutions of a dynamical system described by partial differential equations. More precisely, we revisit the problem of exponential stabilization of the following scalar conservation law proposed in [23, 12]:

$$\partial_t \varphi(t,x) + \lambda\, \partial_x \varphi(t,x) = 0, \quad t \in [0,\infty), \quad x \in (0,L), \tag{2}$$

where $L > 0$ and $\varphi(t,x)$ denotes the system state at position $x \in (0,L)$ and in time $t \in [0,+\infty)$. As considered in [23], the value λ, which denotes the velocity of propagation, is assumed to be a positive constant. Equation (2) comes with a boundary condition under the form of a PI controller:

$$\varphi(t,0) = k_p\, \varphi(t,L) + k_i \int_0^t \varphi(\nu,L)\,\mathrm{d}\nu, \tag{3}$$

where k_p and k_i are the feedback parameters representing “proportional” and “integral” control gains. Applying the Laplace transform to both sides of the boundary condition and multiplying by s, one obtains the closed-loop characteristic function

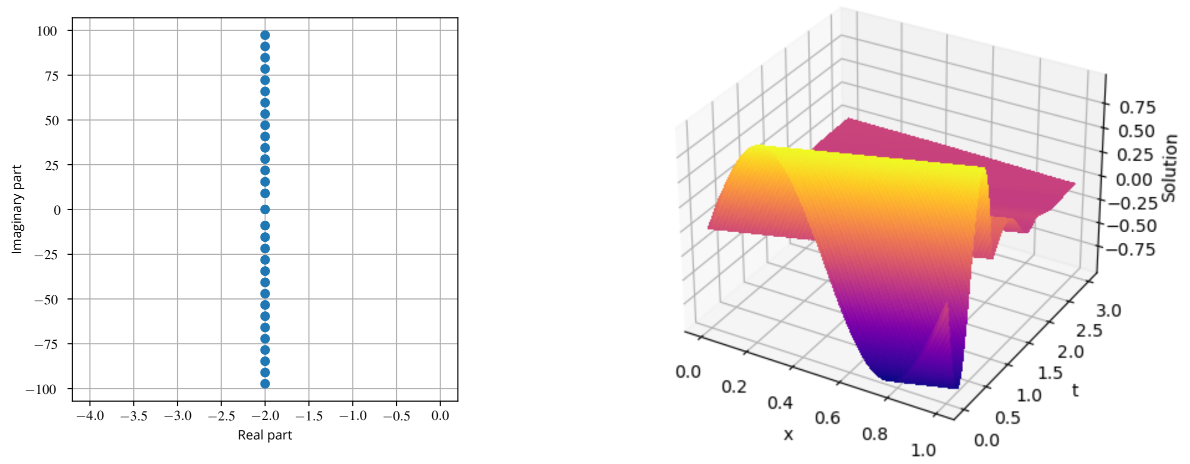
$$\Delta(s) = s - (k_i + k_p\, s)e^{-\frac{L}{\lambda}s}, \tag{4}$$

which corresponds to the characteristic function of a first-order neutral DDE. In this case, the degree $\mathscr{D}_{PS}$ of Δ is equal to 3 and, as mentioned in [12], the maximal multiplicity can be achieved only by a root on the real axis.

Next, by exploiting the results of Theorem 3.1, Theorem 3.6, and Theorem 3.10 from [12], or Theorem 4 from [24], we conclude that forcing a triple spectral value guarantees its dominance as a root of (4), and then the exponential stability of solutions of (2)–(3). More precisely, by tuning the controller gains as

$$k_p = -e^{-2}, \quad k_i = -\frac{4\,e^{-2}\,\lambda}{L}, \tag{5}$$

one achieves the unique admissible triple root, which is $s_0 = -\frac{2\lambda}{L}$ and corresponds to the decay rate of solutions of (2)–(3). Furthermore, as shown in Theorem 3.10 from [12], the set of roots of Δ is $\left\{ s_0 + i\frac{\lambda\zeta}{L} \;\middle|\; \zeta \in \Xi_1 \right\}$ where $\Xi_1 = \left\{ \zeta \in \mathbb{R} \;\middle|\; \tan\left(\frac{\zeta}{2}\right) = \frac{\zeta}{2} \right\}$. Figure 1(a) shows the result of a numerical computation of the roots of (4) with the parameters (5), while Figure 1(b) shows the solution of (2)–(3) in the case $\frac{L}{\lambda} = 1$ with an initial condition $\varphi(0,x) = \sin(2\pi x)$.



(a) (b)

Figure 1: (a) Spectrum distribution of (4) and (b) solution of (2) with initial condition $\varphi(0,x) = \sin(2\pi x)$, with $\frac{L}{\lambda} = 1$ and parameters k_p and k_i satisfying (5). Figure taken from [12].

3. Main Results

3.1. First-order neutral equation

Under appropriate initial conditions, consider the following scalar neutral DDE:

$$\dot{x}(t) + a_0 x(t) + \alpha_1 \dot{x}(t-\tau) + \alpha_0 x(t-\tau) = 0 \tag{6}$$

with *four (real) parameters* $(a_0, \alpha_0) \in \mathbb{R}^2$, $\alpha_1 \in (-1,1)$ and $\tau \in \mathbb{R}_+$.

To the best of the authors' knowledge, the first study in frequency-domain concerning the root location of scalar DDEs of retarded type goes back to the 30s and it concerns the Kalecki' dynamical model of an economic system [25]. Next, the complete characterization of the stability regions in the parametric space for the scalar DDE goes back to the 50s and the works of Hayes [26]. By the end of the 50s, Pinney [19] constructs the stability charts in the scalar and second-orded DDEs, covering both retarded and neutral cases. It should be mentioned that the first analysis in the second-order neutral and retarded cases can be found in the works of Callender *et al.* [27].

Regarding (6), note that the condition imposed to the parameter α_1 corresponds to the stability of the trivial solution of the associated scalar delay-difference equation (in continuous-time), and it is known that its exponential stability is a *necessary condition* for the exponential stability of the null-solution of the DDEs of neutral type. For further discussions and explanations, we refer to [28, 8, 1], [4] and the references therein.

The corresponding characteristic function $\Delta : \mathbb{C} \times \mathbb{R}^2 \times (-1,1) \times \mathbb{R}_+ \mapsto \mathbb{C}$ reads as:

$$\Delta(s; a_0, \alpha_0, \alpha_1, \tau) = s + a_0 + (\alpha_1 s + \alpha_0)\,\mathrm{e}^{-\tau s}. \tag{7}$$

It is easy to observe that $\deg(\Delta) = 3$, and Δ reduces to a polynomial of degree 1 if $\tau = 0$. Thus, the cases of *double*, *triple characteristic roots* correspond to the only situations where we have *over-order multiplicity*. Such situations are specific to dynamical delay systems and have no natural meaning in the finite-dimensional case. Finally, it should be mentioned that the maximal multiplicity of the characteristic roots located on the imaginary axis out from the origin is equal to 1, and therefore such a root (if it exists) is *simple*.

Remark 3.1. *It should be mentioned that, in the 60s, Wright [20] observed that the characteristic function Δ given by* (7) *can have three real roots but he has not explicitly analyzed the link between their multiplicity and the spectral abscissa. His argument was based on appropriate change of parameters and it did not exploit the degree of the corresponding quasipolynomial.*

Theorem 3.1 (over-order multiplicities)**.** *Consider the characteristic function Δ defined by* (7)*.*

1. *GMID [24] : spectral value of maximal admissible multiplicity*
   - *The real s_0 is a root of maximal multiplicity 3 of Δ if, and only if, the coefficients a_0, α_0, α_1, the root s_0 and the delay τ satisfy the following relations*

$$a_0 = -s_0 - \frac{2}{\tau}, \qquad \alpha_0 = \left(-s_0 + \frac{2}{\tau}\right)\mathrm{e}^{s_0\tau}, \qquad \alpha_1 = \mathrm{e}^{\tau s_0}. \tag{8}$$

   - *If relations* (8) *are satisfied then s_0 is necessarily a dominant root of Δ.*
2. *IMID : codimension 2*
   - *The real number s_0 is a root of intermediate multiplicity 2 of Δ if, and only if, the following relations hold*

$$\alpha_0 = \left(\tau a_0 s_0 + \tau {s_0}^2 - a_0\right)\mathrm{e}^{\tau s_0}, \qquad \alpha_1 = (-\tau a_0 - \tau s_0 - 1)\,\mathrm{e}^{\tau s_0}. \tag{9}$$

   - *If the relations are satisfied and a_0 satisfies the lower bound $a_0 > 0$, then s_0 chosen such that*

$$-a_0 - \frac{1}{\tau} \leq s_0 \leq -a_0. \tag{10}$$

   is a dominant root of Δ.

Proof. The proof follows the steps of an algorithm introduced in [29], and it consists of five steps: forcing multiplicity, normalization, appropriate (Fredholm) integral representation, explicit frequency bound estimation, and dominancy.

1. Proof of Item 1: it can be found in [24] and it is summarized as follows:
   (a) Forcing multiplicity: The real s_0 is a root of multiplicity 3 of Δ if, and only if, the coefficients a_0, α_0, α_1, the root s_0 and the delay τ satisfy the following relations

$$a_0 = -s_0 - \frac{2}{\tau}, \qquad \alpha_0 = \left(-s_0 + \frac{2}{\tau}\right)\mathrm{e}^{s_0\tau}, \qquad \alpha_1 = \mathrm{e}^{\tau s_0}. \tag{11}$$

   (b) Normalization: Performing the translation and scaling of the spectrum by the following change of variables

$$\tilde{\Delta}(z) = \tau\Delta(z/\tau + s_0) \tag{12}$$

   for $z \in \mathbb{C}$, we get the following normalized characteristic function $\tilde{\Delta} : \mathbb{C} \mapsto \mathbb{C}$,

$$\tilde{\Delta}(z) = z + b_0 + (\beta_1 z + \beta_0)\,\mathrm{e}^{-z} \tag{13}$$

with relations (11) normalized as follows:

$$b_0 = \tau\,(a_0 + s_0)\,, \qquad \beta_0 = \tau\,(\alpha_1\, s_0 + \alpha_0)\,\mathrm{e}^{-\tau s_0}, \qquad \beta_1 = \alpha_1\,\mathrm{e}^{-\tau s_0}. \tag{14}$$

It is easy to observe that $\deg(\tilde{\Delta}) = \deg(\Delta) = 3$.

(c) Integral representation: The real root s_0 is a root of multiplicity 3 of Δ if, and only if, 0 is a triple root of $\tilde{\Delta}$, that is:

$$\tilde{\Delta}(0) = \tilde{\Delta}'(0) = \tilde{\Delta}''(0) = 0. \tag{15}$$

The latter identities yield a linear system whose unique solution is $(b_0, \beta_0, \beta_1) = (-2, 2, 1)$. From relations (14), one concludes that s_0 is a root of multiplicity 3 of Δ if, and only if, relations (11) hold. Moreover, under the latter conditions, the quasipolynomial (7) reduces to

$$\tilde{\Delta}(z) = \tilde{P}_0(z) + \tilde{P}_1(z)\,\mathrm{e}^{-z}, \quad \tilde{P}_0(z) = z - 2 \quad \text{and} \quad \tilde{P}_1(z) = z + 2. \tag{16}$$

Hence, the quasipolynomial $\tilde{\Delta}$ admits the following Fredholm integral representation

$$\tilde{\Delta}(z) = \int_0^1 q(t)\,\mathscr{K}(z,t)\,\mathrm{d}t, \quad q(t) = t\,(1-t) \quad \text{and} \quad \mathscr{K}(z,t) = z^3 \mathrm{e}^{-tz} \tag{17}$$

which is easily verified via an integration by parts.

(d) Frequency bound: Assume that $z_0 = x_0 + i\,\omega_0 \in \mathbb{R}_+ + i\mathbb{R}_+$ is a root of $\tilde{\Delta}$, so that $\tilde{\Delta}(z_0) = 0$ if, and only if,

$$|\tilde{P}_0(x_0 + i\omega_0)|^2\,\mathrm{e}^{2x_0} = |\tilde{P}_1(x_0 + i\omega_0)|^2. \tag{18}$$

Considering a truncation of order 1 of the exponential term e^{2x}, the latter is lower bounded by $1 + 2x$. Next, define

$$F(x,\omega) = |\tilde{P}_1(x + i\omega)|^2 - (1 + 2x)\,|\tilde{P}_0(x + i\omega)|^2 \tag{19}$$

where $F > 0$ for any $x > 0$. The zeros of F are characterized by the first-order polynomial

$$G(\Omega = \omega^2) = -2x\Omega - 2x^3 + 8x^2. \tag{20}$$

The polynomial function G admits a single real root $\Omega_0(x) = -x\,(x-4)$, which reaches a maximum value at $x^* = 2$. As a consequence, Ω_0 is bounded by $\Omega^* = 4 < \pi^2$. Thus, one obtains the desired frequency bound,

$$0 < \omega \leq 2 < \pi. \tag{21}$$

(e) Dominancy: The goal of the frequency bound is to prove the dominancy by a contradiction argument. For this purpose, assume that there exists $z_0 \in \mathbb{R}_+ + i\mathbb{R}_+$ root of $\tilde{\Delta}$. Then, the integral representation yields

$$\int_0^1 t\,(1-t)\,\mathrm{e}^{-tz_0}\,\mathrm{d}t = 0, \tag{22}$$

the imaginary part of which is

$$\int_0^1 t\,(1-t)\,\mathrm{e}^{-tx}\sin(\omega t)\mathrm{d}t = 0. \tag{23}$$

Now, the frequency bound $0 < \omega \leq \pi$ of the previous step entails that the function

$$t \mapsto t\,(1-t)\,\mathrm{e}^{-xt}\sin(\omega t) \tag{24}$$

is strictly positive in $(0,1)$, thereby contradicting the last equality.

2. Proof of Item 2:

(a) Forcing multiplicity: The real number s_0 is a root of multiplicity 2 of Δ if, and only if, the coefficients α_0, α_1, the root s_0 and the delay τ satisfy the relations below

$$\alpha_0 = \left(\tau a_0 s_0 + \tau {s_0}^2 - a_0\right) \mathrm{e}^{\tau s_0} \quad \text{and} \quad \alpha_1 = (-\tau a_0 - \tau s_0 - 1)\, \mathrm{e}^{\tau s_0}. \tag{25}$$

(b) Normalization: Performing the translation and scaling of the spectrum by the following linear change of variables

$$\tilde{\Delta}(z) = \tau \Delta(z/\tau + s_0) \tag{26}$$

for $z \in \mathbb{C}$, we get the following normalized characteristic function $\tilde{\Delta} : \mathbb{C} \mapsto \mathbb{C}$,

$$\tilde{\Delta}(z) = ((-\rho - 1)\, z - \rho)\, \mathrm{e}^{-z} + z + \rho, \tag{27}$$

where $\rho = \tau\,(s_0 + a_0)$ is a real number.

(c) Integral representation: It can be verified via an integration by parts that the quasipolynomial $\tilde{\Delta}$ defined in (27) can be factorized as

$$\tilde{\Delta}(z) = z^2 \int_0^1 q_\rho(t)\, \mathrm{e}^{-tz}\, \mathrm{dt} \tag{28}$$

where

$$q_\rho(t) = \rho\, t + 1. \tag{29}$$

In our approach, the sign constancy of the polynomial q_ρ for $t \in (0,1)$ is necessary. We easily see that it is guaranteed if, and only if, $\rho \in [-1, +\infty[$.

(d) Frequency bound: In the following, let $z_0 = x_0 + i\omega_0 \in \mathbb{R}_+ + i\mathbb{R}_+$ be a root of

$$\tilde{\Delta}(z) = P_0(z) + P_1(z)\, \mathrm{e}^{-z}, \quad P_0(z) = z + \rho, \quad \text{and} \quad P_1(z) = (-\rho - 1)\, z - \rho, \tag{30}$$

as defined in (27) and z_0 satisfies the following equality

$$|P_0(x_0 + i\omega_0)|^2 \mathrm{e}^{2x_0} = |P_1(x_0 + i\omega_0)|^2. \tag{31}$$

Considering a truncation of order 2 of the exponential term e^{2x}, the latter is lower bounded by $1 + 2x + 2x^2$ for any $x \in \mathbb{R}_+$, see [24, 29]. Next, we define the following function $F_\rho : \mathbb{R}_+^2 \mapsto \mathbb{R}$,

$$F_\rho(x, \omega) = |P_1(x + i\omega)|^2 - |P_0(x + i\omega)|^2\, (1 + 2x), \tag{32}$$

which satisfies $F_\rho(x_0, \omega_0) > 0$. Moreover, the zeros of F_ρ can be characterized by the following linear polynomial in $\Omega = \omega^2$

$$G_\rho(x, \Omega) = \left(\rho^2 - 2x^2 + 2\rho - 2x\right) \Omega - 2x^4 - 2\,(2\rho + 1)\, x^3 - \rho\,(\rho + 2)\, x^2. \tag{33}$$

In our analysis, we are interested in the cases where $G_\rho(x, \Omega)$ is bounded. Hence, let $\Omega_\rho^0(x) = N_\rho(x)/D_\rho(x)$ be the real solution of $G_\rho(x, \Omega)$, where

$$N_\rho(x) = x^2 \left(\rho^2 + 4\rho x + 2x^2 + 2\rho + 2x\right) \quad \text{and} \quad D_\rho(x) = \rho^2 - 2x^2 + 2\rho - 2x.$$

We proceed by setting an upper bound for the solution $\Omega_\rho^0(x)$ with respect to the variation of the parameter ρ in $(-1, +\infty)$. First, note that the denominator $D(x, \rho)$ admits for $\rho \in \left(-1 + \frac{\sqrt{2}}{2}, +\infty\right)$ two real roots:

$$x_D^{\pm}(\rho) = -\frac{1}{2} \pm \frac{\sqrt{2\rho^2 + 4\rho + 1}}{2}. \tag{34}$$

Then, we divide our analysis into two parts:

i. $\rho \in \left(-1, -1+\frac{\sqrt{2}}{2}\right)$:

- The numerator $N_\rho(x)$ admits 4 real roots:

$$\left\{0, 0, x_N^{\pm}(\rho) = -\rho - \frac{1}{2} \pm \frac{\sqrt{2\rho^2+1}}{2}\right\}, \tag{35}$$

  where $N_\rho(x)$ is negative for $x \in (0, x_N^+(\rho))$ and is positive for $x > x_N^+(\rho)$.
- The denominator keeps a constant (negative) sign, since it has no real roots.

Note that, since $\Omega_\rho^0(x)$ is required to be positive, we only consider $x \in (0, x_N^+(\rho))$, case in which $N_\rho(x)$ and $D_\rho(x)$ are both of negative sign. Then, an upper bound for $\Omega_\rho^0(x)$ requires lower bounds for both $N_\rho(x)$ and $D_\rho(x)$:

$$N_\rho(x) \geq -x^2\left(-4x^2+2\sqrt{2}+4x+1\right)/2, \tag{36}$$

$$D_\rho(x) \geq -1/2 - \sqrt{2} - 2x^2 - 2x, \tag{37}$$

which yields the following upper bound for the solution $\Omega_\rho^0(x)$:

$$\Omega_\rho^0(x) \leq \Omega^*(x) = \frac{x^2\left(-4x^2+2\sqrt{2}+4x+1\right)}{4x^2+2\sqrt{2}+4x+1}, \tag{38}$$

which itself is bounded by π^2; see Figure 2-(a).

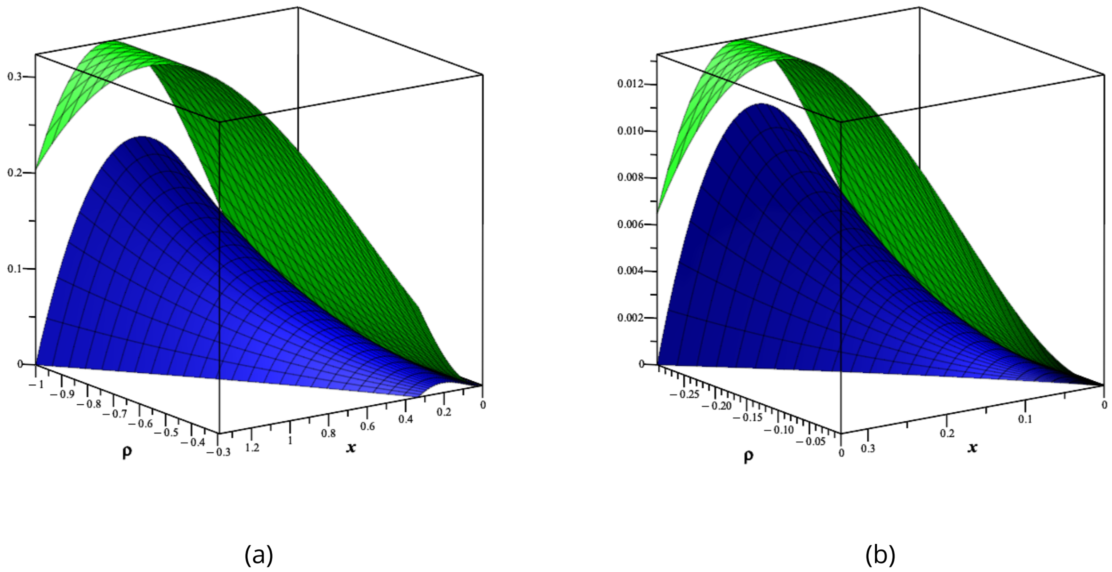


Figure 2: (a) 3D plot showing how $\Omega_0(x,\rho)$ (in blue) is upper bounded by $\Omega_0^*(x,\rho)$ (in green) for $\rho \in \left(-1, -1+\frac{\sqrt{2}}{2}\right)$ and $x \in (0, x_N^+)$ (b) 3D plot showing how $\Omega_0(x,\rho)$ (in blue) is upper bounded by $\Omega_0^*(x,\rho)$ (in green) for $\rho \in \left(-1+\frac{\sqrt{2}}{2}, 0\right)$ and $x \in (0, x_N^+)$.

ii. $\rho \in \left(-1+\frac{\sqrt{2}}{2}, 0\right)$: In this case, one can notice that $x_D^+(\rho) < 0$, which implies that $D_\rho(x) < 0$. Since $\Omega_\rho^0(x)$ is required to be positive, we only consider $x \in (0, x_N^+(\rho))$. Using a similar analysis, in order to set an upper bound for $\Omega^0_\rho(x)$, we first set a lower bound for both $N_\rho(x)$ and $D_\rho(x)$:

$$N_\rho(x) \geq x^2\left(2x\sqrt{2}+2x^2+\sqrt{2}-2x-2\right), \tag{39}$$

$$D_\rho(x) \geq -2-2x^2+\sqrt{2}-2x, \tag{40}$$

which yields the following upper bound of the solution $\Omega^0_\rho(x)$:

$$\Omega^0_\rho(x) \leq \Omega^*(x) = \frac{x^2\left(2x\sqrt{2}+2x^2+\sqrt{2}-2x-2\right)}{-2-2x^2+\sqrt{2}-2x} \tag{41}$$

which itself is bounded by π^2; see Figure 2-(b).

Now, is the solution $\Omega^0_\rho(x)$ is upper bounded with respect to ρ by the following parameter-free function

$$\Omega^*(x) = \begin{cases} \dfrac{x^2\left(-4x^2+2\sqrt{2}+4x+1\right)}{4x^2+2\sqrt{2}+4x+1} & \text{for } \rho \in \left(-1,-1+\frac{\sqrt{2}}{2}\right) \text{ and } x \in \left(0, x_N^+(\rho)\right), \\ \dfrac{x^2\left(2x\sqrt{2}+2x^2+\sqrt{2}-2x-2\right)}{-2-2x^2+\sqrt{2}-2x} & \text{for } \rho \in \left(-1+\frac{\sqrt{2}}{2},0\right) \text{ and } x \in \left(0, x_N^+(\rho)\right). \end{cases} \tag{42}$$

Hence, we have characterized the regions where the frequency is bounded.

(e) Dominancy: By a contradiction argument, assume that there exists $z_0 \in \mathbb{R}_+ + i\mathbb{R}_+$ root of $\tilde{\Delta}$. Then, the integral representation yields

$$\int_0^1 (\rho t+1)\,\mathrm{e}^{-t z_0}\,\mathrm{d}t = 0, \tag{43}$$

the imaginary part of which is $\int_0^1 t\,(\rho t+1)\,\mathrm{e}^{-tx}\sin(\omega t)\mathrm{d}t = 0$. Now, the frequency bound $0 < \omega \leq \pi$ of the previous step entails that the function

$$t \mapsto t\,(\rho t+1)\,\mathrm{e}^{-xt}\sin(\omega t) \tag{44}$$

is strictly positive in $(0,1)$, thereby contradicting the last equality.

To conclude, if relations (25) are verified and a_0 satisfies the lower bound $a_0 > 0$, then the exponential decay s_0 is chosen such that

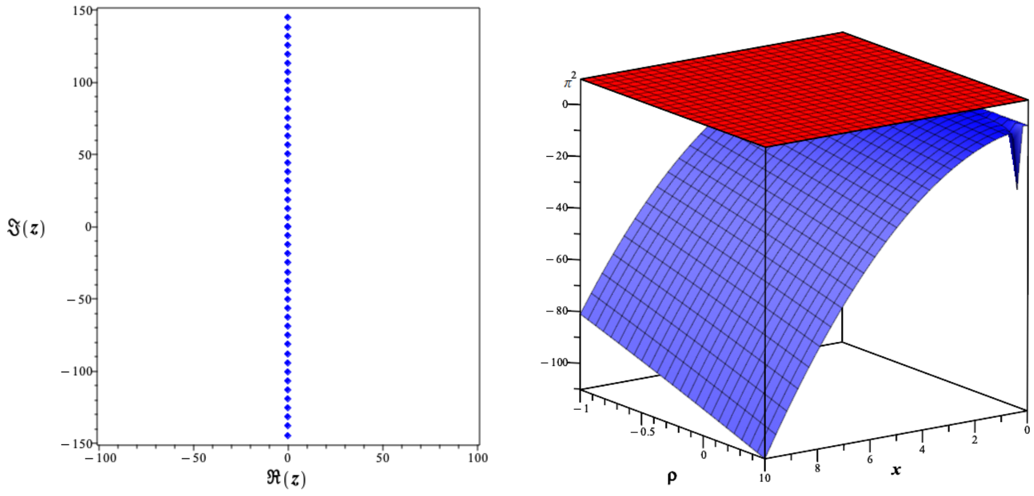
$$-a_0 - \frac{1}{\tau} \leq s_0 \leq -a_0 \tag{45}$$

is necessarily negative and dominant.

□

Remark 3.2. *Note that in the over-order MID (IMID) case (item 2 of Theorem 3.1), one may expect to obtain a larger range to assign the dominant root albeit for truncation orders of the exponential term in the "frequency bound" step that are strictly greater than 2, as illustrated in Figure 3(b).*

Remark 3.3. *As pointed out in [12], the GMID does not allow any degree of freedom in allocating s_0. Indeed, if a_0 and τ are fixed then the assigned spectral abscissa is uniquely determined from* (8) *as $s_0 = -a_0 - \frac{2}{\tau}$. In order to allow some leeway when allocating, as illustrated in* (45)*, one can relax the constraint by forcing the root s_0 to have an over-order multiplicity albeit lower than the maximal one. This fact is important from a robustness perspective.*



(a) (b)

Figure 3: (a) Spectrum distribution of the quasipolynomial function $\hat{\Delta}(z)$ (b) With a truncation order of the exponential term raised to 4 the 3D plot illustrates how $\Omega_0(x,\rho)$ (in blue) is upper bounded by π^2 (in red) for $\rho \in \left(-1, \frac{1}{2}\right)$ and $x > 0$, which enlarges the assignment range.

To illustrate it, reconsider the motivating example, that is the feedback stabilization of a scalar conservation law with PI boundary control presented in section 2.1. Following (10)*, an assignable s_0 as a dominant double root of Δ has to satisfy*

$$-\frac{\lambda}{L} \leq s_0 \leq 0,$$

which occurs if

$$k_i = -\frac{L{s_0}^2}{\lambda}\mathrm{e}^{\frac{Ls_0}{\lambda}}, \qquad k_p = -\frac{\lambda + Ls_0}{\lambda}\mathrm{e}^{\frac{Ls_0}{\lambda}}.$$

Remark 3.4. *As emphasized in [12, 18], the over-order MID property has an interesting link with the well-known* Kummer confluent hypergeometric functions. *As a matter of fact, quasipolynomials with dominant roots of over-order multiplicities can be represented in terms of such special functions. As it was developed by E. Kummer, P. Humbert, E. T. Whittaker, F. Tricomi, L. Erdelyi and others, see, for instance, [30, 31, 32, 33], for every $a,b,z \in \mathbb{C}$ such that $\Re(b) > \Re(a) > 0$, Kummer functions admit the integral representation*

$$\Phi(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 \mathrm{e}^{\lambda t} t^{a-1}(1-t)^{b-a-1}\,\mathrm{d}t. \tag{46}$$

In particular, the quasipolynomial (7) *whose parameters satisfy* (8)*, i.e. the GMID is satisfied, can be written as:*

$$\Delta(s) = \frac{\tau^2}{6}\,(s-s_0)^3\,\Phi(2,4,-\tau\,(s-s_0)).$$

However, the quasipolynomial (7) *whose parameters satisfy* (9) *admits a double root at s_0 arbitrarily chosen in accordance with* (10)*, i.e. the IMID is satisfied, can be written as a linear combination of two Kummer functions:*

$$\Delta(s) = \tau\,(s-s_0)^2\left(\frac{\tau\,(s_0+a_0)}{2}\,\Phi\,(2,3,-\tau\,(s-s_0)) + \Phi\,(1,2,-\tau\,(s-s_0))\right).$$

4. CONCLUSION

In this paper, we provided a complete characterization of the multiplicity-induced-dominancy (MID) property for single-delay linear first-order delay-differential equation (DDE) thanks to the five-step algorithm described in [24, 29]. Despite the complete characterization of the generic multiplicity-induced-dominancy (GMID) property for single-delay systems of arbitrary order [12], the intermediate multiplicity-induced-dominancy (IMID) property, which is more suited for control purposes, remains an open question. Further effort devoted to studying the distribution of zeros of Kummer's confluent hypergeometric functions seems to be the key to answering this question.

References

[1] J. K. Hale, S. Verduyn Lunel, Introduction to functional differential equations, Vol. 99 of Applied Mathematics Sciences, Springer Verlag, New York, 1993.

[2] S.-I. Niculescu, Delay effects on stability. A robust control approach, Vol. 269, Springer: London, LNCIS, 2001.

[3] K. Gu, V. L. Kharitonov, J. Chen, Stability of time-delay systems, Control Engineering, Birkhäuser Boston, Inc., Boston, MA, 2003.

[4] W. Michiels, S.-I. Niculescu, Stability, control, and computation for time-delay systems: an eigenvalue-based approach, SIAM, 2014.

[5] R. K. Brayton, Small-signal stability criterion for electrical networks containing lossless transmission lines, IBM Journ. Res. Develop.

[6] K. L. Cooke, D. W. Krumme, Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations, J. Math. Analysis Appl. 24 (1968) 372–387.

[7] V. Răsvan, Propagation, delays, and stability (robust versus fragile), Int. J. Robust Nonlin. Contr. 32 (6) (2022) 3225–3250.

[8] G. Stépán, Retarded Dynamical Systems: Stability and Characteristic Functions, Pitman research notes in mathematics series, Longman Sci. and Tech., 1989.

[9] H. Logemann, R. Rebarber, G. Weiss, Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop, SIAM J. Contr. Optim. 34 (1996) 572–600.

[10] I. Boussaada, S.-I. Niculescu, A. El-Ati, R. Pérez-Ramos, K. Trabelsi, Multiplicity-induced-dominancy in parametric second-order delay differential equations: analysis and application in control design, ESAIM Cont. Optim. Calc. Var. 26 (2020) Paper No. 57, 34.

[11] G. Mazanti, I. Boussaada, S.-I. Niculescu, Multiplicity-induced-dominancy for delay-differential equations of retarded type, J. Differ. Equ. 286 (2021) 84–118.

[12] I. Boussaada, G. Mazanti, S.-I. Niculescu, The generic multiplicity-induced-dominancy property from retarded to neutral delay-differential equations: When delay-systems characteristics meet the zeros of kummer functions, C. R. Math. Acad. Sci. Paris 360 (2022) 349–369.

[13] I. Boussaada, H. Unal, S.-I. Niculescu, Multiplicity and stable varieties of time-delay systems: A missing link, in: Proceeding of the 22nd International Symposium on Mathematical Theory of Networks and Systems, 2016, pp. 1–6.

[14] S.-I. Niculescu, I. Boussaada, X.-G. Li, G. Mazanti, C.-F. Méndez-Barrios, Stability, delays and multiple characteristic roots in dynamical systems: A guided tour, IFAC-PapersOnLine 54 (18) (2021) 222–239.

[15] T. Balogh, I. Boussaada, T. Insperger, S.-I. Niculescu, Conditions for stabilizability of time-delay systems with real-rooted plant, International Journal of Robust and Nonlinear Control 32 (6) (2022) 3206–3224.

[16] I. Boussaada, S. Tliba, S.-I. Niculescu, H. Unal, T. Vyhlídal, Further remarks on the effect of multiple spectral values on the dynamics of time-delay systems. application to the control of a mechanical system, Linear Algebra and its Applications 542 (2018) 589 – 604, proceedings of the 20th ILAS Conference, Leuven, Belgium 2016.

[17] R. Sipahi, S. Niculescu, C. Abdallah, W. Michiels, K. Gu, Stability and stabilization of systems with time delay: limitations and opportunities, IEEE Control Syst. Mag. 31 (1) (2011) 38–65. doi:10.1109/MCS.2010.939135.

[18] I. Boussaada, G. Mazanti, S.-I. Niculescu, A. Benarab, MID Property for Delay Systems: Insights on Spectral Values with Intermediate Multiplicity, in: Proc. of the 61st IEEE Conf. on Decision and Control, Cancun, Mexico, 2022.

[19] E. Pinney, Ordinary difference-differential equations, Univ. California Press, 1958.

[20] E. M. Wright, Stability criteria and the real roots of a transcedental equation, J. Soc. Indust. Applied Math. 9 (1961) 136–148.

[21] G. Pólya, G. Szegő, Problems and Theorems in Analysis, Vol. I: Series, Integral Calculus, Theory of Functions, Springer-Verlag, New York, Heidelberg, and Berlin, 1972.

[22] I. Boussaada, S.-I. Niculescu, Characterizing the codimension of zero singularities for time-delay systems, Acta Applicandae Mathematicae 145 (1) (2016) 47–88.

[23] J. M. Coron, S. O. Tamasoiu, Feedback stabilization for a scalar conservation law with PID boundary control, Chinese Annals of Mathematics, Series B 36 (5) (2015) 763–776.

[24] G. Mazanti, I. Boussaada, S.-I. Niculescu, Y. Chitour, Effects of roots of maximal multiplicity on the stability of some classes of delay differential-algebraic systems: The lossless propagation case, IFAC-PapersOnLine 54 (9) (2021) 764–769.

[25] R. Frisch, H. Holme, The characteristic solutions of a mixed difference and differential equation occurring in economic dynamics, Econometrica 3 (2) (1935) 225–239.

[26] N. D. Hayes, Roots of the transcendental equation associated with a certain difference-differential equation, Journal of the London Mathematical Society s1-25 (3) (1950) 226–232.

[27] A. Callender, D. R. Hartree, A. Porter, Time-lag in a control system, Phil. Trans. Royal Soc. London (Series A) 235 (756) (1936) 415–444.

[28] R. Bellman, K. Cooke, Differential-difference equations, Academic Press, New York, 1963.
[29] A. Benarab, I. Boussaada, K. Trabelsi, C. Bonnet, Multiplicity–induced–dominancy property for second–order neutral differential equations with application in oscillation damping, European journal of Control (2022) 100721.
[30] H. Buchholz, The confluent hypergeometric function with special emphasis on its applications, Vol. 15 of Springer Tracts in Natural Philosophy, Springer-Verlag, 1969.
[31] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, Higher transcendental functions. Vol. I, Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981.
[32] F. Olver, D. Lozier, R. Boisvert, C. Clark (Eds.), NIST Handbook of Mathematical Functions, U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010.
[33] I. Boussaada, G. Mazanti, S.-I. Niculescu, Some remarks on the location of non-asymptotic zeros of Whittaker and Kummer hypergeometric functions, Bull. Sci. Math. 174 (2022) Paper No. 103093.