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Delay-Dependent Invariance of Polyhedral Sets for Discrete-Time Linear Systems ⋆

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Abstract: In this paper we propose a delay-dependent analysis of the positive invariance property with respect to linear discrete-time systems with delayed states. An appropriate model transformation is employed, together with a matrix parametrization, which allow the derivation of delay-dependent invariance conditions of polyhedral sets with respect to the transformed model. We then show that such conditions imply the confinement of the state trajectories of the original system in the set, as long as the initial states satisfy additional constraints related to the system dynamics. The characterization of this set of admissible initial conditions gives rise to the proposition of a less conservative definition of set-invariance. We illustrate through numerical examples the fact that, under the proposed definition, confinement of state trajectories in the set can be achieved even though it is not invariant according to the classical definition.

Keywords: Linear systems, time-delay, invariant sets, constraints.

1. INTRODUCTION

Positive invariance of sets is a key concept for analysis and control design of dynamic systems subject to state and input constraints. Aside its standalone importance, it gained the interest of the control community also due to its close link with the classical stability theory. The literature on this subject is mature nowadays, mainly for linear systems (Blanchini and Miani, 2015). When time-delay on the system states or inputs enters the picture, the analysis and control design problems become more involved (Gielen et al., 2012; Athanasopoulos and Lazar, 2014; Liu et al., 2020).

It is worth noticing the interest of positive invariance studies in different fields of applications of time-delay systems as, for example, logistics problems (Farraa et al., 2021), autonomous systems (Michel et al., 2019) or network control (Lombardi et al., 2012) to mention just a few.

Linear discrete-time systems affected by delays can be described by finite-dimensional delay difference equations (dDDEs). For this reason, the study of set invariance is easier for such systems than for the continuous-time counterparts, which are infinite-dimensional. Many contributions can be found in the literature on positive invariance for discrete-time delay systems. For a guided tour in this subject, the reader is referred to Laraba et al. (2016).

Concerning, in particular, linear time-delay systems and polyhedral invariant sets, defined by linear constraints on the system states, the classical definition of positive invariance, or $\mathcal{D}$-invariance, (Hennet and Tarbouriech, 1998; Vassilaki and Bitsoris, 1999; Stanković et al., 2014) amounts to imply that the state trajectory remains in the set if the initial conditions belong to it. Since the initial conditions for a discrete-time system with delayed states correspond to the past states in a time interval the size of the delay, this definition results quite conservative, because it accepts past state trajectories that might never exist within the regular behavior of the system. As a consequence of this classical definition, the conditions for set invariance that we find in the literature are independent of the size of the delay. Knowing that the roots of the characteristic polynomial of the system evolve as a function of the delay and that this possibly affects stability, it becomes clear that the conditions for the existence of $\mathcal{D}$-invariant sets are restrictive if they are delay-independent.

For continuous-time systems, a different perspective has been presented in Dórea et al. (2022), under which a delay-dependent analysis of set invariance can be made. From a transformation of the original model, $\mathcal{D}$-invariance conditions were derived depending on the size of the delay. A linear programming approach was proposed to check if a given polyhedron is $\mathcal{D}$-invariant with respect to (w.r.t.) the transformed model. It was established that invariance w.r.t. the transformed model can imply constraint satisfaction for the original model if its initial conditions are tied by the system dynamics.

In the present work, besides extending the results of Dórea et al. (2022) to discrete-time systems, we formally prove

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that $\mathcal{D}$-invariance w.r.t. the transformed model imply constraint satisfaction in the original model provided that the initial conditions belong to an admissible set, which is also formally characterized. That gives rise to the proposition of a new definition of set invariance for time-delay systems, which is much less conservative than the classical definition. We develop numerical examples that show how useful this new definition can be, by certifying constraint satisfaction when the concerned set is not $\mathcal{D}$-invariant. We close the presentation drawing some conclusions.

Notation: $\mathbb{Z}_{[a,b]}$, with $a, b \in \mathbb{Z}$ stands for the set of integers i such that $a \le i \le b$. The *Minkowski sum* of two sets $P, Q \subset \mathbb{R}^n$ is defined by: $P \oplus Q = \{p + q \mid p \in P, q \in Q\}$.

2. LINEAR DISCRETE-TIME DELAY SYSTEM

Consider a linear discrete-time system represented by delay-difference equations of the form:

$$x(k+1) = Ax(k) + A_d x(k-d),\ d > 0, \tag{1}$$

with the initial conditions $x(i),\ i \in \mathbb{Z}_{[-d,0]}$. Here, $x \in \mathbb{R}^n$ denotes the state vector and $d \in \mathbb{Z}_+^*$ the time-delay.

In order to bring more flexibility within the positive invariance analysis performed next, we introduce a *model transformation* of (1) and detail its ingredients in a two stage procedure. First, we introduce an auxiliary variable $K \in \mathbb{R}^{n\times n}$ such that the time-delay model above can be rewritten as:

$$x(k+1) = (A+K)x(k) + (A_d - K)x(k-d) - \\ -K(x(k) - x(k-d)).$$

Now, let us write the difference in the last term as:

$$x(k) - x(k-d) = \sum_{i=-d}^{-1} (x(k+i+1) - x(k+i)),$$

leading to the following transformed model:

$$x(k+1) = (A+K)x(k) + (A_d - K)x(k-d) \\ -K \sum_{i=-d}^{-1} (x(k+i+1) - x(k+i)).$$

Using the model (1) to account for $x(k+i+1)$ leads to:

$$x(k+1) = (A+K)x(k) + (A_d - K)x(k-d) \\ -K \sum_{i=-d}^{-1} (Ax(k+i) + A_d x(k+i-d) - x(k+i))$$

and, finally:

$$x(k+1) = (A+K)x(k) + (A_d - K)x(k-d) \\ - \sum_{i=-d}^{-1} K(A-I)x(k+i) - \sum_{i=-2d}^{-d-1} KA_d x(k+i)) \tag{2}$$

with the initial conditions $x(i),\ i \in \mathbb{Z}_{[-2d,0]}$.

As discussed in (Niculescu, 2001) in the continuous-time framework, the use of this kind of parametrized model transformation allows decoupling delay-independent modes from delay-dependent ones by appropriately exploiting the system's structure. We will show in the sequel how this artifact can be of help in the discrete-time framework as well.

From the derivation of the transformed model, one can see that the equivalence of the original model (1) with respect to possible state trajectories holds only if $x(k+i+1) = Ax(k+i) + A_d x(k+i-d)$ for $i \in \mathbb{Z}_{[-d,-1]}$. This equivalence is formally established as follows:

Theorem 1. Consider systems (1) and (2) and let their state trajectories be denoted, respectively, by $x(k)$, $k \in \mathbb{Z}_+$ with initial conditions $x(i),\ i \in \mathbb{Z}_{[-d,0]}$, and by $x_t(k)$, $k \in \mathbb{Z}_+$ with initial conditions $x_t(i),\ i \in \mathbb{Z}_{[-d,d]}$. If the initial conditions of (2) are given by:

$$\begin{aligned} x_t(i) &= x(i),\ \text{for } i \in \mathbb{Z}_{[-d,0]}, \\ x_t(i+1) &= Ax(i) + A_d x(i-d),\ \text{for } i \in \mathbb{Z}_{[0,d-1]}, \end{aligned} \tag{3}$$

Then, $x_t(k) = x(k)\ \forall k \ge 0$.

Proof: First, we notice that the state trajectories of (1) and (2) coincide in the interval $\mathbb{Z}_{[-d,d]}$ as follows:

- For $-d \le k \le 0$, $x_t(k)$ coincide with the initial conditions of (1).
- For $1 \le k \le d$, $x_t(k)$ is given by the dynamics of (1).

For $k = d+1$, the state x_t is given by (2):

$$x_t(d+1) = (A+K)x_t(d) + (A_d - K)x_t(0) \\ - K \sum_{i=-d}^{-1} (Ax_t(d+i) + A_d x_t(i) - x_t(d+i))$$

From (3), $Ax_t(d+i) + A_d x_t(i) = x_t(d+i+1)$ for $i \in \mathbb{Z}_{[-d,-1]}$. Hence:

$$\begin{aligned} x_t(d+1) &= (A+K)x_t(d) + (A_d - K)x_t(0) \\ &\quad -K \sum_{i=-d}^{-1} (x_t(d+i+1) - x_t(d+i)) \\ &= (A+K)x_t(d) + (A_d - K)x_t(0) - K(x_t(d) - x_t(0)) \\ &= Ax_t(d) + A_d x_t(0) = Ax(d) + A_d x(0) = x(d+1). \end{aligned}$$

Now, the same development can be made for $k = d+2$ and, naturally by induction, for $k = d+i$, $\forall i \ge 2$, proving that the state trajectories $x(k)$ and $x_t(k)$ coincide for $k \ge 0$. □

This Theorem establishes conditions under which a state trajectory of the original model is also a trajectory of the transformed model, and will be a key to analyse positive invariance w.r.t. (1) from results obtained for (2). In the sequel, we study positive invariance w.r.t. (2) and the link of this property with the state trajectories of the original system.

3. POSITIVELY INVARIANT POLYHEDRAL SETS OF THE TRANSFORMED MODEL

We focus next on the analysis of the positive invariance property of given sets in the state space. We adopt the following definition (Lombardi et al., 2011):

Definition 1. Given a scalar $0 \le \lambda \le 1$, a set $\Omega \subset \mathbb{R}^n$ containing the origin is called *positively $\mathcal{D}$-invariant* with respect to the time-delay system (1) if for any initial conditions $x(i) \in \Omega$, $i \in \mathbb{Z}_{[-d,0]}$, it follows that $x(k) \in \lambda\Omega, \forall k \in \mathbb{Z}_+$.

This definition is equivalent to those of *delay-independent positive invariance* of Hennet and Tarbouriech (1998) and of positive invariance w.r.t. an ARMA model of Vassilaki and Bitsoris (1999). If $0 \le \lambda < 1$, Ω is additionally said to be *contractive.*

Necessary and sufficient (*delay-independent*) conditions for $\mathcal{D}$-invariance have been established as follows (Lombardi et al., 2011):

Theorem 2. A set Ω is positively $\mathcal{D}$-invariant w.r.t. (1) if, and only if: $A\Omega \oplus A_d\Omega \subset \lambda\Omega$, $0 \leq \lambda \leq 1$.

The $\mathcal{D}$-invariance definition can be straightforwardly extended to the transformed system (2), but with the initial conditions defined on a larger interval: $x(i)$, with $i \in \mathbb{Z}_{[-2d,0]}$.

Consider now a convex polyhedral set containing the origin in its interior, described by $\Omega = \{x \in \mathbb{R}^n \mid Fx \leq w\}$, with $F \in \mathbb{R}^{f\times n}$, $w \in \mathbb{R}^f$, $w > 0$. The inequalities here are taken component-wise.

We establish necessary and sufficient conditions for positive invariance of the polyhedron Ω *with respect to the transformed system* (2).

Theorem 3. Ω is positively $\mathcal{D}$-invariant w.r.t. system (2) if, and only if, there exist nonnegative matrices $H, L, M, N \in \mathbb{R}^{f\times f}$ such that:

$$HF = F(A + K) \tag{4}$$

$$LF = F(A_d - K) \tag{5}$$

$$MF = -FK(A - I) \tag{6}$$

$$NF = -FKA_d \tag{7}$$

$$(H + L + d(M + N))w \leq w. \tag{8}$$

Proof: (Only if:) Consider a state $x(k) \in \Omega$. At time k, the following conditions are required by Definition 1 of positive $\mathcal{D}$-invariance w.r.t. system (2):

$$\begin{cases} Fx(k) \leq w, \quad Fx(k-d) \leq w, \\ F\displaystyle\sum_{i=-d}^{-1} x(k+i) \leq \sum_{i=-d}^{-1} w = dw, \\ F\displaystyle\sum_{i=-2d}^{-d-1} x(k+i) \leq dw. \end{cases} \tag{9}$$

One can notice that the conditions (9) define a polyhedron on the finite-dimensional space represented by the vector:

$$[x(k)^T \; x(k-d)^T \; \sum_{i=-d}^{-1} x(k+i)^T \; \sum_{i=-2d}^{-d-1} x(k+i)^T]^T. \tag{10}$$

For $\mathcal{D}$-invariance, it is necessary that $x(k+1) \in \Omega$, hence:

$$Fx(k+1) = F(A+K)x(k) + F(A_d - K)x(k-d)$$
$$-FK(A-I)\sum_{i=-d}^{-1} x(k+i) - FKA_d \sum_{i=-2d}^{-d-1} x(k+i) \leq w. \tag{11}$$

This condition defines a polyhedron on the space of (10) as well. Then, a necessary condition for positive $\mathcal{D}$-invariance of Ω is that the polyhedron defined by (9) be included in the polyhedron defined by (11).

According to the so-called extended Farkas' Lemma (see, e.g. Hennet and Tarbouriech (1998)) this inclusion holds if, and only if, there exist non-negative matrices H, L, M, N such that conditions (4)-(8) hold.

(If:) We assume that $x(k+i) \in \Omega$ in a time window of width $2d + 1$, i.e.:

$$Fx(k+i) \leq w, \; i \in \mathbb{Z}_{[-2d,0]}. \tag{12}$$

Then, from (2) we have that:

$$Fx(k+1) = F((A+K)x(k) + (A_d - K)x(k-d)$$
$$- \sum_{i=-d}^{-1} K(A-I)x(k+i) - \sum_{i=-2d}^{-d-1} KA_d x(k+i))).$$

From (4)-(7), (12), and (8):

$$Fx(k+1) = HFx(k) + LFx(k-d)$$
$$+ \sum_{i=-d}^{-1} MFx(k+i) + \sum_{i=-2d}^{-d-1} NFx(k+i))$$
$$\leq Hw + Lw + dMw + dNw \leq w.$$

We have proved that $x(k+1) \in \Omega$ if $x(k+i) \in \Omega$, $i \in \mathbb{Z}_{[-2d,0]}$ for an arbitrary $k \in \mathbb{Z}_+$. Since, from Definition 1, this hypothesis is true for $k = 0$, we conclude, by induction, that $x(k) \in \Omega \; \forall k \in \mathbb{Z}_+$. □

For a given polyhedral set Ω (*a priori* known F and w), conditions (4)-(7) are linear on the matrix variables H, L, M, N and K.

For a given delay d, condition (8) is linear as well. In this case, positive $\mathcal{D}$-invariance of Ω can be checked by solving the following linear programming (LP) problem:

$$\begin{aligned} \min_{\lambda,K,H,L,M,N} \; & \lambda \\ \text{s.t.:} \; & (4)\text{-}(7) \\ & (H + L + d(M + N))w - \lambda w \leq 0 \\ & H, L, M, N \geq 0 \end{aligned} \tag{13}$$

If the optimal solution λ^* is such that $\lambda^* \leq 1$, then Ω is positively $\mathcal{D}$-invariant for the given delay d.

Let $d_r \in \mathbb{R}$ be defined by: $d_r = \min_i \dfrac{w_i - (H_i^* + L_i^*)w}{(M_i^* + N_i^*)w}$. A guaranteed value for the integer $d > 0$ such that positive $\mathcal{D}$-invariance of Ω is satisfied is given by:

$$d_M = \max_{d \in \mathbb{Z}_+^*} d \quad \text{such that } d \leq d_r.$$

Such a d_M may not be the maximal admissible d though, because different values of K, H, L, M, N can be obtained if one chooses another value of d in Problem (13). As such, these matrices can be used to improve the maximal admissible delay granting the positive invariance of the pre-defined polyhedral set Ω.

As usual in the set-invariance literature (Hennet and Tarbouriech, 1998), simplified expressions can be obtained if the polyhedral set Ω is symmetrical w.r.t. to the origin, i.e.:

$$\Omega = \{x \in \mathbb{R}^n \mid |\bar{F}x| \leq \bar{w}\} \tag{14}$$

Corollary 1. A symmetrical polyhedral set Ω is positively $\mathcal{D}$-invariant w.r.t. system (2) if, and only if, there exist matrices $\bar{H}, \bar{L}, \bar{M}, \bar{N}$ of appropriate dimensions such that:

$$\bar{H}\bar{F} = \bar{F}(A + K) \tag{15}$$

$$\bar{L}\bar{F} = \bar{F}(A_d - K) \tag{16}$$

$$\bar{M}\bar{F} = -\bar{F}K(A - I) \tag{17}$$

$$\bar{N}\bar{F} = -\bar{F}KA_d \tag{18}$$

$$(|\bar{H}| + |\bar{L}| + d(|\bar{M}| + |\bar{N}|))\bar{w} \leq 0 \tag{19}$$

This result can be easily obtained from standard manipulation of the conditions obtained for general polyhedra. Linear Programming problems can be easily set up to check for invariance of such symmetrical sets too (Hennet and Tarbouriech, 1998).

The analysis of these simplified expressions allows for the following interpretation of invariance conditions: matrix $\bar{H}$ is obtained from a particular similarity transformation of $(A + K)$ through matrix $\bar{F}$. The same applies to matrix $\bar{L}$ w.r.t. $A_d - K$. In order to satisfy condition (19), the absolute values of the elements of $\bar{H}$ and $\bar{L}$ must be small. The parameter K can act to simultaneously decrease the absolute values of $\bar{H}$ and $\bar{L}$, making it possible to achieve $\mathcal{D}$-invariance w.r.t. (2) when it is not achievable for the original model (1).

4. SET INVARIANCE FOR THE ORIGINAL MODEL

The positive $\mathcal{D}$-invariance conditions for the polyhedron Ω have been established with respect to the transformed model (2). A question now arises on whether the confinement of state trajectories in Ω also holds w.r.t. the original system (1). In what follows, we show that it does hold, but under additional restrictions on the initial conditions.

Corollary 2. If the set Ω is positively $\mathcal{D}$-invariant w.r.t. (2), then, the state trajectory of (1) is such that $x(k) \in \Omega$ $\forall k \geq 0$, provided that:

$$\begin{cases} x(i) \in \Omega, \ \forall i \in \mathbb{Z}_{[-d,d]}, \\ x(i+1) = Ax(i) + A_d x(i-d), \ \forall i \in \mathbb{Z}_{[0,d-1]}. \end{cases} \tag{20}$$

Proof: From Theorem 1, under the conditions of the Corollary's statement, the trajectory of $x(k)$ for (1) is also a trajectory of (2). As such, since the initial conditions of (2) belong to Ω and Ω is $\mathcal{D}$-invariant w.r.t. (2), then $x(k) \in \Omega$ $\forall k \geq 0$. □

The reader will notice that this result is not limited to polyhedral sets Ω.

The classical definition of positive $\mathcal{D}$-invariance w.r.t. system (1) (Definition 1) requires that $x(i) \in \Omega$ $\forall i \in \mathbb{Z}[-d, 0]$. The statement of the preceding Corollary additionally requires that $x(i)$, $i \in \mathbb{Z}_{[1,d]}$ belong to Ω and respect the system dynamics (1). We can interpret these additional restrictions on the initial conditions in two ways:

- the past states of (1), $x(i)$, $i \in \mathbb{Z}_{[-2d,0]}$ can be split into two subsets: for $i \in \mathbb{Z}_{[-2d,-d]}$ the states are "free" and for $i \in \mathbb{Z}_{[-d+1,0]}$, the states must obey the system dynamics. Since the initial conditions of (1) are $x(i)$, $i \in \mathbb{Z}_{[-d,0]}$, these restrictions amount to require that the initial conditions result from a state trajectory started d steps before, i.e., only initial conditions which are consistent with the system dynamics are considered to check the confinement of $x(k)$ in the set Ω.
- If we consider, as usual, that the initial conditions $x(i)$ of (1) are defined for $i \in \mathbb{Z}_{[-d,0]}$, then, the restriction now applies to the interval $i \in \mathbb{Z}_{[-d,d]}$. Hence, $x(i)$ are "free" for $i \in \mathbb{Z}_{[-d,0]}$ and must obey the system dynamics for $i \in \mathbb{Z}_{[1,d]}$. However, the dynamics of (1) imply that $x(i)$ for $i \in \mathbb{Z}_{[1,d]}$ are given by the states for $i \in \mathbb{Z}_{[-d,0]}$. The additional restrictions, then, imply additional constraints to $x(i)$ for $i \in \mathbb{Z}_{[-d,0]}$, i.e., to the initial conditions of (1). We use this interpretation to define a set of admissible initial states w.r.t. trajectory confinement in Ω.

Definition 2. Consider the system (1) and a polyhedral set Ω which is positively $\mathcal{D}$-invariant w.r.t. the transformed system (2). The *set of admissible initial states* of (1) will be defined as follows:

$$\mathcal{I}(\Omega) = \{x(i) \in \Omega, \ i \in \mathbb{Z}_{[-d,0]} \text{ s.t. } x(i) \in \Omega, \ i \in \mathbb{Z}_{[1,d]}\}$$

Now we show that, for a polyhedral set $\Omega = \{x \ : \ Fx \leq w\}$, $\mathcal{I}(\Omega)$ is a polyhedral set defined on the extended state space $x(i)$, $i \in \mathbb{Z}_{[-d,0]}$. The solution of (1) in the interval $j \in \mathbb{Z}_{[1,d]}$ is given by:

$$x(j) = A^j x(0) + \sum_{l=0}^{j-1} A^{j-l-1} A_d x(l-d).$$

From definition 2, $\mathcal{I}(\Omega)$ is then given by $x(i)$, $i \in \mathbb{Z}_{[-d,0]}$, such that:

$$\begin{cases} Fx(i) \leq w, \ i \in \mathbb{Z}_{[-d,0]} \\ F(A^j x(0) + \sum_{l=0}^{j-1} A^{j-l-1} A_d x(l-d)) \leq w, \ j \in \mathbb{Z}_{[1,d]}. \end{cases} \tag{21}$$

The constraints above define a polyhedral set on the extended state space $x(i)$, $i \in \mathbb{Z}_{[-d,0]}$.

The time-delay discrete-time systems (1) can be represented in an extended state space as (Lombardi et al., 2011):

$$X(k+1) = \mathcal{A}X(k) = \begin{bmatrix} A & 0 & \dots & 0 & A_d \\ I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-1) \\ \vdots \\ x(k-d+1) \\ x(k-d) \end{bmatrix}. \tag{22}$$

Under this representation, the numerous tools available for analysis of positive invariance for linear systems without delay can be used. In particular, the relationship between the existence of positively invariant sets and stability can be established. However, a major disadvantage of using the extended model is the complexity of the polyhedral invariant sets, defined over a space of potentially large dimension, which increases with the size of the delay. (Olaru et al., 2014),

A connection between the transformed model (2) and the extended model above is given as follows.

Corollary 3. Consider a polyhedral set Ω_a defined on the extended state space by the following inequalities:

$$\begin{cases} Fx(k+i) \leq w, \ i \in \mathbb{Z}_{[-d,0]} \\ F(A^j x(k) + \sum_{l=0}^{j-1} A^{j-l-1} A_d x(k+l-d)) \leq w, \ j \in \mathbb{Z}_{[1,d]}. \end{cases} \tag{23}$$

If $\Omega = \{x \ : \ Fx \leq w\}$ is positively $\mathcal{D}$-invariant w.r.t. (2), then Ω_a is positively invariant w.r.t. (22).

Proof: Assume that $X(k) \in \Omega_a$. Then, the constraints (23) are satisfied. From Corollary 2, if Ω is $\mathcal{D}$-invariant w.r.t. (2), then, it is clear that conditions (23) are satisfied with k replaced by $k + 1$. □

Since Ω_a is a compact set, then, positive invariance with contraction implies stability of the time-delay system. Hence, $\mathcal{D}$-invariance of Ω w.r.t. (2) can be used as a stability certificate for (1).

Corollary 2 induces a different notion of set-invariance for the original time-delay system (1), that we propose as follows:

Definition 3. Given a scalar $0 \leq \lambda \leq 1$, a set $\Omega \subset \mathbb{R}^n$ containing the origin will be said to be *positively* $\mathcal{I} - \mathcal{D}$-*invariant* with respect to the time-delay system (1) if for any initial conditions $x(i) \in \mathcal{I}(\Omega)$, $i \in \mathbb{Z}_{[-d,0]}$, it follows that $x(k) \in \lambda\Omega, \forall k \in \mathbb{Z}_+$.

This definition is relaxed with respect to that of $\mathcal{D}$-invariance in the sense that it is easier for a given set to be $\mathcal{I} - \mathcal{D}$-invariant because the initial conditions are constrained to result in a trajectory that respects system's dynamics in the interval $\mathbb{Z}_{[1,d]}$. This assumption appears more realistic than accepting, as for $\mathcal{D}$-invariance, any states in Ω as initial conditions.

An immediate consequence of this Definition and Corollary 2 to the links between the original and the transformed model is as follows:

Corollary 4. If the polyhedral set Ω is positively $\mathcal{D}$-invariant w.r.t. (2), then, it is positively $\mathcal{I} - \mathcal{D}$-invariant w.r.t. (1).

Considering, again, the polyhedral case, we notice that $\Omega = \{x : Fx \leq w\}$ is $\mathcal{I} - \mathcal{D}$-invariant if, and only if, (21) implies

$$Fx(d+1) = F(A^{d+1}x(0) + \sum_{l=0}^{d} A^{d-l}A_d x(l-d)) \leq w. \quad (24)$$

Both this condition and (21) define polyhedral sets in the extended space $[x(0)^T x(-1)^T \cdots x(-d)^T]^T$. Hence, Ω is $\mathcal{I} - \mathcal{D}$-invariant if, and only if, the polyhedron defined by (21) is included in the one defined by (24). This condition can be checked via Linear Programming through standard computation based on the so-called extended Farkas' Lemma. In this sense, one can wonder why resorting to $\mathcal{D}$-invariance w.r.t. transformed model, if it is possible to directly test $\mathcal{I}-\mathcal{D}$-invariance w.r.t. (1). There are two main reasons: first, from the numerical point of view, the conditions above involve a number of matrix multiplications, operations known to propagate round-off errors. Second, the conditions above apply for a given value of the delay d, whereas the delay-dependent conditions of Theorem 3 provide values of admissible delays.

We close this section by pointing out that we can use the conditions derived in the previous section to develop an LP-based technique to compute a state feedback controller which makes a polyhedral set positively invariant w.r.t. a input-delayed linear system, similar to that proposed by Dórea et al. (2022) for the continuous-time case.

5. NUMERICAL EXAMPLES

5.1 Example: first-order model

Consider the following system, borrowed from (Olaru et al., 2014): $x(k+1) = 0.8x(k) - 0.4x(k-d)$, for which no $\mathcal{D}$-invariant polyhedron containing the origin in its interior exists. For $d > 4$ this system is unstable, and this is one of the reasons why delay-independent invariance cannot be achieved.

The polyhedral set Ω is given by $\Omega = \{x : |x| \leq 1\}$.

The solution of the LP problem (13) (adapted to the symmetrical case), after a trial-and-error adjustment of the value of d, gives $K^* = -0.4, \quad d_M = 2$, implying that Ω is $\mathcal{D}$-invariant w.r.t. the transformed model (2), for $d = 1$ and $d = 2$.

The possibility of a delay-dependent analysis brought by the proposed model transformation is well illustrated in this example. Ω is not $\mathcal{D}$-invariant w.r.t. (1) because $|A| + |A_d| > 1$ (see Corollary 1, with $\bar{F} = 1$ and $K = 0$). With $K = -0.4$, $|A+K|+|A_d-K| = 0.4$ and there is some space left to accommodate the terms dependent on the delay up to $d = 2$.

From Corollary 4, Ω is $\mathcal{I}-\mathcal{D}$-invariant for $d = 1$ and $d = 2$. The set of admissible initial states $\mathcal{I}(\Omega)$ for $d = 1$ is given by $x(0)$, $x(-1)$ such that $\left| \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.8 & -0.4 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix} \right| \leq \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and is depicted in Figure 1.

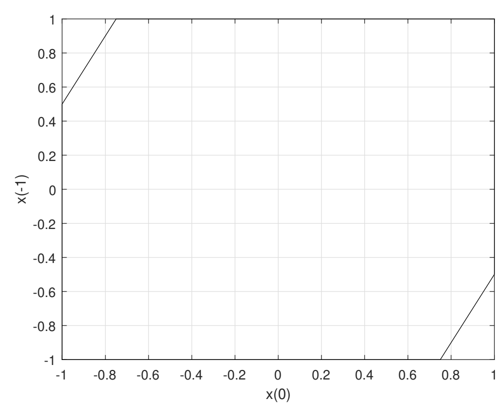

Fig. 1. Example 1: set of admissible initial states for $d = 1$.

One can notice that some initial conditions given by two consecutive extreme values in Ω, for instance, $x(-1) = 1$, $x(0) = -1$, would lead to constraint violation. If $x(-1) = -1$, $x(0)$ cannot be larger than 0.75, and if $x(0) = -1$, $x(-1)$ cannot be larger than 0.5. In this example the restrictions on the initial conditions are quite mild in view of the size of Ω.

5.2 Example: second-order model

Consider system (1) for which:

$$A = \begin{bmatrix} 1.2 & 0.2 \\ -0.4 & 0.6 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.3 & -0.2 \\ 0.4 & 0.2 \end{bmatrix},$$

and a symmetrical polyhedral set Ω with

$$\bar{F} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The solution of the LP problem (13) (adapted to the symmetrical case), after a trial-and-error adjustment of the value of d, gives $K^* = \begin{bmatrix} -0.2036 & -0.1020 \\ 0.2073 & 0.1040 \end{bmatrix}, \quad d_M = 10.$

$$\bar{H}^* = \begin{bmatrix} 0.8004 & -0.0016 \\ 0 & 0.9 \end{bmatrix}, \quad \bar{L}^* = \begin{bmatrix} 0.0996 & 0.0016 \\ 0 & 0 \end{bmatrix}.$$

The elements of matrices $\bar{M}^*$ and $\bar{N}^*$ have small values, which explain the quite large value of d_M.

The polyhedron Ω is depicted in Figure 2 with two state trajectories of (1): the trajectory with $d = 1$, for initial conditions $x(-1) = [2\ \ -3]^T$, $x(0) = [0\ 1]^T$ is represented by red circles, and the (quite oscillatory) trajectory with $d = 10$, for initial conditions $x(-10) = [2\ \ -3]^T$, $x(i) = [0\ 1]^T$, for $i \in \mathbb{Z}_{[-9,0]}$ is represented by blue crosses.

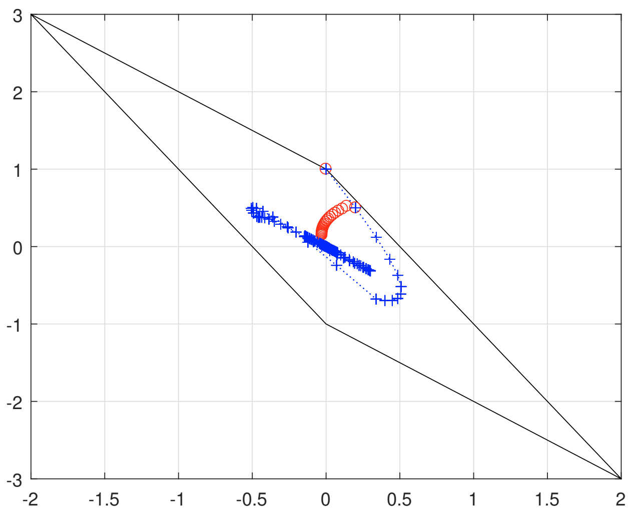

Fig. 2. Example 2: the polyhedron Ω with two state trajectories for $d = 1$ (red) and $d = 10$ (blue).

The choice $K = 0$ in (2) corresponds to the delay-independent case treated in (Hennet and Tarbouriech, 1998). In this case Ω is not positively $\mathcal{D}$-invariant w.r.t. the transformed model (2), which was expected because matrix A has an eigenvalue equal to 1, that was "moved" to 0.9 (see matrix $\bar{H}$) by the parameter K.

Indeed, with $d = 1$, the state trajectory leaves Ω if the initial conditions are $x(-1) = [2\ \ -3]^T$, $x(0) = [-2\ 3]^T$, which are opposite vertices of Ω, confirming that Ω is not $\mathcal{D}$-invariant w.r.t. (1). It appears reasonable that two consecutive states do not move between two opposite vertices, and it illustrates the conservatism of the $\mathcal{D}$-invariance definition.

6. CONCLUSIONS

In this paper we presented a study on set invariance for linear discrete-time systems with delayed states. We argue that the classical delay-independent definition of invariant sets for this class of systems induces conditions which are very hard to be met, because even the stability of the system may depend on the size of the delay. The main contribution of the present paper is, then, the derivation of delay-dependent conditions that guarantee confinement of the state trajectories into a polyhedral set, as long as the initial conditions satisfy some restrictions, that were formally characterized. These conditions gave rise to a new definition of set invariance for time-delay systems that allows to certify constraints satisfaction even when a classical invariant set does not exist, as illustrated through two numerical examples. Future work should focus on the properties of this new type of invariant sets for time-delay systems, and on methods for their practical construction with the associated set of admissible initial states.

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