CRL-Chu correspondences

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Abstract. We continue our study of the general notion of L-Chu correspondence by introducing the category CRL-ChuCors incorporating residuation to the underlying complete lattice L, specifically, on the basis of a residuation-preserving isotone Galois connection λ. Then, the L-bonds are generalized within this same framework, and its structure is related to that of the extent of a suitably defined λ-direct product.

1 Introduction

Morphisms have been suggested [7] as fundamental structural properties for the modelling of, among other applications, communication, data translation, and distributed computing. Our approach can be seen within a research topic linking concept lattices with the theory of Chu spaces [10, 11]; in the latter, it is shown that the notion of state in Scott's information system corresponds precisely to that of formal concepts in FCA with respect to all finite Chu spaces, and the entailment relation corresponds to *association rules* (another link between FCA with database theory) and, specifically, on the identification of the categories associated to certain constructions.

Other researchers have studied as well the relationships between Chu constructions and L-fuzzy FCA. For instance, in [1] FCA is linked to both order-theoretic developments in the theory of Galois connections and to Chu spaces; as a result, not surprisingly from our previous works, they obtain further relationships between formal contexts and topological systems within the category of Chu systems. Recently, Solovyov, in [9], extends the results of [1] to clarify the relationships between Chu spaces, many-valued formal contexts of FCA, lattice-valued interchange systems and Galois connections.

This work is based on the notion, introduced by Mori in [8], of Chu correspondences as morphisms between formal contexts. This categorical approach has been used in previous works [3, 5, 6]. For instance, in [6], the categories associated to L-formal contexts and L-CLLOS were defined and a constructive proof was given of the equivalence between the categories of L-formal contexts with L-Chu correspondences as morphisms and that of completely lattice L-ordered sets and their corresponding morphisms. Similar results can be found in [2], where a

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new notion of morphism on formal contexts resulted in a category equivalent to both the category of complete algebraic lattices and Scott continuous functions, and a category of information systems and approximable mappings.

We are concerned with the category of fuzzy formal contexts and λ-Chu correspondences, built on the basis of a residuation-preserving isotone Galois connection λ. Then, the corresponding extension of the notion of bond between contexts is generalized to this framework, and its properties are studied.

2 Preliminaries

2.1 Residuated lattice

Definition 1. *A* complete residuated lattice *is an algebra* $\langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow\rangle$ *where*

- $\langle L, \wedge, \vee, 0, 1\rangle$ *is a complete lattice with the top* 1 *and the bottom* 0*,*
- $\langle L, \otimes, 1\rangle$ *is a commutative monoid,*
- $\langle \otimes, \rightarrow\rangle$ *is an adjoint pair, i.e. for any* $a, b, c \in L$*:*

$$a \otimes b \leq c \text{ is equivalent to } a \leq b \rightarrow c$$

Definition 2. *A complete residuated lattice* $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow\rangle$ *such that for any value* $k \in L$ *holds* $\neg\neg k = k$ *where* $\neg k = k \rightarrow 0$ *is said to be endowed with* double negation law*.*

Lemma 1. *Let* $\mathcal{L}$ *be a complete residuated lattice satisfying the double negation law. Then for any* $k, m \in L$ *holds* $\neg k \rightarrow m = \neg m \rightarrow k$*.*

2.2 Basics of Fuzzy FCA

Definition 3. *An* L-fuzzy formal context $\mathcal{C}$ *is a triple* $\langle B, A, \mathcal{L}, r\rangle$*, where* B*,* A *are sets,* $\mathcal{L}$ *is a complete residuated lattice, and* $r\colon B \times A \rightarrow L$ *is an* L*-fuzzy binary relation.*

Definition 4. *Let* $\mathcal{C} = \langle B, A, \mathcal{L}, r\rangle$ *be an* L*-fuzzy formal context. A pair of* derivation operators $\langle \uparrow, \downarrow\rangle$ *of the form* $\uparrow\colon L^B \rightarrow L^A$ *and* $\downarrow\colon L^A \rightarrow L^B$*, is defined as follows*

$$\uparrow(f)(a) = \bigwedge_{b \in B} (f(b) \rightarrow r(b, a)) \text{ for any } f \in L^B \text{ and } a \in A,$$
$$\downarrow(g)(b) = \bigwedge_{a \in A} (g(a) \rightarrow r(b, a)) \text{ for any } g \in L^A \text{ and } b \in B.$$

Lemma 2. *Let* $\langle \uparrow, \downarrow\rangle$ *be a pair of derivation operators defined on an* L*-fuzzy formal context* $\mathcal{C}$*. A pair* $\langle \uparrow, \downarrow\rangle$ *forms a Galois connection between complete lattices of all* L*-sets of objects* L^B *and attributes* L^A*.*

Definition 5. *Let $\mathcal{C} = \langle B, A, \mathcal{L}, r\rangle$ be an L-fuzzy formal context. A* formal concept *is a pair of L-sets $\langle f, g\rangle \in L^B \times L^A$ such that $\uparrow(f) = g$ and $\downarrow(g) = f$. The set of all L-concepts of $\mathcal{C}$ will be denoted as* $\mathrm{FCL}(\mathcal{C})$. *The object (resp. attribute) part of any concept is called* extent *(resp.* intent*). The sets of all extents or intents of $\mathcal{C}$ will be denoted as* $\mathrm{Ext}(\mathcal{C})$ *or* $\mathrm{Int}(\mathcal{C})$, *respectively.*

2.3 *L*-Bonds and *L*-Chu correspondences

Definition 6. *Let X and Y be two sets. An L-*multifunction *from X to Y is said to be a mapping from X to L^Y.*

Definition 7. *Let $\mathcal{C}_i = \langle B_i, A_i, \mathcal{L}, r_i\rangle$ for $i \in \{1, 2\}$ be two L-fuzzy formal contexts. A pair of L-multifunctions $\varphi = \langle \varphi_L, \varphi_R\rangle$ such that*

- $\varphi_L : B_1 \longrightarrow \mathrm{Ext}(\mathcal{C}_2)$,
- $\varphi_R : A_2 \longrightarrow \mathrm{Int}(\mathcal{C}_1)$,

where $\uparrow_2(\varphi_L(o_1))(a_2) = \downarrow_1(\varphi_R(a_2))(o_1)$ for any $(o_1, a_2) \in B_1 \times A_2$, is said to be an L-Chu correspondence between $\mathcal{C}_1$ and $\mathcal{C}_2$. A set of all L-Chu corresepondences between $\mathcal{C}_1$ and $\mathcal{C}_2$ will be denoted by L-$\mathrm{ChuCors}(\mathcal{C}_1, \mathcal{C}_2)$.

Definition 8. *Let $\mathcal{C}_i = \langle B_i, A_i, \mathcal{L}, r_i\rangle$ for $i \in \{1, 2\}$ be two L-fuzzy formal contexts. An L-multifunction $\beta : B_1 \longrightarrow \mathrm{Int}(\mathcal{C}_2)$, such that $\beta^{\mathrm{t}} : A_2 \longrightarrow \mathrm{Ext}(\mathcal{C}_1)$, where $\beta^{\mathrm{t}}(a_2)(o_1) = \beta(o_1)(a_2)$ for any $(o_1, a_2) \in B_1 \times A_2$, is said to be an L-*bond. *A set of all L-bonds between $\mathcal{C}_1$ and $\mathcal{C}_2$ will be denoted by* L-$\mathrm{Bonds}(\mathcal{C}_1, \mathcal{C}_2)$.

Lemma 3. *Let $\mathcal{C}_i = \langle B_i, A_i, \mathcal{L}, r_i\rangle$ for $i \in \{1, 2\}$ be two L-fuzzy formal contexts. The sets* L-$\mathrm{Bonds}(\mathcal{C}_1, \mathcal{C}_2)$ *and* L-$\mathrm{ChuCors}(\mathcal{C}_1, \mathcal{C}_2)$ *form complete lattices and, moreover, there exists a dual isomorphism between them.*

3 Residuation-preserving isotone Galois connections

Definition 9. *An* isotone Galois connection *between two complete lattices $\mathcal{L}_1 = (L_1, \leq_1)$ and $\mathcal{L}_2 = (L_2, \leq_2)$ is a pair of monotone mappings $\lambda = \langle \lambda_L, \lambda_R\rangle$ with*

$$\lambda_L : L_1 \longrightarrow L_2 \qquad \text{and} \qquad \lambda_R : L_2 \longrightarrow L_1$$

such that, for any $k_1 \in L_1$ and $k_2 \in L_2$, the following equivalence holds

$$k_1 \leq_1 \lambda_R(k_2) \qquad \Longleftrightarrow \qquad \lambda_L(k_1) \leq_2 k_2. \tag{1}$$

The general theory of adjunctions provides the following result:

Lemma 4. *Let $\langle \lambda_L, \lambda_R\rangle$ be an isotone Galois connection, then for all $k_1 \in L_1$ and $k_2 \in L_2$*

$$\lambda_R(k_2) = \bigvee\{m \in L_1 : \lambda_L(m) \leq_2 k_2\} \tag{2}$$

$$\lambda_L(k_1) = \bigwedge\{m \in L_2 : k_1 \leq_1 \lambda_R(m)\} \tag{3}$$

Definition 10. *An isotone Galois connection λ between two complete residuated lattices $\mathcal{L}_1 = (L_1, \otimes_1, \to_1)$ and $\mathcal{L}_2 = (L_2, \otimes_2, \to_2)$ is said to be a* residuation-preserving isotone Galois connection *if for any $k_1, m_1 \in L_1$ and $k_2, m_2 \in L_2$ the following equalities hold:*

$$\lambda_L(k_1 \otimes_1 m_1) = \lambda_L(k_1) \otimes_2 \lambda_L(m_1) \tag{4}$$
$$\lambda_R(k_2 \otimes_2 m_2) = \lambda_R(k_2) \otimes_1 \lambda_R(m_2) \tag{5}$$
$$k_2 \to_2 \lambda_L(m_1) \geq_2 \lambda_L(\lambda_R(k_2) \to_1 m_1) \tag{6}$$

The set of all residuation-preserving isotone Galois connections from $\mathcal{L}_1$ to $\mathcal{L}_2$ will be denoted as $\mathrm{CRL}(\mathcal{L}_1, \mathcal{L}_2)$.

There is no need to consider other $\to$-preserving rules, since they follow from the previous ones, as stated by the following lemmas.

Lemma 5. *For all $k \in L_1$ and $m \in L_2$ the following equality holds*

$$k \to_1 \lambda_R(m) = \lambda_R(\lambda_L(k) \to_2 m) \tag{7}$$

Proof. Consider the following chain of equivalences

$$\begin{aligned} l \otimes_1 k \leq_1 \lambda_R(m) \quad &\overset{(1)}{\iff} \quad \lambda_L(l \otimes_1 k) \leq_2 m \\ &\overset{(4)}{\iff} \quad \lambda_L(l) \otimes_2 \lambda_L(k) \leq_2 m \\ &\overset{(\mathrm{adj})}{\iff} \quad \lambda_L(l) \leq_2 \lambda_L(k) \to_2 m \end{aligned}$$

As a result, we can write

$$\begin{aligned} k \to_1 \lambda_R(m) &= \bigvee \{ l \in L_1 : l \otimes_1 k \leq \lambda_R(m) \} \\ &= \bigvee \{ l \in L_1 : \lambda_L(l) \leq \lambda_L(k) \to_2 m \} \\ &\overset{(2)}{=} \lambda_R(\lambda_L(k) \to_2 m) \end{aligned}$$

It is worth to note that this proof does not work in the case of (6) because, for the construction of λ_L, one had to use (3) instead of (2). □

Lemma 6. *For all $k_i, m_i \in L_i$ for $i \in \{1, 2\}$, the following inequalities hold*

$$\lambda_L(k_1 \to_1 m_1) \leq_2 \lambda_L(k_1) \to_2 \lambda_L(m_1) \tag{8}$$
$$\lambda_R(k_2 \to_2 m_2) \leq_1 \lambda_R(k_2) \to_1 \lambda_R(m_2) \tag{9}$$

Proof. By the adjoint property and the following chain of inequalities

$$\lambda_L(k_1 \to_1 m_1) \otimes_2 \lambda_L(k_1) \overset{(4)}{=} \lambda_L((k_1 \to_1 m_1) \otimes_1 k_1) \leq_2 \lambda_L(m_1)$$

Similarly, we obtain the other one. □

Below, we recall the notion of fixpoint of a Galois connection, the definition is uniform to the different types of Galois connection, either antitone or isotone, or with any other extra requirement.

Definition 11. *Let λ be a Galois connection between complete residuated lattices $\mathcal{L}_1$ and $\mathcal{L}_2$. The set of all fixpoints of λ is defined as*

$$\mathrm{FP}_\lambda = \{\langle k_1, k_2\rangle \in L_1 \times L_2 : \lambda_L(k_1) = k_2, \lambda_R(k_2) = k_1\}.$$

Lemma 7. *Given $\lambda \in \mathrm{CRL}(\mathcal{L}_1, \mathcal{L}_2)$, the set of its fixpoints can be provided with the structure of complete residuated lattice $\Phi_\lambda = \langle \mathrm{FP}_\lambda, \wedge, \vee, 0, 1, \otimes, \rightarrow\rangle$ where $0 = \langle \lambda_R(0_2), 0_2\rangle$, $1 = \langle 1_1, \lambda_L(1_1)\rangle$, and $\otimes$ and $\rightarrow$ are defined componentwise.*

Proof. We have to check just that the componentwise operations provide a residuated structure to the set of fixed point of λ.

Conditions (4) and (5) allow to prove that componentwise product $\otimes$ is a closed operation in FP_λ, whereas condition (6) allows to prove that the componentwise implication is a closed operation in FP_λ.

It is not difficult to show that, in fact, $\langle \mathrm{FP}_\lambda, \otimes, 1\rangle$ is a commutative monoid: commutativity and associativity follow directly; for the neutral element just consider the following chain of equalities: For any $\langle k_1, k_2\rangle \in \mathrm{FP}_\lambda$ holds

$$\begin{aligned}\langle k_1, k_2\rangle \otimes \langle 1_1, \lambda_L(1_1)\rangle &= \langle k_1 \otimes_1 1_1, \lambda_L(k_1) \otimes_2 \lambda_L(1_1)\rangle \\ &= \langle k_1, \lambda_L(k_1 \otimes_1 1_1)\rangle \\ &= \langle k_1, \lambda_L(k_1)\rangle = \langle k_1, k_2\rangle\end{aligned}$$

The adjoint property follows by definition. □

4 CRL-Chu correspondences and their category

In this section, the notion of L-Chu correspondence is generalized on the basis of a residuation-preserving isotone Galois connection λ. The formal definition is the following:

Definition 12. *Let $\mathcal{C}_i = \langle B_i, A_i, \mathcal{L}_i, r_i\rangle$ for $i \in \{1, 2\}$ be two fuzzy formal contexts, and consider $\lambda \in \mathrm{CRL}(\mathcal{L}_1, \mathcal{L}_2)$. A pair of fuzzy multifunctions $\varphi = \langle \varphi_L, \varphi_R\rangle$ of types*

$$\varphi_L \colon B_1 \longrightarrow \mathrm{Ext}(\mathcal{C}_2) \qquad \textit{and} \qquad \varphi_R \colon A_2 \longrightarrow \mathrm{Int}(\mathcal{C}_1)$$

such that for any $(o_1, a_2) \in B_1 \times A_2$ the following inequality holds

$$\lambda_L(\downarrow_1(\varphi_R(a_2))(o_1)) \leq_2 \uparrow_2(\varphi_L(o_1))(a_2) \tag{10}$$

*is said to be a λ-*Chu correspondence*.*

Note that (10) *is equivalent to $\downarrow_1(\varphi_R(a_2))(o_1) \leq_1 \lambda_R(\uparrow_2(\varphi_L(o_1))(a_2))$.*

It is not difficult to check that the definition of λ-Chu correspondence generalizes the previous one based on a complete (residuated) lattice L; formally, we have the following

Definition 13. *Let X be an arbitrary set. Mapping id^X defined by $\mathrm{id}^X(x) = x$ for any $x \in X$ is said to be an identity mapping on X.*

Lemma 8. *Any L-Chu correspondence is a $\langle \mathrm{id}^L, \mathrm{id}^L \rangle$-Chu correspondence.*

We are now in position to define the category of parameterized fuzzy formal contexts and λ-Chu correspondences between them:

Definition 14. *We introduce a new category whose objects are parameterized fuzzy formal contexts $\langle B, A, \mathcal{L}, r \rangle$ and λ-Chu correspondences between them.*

The identity arrow *of an object $\langle B, A, \mathcal{L}, r \rangle$ is the $\langle \mathrm{id}^L, \mathrm{id}^L \rangle$-Chu correspondence ι such that*

- $\iota_L(o) = \downarrow\uparrow(\chi_o)$ *for any* $o \in B$
- $\iota_R(a) = \uparrow\downarrow(\chi_a)$ *for any* $a \in A$
- *where* $\chi_x(x) = 1$ *and* $\chi_x(y) = 0$ *for any* $y \neq x$.

Composition of arrows[3] $\langle \lambda, \varphi \rangle \colon \mathcal{C}_1 \to \mathcal{C}_2$ *and* $\langle \mu, \psi \rangle \colon \mathcal{C}_2 \to \mathcal{C}_3$ *where* $\mathcal{C}_i = \langle B_i, A_i, \mathcal{L}_i, r_i \rangle$ *for* $i \in \{1, 2, 3\}$ *is defined as:*

- $(\langle \mu, \psi \rangle \circ \langle \lambda, \varphi \rangle)_L(o_1) = \downarrow_3\uparrow_3(\psi_{L+}(\varphi_L(o_1)))$
- $(\langle \mu, \psi \rangle \circ \langle \lambda, \varphi \rangle)_R(a_3) = \uparrow_1\downarrow_1(\varphi_{R+}(\psi_R(a_3)))$

where, for any $(o_i, a_i) \in B_i \times A_i$, $i \in \{1, 3\}$,

$$\psi_{L+}(\varphi_L(o_1))(o_3) = \bigvee_{o_2 \in B_2} \psi(o_2)(o_3) \otimes_3 \mu_L(\varphi_L(o_1)(o_2))$$
$$\varphi_{R+}(\psi_R(a_3))(a_1) = \bigvee_{a_2 \in A_2} \varphi_R(a_2)(a_1) \otimes_1 \lambda_R(\psi_R(a_3)(a_2))$$

Obviously, one has to check that the proposed notions of composition and identity are well-defined, and this is stated in the following lemmas.

Lemma 9. *The identity arrow of any fuzzy formal context $\langle B, A, \mathcal{L}, r \rangle$ is a $\langle \mathrm{id}^L, \mathrm{id}^L \rangle$-Chu correspondence.*

Lemma 10. *Consider $\langle \lambda, \varphi \rangle : \mathcal{C}_1 \to \mathcal{C}_2$ and $\langle \mu, \psi \rangle : \mathcal{C}_2 \to \mathcal{C}_3$, then $\langle \mu, \psi \rangle \circ \langle \lambda, \varphi \rangle$ is a $(\mu \circ \lambda)$-Chu correspondence. Moreover, composition of λ-correspondences is associative.*

[3] Any λ-Chu correspondence φ can be conveniently denoted by $\langle \lambda, \varphi \rangle$.

5 λ-Bonds and λ-direct product of two contexts

We proceed with the corresponding extension of the notion of bond between contexts, and the study of its properties.

Definition 15. *Given a multifunction $\omega\colon X \to (L_1 \times L_2)^Y$, its projections ω^i for $i \in \{1,2\}$ are defined by $\omega^i(x)(y) = k_i$, provided that $\omega(x)(y) = (k_1, k_2)$. Transposition of such multifunction is defined by $\omega^{\mathrm{t}}(y)(x) = \omega(x)(y)$.*

Definition 16. *Given two fuzzy formal contexts $\mathcal{C}_i = \langle B_i, A_i, \mathcal{L}_i, r_i\rangle$, $i \in \{1,2\}$, and $\lambda \in \mathrm{CRL}(\mathcal{L}_1, \mathcal{L}_2)$. A λ-bond is a multifunction $\beta\colon B_1 \to (L_1 \times L_2)^{A_2}$ such that, for any $(o_1, a_2) \in B_1 \times A_2$:*

$$\beta^2(o_1) \text{ is an intent of } \mathcal{C}_2 \tag{11}$$

$$(\beta^{\mathrm{t}})^1(a_2) \text{ is an extent of } \mathcal{C}_1 \tag{12}$$

$$\beta^1(o_1)(a_2) \leq_1 \lambda_R(\beta^2(o_1)(a_2)) \text{ or equivalently } \lambda_L(\beta^1(o_1)(a_2)) \leq_2 \beta^2(o_1)(a_2) \tag{13}$$

The known relation between L-bonds and L-Chu correspondences is preserved in the λ-case. Formally,

Lemma 11. *Let β be a λ-bond between two fuzzy contexts $\mathcal{C}_i = \langle B_i, A_i, \mathcal{L}_i, r_i\rangle$ for $i \in \{1,2\}$. Then φ_β defined as*

$$\varphi_{\beta L}(o_1) = \ {\downarrow_2}(\beta^2(o_1)) \tag{14}$$

$$\varphi_{\beta R}(a_2) = \ {\uparrow_1}((\beta^{\mathrm{t}})^1(a_2)) \tag{15}$$

is λ-Chu correspondence.

Proof. By calculation

$$\begin{aligned}
{\uparrow_2}(\varphi_{\beta L}(o_1))(a_2) &\overset{(14)}{=} {\uparrow_2}{\downarrow_2}(\beta^2(o_1))(a_2) \overset{(11)}{=} \beta^2(o_1)(a_2)\\
&\overset{(13)}{\geq} \lambda_L(\beta^1(o_1)(a_2)) = \lambda_L((\beta^{\mathrm{t}})^1(a_2)(o_1))\\
&\overset{(12)}{=} \lambda_L({\downarrow_1}{\uparrow_1}((\beta^{\mathrm{t}})^1(a_2))(o_1))\\
&\overset{(15)}{=} \lambda_L({\downarrow_1}(\varphi_{\beta R}(a_2))(o_1))
\end{aligned}$$

□

Lemma 12. *Let $\mathcal{L}_1, \mathcal{L}_2$ be two complete residuated lattices satisfying the double negation law and let $\lambda \in \mathrm{CRL}(\mathcal{L}_1, \mathcal{L}_2)$. Then Φ_λ satisfies double negation law.*

Proof. Consider an arbitrary $\langle k_1, k_2\rangle \in \mathrm{FP}_\lambda$. We have that, by definition,

$$\begin{aligned}
\neg\neg\langle k_1, k_2\rangle &= (\langle k_1, k_2\rangle \to 0) \to 0 = (\langle k_1, k_2\rangle \to \langle \lambda_R(0_2), 0_2\rangle) \to \langle \lambda_R(0_2), 0_2\rangle\\
&= \langle (k_1 \to_1 \lambda_R(0_2)) \to_1 \lambda_R(0_2), (k_2 \to_2 0_2) \to_2 0_2\rangle
\end{aligned}$$

The result for the second component is obvious; for the first one, taking into account that $(\lambda_R(0_2), 0_2)$ is a fixed point, we have

$$\begin{aligned}
\lambda_L(k_1) = k_2 &= (k_2 \to_2 0_2) \to_2 0_2 \\
&= \big(k_2 \to_2 \lambda_L\lambda_R(0_2)\big) \to_2 \lambda_L\lambda_R(0_2) \\
&\overset{(6)}{\geq}_2 \lambda_L(\lambda_R\lambda_L(\lambda_R(k_2) \to_1 \lambda_R(0_2)) \to_1 \lambda_R(0_2)) \\
&\overset{(\star)}{=} \lambda_L((\lambda_R(k_2) \to_1 \lambda_R(0_2)) \to_1 \lambda_R(0_2)) \\
&= \lambda_L((k_1 \to_1 \lambda_R(0_2)) \to_1 \lambda_R(0_2)) \\
&\overset{(*)}{\geq}_2 \lambda_L(k_1)
\end{aligned}$$

where equality $(\star)$ follows because, by Lemma 7, $\lambda_R(k_2) \to_1 \lambda_R(0_2)$ is a closed value in L_1 for the composition $\lambda_R\lambda_L$, and inequality $(*)$ follows from the monotonicity of λ_L.

As a result of the previous chain we obtain the following equality

$$\lambda_L(k_1) = \lambda_L((k_1 \to_1 \lambda_R(0_2)) \to_1 \lambda_R(0_2))$$

and, again by Lemma 7, since k_1 and $\lambda_R(0_2)$ are closed for $\lambda_R\lambda_L$, as a result $(k_1 \to_1 \lambda_R(0_2)) \to_1 \lambda_R(0_2)$ is closed too, and $k_1 = (k_1 \to_1 \lambda_R(0_2)) \to_1 \lambda_R(0_2)$. $\square$

We are now ready to include the characterization result on the structure of λ-bonds, but we have to introduce the notion of λ-direct product of contexts.

Definition 17. *Let $\mathcal{C}_i = \langle B_i, A_i, \mathcal{L}_i, r_i\rangle$ for $i \in \{1,2\}$ be two fuzzy formal contexts, $\lambda \in \mathrm{CRL}(\mathcal{L}_1, \mathcal{L}_2)$ and $\mathcal{L}_1$, $\mathcal{L}_2$ satisfy the double negation law. The fuzzy formal context $\langle B_1 \times A_2, B_2 \times A_1, \Phi_\lambda, \Delta_\lambda\rangle$ where $\Delta_\lambda((o_1,a_2),(o_2,a_1)) = \neg(\overline{\lambda_1}(r_1)) \to \overline{\lambda_2}(r_2(o_2,a_2))$ is said to be the* λ-direct product *of $\mathcal{C}_1$ and $\mathcal{C}_2$, where*

$$\overline{\lambda_1}(k) = \langle \lambda_R\lambda_L(k), \lambda_L(k)\rangle \text{ for all } k \in L_1 \tag{16}$$

$$\overline{\lambda_2}(k) = \langle \lambda_R(k), \lambda_L\lambda_R(k)\rangle \text{ for all } k \in L_2 \tag{17}$$

Lemma 13. *Let $\mathcal{C}_1\Delta_\lambda\mathcal{C}_2$ be the λ-direct product of fuzzy formal contexts $\mathcal{C}_1$ and $\mathcal{C}_2$, and $\lambda \in \mathrm{CRL}(\mathcal{L}_1, \mathcal{L}_2)$. For any extent of $\mathcal{C}_1\Delta_\lambda\mathcal{C}_2$ there exists a λ-bond between $\mathcal{C}_1$ and $\mathcal{C}_2$.*

Proof. Let $\langle \beta, \gamma\rangle$ be a concept of $\mathcal{C}_1\Delta_\lambda\mathcal{C}_2$. Then $\beta \in \mathrm{FP}_\lambda^{B_1\times A_2} \subseteq (L_1 \times L_2)^{B_1\times A_2}$.

$$\begin{aligned}
\beta(o_1,a_2) &= \downarrow_{\Delta_\lambda}(\gamma)(o_1,a_2) \\
&= \bigwedge_{o_2\in B_2}\bigwedge_{a_1\in A_1}(\gamma(o_2,a_1) \to \Delta_\lambda((o_1,a_2),(o_2,a_1))) \\
&= \bigwedge_{o_2\in B_2}\bigwedge_{a_1\in A_1}(\gamma(o_2,a_1) \to (\neg\overline{\lambda_1}(r_1(o_1,a_1))) \to \overline{\lambda_2}(r_2(o_2,a_2)))
\end{aligned}$$

Then

$$\begin{aligned}
\beta^1(o_1,a_2) &= \bigwedge_{o_2\in B_2}\bigwedge_{a_1\in A_1}(\gamma^1(o_2,a_1)\to_1(\neg\lambda_R\lambda_L(r_1(o_1,a_1)))\to_1\lambda_R(r_2(o_2,a_2)))\\
&= \bigwedge_{o_2\in B_2}\bigwedge_{a_1\in A_1}((\gamma^1(o_2,a_1)\otimes_1\neg\lambda_R\lambda_L(r_1(o_1,a_1)))\to_1\lambda_R(r_2(o_2,a_2)))\\
&\overset{(7)}{=} \bigwedge_{o_2\in B_2}\bigwedge_{a_1\in A_1}\lambda_R(\lambda_L(\gamma^1(o_2,a_1)\otimes_1\neg\lambda_R\lambda_L(r_1(o_1,a_1)))\to_2 r_2(o_2,a_2))\\
&= \lambda_R(\bigwedge_{o_2\in B_2}\bigwedge_{a_1\in A_1}(\lambda_L(\gamma^1(o_2,a_1)\otimes_1\neg\lambda_R\lambda_L(r_1(o_1,a_1)))\to_2 r_2(o_2,a_2)))\\
&= \lambda_R(\bigwedge_{o_2\in B_2}(\bigvee_{a_1\in A_1}(\lambda_L(\gamma^1(o_2,a_1)\otimes_1\neg\lambda_R\lambda_L(r_1(o_1,a_1))))\to_2 r_2(o_2,a_2)))\\
&= \lambda_R(\bigwedge_{o_2\in B_2}(\sigma(o_1)(o_2)\to_2 r_2(o_2,a_2)))\\
&= \lambda_R(\uparrow_2(\sigma(o_1))(a_2))
\end{aligned}$$

Similarly

$$\begin{aligned}
\beta^2(o_1,a_2) &= \bigwedge_{o_2\in B_2}\bigwedge_{a_1\in A_1}((\gamma^2(o_2,a_1)\otimes_2\neg\lambda_L\lambda_R(r_2(o_2,a_2)))\to_2\lambda_L(r_1(o_1,a_1)))\\
&\overset{(6)}{\geq}_2 \bigwedge_{o_2\in B_2}\bigwedge_{a_1\in A_1}\lambda_L(\lambda_R(\gamma^2(o_2,a_1)\otimes_2\neg\lambda_L\lambda_R(r_2(o_2,a_2)))\to_2 r_1(o_1,a_1))\\
&\geq_2 \lambda_L(\bigwedge_{o_2\in B_2}\bigwedge_{a_1\in A_1}(\lambda_R(\gamma^2(o_2,a_1)\otimes_2\neg\lambda_L\lambda_R(r_2(o_2,a_2)))\to_1 r_1(o_1,a_1)))\\
&= \lambda_L(\bigwedge_{a_1\in A_1}(\bigvee_{o_2\in B_2}(\lambda_R(\gamma^2(o_2,a_1)\otimes_2\neg\lambda_L\lambda_R(r_2(o_2,a_2))))\to_1 r_1(o_1,a_1)))\\
&= \lambda_L(\bigwedge_{a_1\in A_1}(\tau(a_2)(a_1)\to_1 r_1(o_1,a_1))) = \lambda_L(\downarrow_1(\tau(a_2))(o_1))
\end{aligned}$$

Then let us define a multifunction $\widehat{\beta}\colon B_1 \longrightarrow (L_1\times L_2)^{A_2}$ as follows

$$\widehat{\beta}^2(o_1) = \uparrow_2(\sigma(o_1))$$
$$(\widehat{\beta}^{\mathrm{t}})^1(a_2) = \downarrow_1(\tau(a_2))$$

where σ and τ are multifunctions above. We see that $\widehat{\beta}^2(o_1)$ is the intent of $\mathcal{C}_2$, $(\widehat{\beta}^{\mathrm{t}})^1(a_2)$ is the extent of $\mathcal{C}_1$ and moreover $\beta(o_1,a_2)\in\mathrm{FP}_\lambda$, hence $\lambda_L(\beta^1(o_1,a_2)) = \beta^2(o_1,a_2)$ and

$$\begin{aligned}
\lambda_L((\widehat{\beta}^{\mathrm{t}})^1(a_2)(o_1)) &= \lambda_L(\downarrow_1(\tau(a_2)))\\
&\leq_2 \beta^2(o_1,a_2)\\
&= \lambda_L(\beta^1(o_1,a_2))\\
&= \lambda_L\lambda_R(\uparrow_2(\sigma(o_1))(a_2))\\
&\leq_2 \uparrow_2(\sigma(o_1))(a_2) = \widehat{\beta}^2(o_1)(a_2)
\end{aligned}$$

Therefore $\widehat{\beta}$ is a λ-bond between $\mathcal{C}_1$ and $\mathcal{C}_2$. $\square$

6 Motivation example

Lets have two tables of the following data. First table of students, school subjects and study results. Second table of universities (or areas of study) and their requirements for results of students. We would like to find the assignment of students and universities that depends on study results and requirements of universities.

$\mathcal{S}$	Math	Phi	Chem	Bio
Anna	A	A	B	C
Boris	C	B	A	B
Cyril	D	E	B	C

$\mathcal{U}$	CS	Tech	Med
Math	Ex	G	W
Phi	G	Ex	G
Chem	W	Ex	Ex
Bio	W	W	Ex

First table is filled by degrees from well known structure {A,B,C,D,E,F} where A is best and F means failed. Second one is filled by degrees {Ex,G,W} that means Ex-excelent, G-good, W-weak. Now lets define a λ-translation between such truth-degrees structures.

	A	B	C	D	E	F
$\lambda_L(-)$	Ex	G	W	W	W	W

	Ex	G	W
$\lambda_R(-)$	A	B	C

$\langle \lambda_L, \lambda_R \rangle$ is an isotone Galois connection. In fact, $\langle \lambda_L, \lambda_R \rangle$ is a residuation-preserving isotone Galois connection over Łukasiewicz logic, whose set of fix-points is

$$\mathrm{FP}_\lambda = \{(\mathrm{A}, \mathrm{Ex}); (\mathrm{B}, \mathrm{G}); (\mathrm{C}, \mathrm{W})\}$$

The λ-direct product $\mathcal{S}\Delta_\lambda\mathcal{U}$ is the following table that has 510 concepts. Lets simplify the table with translation (A,Ex) as 1, (B,G) as 0.5 and (C,W) as 0.

1	1	0.5	0	1	1	1	0.5	1	1	1	1	1	1	1	1
1	1	1	0.5	1	1	1	1	1	1	1	1	1	1	0.5	0
1	1	1	1	1	1	1	0.5	1	1	0.5	0	1	1	0.5	0
0	0.5	1	0.5	0.5	1	1	1	1	1	1	1	1	1	1	1
0.5	1	1	1	1	1	1	1	1	1	1	1	0	0.5	1	0.5
1	1	1	1	0.5	1	1	1	0	0.5	1	0.5	0	0.5	1	0.5
0	0	0.5	0	0.5	0.5	1	0.5	1	1	1	1	1	1	1	1
0.5	0.5	1	0.5	1	1	1	1	1	1	1	1	0	0	0.5	0
1	1	1	1	0.5	0.5	1	0.5	0	0	0.5	0	0	0	0.5	0

Extents of $\mathcal{S}\Delta_\lambda\mathcal{U}$ are tables of the form students$\times$universities and intents are tables of the form subjects$\times$subjects. One of the concepts that their intents has 1 on diagonal (it means that any subject is assigned to itself) is shown below.

	Med	Tech	CS
Anna	B,G	B,G	A,Ex
Boris	A,Ex	B,G	C,W
Cyril	B,G	B,G	C,W

	Math	Phi	Chem	Bio
Math	A,Ex	A,Ex	B,G	C,W
Phi	B,G	A,Ex	A,Ex	B,G
Chem	C,W	B,G	A,Ex	B,G
Bio	C,W	B,G	A,Ex	B,G

Such concept should be translated into $\{1; 0.5; 0\}$ structure.

	Med	Tech	CS
Anna	0.5	0.5	1
Boris	1	0.5	0
Cyril	0.5	0.5	0

	Math	Phi	Chem	Bio
Math	1	1	0.5	0
Phi	0.5	1	1	0.5
Chem	0	0.5	1	0.5
Bio	0	0.5	1	0.5

Now we can see the result. Due to results of students and requirements of universities we can advise Anna to study Computer science; similarly, we can advise Boris to study Medicine or Technical area; finally, it is hard to advise anything to Cyril. The assignment is right as it is obvious from study results.

Relaxing the connection

Let's change the λ-connection between such truth degrees structures as follows:

	A	B	C	D	E	F
$\lambda_L(-)$	Ex	G	G	W	W	W

	Ex	G	W
$\lambda_R(-)$	A	B	D

The set of fixpoints is $\mathrm{FP}_\lambda = \{(\mathrm{A}, \mathrm{Ex}); (\mathrm{B}, \mathrm{G}); (\mathrm{D}, \mathrm{W})\}$ such that is easy to translate $(\mathrm{A}, \mathrm{Ex})$ as 1, (B, G) as 0.5 and (D, W) as 0. The direct product is shown below, and has 104 concepts.

1	1	1	1	1	1	1	1	1	1	0.5	0.5	1	1	0.5	0.5
1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.5	0.5
1	1	0.5	0.5	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	0.5	0.5	1	0.5	0.5	0.5	1	0.5
1	1	1	1	1	1	1	1	1	1	1	1	0.5	0.5	1	0.5
0.5	0.5	1	0.5	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	0.5	0.5	1	1	0	0	0.5	0.5	0	0	0.5	0.5
0.5	0.5	1	1	1	1	1	1	1	1	1	1	0	0	0.5	0.5
0	0	0.5	0.5	0.5	0.5	1	1	1	1	1	1	1	1	1	1

We have chosen one with 1-diagonal in the intent.

	Med	Tech	CS
Anna	B,G	A,Ex	A,Ex
Boris	A,Ex	A,Ex	B,G
Cyril	B,G	B,G	D,W

	Math	Phi	Chem	Bio
Math	A,Ex	A,Ex	B,G	B,G
Phi	A,Ex	A,Ex	A,Ex	A,Ex
Chem	B,G	B,G	A,Ex	B,G
Bio	B,G	B,G	B,G	B,G

Translating the context into $\{1; 0.5; 0\}$ structure, we obtain

	Med	Tech	CS
Anna	0.5	1	1
Boris	1	1	0.5
Cyril	0.5	0.5	0

	Math	Phi	Chem	Bio
Math	1	1	0.5	0.5
Phi	1	1	1	0.5
Chem	0.5	0.5	1	0.5
Bio	0.5	0.5	0.5	0.5

It can seen that our advice is more generous but still coincide to input data.

7 Conclusion

We continue our study of the general notion of L-Chu correspondence by introducing the category CRL-ChuCors incorporating residuation to the underlying complete lattice L, specifically, on the basis of a residuation-preserving isotone Galois connection λ. Then, the L-bonds are generalized within this same framework, and its structure is related to that of the extent of a suitably defined λ-direct product. A first relationship between extents of λ-direct product have been proved; it is expected to find a proof of the stronger result which states an isomorphism between the extents of the λ-direct product and the λ-bonds between $\mathcal{C}_1$ and $\mathcal{C}_2$.

Potential applications are primary motivations for further future work, for instance, to consider possible classes of formal L-contexts induced from existing datamining notions, and study its associated categories.

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