Gödel $\mathcal{FL}_0$ with Greatest Fixed-Point Semantics*

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Abstract. We study the fuzzy extension of $\mathcal{FL}_0$ with semantics based on the Gödel t-norm. We show that gfp-subsumption w.r.t. a finite set of primitive definitions can be characterized by a relation on weighted automata, and use this result to provide tight complexity bounds for reasoning in this logic.

1 Introduction

Fuzzy Description Logics (DLs) have been introduced as extensions of classical DLs [2] capable of representing and reasoning with vague or imprecise knowledge. The main idea behind these logics is to allow for a set of truth degrees, beyond the standard true and false. The area of fuzzy DLs recently experienced a shift, when it was shown that reasoning in these logics easily becomes undecidable [3,6,8]. To guarantee decidability in fuzzy DLs, one can (i) restrict the semantics to consider finitely many truth degrees [7]; (ii) allow only acyclic or unfoldable ontologies [4,18]; or (iii) restrict to Zadeh or Gödel semantics [5,15,16,17].

In the cases where the Gödel t-norm is used, the complexity of reasoning is typically the same as for its classical version, as shown for $\mathcal{EL}$, which is polynomial [15,16], and $\mathcal{ALC}$, ExpTime-complete [5]. This latter result immediately implies that reasoning in G-$\mathcal{FL}_0$ with general TBoxes is also ExpTime-complete. On the other hand, if TBoxes are restricted to contain only (primitive) definitions, then deciding subsumption in classical $\mathcal{FL}_0$ under the greatest fixed-point semantics is known to be in PSpace [1]. We show that the same complexity bound holds for the Gödel extension of this logic.

To prove this complexity result, we characterize the greatest fixed-point semantics of G-$\mathcal{FL}_0$ by means of weighted automata over lattices. We then show that reasoning with these automata can be reduced to a linear number of inclusion tests between unweighted automata, which can be solved using only polynomial space [10].

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2 Preliminaries

We first introduce some basic notions of lattice theory. For a more comprehensive overview on the topic, refer to [11]. Afterwards, we introduce fuzzy logics based on Gödel semantics, which are studied in more detail in [9,12,14].

A *lattice* is an algebraic structure $(L, \vee, \wedge)$ with two commutative, associative and idempotent binary operations $\vee$ (supremum) and $\wedge$ (infimum) that distribute over each other. It is *complete* if suprema and infima of arbitrary subsets $S \subseteq L$, denoted by $\bigvee_{x \in S} x$ and $\bigwedge_{x \in S} x$ respectively, exist. In this case, the lattice is *bounded* by the greatest element $\mathbf{1} := \bigvee_{x \in L} x$ and the least element $\mathbf{0} := \bigwedge_{x \in L} x$. Lattices induce a natural partial ordering on the elements of L where $x \leq y$ iff $x \wedge y = x$.

Example 1. One common complete lattice used in fuzzy logics (see e.g. [9,12]) is the interval $[0, 1]$ with the usual order on the real numbers. Further complete lattices relevant for this paper can be constructed as follows. Given a complete lattice L and a set S, the set L^S of all functions $f \colon S \to L$ is also a complete lattice, if infimum and supremum are defined component-wise. More precisely, for any two $f_1, f_2 \in L^S$, we define $f_1 \vee f_2$ for all $x \in S$ as $(f_1 \vee f_2)(x) := f_1(x) \vee f_2(x)$. If we similarly define the infimum, we obtain a lattice with the order $f_1 \leq f_2$ iff $f_1(x) \leq f_2(x)$ holds for all $x \in S$. It is easy to verify that infinite infima and suprema can then also be computed component-wise.

We are particularly interested in operators on complete lattices L and their properties.

Definition 2 (fixed-point). *Let L be a complete lattice. A* fixed-point *of an operator $T \colon L \to L$ is an element $x \in L$ such that $T(x) = x$. It is the* greatest fixed-point *of T if for any fixed-point y of T we have $y \leq x$.*

The operator T is monotone *if for all $x, y \in L$, $x \leq y$ implies $T(x) \leq T(y)$. It is* downward ω-continuous *if for every decreasing chain $x_0 \geq x_1 \geq x_2 \geq \dots$ in L we have $T(\bigwedge_{i \geq 0} x_i) = \bigwedge_{i \geq 0} T(x_i)$.*

If it exists, the greatest fixed-point of T is unique and denoted by $\mathsf{gfp}(T)$.

It is easy to verify that every downward ω-continuous operator is also monotone. By a fundamental result from [20], every monotone operator T has a greatest fixed-point. If T is downward ω-continuous, then $\mathsf{gfp}(T)$ corresponds to the infimum of the decreasing chain $\mathbf{1} \geq T(\mathbf{1}) \geq T(T(\mathbf{1})) \geq \dots \geq T^i(\mathbf{1}) \geq \dots$ [13].

Proposition 3. *If L is a complete lattice and T a downward ω-continuous operator on L, then $\mathsf{gfp}(T) = \bigwedge_{i \geq 0} T^i(\mathbf{1})$.*

Our fuzzy DL is based on the well-known Gödel semantics for fuzzy logics, which is one of the main t-norm-based semantics used in Mathematical Fuzzy Logic [9,12]. This semantics is based on the standard interval $[0, 1]$. The *Gödel t-norm* is the binary minimum operator on this set. For consistency, we use the lattice-theoretic notation $\wedge$ instead of min. Two important properties of this operator are that it preserves arbitrary infima and suprema on $[0,1]$, i.e. $\bigwedge_{i\in I}(x_i \wedge x) = \big(\bigwedge_{i\in I} x_i\big) \wedge x$ and $\bigvee_{i\in I}(x_i \wedge x) = \big(\bigvee_{i\in I} x_i\big) \wedge x$ for any index set I and elements $x, x_i \in [0,1]$ for all $i \in I$. In particular, this means that the Gödel t-norm is monotone in both arguments. The *residuum* of the Gödel t-norm is the binary operator $\Rightarrow$ on $[0,1]$ defined for all $x, y \in [0,1]$ by

$$x \Rightarrow y := \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

It is a fundamental property of a t-norm and its residuum that for all values $x, y, z \in [0,1]$ we have $x \wedge y \leq z$ iff $y \leq x \Rightarrow z$. As with the Gödel t-norm, its residuum preserves arbitrary infima in its second component. However, in the first component the order on $[0,1]$ is reversed.

Proposition 4. *For any index set I and values $x, x_i \in [0,1]$, $i \in I$, we have*

$$x \Rightarrow \Big(\bigwedge_{i\in I} x_i\Big) = \bigwedge_{i\in I}(x \Rightarrow x_i) \quad \textit{and} \quad \Big(\bigvee_{i\in I} x_i\Big) \Rightarrow x = \bigwedge_{i\in I}(x_i \Rightarrow x).$$

This shows that the residuum is monotone in the second argument and antitone in the first argument. The following reformulation of nested residua in terms of infima will also prove useful.

Proposition 5. *For all values $x, x_1, \dots, x_n \in [0,1]$, we have*

$$\big((x_1 \wedge \cdots \wedge x_n) \Rightarrow x\big) = \big(x_1 \Rightarrow \dots (x_n \Rightarrow x) \dots\big).$$

Proof. Both values are either x or 1, and they are 1 iff one of the operands x_i, $1 \leq i \leq n$, is smaller than or equal to x. □

3 Fuzzy $\mathcal{FL}_0$

The fuzzy description logic G-$\mathcal{FL}_0$ has the same syntax as classical $\mathcal{FL}_0$. The difference lies in the interpretation of G-$\mathcal{FL}_0$-concepts.

Definition 6 (syntax). *Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be two non-empty, disjoint sets of* concept names *and* role names*, respectively.* Concepts *are built from concept names using the constructors $\top$ (top), $C \sqcap D$ (conjunction), and $\forall r.C$ (value restriction for a role name r).*

A (primitive concept) definition *is of the form $\langle A \sqsubseteq C \geq p\rangle$, where $A \in \mathsf{N_C}$, C is a concept, and $p \in [0,1]$. A* TBox *is a finite set of definitions. Given a TBox $\mathcal{T}$, a concept name is* defined *if it appears on the left-hand side of a definition in $\mathcal{T}$, and* primitive *otherwise.*

We use the expression $\forall w.C$ with $w = r_1 r_2 \dots r_n \in \mathsf{N_R}^*$ to abbreviate the concept $\forall r_1.\forall r_2.\dots.\forall r_n.C$. We also allow $w = \varepsilon$, in which case $\forall w.C$ is simply C. We denote the set of concept names occurring in the TBox $\mathcal{T}$ by $\mathsf{N}_\mathsf{C}^\mathcal{T}$, the set of defined concept names in $\mathsf{N}_\mathsf{C}^\mathcal{T}$ by $\mathsf{N}_\mathsf{D}^\mathcal{T}$, and the set of primitive concept names in $\mathsf{N}_\mathsf{C}^\mathcal{T}$ by $\mathsf{N}_\mathsf{P}^\mathcal{T}$. Likewise, we collect all role names occurring in $\mathcal{T}$ into the set $\mathsf{N}_\mathsf{R}^\mathcal{T}$.

Definition 7 (semantics). *An* interpretation *is a pair* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$*, where* $\Delta^{\mathcal{I}}$ *is a non-empty set, called the* domain *of* $\mathcal{I}$*, and the* interpretation function $\cdot^{\mathcal{I}}$ *maps every concept name* A *to a fuzzy set* $A^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0,1]$ *and every role name* r *to a fuzzy binary relation* $r^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$*. This function is extended to concepts by setting* $\top^{\mathcal{I}}(x) := 1$*,* $(C \sqcap D)^{\mathcal{I}}(x) := C^{\mathcal{I}}(x) \wedge D^{\mathcal{I}}(x)$*, and* $(\forall r.C)^{\mathcal{I}}(x) := \bigwedge_{y \in \Delta^{\mathcal{I}}}(r^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y))$ *for all* $x \in \Delta^{\mathcal{I}}$*.*

The interpretation $\mathcal{I}$ satisfies *(or is a* model *of) the definition* $\langle A \sqsubseteq C \geq p\rangle$ *if* $A^{\mathcal{I}}(x) \Rightarrow C^{\mathcal{I}}(x) \geq p$ *holds for all* $x \in \Delta^{\mathcal{I}}$*. It* satisfies *(or is a* model *of) a TBox if it satisfies all its definitions.*

For an interpretation $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$, $w = r_1 r_2 \dots r_n \in \mathsf{N}_\mathsf{R}^*$, and elements $x_0, x_n \in \Delta$, we set $w^{\mathcal{I}}(x_0, x_n) := \bigvee_{x_1,\dots,x_{n-1} \in \Delta}(r_1^{\mathcal{I}}(x_0,x_1) \wedge \dots \wedge r_n^{\mathcal{I}}(x_{n-1},x_n))$, and can thus treat $\forall w.C$ like an ordinary value restriction with

$$\begin{aligned}
(\forall w.C)^{\mathcal{I}}(x_0) &:= \bigwedge_{x_n \in \Delta} (w^{\mathcal{I}}(x_0,x_n) \Rightarrow C^{\mathcal{I}}(x_n)) \\
&= \bigwedge_{x_1,\dots,x_n \in \Delta} \big(\big(r_1^{\mathcal{I}}(x_0,x_1) \wedge \dots \wedge r_n^{\mathcal{I}}(x_{n-1},x_n)\big) \Rightarrow C^{\mathcal{I}}(x_n)\big) \\
&= \bigwedge_{x_1,\dots,x_n \in \Delta} \big(r_1^{\mathcal{I}}(x_0,x_1) \Rightarrow \dots (r_n^{\mathcal{I}}(x_{n-1},x_n) \Rightarrow C^{\mathcal{I}}(x_n)) \dots\big) \\
&= (\forall r_1. \dots \forall r_n.C)^{\mathcal{I}}(x_0)
\end{aligned}$$

for all $x_0 \in \Delta$ (see Propositions 4 and 5).

It is convenient to consider TBoxes in normal form. The TBox $\mathcal{T}$ is in *normal form* if all definitions in $\mathcal{T}$ are of the form $\langle A \sqsubseteq \forall w.B \geq p\rangle$, where $A, B \in \mathsf{N}_\mathsf{C}$, $w \in \mathsf{N}_\mathsf{R}^*$, and $p \in [0,1]$, and there are no two definitions $\langle A \sqsubseteq \forall w.B \geq p\rangle$, $\langle A \sqsubseteq \forall w.B \geq p'\rangle$ with $p \neq p'$. Every TBox can be transformed into an equivalent TBox in normal form, as follows. First, we distribute the value restrictions over the conjunctions.

Lemma 8. *For every* $r \in \mathsf{N}_\mathsf{R}$*, concepts* C, D*, and interpretation* $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$*, it holds that* $(\forall r.(C \sqcap D))^{\mathcal{I}} = (\forall r.C \sqcap \forall r.D)^{\mathcal{I}}$*.*

Thus, we can equivalently write the right-hand sides of the definitions in $\mathcal{T}$ in the form $\forall w_1.B_1 \sqcap \dots \sqcap \forall w_n.B_n$, where $w_i \in \mathsf{N}_\mathsf{R}^*$ and $B_i \in \mathsf{N}_\mathsf{C} \cup \{\top\}, 1 \leq i \leq n$. Since $\forall r.\top$ is equivalent to $\top$, we can remove all conjuncts of the form $\forall w.\top$ from this representation. After this transformation, all the definitions in the TBox are of the form $\langle A \sqsubseteq \forall w_1.B_1 \sqcap \dots \sqcap \forall w_n.B_n \geq p\rangle$ with $B_i \in \mathsf{N}_\mathsf{C}, 1 \leq i \leq n$, or $\langle A \sqsubseteq \top \geq p\rangle$. The latter axioms are tautologies, and can hence be removed from the TBox without affecting the semantics.

It follows from Proposition 4 that an interpretation $\mathcal{I}$ satisfies the definition $\langle A \sqsubseteq \forall w_1.B_1 \sqcap \dots \sqcap \forall w_n.B_n \geq p\rangle$ iff it satisfies all the axioms $\langle A \sqsubseteq \forall w_i.B_i \geq p\rangle$, $1 \leq i \leq n$. Thus, the former axiom can be equivalently replaced by the latter set of axioms.

After these simplification steps, the TBox contains only axioms of the form $\langle A \sqsubseteq \forall w.B \geq p\rangle$ with $A, B \in \mathsf{N}_\mathsf{C}$, satisfying the first condition of the definition of normal form. Suppose now that $\mathcal{T}$ contains two axioms of the form

$\langle A \sqsubseteq \forall w.B \geq p \rangle$ and $\langle A \sqsubseteq \forall w.B \geq p' \rangle$ with $p > p'$. Then $\mathcal{T}$ is equivalent to the TBox $\mathcal{T} \setminus \{\langle A \sqsubseteq \forall w.B \geq p' \rangle\}$; which means that this axiom can be removed. It is clear that all of these transformations can be done in polynomial time in the size of the original TBox.

Concept definitions can be seen as a restriction of the interpretation of the defined concepts, depending on the interpretation of the primitive concepts. We use this intuition and consider *greatest fixed-point* semantics, as described next.

A *primitive interpretation* is a pair $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$ as in Definition 7, except that $\cdot^{\mathcal{J}}$ is only defined for role names and the *primitive* concept names in $\mathsf{N}_\mathsf{P}^\mathcal{T}$. Given such a $\mathcal{J}$, we use functions $f \in ([0,1]^\Delta)^{\mathsf{N}_\mathsf{D}^\mathcal{T}}$ to describe the interpretation of the remaining (defined) concept names. Recall from Example 1 that these functions form a complete lattice. In the following, we use the abbreviation $L_\mathcal{J}^\mathcal{T} := ([0,1]^\Delta)^{\mathsf{N}_\mathsf{D}^\mathcal{T}}$ for this lattice. Given a primitive interpretation $\mathcal{J}$ and a function $f \in L_\mathcal{J}^\mathcal{T}$, the *induced interpretation* $\mathcal{I}_{\mathcal{J},f}$ has the same domain as $\mathcal{J}$ and extends the interpretation function of $\mathcal{J}$ to the defined concepts names $A \in \mathsf{N}_\mathsf{D}^\mathcal{T}$ by taking $A^{\mathcal{I}_{\mathcal{J},f}} := f(A)$. The interpretation of the remaining concept names, i.e. those that do not occur in $\mathcal{T}$, is fixed to $\mathbf{0}$.

We can describe the effect that the axioms in $\mathcal{T}$ have on $L_\mathcal{J}^\mathcal{T}$ by the operator $T_\mathcal{J}^\mathcal{T} \colon L_\mathcal{J}^\mathcal{T} \to L_\mathcal{J}^\mathcal{T}$, which is defined as follows for all $f \in L_\mathcal{J}^\mathcal{T}$, $A \in \mathsf{N}_\mathsf{D}^\mathcal{T}$, and $x \in \Delta$:

$$T_\mathcal{J}^\mathcal{T}(f)(A)(x) := \bigwedge_{\langle A \sqsubseteq C \geq p \rangle \in \mathcal{T}} (p \Rightarrow C^{\mathcal{I}_{\mathcal{J},f}}(x)).$$

This operator computes new values of the defined concept names according to the old interpretation $\mathcal{I}_{\mathcal{J},f}$ and their definitions in $\mathcal{T}$.

We are interested in using the greatest fixed-point of $T_\mathcal{J}^\mathcal{T}$, for some primitive interpretation $\mathcal{J}$, to define a new semantics for TBoxes $\mathcal{T}$ in $\mathsf{G}\text{-}\mathcal{FL}_0$. Before being able to do this, we have to ensure that such a fixed-point always exists.

Lemma 9. *Given a TBox $\mathcal{T}$ and a primitive interpretation $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$, the operator $T_\mathcal{J}^\mathcal{T}$ on $L_\mathcal{J}^\mathcal{T}$ is downward ω-continuous.*

Proof. Consider a decreasing chain $f_0 \geq f_1 \geq f_2 \geq \dots$ of functions in $L_\mathcal{J}^\mathcal{T}$. We use the abbreviations $f := \bigwedge_{i\geq 0} f_i$, $\mathcal{I} := \mathcal{I}_{\mathcal{J},f}$, and $\mathcal{I}_i := \mathcal{I}_{\mathcal{J},f_i}$ for all $i \geq 0$, and have to show that $T_\mathcal{J}^\mathcal{T}(f) = \bigwedge_{i\geq 0} T_\mathcal{J}^\mathcal{T}(f_i)$ holds.

First, we prove by induction on the structure of C that $C^\mathcal{I} = \bigwedge_{i\geq 0} C^{\mathcal{I}_i}$ holds for all concepts C built from $\mathsf{N}_\mathsf{R}^\mathcal{T}$ and $\mathsf{N}_\mathsf{C}^\mathcal{T}$, where $\bigwedge$ is defined as usual over the complete lattice $[0,1]^\Delta$.

For $A \in \mathsf{N}_\mathsf{P}^\mathcal{T}$, by the definition of $\mathcal{I}_{\mathcal{J},f}$ and $\mathcal{I}_{\mathcal{J},f_i}$ we have $A^\mathcal{I} = A^\mathcal{J} = A^{\mathcal{I}_i}$ for all $i \geq 0$, and thus $A^\mathcal{I} = A^\mathcal{J} = \bigwedge_{i\geq 0} A^{\mathcal{I}_i}$. For $A \in \mathsf{N}_\mathsf{D}^\mathcal{T}$, we have

$$A^\mathcal{I} = f(A) = \Big(\bigwedge_{i\geq 0} f_i\Big)(A) = \bigwedge_{i\geq 0} f_i(A) = \bigwedge_{i\geq 0} A^{\mathcal{I}_i}$$

by the definition of $\mathcal{I}_{\mathcal{J},f}$ and $\mathcal{I}_{\mathcal{J},f_i}$ and the component-wise ordering on the complete lattice $L_\mathcal{J}^\mathcal{T}$.

For concepts of the form $C \sqcap D$, by the induction hypothesis and associativity of $\wedge$ we have

$$(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \wedge D^{\mathcal{I}} = \Big(\bigwedge_{i\geq 0} C^{\mathcal{I}_i}\Big) \wedge \Big(\bigwedge_{i\geq 0} D^{\mathcal{I}_i}\Big) = \bigwedge_{i\geq 0}(C^{\mathcal{I}_i} \wedge D^{\mathcal{I}_i}) = \bigwedge_{i\geq 0}(C \sqcap D)^{\mathcal{I}_i}.$$

Consider now a value restriction $\forall r.C$. Using Proposition 4 we get for all $x \in \Delta$,

$$\begin{aligned}(\forall r.C)^{\mathcal{I}}(x) &= \bigwedge_{y\in\Delta}(r^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)) = \bigwedge_{y\in\Delta}\Big(r^{\mathcal{I}}(x,y) \Rightarrow \Big(\bigwedge_{i\geq 0} C^{\mathcal{I}_i}(y)\Big)\Big)\\ &= \bigwedge_{y\in\Delta}\bigwedge_{i\geq 0}(r^{\mathcal{I}_i}(x,y) \Rightarrow C^{\mathcal{I}_i}(y)) = \bigwedge_{i\geq 0}(\forall r.C)^{\mathcal{I}_i}(x) = \Big(\bigwedge_{i\geq 0}(\forall r.C)^{\mathcal{I}_i}\Big)(x)\end{aligned}$$

by the induction hypothesis, and the component-wise ordering on $[0,1]^{\Delta}$.

Using this, we can now prove the actual claim of the lemma. For all $A \in \mathsf{N}_{\mathsf{D}}^{\mathcal{T}}$ and all $x \in \Delta$, we get, using again Proposition 4 and the previous claim

$$\begin{aligned}T_{\mathcal{J}}^{\mathcal{T}}(f)(A)(x) &= \bigwedge_{\langle A\sqsubseteq C\geq p\rangle\in\mathcal{T}}(p \Rightarrow C^{\mathcal{I}}(x)) = \bigwedge_{\langle A\sqsubseteq C\geq p\rangle\in\mathcal{T}}\Big(p \Rightarrow \Big(\bigwedge_{i\geq 0} C^{\mathcal{I}_i}(x)\Big)\Big)\\ &= \bigwedge_{\langle A\sqsubseteq C\geq p\rangle\in\mathcal{T}}\bigwedge_{i\geq 0}(p \Rightarrow C^{\mathcal{I}_i}(x)) = \Big(\bigwedge_{i\geq 0} T_{\mathcal{J}}^{\mathcal{T}}(f_i)\Big)(A)(x)\end{aligned}$$

by the definition of $T_{\mathcal{J}}^{\mathcal{T}}$, and the component-wise ordering on $L_{\mathcal{J}}^{\mathcal{T}}$. □

By Proposition 3, we know that $\mathsf{gfp}(T_{\mathcal{J}}^{\mathcal{T}})$ exists and is equal to $\bigwedge_{i\geq 0}(T_{\mathcal{J}}^{\mathcal{T}})^i(\mathbf{1})$, where $\mathbf{1}$ is the greatest element of the lattice $L_{\mathcal{J}}^{\mathcal{T}}$ that maps all defined concept names to $\top^{\mathcal{J}}$. In the following, we denote by $\mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$ the interpretation $\mathcal{I}_{\mathcal{J},f}$ for $f := \mathsf{gfp}(T_{\mathcal{J}}^{\mathcal{T}})$. Note that $\mathcal{I} := \mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$ is actually a model of $\mathcal{T}$ since for every $\langle A \sqsubseteq C \geq p\rangle \in \mathcal{T}$ and every $x \in \Delta$ we have

$$A^{\mathcal{I}}(x) = f(A)(x) = T_{\mathcal{J}}^{\mathcal{T}}(f)(A)(x) = \bigwedge_{\langle A\sqsubseteq C'\geq p'\rangle\in\mathcal{T}}(p' \Rightarrow C'^{\mathcal{I}}(x)) \leq p \Rightarrow C^{\mathcal{I}}(x),$$

and thus $p \wedge A^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x)$, which is equivalent to $p \leq A^{\mathcal{I}}(x) \Rightarrow C^{\mathcal{I}}(x)$.

We can now define the reasoning problem in $\mathsf{G}\text{-}\mathcal{FL}_0$ that we want to solve.

Definition 10 (gfp-subsumption). *An interpretation $\mathcal{I}$ is a* gfp-model *of a TBox $\mathcal{T}$ if there is a primitive interpretation $\mathcal{J}$ such that $\mathcal{I} = \mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$. Given $A, B \in \mathsf{N_C}$ and $p \in [0,1]$, we say that A is* gfp-subsumed *by B to degree p w.r.t. $\mathcal{T}$, written $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p\rangle$, if for every gfp-model $\mathcal{I}$ of $\mathcal{T}$ and every $x \in \Delta^{\mathcal{I}}$ we have $A^{\mathcal{I}}(x) \Rightarrow B^{\mathcal{I}}(x) \geq p$.*

Let now $\mathcal{T}$ be a TBox and $\mathcal{T}'$ the result of transforming $\mathcal{T}$ into normal form as described before. It is easy to verify that the operators $T_{\mathcal{J}}^{\mathcal{T}}$ and $T_{\mathcal{J}}^{\mathcal{T}'}$ coincide, and therefore the gfp-models of $\mathcal{T}$ are the same as those of $\mathcal{T}'$. To solve the problem of deciding gfp-subsumptions, it thus suffices to consider TBoxes in normal form.

4 Characterizing Subsumption using Finite Automata

To decide gfp-subsumption between concept names, we employ an automata-based approach following the ideas from [1]. In contrast to that paper, however, we use a *weighted* automata model.

Definition 11 (WWA). *A* weighted automaton with word transitions (WWA) *is a tuple* $\mathcal{A} = (\Sigma, Q, q_0, \mathsf{wt}, q_f)$*, where* Σ *is a finite alphabet of* input symbols*,* Q *is a finite set of* states*,* $q_0 \in Q$ *is the* initial state*,* $\mathsf{wt}\colon Q \times \Sigma^* \times Q \to [0,1]$ *is the* transition weight function *with the property that its* support

$$\mathsf{supp}(\mathsf{wt}) := \{(q,w,q') \in Q \times \Sigma^* \times Q \mid \mathsf{wt}(q,w,q') > 0\}$$

is finite, and $q_f \in Q$ *is the* final state*.*

A finite path *in* $\mathcal{A}$ *is a sequence* $\pi = q_0 w_1 q_1 w_2 \dots w_n q_n$*, where* $q_i \in Q$ *and* $w_i \in \Sigma^*$ *for all* $i \in \{1,\dots,n\}$*, and* $q_n = q_f$*. Its* label *is the finite word* $\ell(\pi) := w_1 w_2 \dots w_n$*. The* weight *of* π *is defined as* $\mathsf{wt}(\pi) := \bigwedge_{i=1}^{n} \mathsf{wt}(q_{i-1}, w_i, q_i)$*. The set of all finite paths with label* w *in* $\mathcal{A}$ *is denoted by* $\mathsf{paths}(\mathcal{A}, w)$*. The* behavior $\|\mathcal{A}\|\colon \Sigma^* \to [0,1]$ *of* $\mathcal{A}$ *is defined as follows for every word* $w \in \Sigma^*$*:* $\|\mathcal{A}\|(w) := \bigvee_{\pi \in \mathsf{paths}(\mathcal{A},w)} \mathsf{wt}(\pi)$*.*

If the image of the transition weight function is included in $\{0,1\}$, then we have a classical finite automaton with word transitions (WA). In this case, wt is usually described as a subset of $Q \times \Sigma^* \times Q$ and the behavior is characterized by the set $L(\mathcal{A})$, called the *language* of $\mathcal{A}$, of all words for which the behavior is 1. The *inclusion problem* for WA is to decide, given two such automata $\mathcal{A}$ and $\mathcal{A}'$, whether $L(\mathcal{A}) \subseteq L(\mathcal{A}')$. This problem is known to be PSPACE-complete [10].

Our goal is to describe the restrictions imposed by a $\mathsf{G}\text{-}\mathcal{FL}_0$ TBox $\mathcal{T}$ using a WWA. For the rest of this paper, we assume w.l.o.g. that $\mathcal{T}$ is in normal form.

Definition 12 (automata $\mathcal{A}^{\mathcal{T}}_{A,B}$). *For concept names* $A, B \in \mathsf{N}_\mathsf{C}^{\mathcal{T}}$*, the WWA* $\mathcal{A}^{\mathcal{T}}_{A,B} = (\mathsf{N}_\mathsf{R}, \mathsf{N}_\mathsf{C}^{\mathcal{T}}, A, \mathsf{wt}_{\mathcal{T}}, B)$ *is defined by the transition weight function*

$$\mathsf{wt}_{\mathcal{T}}(A', w, B') := \begin{cases} p & \text{if } \langle A' \sqsubseteq \forall w.B' \geq p\rangle \in \mathcal{T}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that for a given TBox $\mathcal{T}$ and concept names $A, A', B, B' \in \mathsf{N}_\mathsf{C}^{\mathcal{T}}$, the automata $\mathcal{A}^{\mathcal{T}}_{A,B}$ and $\mathcal{A}^{\mathcal{T}}_{A',B'}$ differ only on the initial and final states they define; their sets of states and transition weight function are identical. Since $\mathcal{T}$ is in normal form, for any two concept names $A', B' \in \mathsf{N}_\mathsf{C}^{\mathcal{T}}$ and $w \in \mathsf{N}_\mathsf{R}^*$, there is at most one axiom $\langle A' \sqsubseteq \forall w.B' \geq p\rangle$ in $\mathcal{T}$, and hence the transition weight function is well-defined. This function has finite support since $\mathcal{T}$ is finite.

We now characterize the gfp-models of $\mathcal{T}$ by properties of the automata $\mathcal{A}^{\mathcal{T}}_{A,B}$.

Lemma 13. *For every gfp-model* $\mathcal{I} = (\Delta, \cdot^{\mathcal{I}})$ *of* $\mathcal{T}$*,* $x \in \Delta$*, and* $A \in \mathsf{N}_\mathsf{C}^{\mathcal{T}}$*,*

$$A^{\mathcal{I}}(x) = \bigwedge_{B \in \mathsf{N}_\mathsf{P}^{\mathcal{T}}} \bigwedge_{w \in \mathsf{N}_\mathsf{R}^*} \left(\|\mathcal{A}^{\mathcal{T}}_{A,B}\|(w) \Rightarrow (\forall w.B)^{\mathcal{I}}(x) \right).$$

Proof. If A is primitive, then the empty path $\pi = A \in \mathsf{paths}(\mathcal{A}^{\mathcal{T}}_{A,A}, \varepsilon)$ has weight $\mathsf{wt}_{\mathcal{T}}(\pi) = 1$, and hence $\|\mathcal{A}^{\mathcal{T}}_{A,A}\|(\varepsilon) = 1$. We also have $(\forall\varepsilon.A)^{\mathcal{I}}(x) = A^{\mathcal{I}}(x)$; thus, $A^{\mathcal{I}}(x) = (1 \Rightarrow A^{\mathcal{I}}(x)) \geq \bigwedge_{B\in\mathsf{N}^{\mathcal{T}}_{\mathsf{P}}} \bigwedge_{w\in\mathsf{N}^*_{\mathsf{R}}} \big(\|\mathcal{A}^{\mathcal{T}}_{A,B}\|(w) \Rightarrow (\forall w.B)^{\mathcal{I}}(x)\big)$. Let now $B \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$ and $w \in \mathsf{N}^*_{\mathsf{R}}$ such that $A \neq B$ or $w \neq \varepsilon$. Since A is primitive, by Definition 12 any finite path π in $\mathcal{A}^{\mathcal{T}}_{A,B}$ with $\ell(\pi) = w$ must have weight 0; i.e. $\|\mathcal{A}^{\mathcal{T}}_{A,B}\|(w) = 0$, and thus $0 \Rightarrow (\forall w.B)^{\mathcal{I}}(x) = 1 \geq A^{\mathcal{I}}(x)$. This shows that the whole infimum is equal to $A^{\mathcal{I}}(x)$.

Consider now the case that $A \in \mathsf{N}^{\mathcal{T}}_{\mathsf{D}}$. Since $\mathcal{I}$ is a gfp-model of $\mathcal{T}$, there is a primitive interpretation $\mathcal{J}$ such that $\mathcal{I} = \mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$; let $f := \mathsf{gfp}(T^{\mathcal{T}}_{\mathcal{J}})$. Thus, we have $A^{\mathcal{I}} = f(A) = T^{\mathcal{T}}_{\mathcal{J}}(f)(A) = \bigwedge_{i\geq 0}(T^{\mathcal{T}}_{\mathcal{J}})^i(\mathbf{1})(A)$ for all $A \in \mathsf{N}^{\mathcal{T}}_{\mathsf{D}}$.

[$\leq$] For the $\leq$-direction, by Proposition 4 it suffices to show that for all $x \in \Delta$, $A \in \mathsf{N}^{\mathcal{T}}_{\mathsf{D}}$, $B \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$, and all finite non-empty paths π in $\mathcal{A}^{\mathcal{T}}_{A,B}$ it holds that

$$A^{\mathcal{I}}(x) \leq \mathsf{wt}_{\mathcal{T}}(\pi) \Rightarrow (\forall w.B)^{\mathcal{I}}(x), \tag{1}$$

where $w := \ell(\pi)$. This obviously holds for $\mathsf{wt}_{\mathcal{T}}(\pi) = 0$, and thus it remains to show this for paths with positive weight. Let $\pi = Aw_1A_1w_2\dots w_nA_n$, where $A_i \in \mathsf{N}^{\mathcal{T}}_{\mathsf{C}}$ and $w_i \in \mathsf{N}^*_{\mathsf{R}}$ for all $i \in \{1,\dots,n\}$ and $A_n = B$ is the only primitive concept name in this path. We prove (1) by induction on n. For $n = 1$, we have $\pi = Aw_1B$ and $\mathsf{wt}_{\mathcal{T}}(A, w_1, B) = \mathsf{wt}_{\mathcal{T}}(\pi) > 0$, and thus $\mathcal{T}$ contains the definition $\langle A \sqsubseteq \forall w_1.B \geq p\rangle$, with $p := \mathsf{wt}_{\mathcal{T}}(A, w_1, B)$. By the definition of $T^{\mathcal{T}}_{\mathcal{J}}$, we obtain

$$A^{\mathcal{I}}(x) = T^{\mathcal{T}}_{\mathcal{J}}(f)(A)(x) \leq p \Rightarrow (\forall w_1.B)^{\mathcal{I}}(x) = \mathsf{wt}_{\mathcal{T}}(\pi) \Rightarrow (\forall w.B)^{\mathcal{I}}(x).$$

For $n > 1$, consider the subpath $\pi' = A_1w_2\dots w_nB$ in $\mathcal{A}^{\mathcal{T}}_{A_1,B}$ with the label $\ell(\pi') = w' := w_2\dots w_n$. For all $y \in \Delta$, the induction hypothesis yields that $A_1^{\mathcal{I}}(y) \leq \mathsf{wt}_{\mathcal{T}}(\pi') \Rightarrow (\forall w'.B)^{\mathcal{I}}(y)$. Again, $p := \mathsf{wt}_{\mathcal{T}}(A, w_1, A_1) \geq \mathsf{wt}_{\mathcal{T}}(\pi) > 0$, and thus $\mathcal{T}$ contains the definition $\langle A \sqsubseteq \forall w_1.A_1 \geq p\rangle$. By the definitions of $T^{\mathcal{T}}_{\mathcal{J}}$, $\mathsf{wt}_{\mathcal{T}}(\pi)$, $w^{\mathcal{I}}$, and Propositions 4 and 5, we have

$$\begin{aligned}
A^{\mathcal{I}}(x) &= T^{\mathcal{T}}_{\mathcal{J}}(f)(A)(x)\\
&\leq p \Rightarrow (\forall w_1.A_1)^{\mathcal{I}}(x)\\
&= \bigwedge_{y\in\Delta} \big(p \Rightarrow (w_1^{\mathcal{I}}(x,y) \Rightarrow A_1^{\mathcal{I}}(y))\big)\\
&\leq \bigwedge_{y\in\Delta} \Big(p \Rightarrow \Big(w_1^{\mathcal{I}}(x,y) \Rightarrow \big(\mathsf{wt}_{\mathcal{T}}(\pi') \Rightarrow (\forall w'.B)^{\mathcal{I}}(y)\big)\Big)\Big)\\
&= \big(p \wedge \mathsf{wt}_{\mathcal{T}}(\pi')\big) \Rightarrow \Big(\bigwedge_{y\in\Delta} \big(w_1^{\mathcal{I}}(x,y) \Rightarrow (\forall w'.B)^{\mathcal{I}}(y)\big)\Big)\\
&= \mathsf{wt}_{\mathcal{T}}(\pi) \Rightarrow (\forall w.B)^{\mathcal{I}}(x)
\end{aligned}$$

[$\geq$] For the $\geq$-direction, we show by induction on i that for all $x \in \Delta$, $A \in \mathsf{N}^{\mathcal{T}}_{\mathsf{D}}$, and $i \geq 0$, it holds that

$$(T^{\mathcal{T}}_{\mathcal{J}})^i(\mathbf{1})(A)(x) \geq \bigwedge_{B\in\mathsf{N}^{\mathcal{T}}_{\mathsf{P}}} \bigwedge_{w\in\mathsf{N}^*_{\mathsf{R}}} \big(\|\mathcal{A}^{\mathcal{T}}_{A,B}\|(w) \Rightarrow (\forall w.B)^{\mathcal{I}}(x)\big). \tag{2}$$

For $i = 0$, we have $(T^{\mathcal{T}}_{\mathcal{J}})^0(\mathbf{1})(A)(x) = \mathbf{1}(A)(x) = 1$, which obviously satisfies (2). For $i > 0$, by Proposition 4 we obtain

$$\begin{aligned}(T^{\mathcal{T}}_{\mathcal{J}})^i(\mathbf{1})(A)(x) &= T^{\mathcal{T}}_{\mathcal{J}}((T^{\mathcal{T}}_{\mathcal{J}})^{i-1}(\mathbf{1}))(A)(x) \\ &= \bigwedge_{\langle A \sqsubseteq \forall w'.A' \geq p\rangle \in \mathcal{T}} (p \Rightarrow (\forall w'.A')^{\mathcal{I}_{i-1}}(x)),\end{aligned} \tag{3}$$

where $\mathcal{I}_{i-1} := \mathcal{I}_{\mathcal{J},(T^{\mathcal{T}}_{\mathcal{J}})^{i-1}(\mathbf{1})}$. Consider now any definition $\langle A \sqsubseteq \forall w'.A' \geq p\rangle \in \mathcal{T}$. Then $\pi' = Aw'A'$ is a finite path in $\mathcal{A}^{\mathcal{T}}_{A,A'}$ with label w' and weight p.

If A' is a primitive concept name, then we have

$$p \Rightarrow (\forall w'.A')^{\mathcal{I}_{i-1}}(x) = \mathsf{wt}_{\mathcal{T}}(\pi') \Rightarrow (\forall w'.A')^{\mathcal{I}}(x) \geq \|\mathcal{A}^{\mathcal{T}}_{A,A'}\|(w') \Rightarrow (\forall w'.A')^{\mathcal{I}}(x)$$

by the definition of $\|\mathcal{A}^{\mathcal{T}}_{A,A'}\|(w')$ and the fact that the interpretation of $\forall w'.A'$ under $\mathcal{I}_{i-1}$ and $\mathcal{I}$ only depends on $\mathcal{J}$. If A' is defined, then we similarly get

$$\begin{aligned}&p \Rightarrow (\forall w'.A')^{\mathcal{I}_{i-1}}(x) \\ &= \bigwedge_{y\in\Delta} \Big(p \Rightarrow \big(w'^{\mathcal{J}}(x,y) \Rightarrow A'^{\mathcal{I}_{i-1}}(y)\big)\Big) \\ &\geq \bigwedge_{y\in\Delta} \bigwedge_{B\in\mathsf{N}^{\mathcal{T}}_{\mathsf{P}}} \bigwedge_{w\in\mathsf{N}^*_{\mathsf{R}}} \Big(p \Rightarrow \big(w'^{\mathcal{I}}(x,y) \Rightarrow (\|\mathcal{A}^{\mathcal{T}}_{A',B}\|(w) \Rightarrow (\forall w.B)^{\mathcal{I}}(y))\big)\Big) \\ &= \bigwedge_{B\in\mathsf{N}^{\mathcal{T}}_{\mathsf{P}}} \bigwedge_{w\in\mathsf{N}^*_{\mathsf{R}}} \Big(\big(p \wedge \|\mathcal{A}^{\mathcal{T}}_{A',B}\|(w)\big) \Rightarrow \Big(\bigwedge_{y\in\Delta} \big(w'^{\mathcal{I}}(x,y) \Rightarrow (\forall w.B)^{\mathcal{I}}(y)\big)\Big)\Big) \\ &= \bigwedge_{B\in\mathsf{N}^{\mathcal{T}}_{\mathsf{P}}} \bigwedge_{w\in\mathsf{N}^*_{\mathsf{R}}} \Big(\Big(\bigvee_{\pi\in\mathsf{paths}(\mathcal{A}^{\mathcal{T}}_{A',B},w)} (\mathsf{wt}_{\mathcal{T}}(\pi') \wedge \mathsf{wt}_{\mathcal{T}}(\pi))\Big) \Rightarrow (\forall w'w.B)^{\mathcal{I}}(x)\Big) \\ &\geq \bigwedge_{B\in\mathsf{N}^{\mathcal{T}}_{\mathsf{P}}} \bigwedge_{w\in\mathsf{N}^*_{\mathsf{R}}} \big(\|\mathcal{A}^{\mathcal{T}}_{A,B}\|(w'w) \Rightarrow (\forall w'w.B)^{\mathcal{I}}(x)\big)\end{aligned}$$

by the induction hypothesis, Propositions 4 and 5, and the definition of $\|\mathcal{A}^{\mathcal{T}}_{A,B}\|$.

In both cases, $p \Rightarrow (\forall w'.A')^{\mathcal{I}_{i-1}}(x)$ is an upper bound for the infimum on the right-hand side of (2), and thus by (3) the same is true for $(T^{\mathcal{T}}_{\mathcal{J}})^i(\mathbf{1})(A)(x)$. □

This allows us to prove gfp-subsumptions by comparing the behavior of WWA.

Lemma 14. *Let $A, B \in \mathsf{N}^{\mathcal{T}}_{\mathsf{C}}$ and $p \in [0,1]$. Then $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p\rangle$ iff for all $C \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$ and $w \in \mathsf{N}^*_{\mathsf{R}}$ it holds that $p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) \leq \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$.*

Proof. Assume that there exist $C \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$ and $w = r_1 \dots r_n \in \mathsf{N}^*_{\mathsf{R}}$ such that $p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) > \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$. We define the primitive interpretation $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$ where $\Delta := \{x_0, \dots, x_n\}$, and for all $D \in \mathsf{N}^{\mathcal{T}}_{\mathsf{P}}$ and $r \in \mathsf{N}_{\mathsf{R}}$, the interpretation function is given by

$$D^{\mathcal{J}}(x) := \begin{cases} \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w) & \text{if } D = C \text{ and } x = x_n, \\ 1 & \text{otherwise; and} \end{cases}$$

$$r^{\mathcal{J}}(x,y) := \begin{cases} 1 & \text{if } x = x_{i-1},\ y = x_i, \text{ and } r = r_i \text{ for some } i \in \{1,\dots,n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Consider now the gfp-model $\mathcal{I} := \mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$ of $\mathcal{T}$. By construction, for all pairs $(w', D) \in \mathsf{N}_\mathsf{R}^* \times \mathsf{N}_\mathsf{P}^{\mathcal{T}} \setminus \{(w,C)\}$ we have $(\forall w'.D)^{\mathcal{I}}(x_0) = 1$. Moreover, we know that $(\forall w.C)^{\mathcal{I}}(x_0)$ is equal to $\|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$, and thus strictly smaller than p and $\|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w)$. By Lemma 13, all this implies that

$$A^{\mathcal{I}}(x_0) = \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w) \Rightarrow (\forall w.C)^{\mathcal{I}}(x_0) = 1 \text{ and}$$
$$B^{\mathcal{I}}(x_0) = \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) \Rightarrow (\forall w.C)^{\mathcal{I}}(x_0) = (\forall w.C)^{\mathcal{I}}(x_0).$$

Thus $A^{\mathcal{I}}(x_0) \Rightarrow B^{\mathcal{I}}(x_0) = (\forall w.C)^{\mathcal{I}}(x_0) < p$, and $\mathcal{T} \not\models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p\rangle$.

Conversely, assume that there are a primitive interpretation $\mathcal{J} = (\Delta, \cdot^{\mathcal{J}})$ and an element $x \in \Delta$ such that $A^{\mathcal{I}}(x) \Rightarrow B^{\mathcal{I}}(x) < p$, where $\mathcal{I} := \mathsf{gfp}_{\mathcal{T}}(\mathcal{J})$. Thus, we have $p \wedge A^{\mathcal{I}}(x) > B^{\mathcal{I}}(x)$, which implies by Lemma 13 the existence of a $C \in \mathsf{N}_\mathsf{P}^{\mathcal{T}}$ and a $w \in \mathsf{N}_\mathsf{R}^*$ with $p \wedge A^{\mathcal{I}}(x) > \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) \Rightarrow (\forall w.C)^{\mathcal{I}}(x)$. Again by Lemma 13, this implies that

$$\begin{aligned} p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) &> A^{\mathcal{I}}(x) \Rightarrow (\forall w.C)^{\mathcal{I}}(x) \\ &\geq \big(\|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w) \Rightarrow (\forall w.C)^{\mathcal{I}}(x)\big) \Rightarrow (\forall w.C)^{\mathcal{I}}(x). \end{aligned}$$

In particular, the latter value cannot be 1, and thus it is equal to $(\forall w.C)^{\mathcal{I}}(x)$. But this can only be the case if $\|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w) \leq (\forall w.C)^{\mathcal{I}}(x)$. To summarize, we obtain $p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) > (\forall w.C)^{\mathcal{I}}(x) \geq \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$, as desired. □

Denote by $\mathcal{V}_{\mathcal{T}} := \{0,1\} \cup \{p \in [0,1] \mid \langle A \sqsubseteq \forall w.B \geq p\rangle \in \mathcal{T}\}$ the set of all values appearing in $\mathcal{T}$, together with 0 and 1. Since $\mathsf{wt}_{\mathcal{T}}$ has finite support and takes only values from $\mathcal{V}_{\mathcal{T}}$, $p \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) > \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$ holds iff $p' \wedge \|\mathcal{A}^{\mathcal{T}}_{B,C}\|(w) > \|\mathcal{A}^{\mathcal{T}}_{A,C}\|(w)$, where p' is the smallest element of $\mathcal{V}_{\mathcal{T}}$ such that $p' \geq p$. This shows that it suffices to be able to check gfp-subsumptions for the values in $\mathcal{V}_{\mathcal{T}}$. We now show how to do this by simulating $\mathcal{A}^{\mathcal{T}}_{B,C}$ and $\mathcal{A}^{\mathcal{T}}_{A,C}$ by polynomially many *unweighted* automata.

Definition 15 (automata $\mathcal{A}_{\geq p}$). *Given a WWA $\mathcal{A} = (\Sigma, Q, q_0, \mathsf{wt}, q_f)$ and a value $p \in [0,1]$, the WA $\mathcal{A}_{\geq p} = (\Sigma, Q, q_0, \mathsf{wt}_{\geq p}, q_f)$ is given by the transition relation $\mathsf{wt}_{\geq p} := \{(q,w,q') \in Q \times \Sigma^* \times Q \mid \mathsf{wt}(q,w,q') \geq p\}$.*

The language of this automaton has an obvious relation to the behavior of the original WWA.

Lemma 16. *Let $\mathcal{A}$ be a WWA over the alphabet Σ and $p \in [0,1]$. Then we have $L(\mathcal{A}_{\geq p}) = \{w \in \Sigma^* \mid \|\mathcal{A}\|(w) \geq p\}$.*

Proof. We have $w \in L(\mathcal{A}_{\geq p})$ iff there is a finite path $\pi = q_0 w_1 q_1 \dots w_n q_n$ in $\mathcal{A}$ with label w such that $\mathsf{wt}(q_{i-1}, w_i, q_i) \geq p$ holds for all $i \in \{1,\dots,n\}$. The latter condition is equivalent to the fact that $\mathsf{wt}(\pi) \geq p$. Thus, $w \in L(\mathcal{A}_{\geq p})$ implies that $\|\mathcal{A}\|(w) \geq p$. Conversely, since wt has finite support, there are only finitely many possible weights for any finite path in $\mathcal{A}$, and thus $\|\mathcal{A}\|(w) \geq p$ also implies that there exists a $\pi \in \mathsf{paths}(\mathcal{A}, w)$ with $\mathsf{wt}(\pi) \geq p$, and thus $w \in L(\mathcal{A}_{\geq p})$. □

We thus obtain the following characterization of gfp-subsumption.

Lemma 17. *Let $A, B \in \mathsf{N}_\mathsf{C}^\mathcal{T}$ and $p \in \mathcal{V}_\mathcal{T}$. Then $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p \rangle$ iff for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $p' \in \mathcal{V}_\mathcal{T}$ with $p' \leq p$ it holds that $L((\mathcal{A}_{B,C}^\mathcal{T})_{\geq p'}) \subseteq L((\mathcal{A}_{A,C}^\mathcal{T})_{\geq p'})$.*

Proof. Assume that we have $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p \rangle$ and consider any $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$, $w \in \mathsf{N}_\mathsf{R}^*$, and $p' \in \mathcal{V}_\mathcal{T} \cap [0, p]$ with $w \in L((\mathcal{A}_{B,C}^\mathcal{T})_{\geq p'})$. By Lemma 16, we obtain $\|\mathcal{A}_{B,C}^\mathcal{T}\|(w) \geq p'$, and by Lemma 14 we have $\|\mathcal{A}_{A,C}^\mathcal{T}\|(w) \geq p \wedge \|\mathcal{A}_{B,C}^\mathcal{T}\|(w) \geq p'$. Thus, $w \in L((\mathcal{A}_{A,C}^\mathcal{T})_{\geq p'})$.

Conversely, assume that $\mathcal{T} \models_{\mathsf{gfp}} \langle A \sqsubseteq B \geq p \rangle$ does not hold. Then by Lemma 14 there are $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $w \in \mathsf{N}_\mathsf{R}^*$ such that $p \wedge \|\mathcal{A}_{B,C}^\mathcal{T}\|(w) > \|\mathcal{A}_{A,C}^\mathcal{T}\|(w)$. For the value $p' := p \wedge \|\mathcal{A}_{B,C}^\mathcal{T}\|(w) \in \mathcal{V}_\mathcal{T} \cap [0, p]$, we have $\|\mathcal{A}_{B,C}^\mathcal{T}\|(w) \geq p'$, but $\|\mathcal{A}_{A,C}^\mathcal{T}\|(w) < p'$, and thus $L((\mathcal{A}_{B,C}^\mathcal{T})_{\geq p'}) \nsubseteq L((\mathcal{A}_{A,C}^\mathcal{T})_{\geq p'})$ by Lemma 16. □

A direct consequence of this lemma is that gfp-subsumption between concept names in G-$\mathcal{FL}_0$ remains in the same complexity class as for classical $\mathcal{FL}_0$.

Theorem 18. *Deciding gfp-subsumption between concept names in G-$\mathcal{FL}_0$ is* PSpace*-complete.*

Proof. By the reductions above, it suffices to decide the language inclusions $L((\mathcal{A}_{B,C}^\mathcal{T})_{\geq p}) \subseteq L((\mathcal{A}_{A,C}^\mathcal{T})_{\geq p})$ for all $C \in \mathsf{N}_\mathsf{P}^\mathcal{T}$ and $p \in \mathcal{V}_\mathcal{T}$. These polynomially many inclusion tests for WA can be done in polynomial space [10]. The problem is PSpace-hard since gfp-subsumption in classical $\mathcal{FL}_0$ is already PSpace-hard [1]. This is a special case of our problem where the input TBox is restricted to the values **0** and **1**, and therefore all relevant WWA are already WA. □

5 Conclusions

We have studied the complexity of reasoning in G-$\mathcal{FL}_0$ w.r.t. primitive concept definitions under greatest fixed-point semantics. More precisely, we have shown that gfp-subsumption between concept names can be reduced to a comparison of the behavior of weighted automata with word transitions. Moreover, the latter can be solved by a polynomial number of inclusion tests on *unweighted* automata. Overall, this shows that gfp-subsumption is PSpace-complete for this logic, just as in the classical case.

This complexity result is consistent with previous work on extensions of description logics with Gödel semantics. Indeed, such extensions of $\mathcal{EL}$ [15,16] and $\mathcal{ALC}$ [5] have been shown to preserve the complexity of their classical counterpart. Since reasoning in classical $\mathcal{FL}_0$ and in G-$\mathcal{ALC}$ w.r.t. general TBoxes is in both cases ExpTime-complete, so is deciding subsumption in G-$\mathcal{FL}_0$ w.r.t. general TBoxes.

We expect our results to generalize easily to any other set of truth degrees that form a total order. However, the arguments used in this paper fail for arbitrary lattices, where incomparable truth degrees might exist [7,19]. Studying these two cases in detail is a task for future work. We also plan to consider fuzzy extensions of $\mathcal{FL}_0$ with semantics based on non-idempotent t-norms, such as the Łukasiewicz or product t-norms [12].

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