Herbrand-satisfiability of a Quantified Set-theoretical Fragment[*]

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Abstract. In the last decades, several fragments of set theory have been studied in the context of the so-called *Computable Set Theory*. In general, the semantics of set-theoretical languages differs from the canonical first-order semantics in that the interpretation domain of set-theoretical terms is fixed to a given universe of sets, as for instance the von Neumann standard cumulative hierarchy of sets, i.e., the class consisting of all the pure sets. Because of this, theoretical results and various machinery developed in the context of first-order logic cannot be easily adapted to work in the set-theoretical realm. Recently, quantified fragments of set-theory which allow one to explicitly handle ordered pairs have been studied for decidability purposes, in view of applications in the field of knowledge representation. Among other results, a NexpTime decision procedure for satisfiability of formulae in one of these fragments, $\forall_0^\pi$, has been provided. In this paper we exploit the main features of such a decision procedure to reduce the satisfiability problem for the fragment $\forall_0^\pi$ to the problem of Herbrand satisfiability for a first-order language extending it. In addition, it turns out that such reduction maps formulae of the *Disjunctive Datalog* subset of $\forall_0^\pi$ into Disjunctive Datalog programs.

1 Introduction

The quantified fragment of set-theory $\forall_0^\pi$ (see [3]) allows the explicit manipulation of ordered pairs. It is expressive enough to include a relevant amount of set-theoretic constructs, in particular map-related ones: in fact, it is characterized by the presence of terms of the form $[\cdot, \cdot]$ (ordered pair) and $\bar{\pi}(\cdot)$ (collection of the non-pair members of its argument). This language has applications in the field of *knowledge representation*. In fact, a large amount of *description logic* constructs are expressible in it. In particular, the very expressive description logic $\mathcal{DL}\langle\forall_0^\pi\rangle$ can be expressed in a sublanguage of $\forall_0^\pi$ which has an NP-complete decision problem, in contrast with the description logic $\mathcal{SROIQ}$ (cf. [9]), which underpins the current standard language for Semantic Web OWL2[1] and whose decision problem is N2ExpTime-complete (see [10]). Despite of these desirable

[1] http://www.w3.org/TR/owl2-primer/

properties of the language $\forall_{\mathbf{0}}^{\pi}$, no decision procedure for it has been implemented yet.

In general, the semantics of set-theoretical languages (see [11] for an introduction to set theory, and [5, 12] for an overview on decidable fragments of set-theory) differs from the *canonical* first-order semantics (see [7]) in that the interpretation domain of set-theoretical terms is based on the *von Neumann standard cumulative hierarchy of sets* $\mathcal{V}$. This is the class consisting of all the *pure* sets, and is recursively defined by

$$
\begin{aligned}
\mathcal{V}_0 &= \emptyset \\
\mathcal{V}_{\gamma+1} &= \mathcal{P}(\mathcal{V}_\gamma), && \text{for each ordinal } \gamma \\
\mathcal{V}_\lambda &= \textstyle\bigcup_{\mu<\lambda} \mathcal{V}_\mu, && \text{for each limit ordinal } \lambda \\
\mathcal{V} &= \textstyle\bigcup_{\gamma\in On} \mathcal{V}_\gamma,
\end{aligned}
$$

where $\mathcal{P}(\cdot)$ is the powerset operator and On denotes the class of all ordinals. Because of such peculiarity of set-theoretical languages, reusing of theoretical results and machinery developed in the context of first-order logic for set-theoretical matters is not straightforward.

In this paper we show that these difficulties can be circumvented for the fragment $\forall_{\mathbf{0}}^{\pi}$ by *encoding* some axioms of set theory (namely regularity and a weak form of extensionality) as $\forall_{\mathbf{0}}^{\pi}$-formulae. More specifically, we will prove that, for every $\forall_{\mathbf{0}}^{\pi}$-formula φ, one can construct in polynomial time a corresponding $\forall_{\mathbf{0}}^{\pi}$-formula χ_φ such that φ admits a set-theoretical model if and only if $\chi_\varphi \wedge \varphi$ admits a Herbrand model (cf. [8]), when set-theoretic predicates in $\chi_\varphi \wedge \varphi$ are regarded as uninterpreted predicates.

It turns out that these formulae χ_φ can be considered as $\mathsf{DATALOG}^{\vee,\neg}$ programs. As a consequence, the reduction we are going to discuss can be seen as a reduction from $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$ to $\mathsf{DATALOG}^{\vee,\neg}$, where $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$ consists of the formulae in $\forall_{\mathbf{0}}^{\pi}$ which satisfy the syntactic constraints required to be regarded also as $\mathsf{DATALOG}^{\vee,\neg}$ programs.

We recall that $\mathsf{DATALOG}^{\vee,\neg}$ (read *Disjunctive Datalog*, see [6]) extends Datalog by allowing disjunctions in the head of program rules. Decidability and complexity of Datalog extensions with various combinations of disjunction and negation has been studied. Also, optimization strategies have been provided for algorithms devised in this context, and a considerable amount of academic and commercial software implementing these algorithms is available (see for example [1]). Thus, the reduction we present in this paper allows one to reuse the machinery available for $\mathsf{DATALOG}^{\vee,\neg}$ in the implementation of an optimized reasoning engine for the language $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$ and, consequently, for the description logic $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ mentioned before, as all the constructs in $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$ are expressible in $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$.

The link between set-theoretical languages and logic programming we start to investigate here is of a certain interest to projects related with the production and usage of *open data* such as $PRISMA^*$: on one hand, it provides a way to implement with small efforts a solid and efficient reasoning and query engine for the very expressive representation language $\mathcal{DL}\langle\forall_{\mathbf{0}}^{\pi}\rangle$; on the other hand, it allows

one to implement typical logic-programming tasks for $\mathcal{DL}\langle\forall_0^\pi\rangle$ knowledge bases such as, for example, answer-set programming and non-monotonic reasoning, which may be of some interest for public utility applications.

The rest of the paper is organized as follows. In Section 2 we review some notions and definitions from first-order logic, including the definition of the Disjunctive Datalog fragment of first-order logic. Then, in Section 3, after recalling the syntax and semantics of the fragment of set-theory $\forall_0^\pi$, we briefly review a decision procedure for $\forall_0^\pi$-formulae, together with some lemmas useful to our needs. A polynomial-time reduction of the satisfiability problem for $\forall_0^\pi$-formulae to the Herbrand satisfiability problem for first-order formulae is described in Section 4. Finally, we draw our conclusions and provide some hints for future research in Section 5.

2 Preliminaries

We briefly review some notations and definitions from first-order logic which will be used throughout the paper.

We shall denote with $Vars = \{x, y, z, \ldots\}$ and $Consts = \{a, b, c \ldots\}$ two denumerably infinite collections of *variables* and *constants*, respectively.[2] In addition, we denote with $Preds = \{P, Q, R, \ldots\}$ a denumerably infinite set of *predicate* symbols, and with $ar : Preds \longrightarrow \mathbb{N}$ the related *arity function*.

Atomic formulae are expressions of the following form

$$P(\nu_1, \ldots, \nu_n),$$

where $P \in Preds$, $ar(P) = n$, and $\{\nu_1, \ldots \nu_n\} \in Vars \cup Consts$. *Literals* are atomic formulae or their negations. *Quantifier-free formulae* are propositional combinations of atomic formulae; thus, in particular, literals are quantifier-free formulae. *Universally quantified prenex-formulae* are expressions of the following form

$$(\forall x_1) \ldots (\forall x_n)\psi,$$

where $n \in \mathbb{N}$, $\{x_1, \ldots x_n\} \subseteq Vars$ and ψ is a quantifier-free formula. In the rest of the paper we will sometime abbreviate quantifier prefixes as $(\forall x_1) \ldots (\forall x_n)$ by $(\forall x_1, \ldots, x_n)$. Here we do not mention existential quantifiers because they do not serve our purposes. However, notice that constants and free variables[3] may be regarded as existentially quantified variables when dealing with satisfiability. In addition, quantifier-free formulae can be considered as universally quantified prenex formulae with an empty quantifier prefix.

We consider also *restricted universal quantifiers*, i.e., quantifiers of the form

$$(\forall x_1, \ldots, x_n | P(x_1, \ldots, x_m)),$$

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[2] We do not mention function symbols here as we are going to consider function-free languages only.

[3] An occurrence of a variable x is free in a formula if it does not fall within the scope of the quantifier $(\forall x)$.

where $1 \leq m \leq n$, $x_1, \ldots, x_n$ are variables, and P is a predicate of arity m, whose intended meaning is that (universal) quantification is restricted to all $x_1, \ldots, x_n$ such that $P(x_1, \ldots, x_m)$ holds. Notice, however, that restricted universal quantifiers can easily be expressed by universally quantified prenex formulae, by way of the following equivalence:

$$(\forall x_1, \ldots, x_n | P(x_1, \ldots, x_m))\psi \; \equiv \; (\forall x_1, \ldots, x_n)(P(x_1, \ldots, x_m) \to \psi).$$

Given any formula φ, we denote with $Preds(\varphi)$, $Consts(\varphi)$, and $Vars(\varphi)$ the sets of the predicate symbols, constant symbols, and variables occurring in φ, respectively. Analogously, for a set Σ of formulae, we denote with $Preds(\Sigma)$, $Consts(\Sigma)$, and $Vars(\Sigma)$ the sets of all the predicate symbols, constant symbols, and variables occurring in any formula $\varphi \in \Sigma$, respectively.

We denote with $\varphi[x \to y]$ the formula obtained from φ by replacing every free occurrence of x with y.

A formula is said to be *ground* if no variable occurs in it. It is said to be *closed* (or to be a *sentence*) if no *free* variable occurs in it, i.e., if every variable x in it occurs ony within the scope of the quantifier $(\forall x)$. Plainly, ground formulae are closed.

Finally, given any set Σ of atomic formulae and any atomic formula γ, with a slight abuse of notation we write $\Sigma \models \gamma$ to express that γ is a member of Σ, i.e., $\gamma \in \Sigma$. Likewise, we write $\Sigma \not\models \gamma$ when $\gamma \notin \Sigma$.

First-order interpretations, whose universe is *Consts* and such that each constant is interpreted by itself,[4] are called *Herbrand interpretations*. Plainly, a Herbrand interpretation is characterized by the set of ground atomic formulae which are evaluated to **true** by it. A Herbrand interpretation $\mathcal{H}$ is said to be a *Herbrand model* for a formula φ if it evaluates φ to **true**, in which case we write $\mathcal{H} \models \varphi$.

A formula is said to be *Herbrand-satisfiable* if and only if it admits a Herbrand model. It is a fundamental result of first-order logic that, for satisfiability purposes, it is not restrictive to limit oneself to Herbrand satisfiability, since a formula is satisfiable if and only if it is Herbrand-satisfiable. Additionally, when considering a (function-free) universally quantified prenex sentence φ, the search space for Herbrand models of φ can be limited to Herbrand interpretations over the constants occurring in it. This result can easily be generalized to finite conjunctions of universally quantified prenex sentences, as stated in the following theorem.

Theorem 1. *Let φ be a finite conjunction of universally quantified prenex sentences. Then φ is satisfiable if and only if it admits a Herbrand model $\mathcal{H}$ whose universe coincides with the set $Consts(\mathcal{H})$, if $Consts(\mathcal{H}) \neq \emptyset$, otherwise is any singleton.* $\square$

We conclude this section by recalling the $\mathsf{DATALOG}^{\vee, \neg}$ first-order fragment. Notice that $\mathsf{DATALOG}^{\vee, \neg}$-formulae are often referred to also as *programs*, so that

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[4] We recall that we are assuming that there are no function symbols.

the expressions "DATALOG$^{\vee,\neg}$-formulae" and "DATALOG$^{\vee,\neg}$-programs" must be regarded as synonyms. A DATALOG$^{\vee,\neg}$ *-formula* is a finite conjunction of *rules*, i.e., closed formulae of the following form

$$(\forall x_1)\ldots(\forall x_n)(\varphi \to \psi)\,,$$

where φ (the rule *body*) is a conjunction of literals, ψ (the rule *head*) is a disjunction of literals, and $Vars(\psi) \subseteq Vars(\varphi)$ (*safety condition*). *Facts* are special ground rules whose body is always true. For this kind of rules, one may omit to indicate the rule body. Thus facts can just be regarded as disjunctions of ground literals. They are used to express facts about real world items, such as for example $childOf(Alice, Bob)$, $isMale(Bob)$, etc.

Finally, we observe that restricted universal quantifiers can easily be embedded in DATALOG$^{\vee,\neg}$ rules and formulae. Indeed, let us consider a closed formula of the form

$$(\forall x_1, \ldots, x_n | P(x_1, \ldots, x_m))(\varphi \to \psi)\,, \tag{1}$$

where φ is a conjunction of literals, ψ is a disjunction of literals, $Vars(\psi) \subseteq Vars(\varphi) \cup \{x_1, \ldots, x_m\}$, and P is a predicate symbol of arity m. Then, as remarked above, (1) is equivalent to the closed formula

$$(\forall x_1, \ldots, x_n)(P(x_1, \ldots, x_m) \to (\varphi \to \psi)),$$

which, in its turn, is equivalent to the closed formula

$$(\forall x_1, \ldots, x_n)((P(x_1, \ldots, x_m) \wedge \varphi) \to \psi). \tag{2}$$

Plainly, (2) is a (standard) DATALOG$^{\vee,\neg}$-rule, with body $(P(x_1, \ldots, x_m) \wedge \varphi)$.

3 The language $\forall_0^\pi$

We recall the syntax and semantics of the set-theoretic language $\forall_0^\pi$, whose decision problem has been studied in [3]. *Atomic $\forall_0^\pi$-formulae* are of the following types

$$\nu \in \bar{\pi}(\mu), \quad [\nu, \nu'] \in \mu, \quad \nu = \mu \tag{3}$$

with $\nu, \nu', \mu \in Vars \cup Consts$. Intuitively. a formula of type $\nu \in \bar{\pi}(\mu)$ expresses that ν is a *non-pair* member of μ, whereas a formula of type $[\nu, \nu'] \in \mu$ expresses that the pair $[\nu, \nu']$ belongs to μ. Atomic $\forall_0^\pi$-formulae and their negations are called $\forall_0^\pi$-*literals*. *Quantifier-free* $\forall_0^\pi$-*formulae* are Boolean combinations of atomic $\forall_0^\pi$-formulae. *Simple-prenex $\forall_0^\pi$-formulae* have the following form:

$$(\forall x_1 \in \bar{\pi}(a_1))\ldots(\forall x_n \in \bar{\pi}(a_n))(\forall [y_1, z_1] \in b_1)\ldots(\forall [y_m, z_m] \in b_m)\psi,$$

where $n, m \geq 0$, $x_i \in Vars$ and $a_i \in Consts$, for $1 \leq i \leq n$, $y_j, z_j \in Vars$ and $b_j \in Consts$, for $1 \leq j \leq m$, and ψ is a quantifier-free $\forall_0^\pi$-formula. The constants $a_1, \ldots, a_n, b_1, \ldots b_m$ are the *domain constants* of the simple-prenex $\forall_0^\pi$-formula under consideration. Finally, $\forall_0^\pi$-*formulae* are finite conjunctions of

closed simple-prenex $\forall_0^\pi$-formulae. The collection of the domain constants of a $\forall_0^\pi$-formula φ, which we denote with $Consts_D(\varphi)$, consists of the domain constants of all of its conjuncts.

Semantics of the language $\forall_0^\pi$ is based on the von Neumann standard cumulative hierarchy of sets, briefly reviewed in Section 1. Other set-theoretic notions particularly useful to our purposes are those of *Cartesian product* and of *ordered pairs*. Let π be a binary operation which associates to each pair of sets $u, v \in \mathcal{V}$ another set in $\mathcal{V}$ (i.e., $\pi(u, v) \in \mathcal{V}$). Then the *Cartesian product* of $u, v \in \mathcal{V}$ with respect to π, denoted by $u \times^{(\pi)} v$, is defined by

$$u \times^{(\pi)} v \ =_{\mathrm{Def}} \{\pi(u', v') \,|\, u' \in u \wedge v' \in v\}.$$

The binary operation π is said to be a *pairing function* if and only if the following two conditions hold for any $u, v, u', v' \in \mathcal{V}$:

1. $\pi(u, v) = \pi(u', v') \iff u = u' \wedge v = v'$,
2. $u \times^{(\pi)} v$ is a set in $\mathcal{V}$.

Let π be a pairing function, and let u be a set in the von Neumann cumulative hierarchy. We denote with $\bar\pi(u)$ the collection of the members of u which are not pairs, with respect to the pairing function π.

The *semantics* for $\forall_0^\pi$-formulae is given in terms of *pair-aware set-theoretical interpretations*. These are first-order interpretations $\mathbf{I}$ whose domain is $\mathcal{V}$ (so that pure sets are assigned to constants and variables) and such that:

- the membership predicate $\in$ is interpreted as the membership relation among sets in $\mathcal{V}$;
- the pairing symbol $[\cdot, \cdot]$ is interpreted by a pairing function, in the sense described above;
- $\mathbf{I}\bar\pi(x) = \{u \in \mathbf{I}x \,|\, (\forall v_1, v_2)(\mathbf{I}[v_1, v_2] \neq u)\}$, for all $x \in Vars \cup Consts$.

Then, a $\forall_0^\pi$-formula φ is said to be *satisfiable* if and only if it admits a pair-aware set-theoretical interpretation which satisfies it (i.e., a pair-aware set-theoretical model). We use the notation $\mathbf{I} \models_s \varphi$ to indicate that $\mathbf{I}$ is a pair-aware set-theoretical model for the $\forall_0^\pi$-formula φ.

The satisfiability test for $\forall_0^\pi$-formulae reported in [3] relies on the existence of finite structures of bounded size, called *skeletal representations*, which witness the existence of certain particular models called *realizations* (see Definitions 1 and 2).

The definition of skeletal representation adopted in this paper differs slightly from the one presented in [3]. In particular, here we extend the notion of skeletal representation so as to also encapsulate the notion of *V-extensionality* and the technical condition *(i)* of Theorem 3 in [3]. It turns out that all the lemmas and theorems in [3] can be adapted to cope with this extended definition of skeletal representation. A reformulation of Theorem 3 in [3], which is central for the satisfiability problem for $\forall_0^\pi$-formulae, is reported in Theorem 2 below.

In the following definition, we shall make use of the *membership closure* relation $\in_\mathcal{S}^+$ of a set $\mathcal{S}$ of atomic $\forall_0^\pi$-formulae. This is the minimal transitive relation on $Consts(\mathcal{S})$ such that, for $a, b, c \in Consts(\mathcal{S})$, we have

$-$ if $\mathcal{S} \models a \in \bar{\pi}(b)$ then $a \in_{\mathcal{S}}^{+} b$, and
$-$ if $\mathcal{S} \models [a,b] \in c$ then $a \in_{\mathcal{S}}^{+} c$ and $b \in_{\mathcal{S}}^{+} c$.

Definition 1 (Skeletal Representation). *Let φ be a $\forall_0^\pi$-formula and let V, T be two disjoint sets of constants. A* skeletal representation $\mathcal{S}$ *relative to* (V,T) *is a finite set of ground atomic $\forall_0^\pi$-formulae such that the following conditions hold:*

(S1) *the membership relation induced by $\mathcal{S}$ is acyclic, i.e., $a \notin_{\mathcal{S}}^{+} a$, for all $a \in V \cup T$;*

(S2) *if $\mathcal{S} \models a = b$, for some a and b, then, for every γ in $\mathcal{S}$, $\gamma[a \to b]$ and $\gamma[b \to a]$ are both in $\mathcal{S}$;*

(S3) *if $\mathcal{S} \not\models a = b$, for some $a, b \in V$, then a and b must be distinguished in $\mathcal{S}$ by some constant c, in the sense that $\mathcal{S} \models c \in \bar{\pi}(a)$ iff $\mathcal{S} \not\models c \in \bar{\pi}(b)$, or by some pair $[c,d]$, in the sense that $\mathcal{S} \models [c,d] \in a$ iff $\mathcal{S} \not\models [c,d] \in b$;*

(S4) *statements in $\mathcal{S}$ of the form $a = b$ can involve only constants occurring in V;*

(S5) *$Consts(\mathcal{S}) \subseteq V \cup T$.* $\qquad\qquad\square$

Condition **(S1)**, which is closely related to the regularity axiom of set theory, guarantees that a skeletal representation $\mathcal{S}$ can be turned into a corresponding pair-aware set-theoretic interpretation (its realization; see below). Conditions **(S2)** and **(S3)** concern the extensionality axiom of set theory, which states that any two sets are equals if and only if they have the same members. Finally, **(S4)** and **(S5)** are technical conditions which, together with **(S1)**–**(S3)** in Definition 1, guarantee some properties of realizations, as reported in Lemma 1 below.

In the following definition of realization of a skeletal representation, taken from from [3] (Definition 1), we shall make use of the family $\{\pi_n\}_{n \in \mathbb{N}}$ of pairing functions, recursively defined by

$$\pi_0(u,v) =_{\mathrm{Def}} \{u, \{u,v\}\}$$
$$\pi_{n+1}(u,v) =_{\mathrm{Def}} \{\pi_n(u,v)\},$$

for $u, v \in \mathcal{V}$.

Definition 2 (Realization [3]). *Let V and $T = \{t_1, t_2, \ldots, t_n\}$, with $n \in \mathbb{N}$, be two finite, nonempty, and disjoint sets of constants, and let $\mathcal{S}$ be a skeletal representation relative to (V,T). Then the* realization *of $\mathcal{S}$ relative to (V,T) is the pair-aware set-theoretical model $\mathcal{R}$ defined as follows:*

$$\mathcal{R}\,[u,v] =_{\mathit{Def}} \pi_h(u,v) \qquad\qquad\qquad \textit{for all } u,v \in \mathcal{V}$$
$$\mathcal{R}\,x =_{\mathit{Def}} \{\mathcal{R}\,y \mid \mathcal{S} \models y \in \bar{\pi}(x)\} \cup \{\mathcal{R}\,[y,z] \mid \mathcal{S} \models [y,z] \in x\} \qquad \textit{for } x \in V$$
$$\mathcal{R}\,t_i =_{\mathit{Def}} \{\mathcal{R}\,y \mid \mathcal{S} \models y \in \bar{\pi}(t_i)\} \cup \{\mathcal{R}\,[y,z] \mid \mathcal{S} \models [y,z] \in t_i\} \cup \qquad \textit{for } t_i \in T$$
$$\{\{k+1, k, i\}\}\,,$$

where $h = |V| + |T|$ and $k = |V| \cdot (|V| + |T| + 3)$.[5] $\qquad\qquad\square$

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[5] We are assuming that integers are represented *à la* von Neumann, namely $0 =_{\mathrm{Def}} \emptyset$ and, recursively, $n + 1 =_{\mathrm{Def}} n \cup \{n\}$.

The following lemma is a direct consequence of Definition 2 and of Lemma 2 in [3].

Lemma 1. *Let V, T be two finite, nonempty, and disjoint sets of constants, $\mathcal{S}$ a skeletal representation relative to (V, T), and $\mathcal{R}$ the realization of $\mathcal{S}$ relative to (V, T). Then*

$$\mathcal{R}\,a \subseteq \{\mathcal{R}\,b \mid \mathcal{S} \models b \in \bar{\pi}(a)\} \cup \{\mathcal{R}\,[b, c] \mid \mathcal{S} \models [b, c] \in a\},$$

for all $a \in V$. In addition, the following conditions hold, for all $a, b, c \in V \cup T$:

- $\mathcal{R}\,a = \mathcal{R}\,b$ *iff* $\mathcal{S} \models a = b$;
- $\mathcal{R}\,a \in \mathcal{R}\,\bar{\pi}(b)$ *iff* $\mathcal{S} \models a \in \bar{\pi}(b)$;
- $\mathcal{R}\,[a, b] \in \mathcal{R}\,c$ *iff* $\mathcal{S} \models [a, b] \in c$. $\qquad\square$

We observe that, in view of Definition 2, the *names* of the variables in the set T do not affect the realization of a skeletal representation relative to (V, T). In other words, if $\mathcal{S}$ is a skeletal representation relative to (V, T), t is a constant in T, and t' is another constant not occurring in $V \cup T$, then the realization of $\mathcal{S}$ relative to (V, T) and the realization of $\mathcal{S}[t \to t']$ (i.e., the set of atomic $\forall_{\mathbf{0}}^{\pi}$-formulae obtained from $\mathcal{S}$ by replacing each occurrence of t by t') relative to $(V, T \setminus \{t\} \cup \{t'\})$ coincide. In addition, if V, T are two disjoint set of constants, and T' is a set of constants disjoint from V and such that $T \subseteq T'$, then

- every skeletal representation $\mathcal{S}$ relative to (V, T) is a skeletal representation relative to (V, T') as well, and
- the realization of $\mathcal{S}$ relative to (V, T) coincides with the realization of $\mathcal{S}$ relative to (V, T').

We conclude this section by restating Theorem 3 of [3], in view of the above observations and of Definition 1.

Theorem 2. *Let φ be a $\forall_{\mathbf{0}}^{\pi}$-formula, let $V = Consts(\varphi)$, and let T be a set of constants disjoint from V and such that $|T| = 2 \cdot |V|$. Then φ is satisfiable if and only if there exists a skeletal representation $\mathcal{S}$ relative to (V, T) such that the realization $\mathcal{R}$ of $\mathcal{S}$ relative to (V, T) is a pair-aware set-theoretical model for φ.* $\qquad\square$

The decidability of the satisfiability problem for $\forall_{\mathbf{0}}^{\pi}$-formulae easily follows from Theorem 2, as the number of possible skeletal representations for any given $\forall_{\mathbf{0}}^{\pi}$-formula is finitely bounded, and it is effectively verifiable whether the realization of a skeletal representation is a pair-aware set-theoretical model for a $\forall_{\mathbf{0}}^{\pi}$-formula.

Next we define the *Disjunctive Datalog subset* $\forall_{\mathbf{0}, \mathbf{D}\vee}^{\pi}$ *of* $\forall_{\mathbf{0}}^{\pi}$ as the collection of the closed $\forall_{\mathbf{0}}^{\pi}$-formulae whose conjuncts are closed simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formulae of the form

$$(\forall x_1 \in \bar{\pi}(a_1)) \ldots (\forall x_n \in \bar{\pi}(a_n))(\forall\,[y_1, z_1] \in b_1) \ldots (\forall\,[y_m, z_m] \in b_m)\psi, \qquad (4)$$

such that ψ has the form

$$(\gamma_1 \wedge \ldots \wedge \gamma_l) \rightarrow (\sigma_1 \vee \ldots \vee \sigma_h),$$

where $l, h \geq 0$ and $\gamma_1, \ldots, \gamma_l, \sigma_1, \ldots \sigma_h$ are $\forall_{\mathbf{0}}^{\pi}$-literals. Since (4) is closed, each variable x which may occur in the head of the rule ψ must be bound, so that x occurs in at least one atom of the rule body, as required for Disjunctive Datalog rules, when restricted quantifiers in (4) are expanded as indicated in (2).

In the next section we present a reduction of the set-theoretic satisfiability for $\forall_{\mathbf{0}}^{\pi}$-formulae to Herbrand satisfiability.

4 Herbrand Satisfiability of $\forall_{\mathbf{0}}^{\pi}$-formulae

In this section we show how to reduce the satisfiability problem for the fragment $\forall_{\mathbf{0}}^{\pi}$ to the problem of first-order Herbrand satisfiability.

For this purpose, we introduce the function-free first-order language $\mathcal{L}_0^{\pi}$ which involves, besides constants, also the following predicate symbols:

binary	ternary	4-ary
$\widehat{\in}, \ P^=, \ P^{\bar{\pi}}, \ dist, \ dist_{\pi}$	$P^{[,]}, \ distBy$	$distBy_{\pi}$

Next, we define a polynomial-time reduction $\varphi \mapsto \psi_{\varphi}$ from $\forall_{\mathbf{0}}^{\pi}$-formulae into $\mathcal{L}_0^{\pi}$-formulae such that any $\forall_{\mathbf{0}}^{\pi}$-formula φ is set-theoretically satisfiable if and only if its image ψ_{φ} in $\mathcal{L}_0^{\pi}$ is Herbrand-satisfiable.

In our reduction, the predicate $\widehat{\in}$ will be used to model the transitive closure of the membership relation among sets, whereas the predicates $dist$, $dist_{\pi}$, $distBy$, and $distBy_{\pi}$ will be used to model the fact that two sets are distinct. In particular, the predicates $dist$ and $distBy$ will take care of the case in which two sets x, y are distinguished by a set z that is not a pair, whereas the predicates $dist_{\pi}$ and $distBy_{\pi}$ will take care of the case in which two sets x, y are distinguished by a $[\nu_1, \nu_2]$. Finally, the predicate $P^=(\nu_1, \nu_2)$ will be used to model equality between ν_1 and ν_2, the predicate $P^{\bar{\pi}}(\nu_1, \nu_2)$ will stand for the set-theoretic formula $\nu_1 \in \bar{\pi}(\nu_2)$, and $P^{[,]}(\nu_1, \nu_2, \nu_3)$ will stand for $[\nu_1, \nu_2] \in \nu_3$, where $\nu_1, \nu_2, \nu_3 \in Vars \cup Consts$.

For the sake of clarity, with a slight abuse of notation, we shall write $\nu_1 = \nu_2$, $\nu_1 \in \bar{\pi}(\nu_2)$, and $[\nu_1, \nu_2] \in \nu_3$ in place of $P^=(\nu_1, \nu_2)$, $P^{\bar{\pi}}(\nu_1, \nu_2)$ and $P^{[,]}(\nu_1, \nu_2, \nu_3)$, respectively, for all $\nu_1, \nu_2, \nu_3 \in Vars \cup Consts$. With such an understanding, $\forall_{\mathbf{0}}^{\pi}$ can be regarded as a sublanguage of $\mathcal{L}_0^{\pi}$, and skeletal representations as Herbrand interpretations (over $\mathcal{L}_0^{\pi}$) subject to the conditions reported in Definition 1.

In the rest of the paper, we shall refer to the subset of a Herbrand interpretation $\mathcal{H}$ consisting of the atomic $\forall_{\mathbf{0}}^{\pi}$-formulae in $\mathcal{H}$ as the $\forall_{\mathbf{0}}^{\pi}$-*subset* of $\mathcal{H}$.

As a first step in our reduction, we provide a polynomial-time procedure to construct an $\mathcal{L}_0^{\pi}$-formula $\chi^{(V,T)}$ from two disjoint sets of constants V and T, which enforces the conditions of Definition 1 on its models, in such a way that

- the $\forall_{\mathbf{0}}^{\pi}$-subset of every Herbrand model $\mathcal{H}$ for $\chi^{(V,T)}$ is a skeletal representation relative to (V, T), and

    – every skeletal representation relative to (V,T) can easily be extended to a Herbrand model for $\chi^{(V,T)}$.

    Then, to complete the reduction of $\forall_{\mathbf{0}}^{\pi}$-satisfiability to Herbrand satisfiability, we shall prove that

$$\mathcal{S} \models \chi^{(V,T)} \wedge \varphi \iff \mathcal{R} \models_s \varphi,$$

where V, T are two disjoint sets of constants, φ is a $\forall_{\mathbf{0}}^{\pi}$-formula such that $Consts(\varphi) = V$, $\mathcal{S}$ is a skeletal representation relative to (V,T), and $\mathcal{R}$ is the realization of $\mathcal{S}$ relative to (V,T).

    Thus, let V, T be two disjoint sets of constants. The formula $\chi^{(V,T)}$ is defined as follows:

$$\chi^{(V,T)} =_{\text{Def}} \chi_1 \wedge \chi_2 \wedge \chi_3^{(V,T)} \wedge \chi_4^{(V,T)},$$

where

$$
\begin{aligned}
\chi_1 =_{\text{Def}} \ & (\forall x, y)(x \in \bar{\pi}(y) \to x\widehat{\in}y) \\
& \wedge (\forall x, y)([x, y] \in z \to x\widehat{\in}z) \\
& \wedge (\forall x, y)([x, y] \in z \to y\widehat{\in}z) \\
& \wedge (\forall x, y, z)(x\widehat{\in}y \wedge y\widehat{\in}z \to x\widehat{\in}z) \\
& \wedge (\forall x)\neg(x\widehat{\in}x)
\end{aligned}
$$

$$
\begin{aligned}
\chi_2 =_{\text{Def}} \ & (\forall x, y)(x = y \to y = x) \\
& \wedge (\forall x, y, z)(x = y \wedge y = z \to x = z) \\
& \wedge (\forall x, y, z)(x \in \bar{\pi}(y) \wedge x = z \to z \in \bar{\pi}(y)) \\
& \wedge (\forall x, y, z)(x \in \bar{\pi}(y) \wedge y = z \to x \in \bar{\pi}(z)) \\
& \wedge (\forall x, y, z, v)([x, y] \in z \wedge x = v \to [v, y] \in z) \\
& \wedge (\forall x, y, z, v)([x, y] \in z \wedge y = v \to [x, v] \in z) \\
& \wedge (\forall x, y, z, v)([x, y] \in z \wedge z = v \to [x, y] \in v)
\end{aligned}
$$

$$
\begin{aligned}
\chi_3^{(V,T)} =_{\text{Def}} \ & \bigwedge_{x,y\in V} \left(\neg(x = y) \to dist(x, y) \vee dist_{\pi}(x, y) \right) \\
& \wedge (\forall x, y) \left(dist(x, y) \to \bigvee_{z\in V\cup T} \left(distBy(x, y, z) \vee distBy(y, x, z) \right) \right) \\
& \wedge (\forall x, y) \left(dist_{\pi}(x, y) \to \bigvee_{z,v\in V\cup T} \left(distBy_{\pi}(x, y, z, v) \vee distBy_{\pi}(y, x, z, v) \right) \right) \\
& \wedge (\forall x, y, z) \left(distBy(x, y, z) \to (z \in \bar{\pi}(x) \wedge \neg(z \in \bar{\pi}(y))) \right) \\
& \wedge (\forall x, y, z, v) \left(distBy_{\pi}(x, y, z, v) \to ([z, v] \in x \wedge \neg([z, v] \in y)) \right)
\end{aligned}
$$

$$
\chi_4^{(V,T)} =_{\text{Def}} \bigwedge_{x\in V\cup T, t\in T, x\neq t} \neg(x = t).
$$

It can be easily verified that $\chi^{(V,T)}$ is a $\mathsf{DATALOG}^{\vee,\neg}$-formula. The subformulae χ_1, χ_2, $\chi_3^{(V,T)}$, and $\chi_4^{(V,T)}$ formalize the conditions **(S1)**, **(S2)**, **(S3)**, and **(S4)** of Definition 1, as clarified by the following lemma.

Lemma 2. *Let V,T be two disjoint sets of constants. Let $\mathcal{H}$ be a model for $\chi^{(V,T)}$ such that $Consts(\mathcal{H}) \subseteq V \cup T$, and let $\mathcal{S}$ be the $\forall_{\mathbf{0}}^{\pi}$-subset of $\mathcal{H}$. Then $\mathcal{S}$ is a skeletal representation relative to (V,T).*

Proof. Plainly, $\mathcal{S}$ is a set of atomic $\forall_{\mathbf{0}}^{\pi}$-formulae, since it is the $\forall_{\mathbf{0}}^{\pi}$-subset of $\mathcal{H}$. We must prove that $\mathcal{S}$ satisfies all the conditions reported in Definition 1.

We first observe that the relation $\widehat{\in}_{\mathcal{H}} = \{[a,b] \,|\, \mathcal{H} \models a \widehat{\in} b\}$ is acyclic, as $\mathcal{H}$ models correctly the conjunct χ_1 of $\chi^{(V,T)}$. Hence, $\in_{\mathcal{S}}^{+}$ must be acyclic as well, as required by Condition **(S1)**, since $\in_{\mathcal{S}}^{+} \subseteq \widehat{\in}_{\mathcal{H}}$.[6] Thus, we can conclude that $\mathcal{S}$ satisfies **(S1)**, since $\mathcal{S} \subseteq \mathcal{H}$.

Next, suppose that $\mathcal{S} \models x = y$, for some constants $x,y \in V \cup T$. Plainly, $\mathcal{H} \models x = y$. Let us consider any $\forall_{\mathbf{0}}^{\pi}$-formula γ in $\mathcal{S} \subseteq \mathcal{H}$ involving either x or y, say $x \in \bar{\pi}(z)$. The definition of $\mathcal{S}$ yields that $z \in V \cup T$, as we are assuming that $\mathcal{S} \models x \in \bar{\pi}(z)$. Then, $\mathcal{H} \models \gamma[x \to y]$, where $\gamma[x \to y]$ is the literal $y \in \bar{\pi}(z)$, easily follows, since $\mathcal{H}$ models correctly the conjunct $(\forall x, y, z)(x \in \bar{\pi}(y) \wedge x = z \to z \in \bar{\pi}(y))$ of χ_2. In addition, $\mathcal{S} \models y \in \bar{\pi}(z)$ holds as well, as $y \in \bar{\pi}(z)$ is a $\forall_{\mathbf{0}}^{\pi}$-formula with $y, z \in V \cup T$. Property **(S2)** for $\mathcal{S}$ can be proved by reasoning in similar way, for all the $\forall_{\mathbf{0}}^{\pi}$-formulae occurring in $\mathcal{S}$.

Conversely, let $x, y \in V$ be such that $\mathcal{S} \not\models x = y$ and, consequently, $\mathcal{H} \not\models x = y$. Thus, $\mathcal{H} \models \neg(x = y)$, so that $\mathcal{H} \models dist(x,y) \vee dist_{\pi}(x,y)$, as we are supposing that $\mathcal{H}$ is a model for $\chi_3^{(V,T)}$. It can easily be verified that if $\mathcal{H} \models dist(x,y)$, then x and y are distinguished in $\mathcal{S}$ by a constant (in the sense of Definition 1), whereas, if $\mathcal{H} \models dist_{\pi}(x,y)$, then x and y are distinguished in $\mathcal{S}$ by a pair term. Let us assume that $\mathcal{H} \models dist(x,y)$. Thus $\mathcal{H} \models distBy(x,y,z) \vee distBy(y,x,z)$, for some constant $z \in V \cup T$, so that either $\mathcal{H} \models z \in \bar{\pi}(x)$ and $\mathcal{H} \not\models z \in \bar{\pi}(y)$, or $\mathcal{H} \models z \in \bar{\pi}(y)$ and $\mathcal{H} \not\models z \in \bar{\pi}(x)$. Then x and y are distinguished in $\mathcal{H}$ by z, and, consequently, they are distinguished in $\mathcal{S}$ by z too, since $z \in \bar{\pi}(x)$ and $z \in \bar{\pi}(y)$ are $\forall_{\mathbf{0}}^{\pi}$-formulae and $x, y, z \in V \cup T$. By reasoning as above, it can be proved that if $\mathcal{H} \models dist_{\pi}(x,y)$ then x and y are distinguished in $\mathcal{S}$ by the pair $[z,v]$ if $\mathcal{H} \models dist_{\pi}(x,y)$, thus concluding the proof that $\mathcal{H}$ satisfies **(S3)**.

Concerning **(S4)**, by way of contradiction we prove that $\mathcal{S} \models x = y$ implies that x and y are constants in V. Thus, let us assume that $y \notin V$. Then y must be in T, as T and V are disjoint, and $Consts(\mathcal{S}) \subseteq Consts(\mathcal{H}) \subseteq V \cup T$. But $\mathcal{H}$ must model the conjunct $\neg(x = y)$ of $\chi_4^{(V,T)}$. Hence, we have $\mathcal{H} \not\models x = y$ and, *a fortiori*, $\mathcal{S} \not\models x = y$, since $\mathcal{S} \subseteq \mathcal{H}$, contradicting our initial assumption. Thus, $y \in V$. In addition, $\mathcal{S} \models x = y$ yields that $\mathcal{S} \models y = x$, as $\mathcal{H}$ models the conjunct $(\forall x)(x = y \to y = x)$ of χ_2, x and y are in $V \cup T$, and $y = x$ is a $\forall_{\mathbf{0}}^{\pi}$-formula. Thus, $x \in V$ can be proved analogously.

Finally, **(S5)** holds as well, since, by assumption, $Consts(\mathcal{S}) \subseteq Consts(\mathcal{H}) \subseteq V \cup T$. $\square$

In addition, every skeletal interpretation can be extended to comply with the constraints imposed by $\chi^{(V,T)}$.

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[6] The relation $\in_{\mathcal{H}}^{+}$ does not necessarily coincide with $\widehat{\in}_{\mathcal{H}}$, as the minimality of $\widehat{\in}_{\mathcal{H}}$ is not enforced in χ_1.

Lemma 3. *Given two disjoint sets of constants V, T and a skeletal represen-tation $\mathcal{S}$ relative to (V, T), there always exists a Herbrand model $\mathcal{H}$ (over $\mathcal{L}_0^\pi$) having $\mathcal{S}$ as $\forall_0^\pi$-subset.*

Proof. Let $\mathcal{S}$ be a skeletal representation relative to (V, T), where V and T are disjoint sets of constants. We can extend $\mathcal{S}$ to a Herbrand model $\mathcal{H}$ for $\chi^{(V,T)}$.

Initially, we put $\mathcal{H} =_{\mathrm{Def}} \mathcal{S}$ and then we enrich $\mathcal{H}$ with new atomic formulae according to the following rules. If $x \in \bar\pi(y)$ is in $\mathcal{S}$, we add $x\widehat{\in}y$ to $\mathcal{H}$. Likewise, if $[x, y] \in z$ is in $\mathcal{S}$, we add the atoms $x\widehat{\in}z$ and $y\widehat{\in}z$ to $\mathcal{H}$. Then we close transitively $\mathcal{H}$ with respect to the $\widehat{\in}$-atoms, i.e., for each sequence of atoms in $\mathcal{H}$ of the form

$$x_1\widehat{\in}x_2, \ x_2\widehat{\in}x_3, \ \ldots, \ x_{n-1}\widehat{\in}x_n \,,$$

we add the atom $x_1\widehat{\in}x_n$ to $\mathcal{H}$.

Next, for every $x, y \in V$, if $\mathcal{S} \not\models x = y$ and x and y are distinguished in $\mathcal{S}$ by a $z \in V \cup T$, we add to $\mathcal{H}$ the atoms $dist(x, y)$ and $distBy(x, y, z)$ (resp., $dist(y, x)$ and $distBy(y, x, z)$), provided that $z \in \bar\pi(x)$ is in $\mathcal{S}$ and $z \in \bar\pi(y)$ is not in $\mathcal{S}$ (resp., $z \in \bar\pi(y)$ is in $\mathcal{S}$ and $z \in \bar\pi(x)$ is not in $\mathcal{S}$). Otherwise, if x and y are distin-guished in $\mathcal{S}$ by a pair $[z, v]$, with $z, v \in V \cup T$, we add to $\mathcal{H}$ the atoms $dist_\pi(x, y)$ and $distBy_\pi(x, y, z, v)$ (resp., $dist_\pi(y, x)$ and $distBy_\pi(y, x, z, v)$), provided that $[z, v] \in x$ is in $\mathcal{S}$ and $[z, v] \in y$ is not in $\mathcal{S}$ (resp., $[z, v] \in y$ is in $\mathcal{S}$ and $[z, v] \in x$ is not in $\mathcal{S}$).

In order to show that the set $\mathcal{H}$ so obtained is a Herbrand model for $\chi^{(V,T)}$, we have to prove that $\mathcal{H}$ is a Herbrand model for the formulae χ_1, χ_2, $\chi_3^{(V,T)}$, and $\chi_4^{(V,T)}$.

By the very construction process, $\mathcal{H}$ satisfies the first four conjuncts of χ_1. In addition, $\mathcal{H}$ satisfies also the last conjunct $(\forall x)\neg(x\widehat{\in}x)$ of χ_1. Indeed, if this were not the case, $\mathcal{H}$ would contain an atom of the form $z\widehat{\in}z$, for some $z \in V \cup T$. But this would be possible only if one of the following situations occurs:

(a) $\mathcal{S}$ contains an atom of one of the following three forms

$$z \in \bar\pi(z), \ [z, y] \in z, \ [x, z] \in z \,,$$

for some $x, y \in V \cup T$;

(b) $\mathcal{H}$ contains a sequence of atoms of the form

$$z\widehat{\in}x_1, \ x_1\widehat{\in}x_2, \ \ldots, \ x_{n-1}\widehat{\in}x_n, \ x_n\widehat{\in}z \,,$$

with $x_i \in V \cup T$ for $1 \le i \le n$.

However, case (a) cannot occur, since, by condition **(S1)**, the membership rela-tion induced by $\mathcal{S}$ is acyclic. We show that also case (b) cannot occur, thereby proving that $\mathcal{H}$ cannot contain any atom of the form $z\widehat{\in}z$. Indeed, if (b) were true, then $\mathcal{H}$ would contain a maximal sequence of atoms of the form

$$x_0\widehat{\in}x_1, \ x_1\widehat{\in}x_2, \ \ldots, \ x_{n-1}\widehat{\in}x_n, \ x_n\widehat{\in}x_{n+1} \,,$$

where x_0 and x_{n+1} coincide. But then, for each $i = 0, 1, \ldots, n$, $\mathcal{S}$ would contain at least an atom of one of the following types

$$x_i \in \bar{\pi}(x_{i+1}), \ \ [x_i, w] \in x_{i+1}, \ \ [w, x_i] \in x_{i+1},$$

with $w \in V \cup T$, and therefore the membership relation $\in_{\mathcal{S}}^+$ induced by $\mathcal{S}$ would contain a cycle, contradicting condition **(S1)**.

Summing up, $\mathcal{H}$ satisfies also the last conjunct of χ_1, and hence it satisfies the whole formula χ_1.

To show that $\mathcal{H}$ satisfies χ_2 and $\chi_3^{(V,T)}$, it is enough to observe that $\mathcal{S} \subseteq \mathcal{H}$ and that $\mathcal{S}$ satisfies conditions **(S2)** and **(S3)**, respectively.

Finally, since (again) $\mathcal{S} \subseteq \mathcal{H}$ and $\mathcal{S}$ satisfies condition **(S4)**, it follows that $\mathcal{H}$ satisfies also the formula $\chi_4^{(V,T)}$, as no new atom of the form $x = y$ can possibly be introduced into $\mathcal{H}$ during its construction.

Hence, in conclusion, $\mathcal{H}$ is a Herbrand model for the formula $\chi^{(V,T)}$. $\qquad\square$

As already remarked, the following lemma exploits a semantical correspondence between pair-aware set-theoretical models and Herbrand models of $\forall_{\mathbf{0}}^{\pi}$-formulae.

Lemma 4. *Let V, T be two disjoint sets of constants and let $\mathcal{H}$ be a Herbrand interpretation such that $\mathcal{H} \models \chi^{(V,T)}$ and $Consts(\mathcal{H}) \subseteq V \cup T$. Let $\mathcal{S}$ be the $\forall_{\mathbf{0}}^{\pi}$-subset of $\mathcal{H}$, and let $\mathcal{R}$ be the realization of $\mathcal{S}$ relative to V, T. Then*

$$\mathcal{H} \models \varphi \ \iff \ \mathcal{R} \models_s \varphi,$$

for every $\forall_{\mathbf{0}}^{\pi}$-formula φ such that $Consts(\varphi) \subseteq V \cup T$ and $Consts_D(\varphi) \subseteq V$.

Proof. It will be enough to prove the lemma in the case of simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formulae, as general $\forall_{\mathbf{0}}^{\pi}$-formulae are just finite conjunctions of simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formulae. Accordingly, in the rest of this proof we will assume that φ is a simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formula.

We proceed by induction on the quantifier prefix length of φ. Preliminarily, we observe that

$$\mathcal{H} \models \gamma \ \iff \ \mathcal{R} \models_s \gamma,$$

for each atomic $\forall_{\mathbf{0}}^{\pi}$-formula γ such that $Consts(\gamma) \subseteq V \cup T$ (see Lemma 1), so that the lemma follows easily from propositional logic when φ is quantifier-free.

For the inductive case, let us first assume that $\varphi = (\forall x \in \bar{\pi}(a))\psi$, for some $x \in Vars$ and $a \in Consts$, where ψ is a simple-prenex $\forall_{\mathbf{0}}^{\pi}$-formula with one less quantifier than φ. Since $\mathcal{R}$ is a realization, we plainly have $\mathcal{R}\,\bar{\pi}(a) = \{\mathcal{R}\,b \mid \mathcal{H} \models b \in \bar{\pi}(a)\}$, so that $\mathcal{R}$ models correctly $(\forall x \in \bar{\pi}(a))\psi$ if and only if it correctly models all the formulae $\psi[x \to b]$ such that $\mathcal{H} \models b \in \bar{\pi}(a)$. Observe that, for all the constants b such that $\mathcal{H} \models b \in \bar{\pi}(a)$, we have $Consts_D(\psi[x \to b]) = Consts_D(\psi) = Consts_D(\varphi) \setminus \{a\} \subseteq V$, and $Consts(\psi[x \to b]) \subseteq Consts(\psi) \cup \{b\} \subseteq Consts(\varphi) \cup \{b\} \subseteq V \cup T$, as we have assumed that $Consts(\varphi) \cup Consts(\mathcal{H}) \subseteq V \cup T$. Thus

$$\mathcal{R} \models_s \psi[x \to b] \ \iff \ \mathcal{H} \models \psi[x \to b],$$

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for each b such that $\mathcal{H} \models b \in \bar{\pi}(a)$, and, consequently,

$$\mathcal{R} \models_s (\forall x \in \bar{\pi}(a))\psi \iff \mathcal{H} \models (\forall x \in \bar{\pi}(a))\psi$$

follows by applying the inductive hypothesis to the formulae $\psi[x \to b]$, for all b such that $\mathcal{H} \models b \in \bar{\pi}(a)$.

Next, let us consider the case in which φ has the form $(\forall[x, y] \in a)\psi$, with $x, y \in \textit{Vars}$, $a \in \textit{Consts}$, and ψ a simple-prenex $\forall_0^\pi$-formula with one less quantifier than φ. In this case we must consider the set $\mathcal{R}\,a \setminus \mathcal{R}\,\bar{\pi}(a)$ of the pair members of $\mathcal{R}\,a$. But, $\mathcal{R}\,a \setminus \mathcal{R}\,\bar{\pi}(a) = \{\mathcal{R}\,[b, c] \,|\, \mathcal{H} \models [b, c] \in a\}$, by Lemma 1, and thus $\mathcal{R}$ models $(\forall[x, y] \in a)\psi$ if and only if it correctly models the formula $\psi[x \to b][y \to c]$, for all a, b such that $\mathcal{H} \models [b, c] \in a$. Thus, the thesis follows by reasoning much as in the previous case, by applying the inductive hypothesis to the formulae $\psi[x \to b][y \to c]$, for all a, b such that $\mathcal{H} \models [b, c] \in a$ $\qquad\square$

Finally, the next theorem states that the satisfiability of every $\forall_0^\pi$-formula φ can be decided by checking the Herbrand satisfiability of the corresponding $\mathcal{L}_0^\pi$-formula $\chi^{(V,T)} \wedge \varphi$, thus concluding the verification of the correctness of our reduction.

Theorem 3. *Let φ be a $\forall_0^\pi$-formula, let $V = \textit{Consts}(\varphi)$, and let T be any set of constants disjoint from V such that such that $|T| = 2 \cdot |V|$. Then φ is satisfiable if and only if $\chi^{(V,T)} \wedge \varphi$ is Herbrand-satisfiable.*

Proof. To begin with, let us assume that φ is satisfiable. Then, by Theorem 2, there exists a skeletal representation $\mathcal{S}$ (relative to (V, T)) such that its realization $\mathcal{R}$ relative to (V, T) is a pair-aware set-theoretical model for φ. In addition, by Lemma 2, there exists a Herbrand interpretation $\mathcal{H}$ with $\mathcal{S}$ as its $\forall_0^\pi$-subset, which correctly models $\chi^{(V,T)}$, and such that $\textit{Consts}(\mathcal{H}) = \textit{Consts}(\mathcal{S})$. Thus, $\mathcal{H} \models \varphi$ directly follows from Lemma 4.

Conversely, let $\mathcal{H}$ be a Herbrand model for $\chi^{(V,T)} \wedge \varphi$, and let $\mathcal{S}$ be the $\forall_0^\pi$-subset of $\mathcal{H}$. We have $\textit{Consts}(\mathcal{H}) \subseteq V \cup T$, as $\textit{Consts}\left(\chi^{(V,T)}\right) = V \cup T$ and $\textit{Consts}(\varphi) = V$. In addition, from Lemma 2 and from $\mathcal{H} \models \chi^{(V,T)}$, $\mathcal{S}$ is a skeletal representation. Thus, by Lemma 4, the realization of $\mathcal{S}$ relative to (V, T) must be a model for φ. $\qquad\square$

We conclude this section by observing that, if φ is a $\forall_{0,\mathbf{D}\vee}^\pi$-formula, then $\chi^{(V,T)} \wedge \varphi$ is a $\mathsf{DATALOG}^{\vee,\neg}$-formula, as $\chi^{(V,T)}$ is a $\mathsf{DATALOG}^{\vee,\neg}$-formula. Thus the satisfiability problem for $\forall_{0,\mathbf{D}\vee}^\pi$-formulae can be reduced in polynomial time to the Herbrand satisfiability problem for $\mathsf{DATALOG}^{\vee,\neg}$-formulae.

Corollary 1. *Let φ be a $\forall_{0,\mathbf{D}\vee}^\pi$-formula. Let $V = \textit{Consts}(\varphi)$ and let T be any set of constants disjoint from V and such that $|T| = 2 \cdot |V|$. Then φ is set-theoretically satisfiable if and only if the corresponding $\mathsf{DATALOG}^{\vee,\neg}$-formula $\chi^{(V,T)} \wedge \varphi$ is satisfiable, in the sense of Disjunctive Datalog.* $\qquad\square$

5   Conclusions and Future Works

In this paper we have identified a correspondence between the fragment of set-theory $\forall_{\mathbf{0}}^{\pi}$ and first-order logic (in particular Herbrand logic) by providing a polynomial-time reduction of $\forall_{\mathbf{0}}^{\pi}$-formulae to formulae in a first-order language, called $\mathcal{L}_0^{\pi}$, suitable for this purpose. In addition, we have shown that if we limit ourselves to the Disjunctive Datalog restriction of $\forall_{\mathbf{0}}^{\pi}$ called $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$, then our reduction maps formulae in this subfragment to $\mathsf{DATALOG}^{\vee,\neg}$-formulae.

Such correspondence, and its consequences, has to be further investigated. For instance, applications of techniques and results devised in the context of logic programming, such as, for example, answer-set programming and negation-as-failure, to the $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$ subfragment need to be studied. In view of our reduction, a satisfiability checker for $\forall_{\mathbf{0},\mathbf{D}\vee}^{\pi}$-formulae can be implemented by reusing some machinery from logic programming, for example the Disjunctive Datalog system *DLV* described in [1].[7]

We intend to develop analogous correspondences for other decidable fragments of set-theory, such as, for instance, the quantified fragment $\forall_{0,2}^{\pi}$, whose decision procedure presented in [2] is based on a reduction to $\forall_{\mathbf{0}}^{\pi}$, and the unquantified fragment $\mathsf{MLSS}_{2,m}^{\times}$ (see [4]), whose satisfiability problem can be in turn reduced to the satisfiability problem for $\forall_{0,2}^{\pi}$.

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