Adding two equivalence relations to the interval temporal logic AB

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Abstract. The interval temporal logic AB features two modalities that make it possible to access intervals which are adjacent to the right of the current interval (modality $\langle A \rangle$) and proper subintervals that have the same left endpoint of it (modality $\langle B \rangle$). AB is one of the most significant interval logics, as it allows one to express meaningful (metric) properties, while maintaining decidability (undecidability rules over interval logics, AB is EXPSPACE-complete [14]). In an attempt to capture ωS-regular languages with interval logics [15], it was proved that AB extended with an equivalence relation, denoted AB$\sim$, is decidable (non-primitive recursive) on the class of finite linear orders and undecidable on $\mathbb{N}$. The question whether the addition of two or more equivalence relations makes finite satisfiability for AB undecidable was left open. In this paper, we answer this question proving that AB$\sim_1\sim_2$ is undecidable.

1 Introduction

Interval temporal logics (ITL) are temporal logics where time intervals/periods, instead of time points/instants as in the standard framework, are used as basic building blocks. ITL are characterized by high expressiveness and high computational complexity. The main formalization of these logics is known as HS which features one modality for each interval order relation [7] (the so-called Allen's relations [1]). In this paper, we analyze the complexity of the finite satisfiability problem for the interval temporal logic AB of Allen's relations *meets* and *begun by* extended with two equivalence relations. AB is one of the most significant fragments of HS since it is decidable [14] (undecidability rules over interval logics [5]) and it can express various important (metric) temporal properties. As an example, it allows one to encode the standard until operator of point-based linear temporal logic as well as to constrain the length of an interval to be less than/equal to/greater than a given value [14].

The trade-off between the increase in expressiveness and the complexity blow-up induced by the addition of one or more equivalence relations to a logic has been already highlighted in the literature (see, for instance, the logics for semi-structured data [3], temporal logics [6], and timed automata [16]). Finite satisfiability of the two-variable fragment of first-order logic FO^2 extended with one, two, or more equivalence relations has been systematically explored in [8–10], while the extension of FO^2, interpreted over finite or infinite data words, with an equivalence relation has been investigated by Bojańczyk et al. in [3]. Similar results have been obtained by Demri and Lazic [6], that studied the extension of linear temporal logic over data words with freeze quantifiers, which allow one to store elements at the current word position into a register and then to use them in equality comparisons deeper in the formula, and by Ouaknine and Worrell [16], who showed that both satisfiability and model checking for metric temporal logic over finite timed words are decidable with a non-primitive recursive complexity.

The addition of an equivalence relation to an interval temporal logic was first investigated by Montanari and Sala in [15]. They focused their attention on the interval logic AB of Allen's relations *meets* and *begins* extended with an equivalence relation, denoted $\mathsf{AB}\sim$, interpreted over finite linear orders and $\mathbb{N}$, and they showed that the resulting increase in expressive power makes it possible to establish an original connection between interval temporal logics and extended regular languages of finite and infinite words [2]. As for the computational complexity, they proved that $\mathsf{AB}\sim$ is decidable (non-primitive recursive) on the class of finite linear orders and undecidable on $\mathbb{N}$. Recently, the interval logic of temporal neighborhood PNL, which features two modalities for Allen's relations *meets* and *met by*, and its metric variant MPNL, both extended with one equivalence relation, have been proved to be decidable over finite linear orders [13] (NEXPTIME-complete the former, EXPSPACE-hard the latter). In this paper, we answer a question left open in [15], showing that the addition of two (or more) equivalence relations makes AB undecidable also over finite linear orders.

The paper is organized as follows. Section 2 illustrates in some detail previous work on the interval temporal logic AB extended with one equivalence relation and some related work. Section 3 introduces syntax and semantics of the logic $\mathsf{AB}\sim_1\sim_2$ and gives some background knowledge about counter machines. The next two sections provide a reduction from the undecidable 0-0 reachability problem for Minsky counter machines to the finite satisfiability problem for $\mathsf{AB}\sim_1\sim_2$. More precisely, Section 4 outlines the general structure forced on each model (if any) by the formulas given in Section 5. Finally, in Section 6 we prove that the proposed encoding is correct. Conclusions provide an assessment of the work and outline future research directions.

2 Related work

The present paper can be viewed as the natural completion of the work reported in [15], where Montanari and Sala proved that the satisfiability problem for $\mathsf{AB}\sim$ is decidable over finite linear orders, with non-primitive recursive complexity, and undecidable over $\mathbb{N}$. Undecidability has been proved by a reduction from the (undecidable) 0-n reachability problem for lossy counter machines [11]. As for finite satisfiability, they initially reduced the problem of finding a model for a given $\mathsf{AB}\sim$ formula φ to the existence of a particular *compass structure* exploiting the correspondence that can be established between intervals and points in the positive octant of the Cartesian plane, that is, the map that links any interval $[x, y]$ to the corresponding point (x, y). Then, by exploiting a suitable model contraction technique, they showed that if φ is finitely satisfiable, then a structure satisfying it can be obtained via a bottom-up generation of candidate (finite) compass structures. Since the number of pairwise incomparable rows of any candidate structure can be proved to be finite, termination easily follows. The complexity bound has been obtained by encoding in $\mathsf{AB}\sim$ the 0-0 reachability for *lossy* counter machines (LCM) which is known to be a non-primitive recursive (decidable) problem [17].

The general structure of the $\mathsf{AB}\sim$ encoding of the 0-0 reachability problem for LCM is similar to the one provided in this paper for (non-lossy) counter machines (CM). However, when only one equivalence relation is available, it is possible to enforce the presence of certain points in a configuration (depending on the points of the previous configuration), but not to restrict the number of points in it. This last constraint is necessary for the encoding of CM and it can be enforced only by making use of two equivalence relations.

From a technical point of view, our work presents some similarities to the one by Bojańczyk et al. in [4], where it is shown, among other results, that the logic $\mathsf{FO}^2(+1, \sim_1, \sim_2)$ over finite data words is undecidable. To prove it, they provide a reduction from the Post correspondence problem to finite satisfiability for $\mathsf{FO}^2(+1, \sim_1, \sim_2)$, that exploits the interconnections between equivalence relations in a way that is similar to what we do here. However, their encoding strongly depends on constraints of the form $\forall\exists$ which are not expressible in AB. To overcome these limitations, we will exploit some metric properties definable in AB.

3 Preliminaries

In this section, we first introduce syntax and semantics of $\mathsf{AB}\sim_1\sim_2$ and then we provide background knowledge about Minsky counter machines.

3.1 The interval temporal logic $\mathsf{AB}\sim_1\sim_2$

The interval temporal logic $\mathsf{AB}\sim_1\sim_2$ features two modalities $\langle \mathsf{A} \rangle$ and $\langle \mathsf{B} \rangle$ corresponding to Allen's relations *meets* and *begun by*, respectively, and two special binary relation symbols $\sim_1$ and $\sim_2$. Formally, given a set $\mathcal{P}rop$ of propositional variables, formulas of $\mathsf{AB}\sim_1\sim_2$ are built up from $\mathcal{P}rop$ and $\sim_1, \sim_2$ using the Boolean connectives $\neg$, $\vee$ and the modalities $\langle \mathsf{A} \rangle$ and $\langle \mathsf{B} \rangle$. Moreover, we make use of shorthands $\varphi_1 \wedge \varphi_2$ for $\neg(\neg\varphi_1 \vee \neg\varphi_2)$, $[\mathsf{A}]\varphi$ for $\neg\langle \mathsf{A} \rangle\neg\varphi$, $[\mathsf{B}]\varphi$ for $\neg\langle \mathsf{B} \rangle\neg\varphi$, $\top$ for $p \vee \neg p$, and $\bot$ for $p \wedge \neg p$, with $p \in \mathcal{P}rop$.

We interpret formulas of $\mathsf{AB}{\sim_1}{\sim_2}$ in interval temporal structures over (prefixes of) $\mathbb{N}$ endowed with the ordering relations *meets* (denoted by A) and *begun by* (denoted by B), and two equivalence relations $\sim_1$ and $\sim_2$. More precisely, we identify any given ordinal $N < \omega$ with the prefix of length N of $\mathbb{N}$ and we accordingly define $\mathbb{I}(N)$ as the set of all closed intervals $[i, j]$, with $i, j \in N$ and $i \leq j$. A special role will be played by point-intervals (intervals of the form $[i, i]$, with $i \in N$) and unit-intervals (intervals of the form $[i, i+1]$), which can be respectively defined as $[\mathsf{B}]\perp$ (abbreviated π) and $[\mathsf{B}][\mathsf{B}]\perp$ (abbreviated *unit*). For any pair of intervals $[i, j], [i', j'] \in \mathbb{I}(N)$, relations A and B are defined as follows:

- *meets* relation: $[i, j]\ A\ [i', j']$ iff $j = i'$;
- *begun by* relation: $[i, j]\ B\ [i', j']$ iff $i = i'$ and $j' < j$.

The (non-strict) semantics of $\mathsf{AB}{\sim_1}{\sim_2}$ is given in terms of interval models $\mathcal{S} = \langle \mathbb{I}(N), A, B, \sim_1, \sim_2, V\rangle$, where $\sim_1$ and $\sim_2$ are two equivalence relations over N and $V \colon \mathbb{I}(N) \to \wp(\mathcal{P}rop)$ is a valuation function that assigns to every interval $[i, j] \in \mathbb{I}(N)$ the set of propositional variables $V([i, j])$ that are true on it. The truth of an $\mathsf{AB}{\sim_1}{\sim_2}$ formula over a given interval $[i, j]$ in a model $\mathcal{S}$ is defined by structural induction as follows:

- $\mathcal{S}, [i, j] \vDash p$ iff $p \in V([i, j])$, for all $p \in \mathcal{P}rop$;
- $\mathcal{S}, [i, j] \vDash \neg\psi$ iff $\mathcal{S}, [i, j] \nvDash \psi$;
- $\mathcal{S}, [i, j] \vDash \varphi \vee \psi$ iff $\mathcal{S}, [i, j] \vDash \varphi$ or $\mathcal{S}, [i, j] \vDash \psi$;
- $\mathcal{S}, [i, j] \vDash \langle \mathsf{X}\rangle\psi$ iff there exists an interval $[i', j']$ such that $[i, j]\ X\ [i', j']$, and $\mathcal{S}, [i', j'] \vDash \psi$, for $X \in \{A, B\}$;
- $\mathcal{S}, [i, j] \vDash \sim_k$ iff $i \sim_k j$, for $k \in \{1, 2\}$.

Given an interval structure $\mathcal{S}$ and a formula φ, we say that $\mathcal{S}$ *satisfies* φ if there is an interval I in $\mathcal{S}$ such that $\mathcal{S}, I \vDash \varphi$. We say that φ is *(finitely) satisfiable* if there exists a (finite) interval structure that satisfies it. We define the *(finite) satisfiability problem* for $\mathsf{AB}{\sim_1}{\sim_2}$ as the problem of establishing whether a given $\mathsf{AB}{\sim_1}{\sim_2}$ formula φ is (finitely) satisfiable.

3.2 Counter machines

A k *counter machine* (kCM) is a triple of the form $M = (Q, k, \delta)$, where Q is a finite set of control states, k is the number of counters, whose values range over $\mathbb{N}$, and δ is a function that maps each state $q \in Q$ to a transition rule having one of the following forms:

1. $\mathit{value}(h) \leftarrow \mathit{value}(h) + 1;\ \mathit{goto}\ q'$, for some $1 \leq h \leq k$ and $q' \in Q$ (abbreviated $(q, h{+}{+}, q')$), meaning that, whenever M is at state q, it increases the counter h and it moves to state q';
2. if $\mathit{value}(h) = 0$ then goto q' else $\mathit{value}(h) \leftarrow \mathit{value}(h) - 1$; goto q'', for some $1 \leq h \leq k$ and $q', q'' \in Q$ (abbreviated $(q, h?0, q', q'')$), meaning that, whenever M is at state q and the value of the counter h is 0 (resp., greater than 0), it moves to state q' (resp., it decrements the counter h and it moves to state q'').

A computation of M is any sequence of configurations that conforms to the transition relation. The reachability problem for a counter machine $M = (Q, k, \delta)$ is the problem of deciding, given two configurations $(q_0, \overline{z}_0)$ and $(q_f, \overline{z}_f)$, whether there is a computation that takes M from $(q_0, \overline{z}_0)$ to $(q_f, \overline{z}_f)$. The reachability problem for counter machines is undecidable even for machines with only two counters [12]. For convenience, we will use a restricted, but equally undecidable, form of this problem, called 0-0 *reachability* problem, where z_0 and z_f are both $\overline{0}$. Moreover, without loss of generality, we restrict our attention to computations where q_0 and q_f occur only at the beginning and at the end, respectively.

4 The structure of intended models

The main contribution of the paper is the proof of the following theorem.

Theorem 1. *The satisfiability problem for* $\mathsf{AB}{\sim_1}{\sim_2}$ *over the class of finite linear orders is undecidable.*

To prove it, we provide a reduction from the 0-0 reachability problem for Minsky two-counter machines to the satisfiability problem for $\mathsf{AB}{\sim_1}{\sim_2}$. More precisely, given a two-counter machine $M = (Q, 2, \delta)$ and two states $q_0, q_f \in Q$, we build a formula ψ_{M,q_0,q_f} such that there exists a computation in M from the configuration $(q_0, 0, 0)$ to the configuration $(q_f, 0, 0)$ if and only if ψ_{M,q_0,q_f} is satisfiable. First, in the present section, we delineate the structure that we want to give to each model (if any) of ψ_{M,q_0,q_f} (the intended model). Then, in Section 5, we show how to encode in $\mathsf{AB}{\sim_1}{\sim_2}$ such a structure. We conclude the paper with the proof of the correctness of the encoding.

The structure of the models of the encoding formula can be described as follows. To start with, we partition the set of point-intervals (points for short) in two subsets: points labeled by a state in Q (*state-points*) and points labeled by c_1 or c_2 (*counter-points*). A configuration (q, v_1, v_2) is represented as a set of contiguous points, where the first point is a state-point with label q and the remaining ones are counter-points such that exactly v_1 points have label c_1 and v_2 points have label c_2, in any order. Counter-points with label *del* are points which have been "deleted" and thus do not count for the value of a counter in a configuration.

The computation of the two-counter machine M (from q_0 to q_f) consists of a sequence of contiguous configurations. Counter-points with label *plus* and *minus* indicate, respectively, points added in a configuration as a result of a counter increment or eliminated in the next configuration as a result of a counter decrement. Each configuration, but the first one, is obtained from the previous one by applying a transition of M, which amounts to say that the sequence of configurations is a valid computation of M.

In Figure 1, we give an example of the proposed model representation for the computation $(q, 1, 1) \to (q', 1, 0) \to (q'', 2, 0)$. The second configuration $(q', 1, 0)$ is obtained from the initial configuration $(q, 1, 1)$ by decrementing the second counter; the third configuration is obtained from the second one by incrementing the first counter. Notice that points with label *minus* are deleted, that is, labeled by *del*, in the configuration that immediately follows the one in which the decrement of the counter takes place.

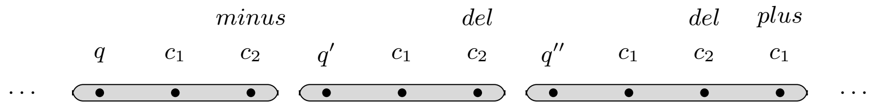

Fig. 1. Labels of the points in a model representing the partial computation $(q, 1, 1) \to (q', 1, 0) \to (q'', 2, 0)$.

5 Encoding of 0-0 reachability in AB$\sim_1\sim_2$

In this section, we show how to encode intended models, representing valid computations of the two-counter machine, by an AB$\sim_1\sim_2$ formula.

Let us consider the AB$\sim_1\sim_2$ formula ψ_{M,q_0,q_f}, which is defined as follows:

$$\psi_{M,q_0,q_f} \equiv \psi_{0\to 0} \wedge [\mathsf{G}](\psi_{points} \wedge \psi_\delta \wedge \psi_\sim),$$

where $[\mathsf{G}]\varphi$ is an shorthand for $[\mathsf{B}]\varphi \wedge \varphi \wedge [\mathsf{B}][\mathsf{A}]\varphi \wedge [\mathsf{A}][\mathsf{A}]\varphi$ (universal modality) and

- $\psi_{0\to 0}$ forces the initial and final configurations of the computation to be respectively $(q_0, 0, 0)$ and $(q_f, 0, 0)$;
- ψ_{points} specifies conditions on points: (i) points are partitioned in state-points and counter-points, (ii) labels *plus* and *minus* can label counter-points only, and (iii) in a configuration, there is at most one point with label *minus* and at most one point with label *plus*;
- ψ_δ ensures the consistency of state-points and *plus*/*minus* points with the transitions in M, that is, if a configuration τ', with state-point q', immediately follows a configuration τ, with state-point q, then one of the following three cases must hold: (i) there is a transition $\delta_i = (q, h++, q') \in \delta$, there is no point with label *minus* in τ, and there is one point with labels c_h and *plus* in τ'; (ii) there is a transition $\delta_i = (q, h?0, q', q'')$, there is no point with label c_h and without label *del* in τ, and there is no point with label *plus* in τ'; (iii) there is a transition $\delta_i = (q, h?0, q'', q')$, there is one point with label c_h and without label *del* in τ, and there is no point with label *plus* in τ';
- $\psi_\sim$ guarantees that each configuration is obtained from the previous one by an increment/decrement/no-action transition (notice that the fact that the transition actually belongs to M is checked by ψ_δ and not by $\psi_\sim$).

Formulas $\psi_{0\to 0}$, ψ_{points}, and ψ_δ can be easily expressed in AB, that is, equivalence relations play no role in the encoding of the corresponding conditions.

The main component of the encoding is the formula $\psi_\sim$, which acts like a controller of the form of configurations by preventing the addition of any unwanted counter-point. More precisely, it forces each configuration to be an isomorphic copy of the previous configuration, that is, to feature the same counter-points in the same order, plus at most one extra point with label *plus*.

We now show how to express each of above conditions in $\mathsf{AB}{\sim_1}{\sim_2}$. To facilitate the reading of the formulas, we make use of the following abbreviations: we denote the formula $\bigvee_{i=1}^{n} q_i$ by the symbol q and the formula $\bigvee_{i=1}^{2} c_i$ by the symbol c. Moreover, we define the following formula, parametric in φ:

$$succ(\varphi) \equiv \langle \mathsf{A} \rangle (unit \wedge \langle \mathsf{A} \rangle (\pi \wedge \varphi))$$

which states the truth of φ at the point immediately after the current interval, that is, at the successor of the right endpoint of the current interval. Finally, we introduce a derived modality $\langle \mathsf{P} \rangle$, which is defined in terms of modalities $\langle \mathsf{A} \rangle$ and $\langle \mathsf{B} \rangle$, that allows one to force a given formula φ to be true at some point of the current interval, endpoints included. The modality $\langle \mathsf{P} \rangle$ is formally defined as follows:

$$\langle \mathsf{P} \rangle \varphi \equiv \langle \mathsf{B} \rangle \langle \mathsf{A} \rangle (\pi \wedge \varphi) \vee \langle \mathsf{A} \rangle (\pi \wedge \varphi).$$

The dual of the above modality is defined as usual, that is, $[\mathsf{P}]\varphi \equiv \neg \langle \mathsf{P} \rangle \neg \varphi$, and it states that φ holds at each point of the current interval (endpoints included).

The initial and final configurations are encoded by the formula:

$$\psi_{0 \to 0} \equiv q_0 \wedge succ(q) \wedge \langle \mathsf{A} \rangle \langle \mathsf{A} \rangle q_f \wedge [\mathsf{A}][\mathsf{A}](q_f \to succ([\mathsf{A}][\mathsf{A}](\neg q \wedge (c \to del)))).$$

It states that the initial configuration is $(q_0, 0, 0)$ (the first state-point q_0 is followed by a state-point) and that the final configuration is $(q_f, 0, 0)$ (the last state-point is q_f and every counter-point after it, if any, is deleted).

The formula ψ_{points} is defined as the conjunction of the following conditions (hereafter, we explicitly provide the encoding of the most complex conditions only):

(A1) every point of the domain has one and only one label from the set $Q \cup \{c_1, c_2\}$;

(A2) labels in $Q \cup \{c_1, c_2\} \cup \{plus, minus, last, del\}$ are given to points only;

(A3) at most one counter-point per configuration can be labeled with *plus* and at most one with *minus*;

(A4) the label *last* is associated with the points of the final configuration only:

$$(last \to \pi) \wedge (last \leftrightarrow [\mathsf{A}](\neg \pi \to [\mathsf{A}]\neg q));$$

(A5) only counter-points can be labeled with *del* and a point can not have both label *del* and label *plus* (or *minus*).

The formula ψ_δ is defined as the conjunction of the following conditions:

(B1) all configurations but the initial one (and only them) have one (and only one) label in $\delta = \{\delta_1, ..., \delta_m\}$. We label the first configuration with a dummy transition δ_0. If a configuration is labeled with δ_i, for some $i > 0$, it means that it is obtained from the previous configuration by the application of the transition δ_i;

(B2) transition labels are consistent with state-points of each configuration:

$$\bigwedge_{\delta_i=(q,h{+}{+},q')\in\delta} (\delta \wedge succ(\langle \mathsf{A}\rangle \delta_i)) \rightarrow$$
$$(\langle \mathsf{B}\rangle q \wedge succ(q' \wedge \langle \mathsf{A}\rangle(\delta_i \wedge \langle \mathsf{P}\rangle(c_h \wedge plus))))$$
$$\wedge \bigwedge_{\delta_i=(q,h?0,q',q'')\in\delta} (\delta \wedge [\mathsf{P}](c_h \rightarrow del) \wedge succ(\langle \mathsf{A}\rangle \delta_i)) \rightarrow (\langle \mathsf{B}\rangle q \wedge succ(q'))$$
$$\wedge \bigwedge_{\delta_i=(q,h?0,q',q'')\in\delta} (\delta \wedge \langle \mathsf{P}\rangle(c_h \wedge \neg del) \wedge succ(\langle \mathsf{A}\rangle \delta_i)) \rightarrow$$
$$(\langle \mathsf{B}\rangle q \wedge succ(q'') \wedge \langle \mathsf{P}\rangle(c_h \wedge minus)),$$

where δ is a shorthand for the formula $\bigvee_{i=0}^{m} \delta_i$;

(B3) there are no points labeled with *plus* in configurations labeled with non-increment transitions or point labeled with *minus* in configurations that precede configurations labeled with non-decrement transitions;

(B4) every (non-final) configuration devoid of counter-points is followed by a configuration with at most one counter-point (labeled with *plus*).

In order to define the formula $\psi_\sim$, it turns out to be useful to introduce the following formula:

$$\psi_{\exists!q} \equiv ([\mathsf{B}](\langle \mathsf{A}\rangle q \rightarrow [\mathsf{B}]\langle \mathsf{A}\rangle \neg q) \wedge \langle \mathsf{B}\rangle\langle \mathsf{A}\rangle q \wedge \langle \mathsf{A}\rangle \neg q) \vee (\langle \mathsf{A}\rangle q \wedge [\mathsf{B}][\mathsf{A}]\neg q),$$

which guarantees the existence of a unique state-point inside the current interval (endpoints included).

We call an interval labeled with both $\sim_1$ and $\sim_2$ and containing exactly one state-point a *linking* interval, that is, a linking interval is an interval that satisfies the formula $\sim_1 \wedge \sim_2 \wedge \psi_{\exists!q}$, and we say that its endpoints are *linked* together (the use of linking intervals in the encoding will be explained in Section 6).

The formula $\psi_\sim$ is defined as the conjunction of the following conditions:

(C1) a unit interval can not be labeled with both $\sim_1$ and $\sim_2$:

$$unit \rightarrow \neg(\sim_1 \wedge \sim_2);$$

(C2) there is no interval, whose endpoints belong to the same configuration, which is neither a point interval nor a unit interval and is labeled with $\sim_1$ or $\sim_2$:

$$(\neg\pi \wedge \neg unit \wedge [\mathsf{P}]\neg q) \rightarrow \neg(\sim_1 \vee \sim_2);$$

(C3) two consecutive counter-points are in relation $\sim_1$ or $\sim_2$:

$$(unit \wedge \langle \mathsf{B} \rangle c \wedge \langle \mathsf{A} \rangle c) \rightarrow (\sim_1 \vee \sim_2)$$

(C4) each counter-point, which does not belong to the last configuration, starts a linking interval:

$$(c \wedge \neg last) \rightarrow \langle \mathsf{A} \rangle (\sim_1 \wedge \sim_2 \wedge \psi_{\exists ! q});$$

(C5) the labels of the endpoints of a linking interval must satisfy the following constraints:

$$\begin{aligned}(\sim_1 \wedge \sim_2 \wedge \psi_{\exists ! q}) \rightarrow \big(& ((\langle \mathsf{B} \rangle c_1 \wedge \langle \mathsf{A} \rangle c_1) \vee (\langle \mathsf{B} \rangle c_2 \wedge \langle \mathsf{A} \rangle c_2)) \\ & \wedge (\langle \mathsf{B} \rangle del \rightarrow \langle \mathsf{A} \rangle del) \\ & \wedge (\langle \mathsf{B} \rangle minus \rightarrow \langle \mathsf{A} \rangle del) \\ & \wedge (\langle \mathsf{B} \rangle (\neg minus \wedge \neg del) \rightarrow \langle \mathsf{A} \rangle \neg del) \\ & \wedge (\langle \mathsf{A} \rangle \neg plus) \big);\end{aligned}$$

(C6) the first point of all configurations, but the final one, is linked to the first point of the next configuration:

$$\begin{aligned}&(unit \wedge \langle \mathsf{B} \rangle (q \wedge \neg last) \wedge \langle \mathsf{A} \rangle c) \\ &\qquad \rightarrow \langle \mathsf{A} \rangle (\sim_1 \wedge \sim_2 \wedge \psi_{\exists ! q} \wedge [\mathsf{B}](\langle \mathsf{A} \rangle q \vee [\mathsf{P}] \neg q)).\end{aligned}$$

We would like to observe that the formula $[\mathsf{B}](\langle \mathsf{A} \rangle q \vee [\mathsf{P}] \neg q))$ enforces the second to last point of the linking interval to be the only state-point. Let $[x, y]$ be this interval. If y', with $x < y' < y - 1$, were a state-point (it can not be the final point y since $[x, y]$ is a linking-interval, and, by C5, linking intervals have counter-points as their endpoints), then $[x, y - 1]$ would satisfy neither $\langle \mathsf{A} \rangle q$ (there can only be one state-point between x and y) nor $[\mathsf{P}] \neg q$ (since y' is between x and $y - 1$ and it is a state-point);

(C7) the last point of every configuration, but the final one, is linked to a point that is followed by a state-point or by a counter-point with label *plus* followed by a state-point:

$$\begin{aligned}(\delta_i \wedge \langle \mathsf{B} \rangle \neg last) \rightarrow \langle \mathsf{A} \rangle \big(& \sim_1 \wedge \sim_2 \wedge \psi_{\exists ! q} \wedge \\ & (succ(q) \vee succ(plus \wedge succ(q))) \big).\end{aligned}$$

We would like to emphasize the crucial role of condition (C5). First of all, it constrains linking intervals to connect counter-points with the same label (either c_1 or c_2). Moreover, it transfers logical deletion (counter-points labeled by *del*) from one configuration to the next one, it forces the actual execution of a new deletion by connecting a counter-point labeled by *minus* to a counter-point labeled by *del*, and it prevents unwanted deletions or insertions to take place.

6 Proof of correctness of the encoding

The proof of correctness for the formulas $\psi_{points}, \psi_\delta$, and $\psi_{0\to 0}$ is straightforward. The only thing that we really need to show is that the behavior of the formula $\psi_\sim$ is actually the one we described in Section 5, that is, we need to prove that $\psi_\sim$ forces each configuration to be the result of the application of a transition of a (generic) counter machine to the previous configuration. To this end, we show that each configuration τ_{i+1} contains an exact copy of the counter-points of the previous configuration τ_i in its initial part plus possibly an additional point labeled with *plus* at the end (if it is obtained from the previous configuration by an increment transition).

Let $\mathcal{S} = (\mathbb{I}(N), A, B, \sim_1, \sim_2, V)$ be a model of ψ_{M,q_0,q_f} and $q^0, q^1, \dots, q^m$ be the enumeration of its state-points according to the order of the domain (therefore, by $\psi_{0\to 0}$, $q^0 = q_0$ and $q^m = q_f$). The configuration τ_i is defined as the set of points from q^i (included) to q^{i+1} (excluded). τ_m consists of q^m and the points that follows it, till the end of the domain $\mathbb{I}(N)$. We denote by $\sim_k$ any of the relations $\sim_1$ or $\sim_2$. If τ is an ordered set of points, we write τ^j to denote the j-th point of τ.

Let f be the set of all and only those pairs of points, belonging to two consecutive configurations τ_i and τ_{i+1}, which are connected by a linking interval, that is, $(x, y) \in f$ if and only if there exist τ_i, τ_{i+1} such that $x \in \tau_i$, $y \in \tau_{i+1}$, $x \sim_1 y$, and $x \sim_2 y$.

First of all, we observe that for each counter-point x in τ_i there exists a counter-point y in τ_{i+1} such that $(x, y) \in f$ (by (C4)). Moreover, by condition (C5), every pair in f consists of counter-points with the same label c_k. We now prove that f is (the map of) an injective function that preserves adjacency between points, that is, if x, x' are consecutive counter-points, then $f(x), f(x')$ are consecutive counter-points as well:

- f is a function: suppose that there exist y, y', with $y < y'$, in τ_{i+1} which have the same counter-image x, with $x \in \tau_i$, in f, then $y \sim_1 x \sim_1 y'$ and $y \sim_2 x \sim_2 y'$, and thus $y \sim_1 y'$ and $y \sim_2 y'$, which violates condition (C1) or condition (C2), depending on the distance between y and y' (if $|y' - y| = 1$, then it violates (C1); otherwise, it violates (C2));
- f is injective: the proof is similar to the one given for the previous point. Suppose that there exist x, x', with $x < x'$, in τ_i which have the same image y, with $y \in \tau_{i+1}$, in f, then $x \sim_1 y \sim_1 x'$ and $x \sim_2 y \sim_2 x'$, and thus $x \sim_1 x'$ and $x \sim_2 x'$, that violates condition (C1) or condition (C2);
- f preserves adjacency: let x, x' be two consecutive points in τ_i, that is, x' is the successor of x, and let y and y' be their respective images (since f is a injective function, it immediately follows that they are unique). By condition (C3), it holds that $\sim_k \in V([x, x'])$, and thus $y \sim_k x \sim_k x' \sim_k y'$ and $\sim_k \in V([y, y'])$, which, by condition (C2), implies that y, y' are consecutive.

By the properties of f and condition (C6), it follows that $f(\tau_i^j) = \tau_{i+1}^j$ for $j = 2, \dots, |\tau_i|$ (order preservation). A graphical account of the relationships between pairs of counter-points belonging to two consecutive configurations is given in

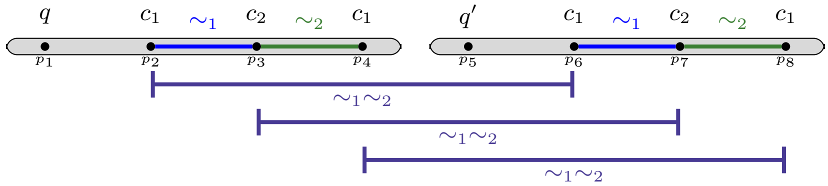

Fig. 2. Correspondence between counter-points of two consecutive configurations.

Figure 2. Moreover, thanks to condition (C7), the inequality $|\tau_i| \leq |\tau_{i+1}| \leq |\tau_i|+1$ holds, and thus the possible extra point of τ_{i+1} must have label *plus*. By this fact and the consistency of the labels *plus*, *minus*, and *del* guaranteed by condition (C5), it follows that τ_{i+1} is obtained from τ_i by an increment transition (if it has an extra point labeled with *plus*) or by a decrement transition (if it has no extra point), as desired.

It is worth emphasizing that the role of $\psi_\sim$ is not to guarantee that the transition applied to τ_i to obtain τ_{i+1} is a valid transition for M (which is the job of ψ_δ), but to ensure that τ_{i+1} *could* be obtained from τ_i by a transition of a counter machine.

7 Conclusions

In [15], Montanari and Sala studied complexity and expressiveness of the interval temporal logic AB$\sim$, that extends AB with an equivalence relation. Complexity and (un)decidability results are given by means of suitable reductions from reachability problems for lossy counter machines. The resulting picture is as follows: one gets decidability over finite linear orders with nonprimitive recursive complexity and undecidability over $\mathbb{N}$. In addition, they showed that decidability can be recovered by suitably restricting the class of models over which AB$\sim$ formulas are interpreted. In this paper, we proved that decidability of finite satisfiability is lost when two or more equivalence relations are added to AB.

As for future work, in analogy to what Montanari and Sala did for AB$\sim$ over the natural numbers, we are thinking of possible ways of restricting the class of models over which AB$\sim_1\sim_2$ formulas are interpreted in order to recover finite satisfiability. We are also studying the effects of the addition of two or more equivalence relations to other interval logics, such as (metric) propositional neighborhood logic, which preserves decidability when extended with one equivalence relation only [13].

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