A graph-easy class of mute lambda-terms

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Abstract. Among the unsolvable terms of the lambda calculus, the *mute* (or *root-active*) ones are those having the highest degree of undefinedness. In this paper, we define an infinite set S of mute terms, and show that it is *graph-easy*: for any closed term t of the lambda calculus there exists a graph model equating all the terms of S to t.
Keywords: Lambda-calculus, mute terms, graph models, forcing, ultraproducts.

1 Introduction

It is a well known result by Jacopini [15] that Ω can be consistently equated to any closed term t of the (untyped) lambda-calculus, where Ω is the paradigmatic unsolvable term $(\lambda x.xx)(\lambda x.xx)$ (this is called *the easiness of* Ω). Baeten and Boerboom [3] gave the first semantic proof of this result by showing that for all closed terms t one can build a graph model satisfying the equation $\Omega = t$. This semantic result extends to other classes of models and to some other terms which share with Ω enough of its good will (cf. [7] for a survey of such results).

Mute lambda terms have been introduced by Berarducci [5], for defining models of the lambda calculus that do not identify all the unsolvable terms. Mute terms are somehow the "most undefined" lambda terms, as they are unsolvable of order 0 (zero terms), which are not β-convertible to a zero term applied to something else. For instance, Ω is mute, and $\Omega_3 = (\lambda x.xxx)(\lambda x.xxx)$ is a zero term that is not mute, since it reduces to $\Omega_3(\lambda x.xxx)$.

Berarducci proved that the set of mute terms is easy, in the sense that it is consistent with the lambda calculus to simultaneously equate all the mute terms to a fixed arbitrary closed term. Hereafter, a set of lambda terms that can be simultaneously, consistently equated to a fixed arbitrary closed term is called an *easy set*.

Given a class $\mathcal{C}$ of models of the lambda calculus, and an easy set S, we say that S is $\mathcal{C}$-easy if, for every closed term t, there exists a model in $\mathcal{C}$ that equates all the terms in S to t.

Studying $\mathcal{C}$-easiness gives insights on the expressive power of the class $\mathcal{C}$. Concerning filter lambda models, for instance, it had been conjectured [2] that they have full expressive power for singletons, in the sense that any easy singleton set is filter-easy. Carraro and Salibra [13] showed that this is not the case: there exists a co-r.e. set of easy terms that are not filter-easy. The first negative semantic result was obtained by Kerth [19]: $\Omega_3 I$, where $I = \lambda x.x$, is an easy term, but no graph model satisfies the identity $\Omega_3 I = I$. This result shows a limitation of graph models. The easiness of $\Omega_3 I$ was proven syntactically in [16] (see also [6]), but it was only given a semantic proof in [1], where the authors build, for each closed t, a filter model of $\Omega_3 I = t$.

Graph models are arguably the simplest models of the lambda calculus. There are two known methods for building graph models, namely: by *forcing* or by *canonical completion*. Both methods consist in completing a partial model into a total one.

The canonical completion method was introduced by Plotkin and Engeler and then systematized by Longo [21] for graph models. The word "canonical" refers here to the fact that the graph model is built inductively from the partial one and completely determined by it. This method was then used by Kerth [18] to prove the existence of 2^ω pairwise inconsistent graph theories, and by Bucciarelli–Salibra [11, 9, 10] to characterize minimal and maximal graph theories. In particular [11] shows that the minimal graph theory is not equal to the minimal lambda theory $\lambda\beta$, and that the lambda theory $\mathcal{B}$ (generated by equating lambda terms with the same Böhm tree) is the greatest sensible graph theory.

The forcing method originates with Baeten–Boerboom [3], and it is more flexible than canonical completions. In fact, the inductive construction depends here not only on the initial partial model but also on the consistency problem one is interested in. The method was afterwards generalized to other classes of webbed models by Jiang [17] and Kerth [20]. It was also generalized to families of terms similar to Ω by Zylberajch [23] and Berline–Salibra [8].

One more difference between these methods is that if we start with a recursive partial web, the canonical completion builds a recursive total web, while forcing always generates a non recursive web.

In this paper we define an infinite and recursive set of mute terms, the *regular mute* terms. A regular mute term has the form $s_0 s_1 \dots s_n$, for some n, and it has the property that, in n steps of head reduction, it reduces to a term of the same shape $t_0 t_1 \dots t_n$, where $t_0 = s_i$ for some $1 \leq i \leq n$. As regular mute terms are mute, we know that the set of all regular mute terms is easy, since each subset of an easy set is itself easy. We show that it is actually graph-easy by generalizing the forcing technique used in [8].

More precisely, given a closed λ-term t and a finite set $\{n_1, \dots, n_k\}$ of natural numbers, we construct a graph model which equates to t all the regular mute terms of the form $s_0 s_1 \dots s_{n_j}$, $1 \leq j \leq k$, using forcing.

Then we glue together these graph models in an ultraproduct, using a technique introduced in [12]. This gives rise to a graph model that is an expansion of the ultraproduct, where all the regular mute terms are equated to t, thus concluding the proof that the set of regular mute terms is graph-easy.

2 Theories and models of λ-calculus

With regard to the lambda-calculus we follow the notation and terminology of [4]. By Λ and Λ^o, respectively, we indicate the set of λ-terms and of closed λ-

terms. We denote $\alpha\beta$-conversion by $\lambda\beta$. A λ-*theory* is a congruence on Λ (with respect to the operators of abstraction and application) which contains $\lambda\beta$. A λ-theory is *consistent* if it does not equate all λ-terms, *inconsistent* otherwise.

It took some time, after Scott gave his model construction, for consensus to arise on the general notion of a model of the λ-calculus. There are mainly two descriptions that one can give: the category-theoretical and the algebraic one. The categorical notion of model, that of reflexive object in a Cartesian closed category (ccc), is well-suited for constructing concrete models, while the algebraic one is rather used to understand global properties of models (constructions of new models out of existing ones, closure properties, etc.) and to obtain results about the structure of the lattice of λ-theories. The algebraic description of models of λ-calculus proposes two kinds of structures, viz. the λ-*algebras* and the λ-*models*, both based on the notion of *combinatory algebra*. We will focus on λ-models.

A *combinatory algebra* $\mathbf{A} = (A, \cdot, \mathbf{k}, \mathbf{s})$ is a structure with a binary operation called *application* and two distinguished elements $\mathbf{k}$ and $\mathbf{s}$ called *basic combinators*. The symbol "$\cdot$" is usually omitted from expressions and by convention application associates to the left, allowing to leave out superfluous parentheses. The class of combinatory algebras is axiomatized by the equations $\mathbf{k}xy = x$ and $\mathbf{s}xyz = xz(yz)$. A function $f : A \to A$ is *representable* in $\mathbf{A}$ if there exists an element $a \in A$ such that $f(b) = ab$ for all $b \in A$. For example, the identity function is represented by the combinator $\mathbf{i} = \mathbf{skk}$.

The axioms of an elementary subclass of combinatory algebras, called λ-*models*, were expressly chosen to make coherent the interpretation of the λ-terms (see Barendregt [4, Def. 5.2.7]). In addition to five axioms due to Curry (see [4, Thm. 5.2.5]), the *Meyer-Scott axiom* is the most important one in the definition of a λ-model. In the first-order language of combinatory algebras it is formulated as $\forall xy.(\forall z.\ xz = yz) \Rightarrow \boldsymbol{\varepsilon} x = \boldsymbol{\varepsilon} y$, where the combinator $\boldsymbol{\varepsilon} = \mathbf{s}(\mathbf{ki})$ is made into an inner choice operator. Indeed, given any a, the element $\boldsymbol{\varepsilon} a$ represents the same function as a; by the Meyer-Scott axiom, $\boldsymbol{\varepsilon} c = \boldsymbol{\varepsilon} d$ for all c, d representing the same function.

Given a set A, we denote by Env_A the set of A-environments, i.e., the functions from the set Var of λ-calculus variables to A. For every $x \in \mathrm{Var}$ and $a \in A$ we denote by $\rho[x := a]$ the environment ρ' which coincides with ρ everywhere except on x, where ρ' takes the value a.

Given a λ-model $\mathbf{A}$, the interpretation $|t|^{\mathbf{A}} : Env_A \to A$ of a λ-term is defined by induction on the complexity of t in such a way that

$$|x|^{\mathbf{A}}_{\rho} = \rho(x); \qquad |tu|^{\mathbf{A}}_{\rho} = |t|^{\mathbf{A}}_{\rho}|u|^{\mathbf{A}}_{\rho}; \qquad |\lambda x.t|^{\mathbf{A}}_{\rho} = \boldsymbol{\varepsilon} b$$

where b is any element satisfying $ba = |t|^{\mathbf{A}}_{\rho[x:=a]}$ for every $a \in A$.

It is important to stress that the class of λ-models is axiomatized by first-order axioms expressed in terms of Horn formulas, so that it is closed under direct products; it is not axiomatized by equations only, so that it is not closed neither under substructures nor under homomorphic images.

3 Graph models

The class of graph models belongs to Scott's continuous semantics. Graph models owe their name to the fact that continuous functions are encoded in them via (a sufficient fragment of) their graphs, namely their traces.

A *graph model* is a model of untyped λ-calculus, which is generated from a web in a way that will be recalled below. Historically, the first graph model was Plotkin and Scott's P_ω (see e.g. [4]), which is also known in the literature as "the graph model". The simplest graph model, $\mathcal{E}$, was introduced soon afterwards, and independently, by Engeler [14] and Plotkin [22]. More examples can be found in [7].

As a matter of notation, we denote by D^* the set of all finite subsets of a set D. Elements of D^* will be denoted by small roman letters $a, b, c, \ldots$, while elements of D by greek letters $\alpha, \beta, \gamma, \ldots$.

For short we will confuse the model and its web and so we define:

Definition 1. *A graph model is a pair (D, p), where D is an infinite set and $p : D^* \times D \to D$ is an injective total function.*

Such a pair will also be called a *total pair*. In the setting of graph models a *partial pair* (see [7]) is a pair (A, q) where A is any set and $q : A^* \times A \rightharpoonup A$ is a partial (possibly total) injection. Examples of partial pairs are: the empty pair $(\emptyset, \emptyset)$ and all the graph models.

If (D, p) is a partial pair, we write $a \to_p \alpha$ (or $a \to \alpha$ if p is evident from the context) for $p(a, \alpha)$. Moreover, $\beta \to \alpha$ means $\{\beta\} \to \alpha$. $a_1 \to a_2 \to \cdots \to a_{n-1} \to a_n \to \alpha$ stands for $(a_1 \to (a_2 \to \ldots (a_{n-1} \to (a_n \to \alpha)) \ldots))$. If $\bar{a} = a_1, a_2, \ldots, a_n$, then $\bar{a} \to \alpha$ stands for $(a_1 \to (a_2 \to \ldots (a_{n-1} \to (a_n \to \alpha)) \ldots))$.

A total pair (D, p) generates a *λ-model* of universe $\mathcal{P}(D)$, called *graph λ-model*. In particular $\mathcal{P}(D)$ is endowed with an application operator that makes it a λ-model. The interpretation $|t|^p : Env_{\mathcal{P}(D)} \to \mathcal{P}(D)$ of a λ-term t relative to (D, p) can be described inductively as follows (see Section 2):

- $|x|^p_\rho = \rho(x)$
- $|tu|^p_\rho = \{\alpha : (\exists a \subseteq |u|^p_\rho) \;\; a \to \alpha \in |t|^p_\rho\}$
- $|\lambda x.t|^p_\rho = \{\, a \to \alpha : \alpha \in |t|^p_{\rho[x:=a]}\}$

Since $|t|^p_\rho$ only depends on the value of ρ on the free variables of t, we only write $|t|^p$ if t is closed.

A graph model (D, p) *satisfies* $t = u$, written $(D, p) \models t = u$, if $|t|^p_\rho = |u|^p_\rho$ for all environments ρ. The λ-theory $Th(D, p)$ induced by (D, p) is defined as

$$Th(D, p) = \{t = u : t, u \in \Lambda \text{ and } |t|^p = |u|^p\}.$$

A λ-theory induced by a graph model will be called a *graph theory*.

4 The regular mute λ-terms

A first step towards the definition of regular mute terms are the hereditarily n-ary terms, defined below.

Definition 2. *Let $n > 0$ and $\bar{x} \equiv x_1, \dots x_k$ be distinct variables. The set of hereditarily n-ary λ-terms over $\bar{x}$, written $H_n[\bar{x}]$, is the smallest set of λ-terms containing $x_1, \dots, x_k$ and satisfying the following property, for all fresh distinct variables $\bar{y} \equiv y_1, \dots, y_n$ and all terms $t_1, \dots, t_n$:*

$$t_1, \dots, t_n \in H_n[\bar{x}, \bar{y}] \ \Rightarrow \ \lambda\bar{y}.y_i t_1 \dots t_n \in H_n[\bar{x}].$$

We write H_n for $H_n[\,]$.

Example 1. Some unary and binary hereditary λ-terms:

- $\lambda x.xx \in H_1$
- $\lambda y.yx \in H_1[x]$
- $\lambda x.x(\lambda y.yx) \in H_1$
- $\lambda zy.yzx \in H_2[x]$
- $\lambda xy.x(\lambda zy.yzx)y \in H_2$.

Given a natural number n and variables $\bar{x}$ we define inductively an increasing sequence of sets of λ-terms, starting at $H_n[\bar{x}]$:

Definition 3. *Let $\bar{x} \equiv x_1, \dots x_k$ and $\bar{y} \equiv y_1, \dots, y_n$ be distinct fresh variables.*

- $H_n^0[\bar{x}] = H_n[\bar{x}]$
- $H_n^{m+1}[\bar{x}] = \{s[\overline{u}/\overline{y}] : s \in H_n^m[\bar{x}, \bar{y}], \bar{u} \equiv u_1, \dots, u_n \in H_n^m[\bar{x}]\}$
- $S_n[\bar{x}] = \bigcup_m H_n^m[\bar{x}]$.

We write S_n for $S_n[\,]$. For $t \in S_n[\bar{x}]$, we denote by $rk(t)$ the smallest number such that $t \in H_n^{rk(t)}[\bar{x}]$.

Lemma 1. *If $\bar{y}$ is a sequence of n distinct variables, $s \in S_n[\bar{x}, \bar{y}]$ and $\bar{t} \equiv t_1, \dots, t_n \in S_n[\bar{x}]$, then $s[\bar{t}/\bar{y}] \in S_n[\bar{x}]$.*

Lemma 2. *Let t be a λ-term. Then $t \in H_n^m[\bar{x}]$ if, and only if, there exist*

- *$s \in H_n^0[\bar{x}, \bar{z}^1, \dots, \bar{z}^m]$,*
- *sequences $\bar{z}^i$ $(i = 1, \dots, m)$ of n distinct variables,*
- *sequences $\bar{t}^i$ $(i = 1, \dots, m)$ of n terms $\bar{t}^i \equiv t_1^i, \dots, t_n^i \in H_n^{m-i}[\bar{x}, \bar{z}^1, \dots, \bar{z}^{i-1}]$*

such that $t \equiv s[\overline{t^m}/\overline{z^m}] \cdots [\overline{t^1}/\overline{z^1}]$.

Proof. Just an unfolding of the previous definition.

Proposition 1. *For all $n > 0$, $s_0, \dots, s_n \in S_n$, there exist $r_0, \dots, r_n \in S_n$ and $i \leq n$ such that*

$$s_0 s_1 \dots s_n \to_\beta^n r_0 r_1 \dots r_n \ \textit{and} \ r_0 \equiv s_i$$

Proof. (1) $rk(s_0) = 0$.

Since $s_0 \in H_n$, then $s_0 \equiv \lambda y_1 \dots y_n.y_i r_1 \dots r_n$ with $r_1, \dots, r_n \in H_n[y_1, \dots, y_n]$. Hence $s_0 s_1 \dots s_n \to_\beta^n s_i r_1[\bar{s}/\bar{y}] \dots r_n[\bar{s}/\bar{y}]$. By Lemma 1 the term $r_i[\bar{s}/\bar{y}] \in S_n$, and we are done.

(2) $rk(s_0) = m > 0$.

By Lemma 2 there exists $u \in H_n[\bar{z}^1, \dots, \bar{z}^m]$ such that $s_0 \equiv u[\bar{t}^m/\bar{z}^m] \dots [\bar{t}^1/\bar{z}^1]$, for some terms $\bar{t}^i \in H_n^{m-i}[\bar{z}^1, \dots, \bar{z}^{i-1}]$, for $1 \le i \le m$. The term u cannot be a variable because of the rank of s_0. Then by definition $u \equiv \lambda\bar{y}.y_i u_1 \dots u_n$ with $u_i \in H_n[\bar{z}^1, \dots, \bar{z}^m, \bar{y}]$. Then

$$s_0 = \lambda\bar{y}.y_i(u_1[\bar{t}^m/\bar{z}^m] \dots [\bar{t}^1/\bar{z}^1]) \dots (u_n[\bar{t}^m/\bar{z}^m] \dots [\bar{t}^1/\bar{z}^1])$$

and, if $\bar{s} = s_1, \dots, s_n$

$$s_0 s_1 \dots s_n \to_\beta^n s_i(u_1[\bar{t}^m/\bar{z}^m] \dots [\bar{t}^1/\bar{z}^1][\bar{s}/\bar{y}]) \dots (u_n[\bar{t}^m/\bar{z}^m] \dots [\bar{t}^1/\bar{z}^1][\bar{s}/\bar{y}]).$$

Theorem 1. *For all $s_0, \dots, s_n \in S_n$, the term $s_0 s_1 \dots s_n$ is mute.*

Hereafter, a term $s_0 s_1 \dots s_n$ ($s_i \in S_n$) is called a *n-regular mute term*; $\mathcal{M}_n$ will denote the set of all n-regular mute terms.

Example 2. Some unary and binary regular mute terms:

- $(\lambda x.xx)(\lambda x.xx) \in \mathcal{M}_1$
- $(\lambda x.x(\lambda y.yx))(\lambda x.xx) \in \mathcal{M}_1$
- $AAA \in \mathcal{M}_2$, where $A := \lambda xy.x(\lambda zt.tzx)y$.

Example 3. Let $B := \lambda x.x(\lambda y.xy)$. Then BB is a mute term that is not regular:

$$BB = (\lambda x.x(\lambda y.xy))B \to_\beta B(\lambda y.By) \to_\beta BB$$

5 Forcing for regular mute terms

In this section we show that, given a closed λ-term t and a finite set $\{n_1, \dots, n_k\}$ of natural numbers, there exists a graph model which equates all the regular mute terms of the form $s_0 s_1 \dots s_{n_j}$, $1 \le j \le k$, to t, using forcing.

5.1 Some useful lemmas

Lemma 3. *Let (D, p) be a graph model, ρ be D-environment and $\bar{\beta} = \beta, \beta, \dots, \beta$ (n-times). If $\beta = \bar{\beta} \to \alpha$, $t \in S_n[\bar{x}]$ and $\beta \in \rho(x_i)$ $(i = 1, \dots, k)$ then $\beta \in |t|^p_\rho$.*

Proof. Base case: $t \in H_n[\bar{x}]$. Let $\bar{u} \in H_n[\bar{x}, \bar{y}]$ and $z \in \{\bar{x}, \bar{y}\}$ such that $t = \lambda\bar{y}.z\bar{u}$.

$$\beta = \bar{\beta} \to \alpha \in |\lambda\bar{y}.z\bar{u}|^p_\rho \;\Leftrightarrow\; \alpha \in |z\bar{u}|^p_{\rho[\bar{y}:=\bar{\beta}]}$$

Since $\beta \in \rho[\bar{y} := \bar{\beta}]$ and by induction hypothesis $\beta \in |u_i|^p_{\rho[\bar{y}:=\bar{\beta}]}$, then $\alpha \in |z\bar{u}|^p_{\rho[\bar{y}:=\bar{\beta}]}$.

Let $t \in H_n^{m+1}[\bar{x}]$. Then $t \equiv s[\overline{u}/\overline{y}]$, where $s \in H_n^m[\bar{x}, \bar{y}]$ and $\bar{u} \equiv u_1, \dots, u_n \in H_n^m[\bar{x}]$. By induction hypothesis we have $\beta \in |u_i|^p_\rho$. Since $|s[\overline{u}/\overline{y}]|^p_\rho = |s|^p_{\rho[\bar{y}:=|\bar{u}|^p_\rho]}$ and $\beta \in \rho[\bar{y} := |\bar{u}|^p_\rho](y_i)$, then by induction hypothesis $\beta \in |s|^p_{\rho[\bar{y}:=|\bar{u}|^p_\rho]}$ and we get the conclusion.

Lemma 4. *Let (D, p) be a graph model, $s_0^0 s_1^0 \dots s_n^0 \in \mathcal{M}_n$ ($s_i^0 \in S_n$) and $\gamma \in |s_0^0 s_1^0 \dots s_n^0|^p$. Then there exist a sequence $\beta_i \equiv a_1^i \to \dots \to a_n^i \to \gamma$ ($i \in \omega$) of elements of D and a sequence d_i ($i \in \omega$) of natural numbers $\leq n$ such that $\beta_{i+1} \in a^i_{d_i}$.*

Proof. By Proposition 1 there exists an infinite sequence of mute terms such that

$$s_0^0 s_1^0 \dots s_n^0 \ \to_\beta^n \ s_0^1 s_1^1 \dots s_n^1 \ \to_\beta^n \ \dots \ \to_\beta^n \ s_0^k s_1^k \dots s_n^k \ \to_\beta^n \dots$$

and $s_0^k \equiv s^{k-1}_{d_{k-1}}$ for some $1 \leq d_{k-1} \leq n$. The number d_{k-1} is the order of the head variable of the term s_0^{k-1}. By $\gamma \in |s_0^0 s_1^0 \dots s_n^0|^p$ there exists $a_1^0 \to \dots \to a_n^0 \to \gamma \in |s_0^0|^p$ such that $a_i^0 \subseteq |s_i^0|^p$. We define

$$\beta_0 = a_1^0 \to \dots \to a_n^0 \to \gamma.$$

Assume $\beta_k = a_1^k \to \dots \to a_n^k \to \gamma \in |s_0^k|^p$ and $a_j^k \subseteq |s_j^k|^p$ for every $j \leq n$.

Since $s_0^k = \lambda \bar{y}.y_{d_k} u_1 \dots u_n$ for some terms u_i and $\beta_k \in |s_0^k|^p$, then, if $\bar{a} = a_1^k, \dots, a_n^k$

$$\gamma \in a^k_{d_k}(u_1[\bar{a}/\bar{y}]) \dots (u_n[\bar{a}/\bar{y}]).$$

Then there exists $\beta_{k+1} = a_1^{k+1} \to \dots \to a_n^{k+1} \to \gamma \in a^k_{d_k} \subseteq |s_0^{k+1}|^p = |s^k_{d_k}|^p$ and $a_j^{k+1} \subseteq |s_j^{k+1}|^p$.

5.2 Forcing at work

We recall the notion of weakly continuous operator from [8].

Definition 4. *Let D be an infinite countable set. By $\mathcal{I}(D)$ we indicate the cpo of partial injections $q : D^* \times D \rightharpoonup D$, ordered by inclusion of their graphs.*

By a "*total p*" we will mean "an element of $\mathcal{I}(D)$ which is a total map" (equivalently: which is a maximal element of $\mathcal{I}(D)$). The domain and range of $q \in \mathcal{I}(D)$ are denoted by $dom(q)$ and $rg(q)$. We will also confuse the partial injections and their graphs.

Definition 5. *A function $F : \mathcal{I}(D) \to \mathcal{P}(D)$ is weakly continuous if it is monotone with respect to inclusion and if furthermore, for all total $p \in \mathcal{I}(D)$,*

$$F(p) = \bigcup_{q \subseteq_{\text{fin}} p} F(q).$$

Let $p \in \mathcal{I}(D)$. The universe $U(p)$ of p is defined as follows:

$$U(p) = \bigcup_{(a,\alpha)\in dom(p)} (a \cup \{\alpha, p(a,\alpha)\}).$$

If p is finite, then the universe of p is also finite.

Let $p \in \mathcal{I}(D)$ be finite, $\alpha \in D$, $\bar{\epsilon} \equiv \epsilon_1, \dots, \epsilon_k \in D \setminus U(p)$ and $k \in \mathbb{N}$. Then we denote $p_{\bar{\epsilon},\alpha}$ the extension of p such that

$$\epsilon_2 = \epsilon_1 \to \alpha; \qquad \epsilon_{j+1} = \epsilon_1 \to \epsilon_j \ (j = 2, \dots, k-1); \qquad \epsilon_1 = \epsilon_1 \to \epsilon_k.$$

Notice that

$$\epsilon_1 = \epsilon_1 \to \epsilon_1 \to \cdots \to \epsilon_1 \to \alpha \quad (k\text{-times})$$

and $p_{\bar{\epsilon},\alpha}$ is also finite.

Let $e \subseteq_{\text{fin}} \mathbb{N}$. We let $\mathcal{M}_e = \bigcup_{i\in e} \mathcal{M}_i$ the set of n-regular mute terms for $n \in e$.

The next theorem is the main technical tool for proving the easiness of the full set of n-regular mute terms. It generalizes [8, Thm. 11].

Theorem 2. *Let $F : \mathcal{I}(D) \to \mathcal{P}(D)$ be a weakly continuous function and let $e \subseteq_{\text{fin}} \mathbb{N}$. Then there exists a total $p_e : D^* \times D \to D$ such that $(D, p_e) \models t = F(p_e)$ for all terms $t \in \mathcal{M}_e$.*

Proof. We are going to build an increasing sequence of finite injective maps p_n, starting from $p_0 = \emptyset$, and a sequence of elements $\alpha_n \in D \cup \{*\}$, where $*$ is a new element, such that: $p_e =_{def} \cup p_n$ is a total injection, and $(D, p_e) \models t = A = F(p_e)$ for all $t \in \mathcal{M}_e$, where $A =_{def} \{\alpha_n : n \in \omega\} \cap D$.

We fix an enumeration of D and an enumeration of $D^* \times D$.

We start from $p_0 = \emptyset$.

Assume that p_n and $\alpha_0, \dots, \alpha_{n-1}$ have been built. We let

- $\alpha_n =$ First element of $F(p_n) \setminus \{\alpha_0, \dots, \alpha_{n-1}\}$ in the enumeration of D, if this set is non-empty, and $\alpha_n = *$ otherwise;
- $(b_n, \delta_n) =$ "the first element in $(D^* \times D) \setminus dom(p_n)$";
- $\gamma_n =$ "the first element in $D \setminus (U(p_n) \cup b_n \cup \{\delta_n\} \cup \{\alpha_0, \dots, \alpha_{n-1}, \alpha_n\})$".

Let $r = p_n \cup \{\gamma_n = b_n \to_r \delta_n\}$.

Case 1: $\alpha_n = *$. We let $p_{n+1} = r$.

Case 2: $\alpha_n \in D$.

Let $e = \{k_1, \dots, k_m\}$. We define $q_0 \subseteq q_1 \subseteq \cdots \subseteq q_m \in \mathcal{I}(D)$ as follows: $q_0 = r$ and $p_{n+1} = q_m$. Assume we have defined q_i. We define $q_{i+1} = (q_i)_{\bar{\epsilon}^{n,i},\alpha_n}$ (see above), where

$$\bar{\epsilon}^{n,i} \equiv \epsilon_1^{n,i}, \dots, \epsilon_{k_{i+1}}^{n,i} \in D \setminus (U(q_i) \cup \{\alpha_n\})$$

are distinct elements.

It is clear that p_n is a strictly increasing sequence of well-defined finite injective maps and that $p_e = \cup p_n$ is total.

It is also clear that each p_n (and p_e) is partitioned into two disjoint sets: $p_n = p_n^1 \cup p_n^2$, where $p_n^1 = \{b_i \to \delta_i = \gamma_i : 1 \leq i \leq n-1\}$ is called the gamma part of p_n and $p_n^2 = p_n \setminus p_n^1$ is called the epsilon part.

For every $\gamma \in D$, we define

$$deg(\gamma) = \begin{cases} 0 & \text{if } \gamma \notin rg(p_e) \\ min\{n : \gamma \in rg(p_n)\} & \text{if } \gamma \in rg(p_e) \end{cases}$$

Moreover, $deg(c) = max\{deg(x) : x \in c\}$ for every $c \subseteq_{\text{fin}} D$.

The following lemmas easily derive from the construction of p_e since $(rg(p_{n+1}) \setminus rg(p_n)) \cap U(p_n) = \emptyset$.

Lemma 5. *If $deg(a \to \alpha) = n$ and $\alpha \notin rg(p_n)$, then $\alpha \notin rg(p_e)$.*

Lemma 6. *(i) $deg(a \to \alpha) \geq deg(a), deg(\alpha)$.*
(ii) If $a \to \alpha$ is in the gamma part of p_e, then $deg(a \to \alpha) > deg(a), deg(\alpha)$.

Lemma 7. *If $\alpha_n \in rg(p_e)$ then $deg(\alpha_n) \leq n$.*

Lemma 8. *There exists no cycle $\beta = c_1 \to c_2 \to \dots c_m \to \beta$.*

Proof. Consider a minimal cycle $\beta_i = c_i \to \beta_{i+1}$ $(1 \leq i \leq m-1)$ and $\beta_m = c_m \to \beta_1$. By Lemma 6 we have $deg(\beta_1) \geq deg(\beta_2) \geq \dots \geq deg(\beta_m) \geq deg(\beta_1)$. Let us set this common degree equal to $k+1$. If $\beta_1 = \gamma_k = b_k \to_{p_{k+1}} \delta_k$ then $\delta_k = \beta_2$ has degree $k+1$. This is not possible by Lemma 6(ii). If $\beta_1 = \epsilon_j^{k,i}$ then $\epsilon_j^{k,i} = c_1 \to c_2 \to \dots c_m \to \epsilon_j^{k,i}$. From this it follows that either α_k has degree $k+1$ (contradicting Lemma 7) or $\epsilon_j^{k,i} = \epsilon_{j-l}^{k,i}$ (contradicting that the epsilon elements are distinct) or $\epsilon_j^{k,i} = \alpha_k$ (contradicting the definition of epsilon elements). This concludes the proof of the lemma.

There remains to see that $(D, p_e) \models t = A = F(p_e)$ for every $t \in \mathcal{M}_e$.

$A \subseteq F(p_e)$: it follows from the definition of α_n and from the fact that $F(p_n) \subseteq F(p_e)$.

$F(p_e) \subseteq A$: suppose $\gamma \in F(p_e)$; then, since F is weakly continuous, $\gamma \in F(p_m)$ for some m (and for all the larger ones). If $\gamma \notin A$ then, for all $n \geq m$, $\alpha_n \in D$ has smaller rank than γ in the enumeration of D, contradicting the fact that there is only a finite number of such elements.

Let $m \in e$ and $t \equiv s_0 s_1 \dots s_m \in \mathcal{M}_m$.
$A \subseteq |t|^{p_e}$: Let $\alpha_n \neq *$. The condition $(D, p_e) \models \alpha_n \in |t|^{p_e}$ follows immediately from Lemma 3 and the fact that

$$\epsilon_1^{n,m} = \epsilon_1^{n,m} \to \epsilon_1^{n,m} \to \dots \to \epsilon_1^{n,m} \to \alpha_n \quad (m\text{-times}).$$

$|t|^{p_e} \subseteq A$: Assume by contraposition that $\gamma \in |t|^{p_e}$ and $\gamma \neq \alpha_n$ for every n. Then by Lemma 4 there exist a sequence $\beta_j \equiv a_1^j \to \dots \to a_m^j \to \gamma$ $(j \in \omega)$ of elements of D and a sequence d_j $(j \in \omega)$ of natural numbers $\leq m$ satisfying the property $\beta_{j+1} \in a_{d_j}^j$.

By Lemma 6 and by $\beta_{j+1} \in a^j_{d_j}$ the sequence $deg(\beta_j)$ is an infinite decreasing sequence of natural numbers. Then there exists j such that $deg(\beta_{j+i}) = deg(\beta_j) = n$ for all $i \geq 0$. Since p_n is finite, it must exist $k \geq j$ and $l > 0$ such that $\beta_k = \beta_{k+l}$.

Moreover, $n = deg(\beta_k) \geq deg(a^k_{d_k} \to a^k_{d_k+1} \to \cdots \to \gamma) \geq deg(\beta_{k+1}) = n$ because $\beta_{k+1} \in a^k_{d_k}$. Then $deg(a^{k+i}_{d_{k+i}} \to a^k_{d_k+1} \to \dots \gamma) = n$ for every $i \leq l$. Since $a^{k+i}_{d_{k+i}}$ cannot be $\{\epsilon_1\}$ (otherwise $\beta_{k+i+1} = \epsilon_1$ and $\gamma = \alpha_n$) and there is exactly one pair (b_{n-1}, δ_{n-1}) such that $((b_{n-1}, \delta_{n-1}), \gamma_{n-1}) \in p^1_n \setminus p_{n-1}$, then

$$a^{k+i}_{d_{k+i}} \to (a^k_{d_{k+i}+1} \to \dots \gamma) = a^{k+j}_{d_{k+j}} \to (a^k_{d_{k+j}+1} \to \dots \gamma), \quad \text{for every } i, j \leq l.$$

This implies that $a^{k+i}_{d_{k+i}} = a^{k+j}_{d_{k+j}}$, etc. Since by Lemma 8 there are no cycles, then we get $\beta_k = \beta_{k+1} = \cdots = \beta_{k+l-1} = \beta_{k+l}$. It follows that $\beta_k \in a^k_{d_k}$. Since $a^k_{d_k} \to a^k_{d_k+1} \to \cdots \to \gamma$ belongs to the gamma part of p_e, this contradicts Lemma 6(ii).

Definition 6. *(Forcing) For a term M, a partial pair (D, q), a D-environment ρ and $\alpha \in D$, the abbreviation $q \Vdash_\rho \alpha \in M$ means that for all total injections $p \supseteq q$ we have that $(D, p) \models \alpha \in |M|^p_\rho$. Furthermore $q \Vdash_\rho X \subseteq M$ means that $q \Vdash_\rho \alpha \in M$ for all $\alpha \in X$.*

If M is closed we write $q \Vdash \alpha \in M$ for $q \Vdash_\rho \alpha \in M$.

Thus, for p is total, $p \Vdash \alpha \in M$ if and only if $\alpha \in |M|^p$.

Lemma 9. *For every term M and environment ρ the function $F_{M,\rho} : \mathcal{I}(D) \to \mathcal{P}(D)$ defined by $F_{M,\rho}(q) = \{\, \alpha \in D : q \Vdash_\rho \alpha \in M\}$ is weakly continuous, and we have $F_{M,\rho}(p) = |M|^p_\rho$ for each total p.*

Proof. The proof of the weak continuity of $F_{M,\rho}$ is a straightforward induction on the complexity of M. Let $p \in Q$ be total. We have to show that $F_{M,\rho}(p) = \bigcup_{q \subseteq_{\text{fin}} p} F_{M,\rho}(q) = |M|^p_\rho$.

If M is a variable x then $F_{x,\rho}(q) = \{\, \alpha \in D : q \Vdash \alpha \in \rho(x)\}$ is the constant function with value $\rho(x)$.

If $M = PQ$ and $\alpha \in |M|^p_\rho$, then there exists $a \subseteq |Q|^p_\rho$ such that $p(a, \alpha) \in |P|^p_\rho$. Choose such an a and let $\gamma = p(a, \alpha)$. By induction hypothesis there is a finite $q \subseteq p$ such that $q \Vdash_\rho a \subseteq Q$ and a finite $r \subseteq p$ such that $r \Vdash_\rho \gamma \in P$; then it is clear that $q \cup r \cup \{((a, \alpha), \gamma)\} \Vdash \alpha \in M$.

If $M = \lambda x.P$ and $\alpha \in |M|^p_\rho$ then there is a unique pair (b, β) such that $\alpha = p(b, \beta)$ and $\beta \in |P|^p_{\rho[x:=b]}$. By induction hypothesis there is a finite $q \subseteq p$ such that $q \Vdash_{\rho[x:=b]} \beta \in P$; then it is clear that $q \cup \{((b, \beta), \alpha)\} \Vdash_\rho \alpha \in M$.

Theorem 3. *Let M be a closed term. Then, for every $e \subseteq_{\text{fin}} \omega$ there exists a graph model (D, p_e) such that $(D, p_e) \models t = M$ for all regular mute terms $t \in \mathcal{M}_e$.*

Proof. It is sufficient to consider an arbitrary environment ρ, the weakly continuous map $F_{M,\rho} : \mathcal{I}(D) \to \mathcal{P}(D)$ defined in Lemma 9 and the graph model (D, p_e) defined in Theorem 2.

6 Ultraproducts

Ultraproducts result from a suitable combination of the direct product and quotient constructions. They were introduced in the 1950's by Loś.

Let I be a non-empty set and let $\{\mathbf{A}_i\}_{i\in I}$ be a family of combinatory algebras. Let U be a proper ultrafilter of the boolean algebra $\mathcal{P}(I)$. The relation $\sim_U$, given by $a \sim_U b \iff \{i \in I : a(i) = b(i)\} \in U$, is a congruence on the combinatory algebra $\prod_{i\in I} \mathbf{A}_i$. The *ultraproduct* of the family $\{\mathbf{A}_i\}_{i\in I}$, noted $(\prod_{i\in I} \mathbf{A}_i)/U$, is defined as the quotient of the product $\prod_{i\in I} \mathbf{A}_i$ by the congruence $\sim_U$. If $a \in \prod_{i\in I} \mathbf{A}_i$, then we denote by a/U the equivalence class of a with respect to the congruence $\sim_U$. If all members of $\{\mathbf{A}_i\}_{i\in I}$ are λ-models, by a celebrated theorem of Loś we have that $(\prod_{i\in I} \mathbf{A}_i)/U$ is a λ-model too, because λ-models are axiomatized by first-order sentences. The basic combinators of the λ-model $(\prod_{i\in I} \mathbf{A}_i)/U$ are $\mathbf{k}/U$ and $\mathbf{s}/U$, and application is given by $x/U \cdot y/U = (x\cdot y)/U$, where the application $x \cdot y$ is defined pointwise.

We now recall the famous Loś theorem.

Theorem 4 (Loś). *Let $\mathcal{L}$ be a first-order language and $\{\mathbf{A}_i\}_{i\in I}$ be a family of $\mathcal{L}$-structures indexed by a non-empty set I an let U be a proper ultrafilter of $\mathcal{P}(I)$. Then for every $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$ and for every tuple $(a_1, \ldots, a_n) \in \prod_{i\in I} \mathbf{A}_i$ we have that*

$$(\prod_{i\in I} \mathbf{A}_i)/U \models \varphi(a_1/U, \ldots, a_n/U) \iff \{i \in I : \mathbf{A}_i \models \varphi(a_1(i), \ldots, a_n(i))\} \in U.$$

The following theorem is [12, Theorem 4.5].

Theorem 5. *Let $(D_j, p_j)_{j\in J}$ be a family of total pairs, $\mathbf{A} = (\mathbf{A}_j : j \in J)$ be the corresponding family of graph λ-models, where $\mathbf{A}_j = (\mathcal{P}(D_j), \cdot, \mathbf{k}, \mathbf{s})$, and let $\mathcal{F}$ be an ultrafilter on J. Then there exists a graph model (E, q) such that the ultraproduct $(\Pi_{j\in J}\mathbf{A}_j)/\mathcal{F}$ can be embedded into the graph λ-model determined by (E, q).*

Theorem 6. *Let M be a closed term and $\mathcal{M} = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$ be the set of all regular mute λ-terms. Then there exists a total pair (E, q) such that*

$$(E, q) \models M = t, \quad \textit{for every } t \in \mathcal{M}.$$

Proof. Let

$$K := \{e \subseteq \mathbb{N} : e \text{ is finite}\}$$

and $\mathcal{F}$ be a non-principal ultrafilter on $\mathcal{P}(K)$ that contains the set

$$K_n = \{e : n \in e\}, \quad \text{for each } n \in \mathbb{N}.$$

Hence $\mathcal{F}$ contains

$$K_e = \{d : e \subseteq d\} \quad \text{for each } e \subseteq_{\text{fin}} \mathbb{N}.$$

For every $e \subseteq \mathbb{N}$, let (D, p_e) be the total pair determined by Theorem 3 and define $\mathbf{A}_e$ be the corresponding graph λ-model. We show that $(\Pi_{e\in K}\mathbf{A}_e)/\mathcal{F} \models M = t$ for every $t \in \mathcal{M}$. Let $t \in \mathcal{M}_n$. Since

$$K_n \subseteq \{e : \mathbf{A}_e \models M = t\}.$$

and $K_n \in \mathcal{F}$ then we have that $(\Pi_{e\in K}\mathbf{A}_e)/\mathcal{F} \models M = t$ and the conclusion is obtained.

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