Spans of Delta Lenses

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Abstract

As part of an ongoing project to unify the treatment of symmetric lenses (of various kinds) as equivalence classes of spans of asymmetric lenses (of corresponding kinds) we relate the symmetric delta lenses of Diskin et al, with spans of asymmetric delta lenses. Because delta lenses are based on state spaces which are categories rather than sets there is further structure that needs to be accounted for and one of the main findings in this paper is that the required equivalence relation among spans is compatible with, but coarser than, the one expected. The main result is an isomorphism of categories between a category whose morphisms are equivalence classes of symmetric delta lenses (here called fb-lenses) and the category of spans of delta lenses modulo the new equivalence.

1 Introduction

In their 2011 POPL paper [4] Hoffmann, Pierce and Wagner defined and studied (set-based) symmetric lenses. Since then, with the study of variants of asymmetric lenses (set-based or otherwise), there has been a need for more definitions of corresponding symmetric variants. This paper is part of an ongoing project by the authors to develop a unified theory of symmetric and asymmetric lenses of various kinds. The goal is to make it straightforward to define the symmetric version of any, possibly new, asymmetric lens variant (and conversely). Once an asymmetric lens is defined the unified theory should provide the symmetric version (and vice-versa).

In [4] Hoffmann et al noted that there were two approaches that they could take to defining symmetric lenses. One involved studying various *right* and *left* (corresponding to what other authors call *forwards* and *backwards*) operations. The other would be based on spans of asymmetric lenses. In both cases an equivalence relation was needed to define composition of symmetric lenses, to ensure that that composition is associative, and to identify lenses which were equivalent in their updating actions although they might differ in "hidden" details such as their *complements* (see [4]) or the head (peak) of their spans. Hoffmann et al gave their definition of symmetric lens in terms of left and right update operations, noting that "in the span presentation there does not seem to be a natural and easy-to-use candidate for . . . equivalence".

In [6] the present authors developed the foundations needed to work with spans of lenses of various kinds and proposed an equivalence for spans of well-behaved set-based asymmetric lenses (called here HPW-lenses) and also for several other set-based variants. Our goal was to find the finest equivalence among spans of HPW lenses that would satisfy the requirements of the preceding paragraph. Such an equivalence needed to include, and be coarser than, span equivalence (an isomorphism between the heads of the spans commuting with the

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legs of the spans). Furthermore pre-composing the legs of a span of HPW lenses with a non-trivial HPW lens gives a new span which differs from the first only in the "hidden" details — the head would be different but the updating actions at the extremities would be the same — so such pairs of spans should also be equivalent. In [6] we were able to show that the equivalence generated by such non-trivial HPW lenses (commuting with the legs of the spans) worked well, and that result was very satisfying because it demonstrated, as so much work in the theory of lenses does, that lenses and bidirectional transformations more generally are valuable generalisations of isomorphisms.

Of course, the work so far, being entirely set-based, is still far from a unified theory, so in this paper we turn to the category-based delta-lenses of Diskin, Xiong and Czarnecki [1] and study the symmetric version derived from spans of such lenses and compare it with the symmetric (forwards and backwards style) version that Diskin et al propose in [2].

The paper is structured as follows. In Sections 2 and 3 we review and develop the basic mathematical properties of delta-lenses (based on [1] and referred to here as d-lenses) and symmetric delta lenses (based on [2], and called here fb-lenses after their basic operations called "forwards" and "backwards", thus avoiding clashing with the general use of "symmetric" for equivalence reduced spans of lenses). As in the work of Hoffmann et al, both fb-lenses and spans of d-lenses need to be studied modulo an equivalence relation and the two equivalence relations we propose are introduced in Section 4. In Section 5 we show that using the two equivalences does indeed yield a category of (equivalence classes of) fb-lenses and a category of (equivalence classes of) spans of d-lenses respectively (and of course, we need to show that the equivalence relations we have introduced are congruences in order to construct the categories). Finally in Section 6 we explore the relationship between the two categories and show that there is an equivalence of categories, indeed in this case an isomorphism of categories, between them.

Because of the usefulness of category-based lenses (in particular delta-lenses) in applications, the work presented here lays important mathematical foundations. Furthermore the extra mathematical structure provided in the category-based variants has revealed a surprise — an equivalence generated by non-trivial lenses is not coarse enough to ensure that two spans of d-lenses with the same fb-behaviour are always identified. The difficulty that arises is illustrated in a short example and amounts to "twisting" the structures so that no single lens can commute with the lenses on the left side of the spans, and at the same time commute with the lenses on the right side of the spans. The solution, presented as one of the equivalences in Section 4, relaxes the requirement that the comparison be itself a lens, and asks that it properly respect the put operations on both sides (rather than having its own put operation commuting with both sides).

2 Asymmetric delta lenses

For any category $\mathbf{C}$, we write $|\mathbf{C}|$ for the set (discrete category) of objects of $\mathbf{C}$ and $\mathbf{C^2}$ for the category whose objects are arrows of $\mathbf{C}$. For a functor $G : \mathbf{S} \longrightarrow \mathbf{V}$, denote the "comma" category, whose objects are pairs consisting of an object S of $\mathbf{S}$ and an arrow $\alpha : GS \longrightarrow V$, by $(G, 1_{\mathbf{V}})$. We recall the definition of a delta lens (or d-lens) [1, 5]:

Definition 1 *A (very well-behaved)* delta lens (d-lens) *from* $\mathbf{S}$ *to* $\mathbf{V}$ *is a pair* (G, P) *where* $G : \mathbf{S} \longrightarrow \mathbf{V}$ *is a functor (the "Get") and* $P : |(G, 1_{\mathbf{V}})| \longrightarrow |\mathbf{S^2}|$ *is a function (the "Put") and the data satisfy:*

(i) d-PutInc: the domain of $P(S, \alpha : GS \longrightarrow V)$ *is* S

(ii) d-PutId: $P(S, 1_{GS} : GS \longrightarrow GS) = 1_S$

(iii) d-PutGet: $GP(S, \alpha : GS \longrightarrow V) = \alpha$

(iv) d-PutPut: $P(S, \beta\alpha : GS \longrightarrow V \longrightarrow V') = P(S', \beta : GS' \longrightarrow V')P(S, \alpha : GS \longrightarrow V)$ *where* S' *is the codomain of* $P(S, \alpha : GS \longrightarrow V)$

For examples of d-lenses, we refer the reader to [1]. Meanwhile, we offer a few sentences here to help orient the reader to the notations used. Both the categories $\mathbf{S}$ and $\mathbf{V}$ represent state spaces. Objects of $\mathbf{S}$ are states and an arrow $S \longrightarrow S'$ of $\mathbf{S}$ represents a specific transition from the state S which is its domain to the state S' which is its codomain. Such specified transitions are often called "deltas". Similarly for $\mathbf{V}$. A functor $G : \mathbf{S} \longrightarrow \mathbf{V}$ maps states of $\mathbf{S}$ to states of $\mathbf{V}$ — S is sent to GS. Furthermore, being a functor it maps deltas $S \longrightarrow S'$ in $\mathbf{S}$ to deltas $GS \longrightarrow GS'$ in $\mathbf{V}$. The objects of $(G, 1_{\mathbf{V}})$ are important because, being a pair $(S, \alpha : GS \longrightarrow V)$, they encapsulate both an object of **S** and a delta starting at GS. Such a pair is the basic input for a Put operation. The Put operation P itself, in the case of a d-lens, is just a function (not a functor) and it takes such an "anchored delta" $(S, \alpha : GS \longrightarrow V)$ in **V** to a delta in **S** which, by d-PutInc, starts at S. The axioms d-PutId and d-PutPut ensure that the P operation respects composition, including identities. Finally, the axiom d-PutGet ensures that, as expected, the Put operation results in a delta in **S** which is carried by G to α, the given input delta.

In [5], we proved that d-lenses compose, that a c-lens as defined there is a special case of d-lens and that their composition is as for d-lenses, and finally that d-lenses are strictly more general than c-lenses. We also proved that d-lenses are certain algebras for a semi-monad.

Furthermore, in [6] we developed the technique to compose spans of lenses in general. For d-lenses, this specializes to Definition 3. We first need a small but important proposition, and we formally remind the reader about our notations for spans and cospans.

A *span* is a pair of morphisms, with common domain:

$$\begin{array}{ccccc} & & S & & \\ & {\scriptstyle u}\swarrow & & \searrow{\scriptstyle v} & \\ X & & & & Y \end{array}$$

Despite the symmetry, such a span is often described as a "span from X to Y", and is distinguished from the same two arrows viewed as a span from Y to X. The illustrated span above is often denoted for brevity's sake $u : X \longleftarrow S \longrightarrow Y : v$ and, when X, S and Y are understood or easily derived, we sometimes just refer to it as the span u, v. The object S is sometimes called the *head* or *peak* of the span and the arrows u and v are called the *legs* of the span. The objects X and Y are, naturally enough, called the *feet* of the span. Cospans are described and notated in the same way but the arrows u and v are reversed. Finally, if, as sometimes is necessary, a span is drawn upside down, the common domain is still called the head despite being drawn below the feet.

When working with spans it is often necessary to calculate pullbacks. For simplicity of presentation we will usually assume that the pullback has been chosen so that its objects and morphisms are pairs of objects from the categories from which it has been constructed (so, for example, in the diagram below, objects of **T** are pairs of objects (S, W) from the categories **S** and **W** respectively, with the property that $G(S) = H(W)$, and similarly for morphisms of **T**).

Proposition 2 *Let* $G : \mathbf{S} \longrightarrow \mathbf{V} \longleftarrow \mathbf{W} : H$ *be a cospan of functors. Suppose that* $P : |(G, 1_{\mathbf{V}})| \longrightarrow |\mathbf{S^2}|$ *is a function which, with* G*, makes the pair* (G, P) *a d-lens. Then, in the pullback square in* **cat***:*

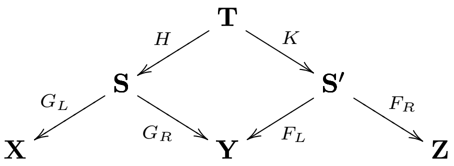
$$\begin{array}{ccccc} & & \mathbf{T} & & \\ & {\scriptstyle H'}\swarrow & & \searrow{\scriptstyle G'} & \\ \mathbf{S} & & & & \mathbf{W} \\ & {\scriptstyle G}\searrow & & \swarrow{\scriptstyle H} & \\ & & \mathbf{V} & & \end{array}$$

the functor G' *together with* $P' : |(G', 1_{\mathbf{W}})| \longrightarrow |\mathbf{T^2}|$ *defined by* $P'((S, W), \beta : G'(S, W) \longrightarrow W') = (P(S, H(\beta)), \beta) : (S, W) \longrightarrow (S', W')$ *define a d-lens from* **T** *to* **W**.

Proof. Note first that P' makes sense since $H(\beta)$ is a morphism $HG'(S, W) \longrightarrow H(W')$ but $HG'(S, W) = GH'(S, W) = GS$ so it is in fact a morphism $GS \longrightarrow H(W')$. Furthermore we denote the codomain of $P(S, H(\beta))$ by S' so that $G(S') = H(W')$ and thus (S', W') is an object of **T**. The d-PutInc, d-PutId and d-PutGet conditions on (G', P') are satisfied by construction. The d-PutPut condition follows immediately from d-PutPut for (G, P). ∎

This means we can talk about the "pullback" of a d-lens along an arbitrary functor, in particular along the Get of another d-lens. This is similar to the situations described in [6]. The inverted commas around "pullback" are deliberate because the constructed d-lens may not be an actual pullback in the category of d-lenses (the category whose objects are categories and whose arrows $\mathbf{S} \longrightarrow \mathbf{V}$ are d-lenses from **S** to **V**).

Definition 3 *Suppose that in*



the functors G_L, G_R, F_L, and F_R are the Gets of d-lenses with corresponding Puts P_L, P_R, Q_L, and Q_R, and $\mathbf{T}$ is the pullback of G_R and F_L. For the "pullback" d-lenses with Gets H and K, denote the Puts by P_H and P_K. Then the span composite *of the span of d-lenses (G_L, P_L), (G_R, P_R) from $\mathbf{X}$ to $\mathbf{Y}$ with the span of d-lenses (F_L, Q_L), (F_R, Q_R) from $\mathbf{Y}$ to $\mathbf{Z}$, denoted*

$$((G_L, P_L), (G_R, P_R)) \circ ((F_L, Q_L), (F_R, Q_R))$$

is the span of d-lenses from $\mathbf{X}$ to $\mathbf{Z}$ specified as follows. The Gets are $G_L H$ and $F_R K$. The Puts are those for the composite d-lenses $(G_L, P_L)(H, P_H)$ and $(F_R, Q_R)(K, P_K)$.

In a sense, the composite just defined corresponds to the ordinary composite of spans in a category with pullbacks. In the category of categories, the ordinary span composition of the span G_L, G_R and with the span F_L, F_R is the span $G_L H, F_R K$. As usual for such composites, the operation is not associative without introducing an equivalence relation and we do so later in this paper.

3 Symmetric delta lenses

A symmetric delta lens (called an "fb-lens" below) is between categories, say $\mathbf{X}$ and $\mathbf{Y}$. It consists of a set of synchronizing "corrs", so named because the make explicit intended correspondences between objects of $\mathbf{X}$ and objects of $\mathbf{Y}$, together with "propagation" operations. In the forward direction, given objects X and Y synchronized by a corr r and an arrow x with domain X, the propagation returns an arrow y with domain Y and a corr synchronizing the codomains of x and y. This is made precise in the following definition and is based on definitions in [2] and [3]. We denote the domain and codomain of an arrow x by $d_0(x), d_1(x)$.

Definition 4 *Let $\mathbf{X}$ and $\mathbf{Y}$ be categories. An* fb-lens *from $\mathbf{X}$ to $\mathbf{Y}$ is $M = (\delta_\mathbf{X}, \delta_\mathbf{Y}, \mathsf{f}, \mathsf{b}) : \mathbf{X} \longleftrightarrow \mathbf{Y}$ specified as follows. The data $\delta_\mathbf{X}, \delta_\mathbf{Y}$ are a span of sets*

$$\delta_\mathbf{X} : |\mathbf{X}| \longleftarrow R_{\mathbf{XY}} \longrightarrow |\mathbf{Y}| : \delta_\mathbf{Y}$$

An element r of $R_{\mathbf{XY}}$ is called a corr. *For r in $R_{\mathbf{XY}}$, if $\delta_\mathbf{X}(r) = X, \delta_\mathbf{Y}(r) = Y$ the corr is denoted $r : X \leftrightarrow Y$. The data f and b are operations called* forward *and* backward propagation*:*

$$\mathsf{f} : Arr(\mathbf{X}) \times_{|\mathbf{X}|} R_{\mathbf{XY}} \longrightarrow Arr(\mathbf{Y}) \times_{|\mathbf{Y}|} R_{\mathbf{XY}}$$

$$\mathsf{b} : Arr(\mathbf{Y}) \times_{|\mathbf{Y}|} R_{\mathbf{XY}} \longrightarrow Arr(\mathbf{X}) \times_{|\mathbf{X}|} R_{\mathbf{XY}}$$

where the pullbacks (also known as fibered products) mean that if $\mathsf{f}(x, r) = (y, r')$, we have $d_0(x) = \delta_\mathbf{X}(r), d_1(y) = \delta_\mathbf{Y}(r')$ and similarly for b. We also require that $d_0(y) = \delta_\mathbf{Y}(r)$ and $\delta_\mathbf{X}(r') = d_1(x)$, and the similar equations for b.

Furthermore, we require that both propagations respect both the identities and composition in $\mathbf{X}$ and $\mathbf{Y}$, so that we have:

$$r : X \leftrightarrow Y \quad \textit{implies} \quad \mathsf{f}(\mathrm{id}_X, r) = (\mathrm{id}_Y, r) \quad \textit{and} \quad \mathsf{b}(\mathrm{id}_Y, r) = (\mathrm{id}_X, r)$$

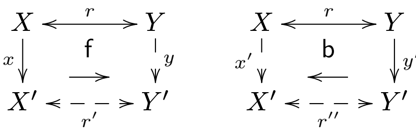
and

$$\mathsf{f}(x, r) = (y, r') \text{ and } \mathsf{f}(x', r') = (y', r'') \quad \textit{imply} \quad \mathsf{f}(x'x, r) = (y'y, r'')$$

and

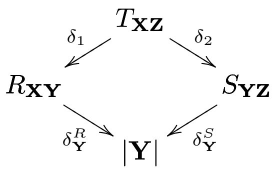
$$\mathsf{b}(y, r) = (x, r') \text{ and } \mathsf{b}(y', r') = (x', r'') \quad \textit{imply} \quad \mathsf{b}(y'y, r) = (x'x, r'')$$

If $\mathsf{f}(x,r) = (y,r')$ and $\mathsf{b}(y',r) = (x',r'')$, we display instances of the propagation operations as:



For examples of fb-lenses we refer the reader to [2].

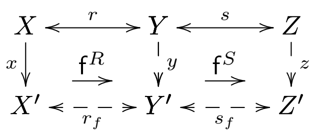
Definition 5 *Let* $M = (\delta^R_{\mathbf{X}}, \delta^R_{\mathbf{Y}}, \mathsf{f}^R, \mathsf{b}^R)$ *and* $M' = (\delta^S_{\mathbf{Y}}, \delta^S_{\mathbf{Z}}, \mathsf{f}^S, \mathsf{b}^S)$ *be two fb-lenses. We define the* composite fb-lens $M'M = (\delta_{\mathbf{X}}, \delta_{\mathbf{Z}}, \mathsf{f}, \mathsf{b})$ *as follows. Let* $T_{\mathbf{XZ}}$ *be the pullback of categories in*



Let $\delta_{\mathbf{X}} = \delta^R_{\mathbf{X}}\delta_1 : T_{\mathbf{XZ}} \longrightarrow \mathbf{X}$ *and* $\delta_{\mathbf{Z}} = \delta^S_{\mathbf{Z}}\delta_2$*. The operations for* $M'M$ *are defined as follows. Denote* $\mathsf{f}^R(x,r) = (y, r_f), \mathsf{f}^S(y,s) = (z, s_f)$ *and* $\mathsf{b}^S(z,s') = (y, s_b), \mathsf{b}^R(y,r') = (x, r_b)$*. Then*

$$\mathsf{f}(x,(r,s)) = (z,(r_f,s_f)) \text{ and } \mathsf{b}(z,(r',s')) = (x,(r_b,s_b))$$

The diagram



shows that the arities are correct for f in the forward direction. That is, we have

$$\mathsf{f} : Arr(\mathbf{X}) \times_{|\mathbf{X}|} T_{\mathbf{XZ}} \longrightarrow Arr(\mathbf{Z}) \times_{|\mathbf{Z}|} T_{\mathbf{XZ}}$$

and similarly

$$\mathsf{b} : Arr(\mathbf{Z}) \times_{|\mathbf{Z}|} T_{\mathbf{XZ}} \longrightarrow Arr(\mathbf{X}) \times_{|\mathbf{X}|} T_{\mathbf{XZ}}$$

It is easy to show that the f and b just defined respect composition and identities in $\mathbf{X}$ and $\mathbf{Z}$ and we record:

Proposition 6 *The composite* $M'M$ *just defined is an fb-lens from* $\mathbf{X}$ *to* $\mathbf{Z}$*.*

We note that because it is defined using a pullback, this construction of the composite of a pair of fb-lenses is not associative, and when we later define a category of fb-lenses the arrows will be equivalence classes of fb-lenses.

Next we define two constructions relating spans of d-lenses with fb-lenses.

We first consider a span of d-lenses. Let $L = (G_L, P_L)$ where $G_L : \mathbf{S} \longrightarrow \mathbf{V}$ and $K = (G_K, P_K)$ where $G_K : \mathbf{S} \longrightarrow \mathbf{W}$ be (a span of) d-lenses.

Construct the fb-lens $M_{L,K} = (\delta_{\mathbf{V}}, \delta_{\mathbf{W}}, \mathsf{f}, \mathsf{b})$ as follows:

- the corrs are $R_{\mathbf{V},\mathbf{W}} = |\mathbf{S}|$ with $\delta_{\mathbf{V}} S = G_L S$ and $\delta_{\mathbf{W}} S = G_K S$;

- forward propagation f for $v : V \longrightarrow V'$ and $S : V \leftrightarrow W$ is defined by $\mathsf{f}(v,S) = (w,S')$ where $w = G_K(P_L(S,v))$ and S' is the codomain of $P_L(S,v)$;

- backward propagation b is defined analogously.

Lemma 7 $M_{L,K}$ *is an fb-lens.*

Proof. Identity and compositionality for $M_{L,K}$ follow from functoriality of the Gets for L and K and the *d-PutId* and *d-PutPut* equations in Definition 1. ∎

In the other direction, suppose that $M = (\delta_{\mathbf{V}}, \delta_{\mathbf{W}}, \mathsf{f}, \mathsf{b})$ is an fb-lens from $\mathbf{V}$ to $\mathbf{W}$ with

$$\delta_{\mathbf{V}} : |\mathbf{V}| \longleftarrow R \longrightarrow |\mathbf{W}| : \delta_{\mathbf{W}}$$

We now construct a span of d-lenses $L_M : \mathbf{V} \longleftarrow \mathbf{S} \longrightarrow \mathbf{W} : K_M$ from $\mathbf{V}$ to $\mathbf{W}$. The first step is to define the head $\mathbf{S}$ of the span. The set of objects of $\mathbf{S}$ is the set R of corrs of M. The morphisms of $\mathbf{S}$ are defined as follows: For objects r and r', $\mathbf{S}(r, r') = \{(v, w) \mid d_0 v = \delta_{\mathbf{V}}(r), d_1 v = \delta_{\mathbf{V}}(r'), d_0 w = \delta_{\mathbf{W}}(r), d_1 v = \delta_{\mathbf{W}}(r')\}$ (where we write, as usual, $\mathbf{S}(r, r')$ for the set of arrows of $\mathbf{S}$ from r to r'). Thus an arrow may be thought of as a formal square:

$$\begin{array}{ccc} V & \overset{r}{\longleftrightarrow} & W \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ V' & \underset{r'}{\longleftrightarrow} & W' \end{array}$$

Composition is inherited from composition in $\mathbf{V}$ and $\mathbf{W}$ at boundaries, or more precisely, for $(v, w) \in \mathbf{S}(r, r')$ and $(v', w') \in \mathbf{S}(r', r'')$ we define:

$$(v', w')(v, w) = (v'v, w'w)$$

in $\mathbf{S}(r, r'')$. The identities are pairs of identities. It is easy to see that $\mathbf{S}$ is a category.

Next we define the d-lens L_M to be the pair (G_L, P_L) where we define $G_L : \mathbf{S} \longrightarrow \mathbf{V}$ on objects by $\delta_{\mathbf{V}}$, and on arrows by projection, that is $G_L(v, w) = v$. The Put for L_M, $P_L : |(G_L, 1_{\mathbf{V}})| \longrightarrow |\mathbf{S^2}|$, is defined on objects $(r, v : G_L(r) \longrightarrow V')$ of the category $(G_L, 1_{\mathbf{V}})$ by $P_L(r, v) = (v, \pi_0 \mathsf{f}(v, r))$ which is indeed an arrow of $\mathbf{S}$ from r to $\pi_1 \mathsf{f}(v, r)$. (As is usual practice, we write π_0 and π_1 for the projection from any pair onto its first and second factors respectively.) We define $K_M = (G_K, P_K)$ similarly.

Lemma 8 *$L_M = (G_L, P_L)$ and $K_M = (G_K, P_K)$ is a span of d-lenses.*

Proof. G_L and G_K are evidently functorial. We need to show that P_L and P_K satisfy (i)-(iv) of Definition 1. These follow immediately from the properties of of the fb-lens M. ∎

The two constructions above are related. One composite of the constructions is actually the identity.

Proposition 9 *For any fb-lens M, with the notation of the constructions above*

$$M = M_{L_M, K_M}$$

Proof. By inspection, the corrs and δ's of M_{L_M, K_M} are those of M. Further, it is easy to see that, for example, the forward propagation of M_{L_M, K_M} is identical to that of M. ∎

However, the other composite of the constructions above, namely the span of d-lenses $L_{M_{L.K}}, K_{M_{L.K}}$ is not equal to the original span L, K (because the arrows of the original $\mathbf{S}$ have been replaced by the formal squares described above). We have yet to consider the appropriate equivalence for spans of d-lenses, and we do so now. We will see that $L_{M_{L.K}}, K_{M_{L.K}}$ is indeed equivalent to L, K.

4 Two equivalence relations

Our first equivalence relation is on spans of d-lenses from $\mathbf{X}$ to $\mathbf{Y}$.

Suppose that

$$\mathbf{X} \xleftarrow{(G_L, P_L)} \mathbf{S} \xrightarrow{(G_R, P_R)} \mathbf{Y} \quad \text{and} \quad \mathbf{X} \xleftarrow{(G'_L, P'_L)} \mathbf{S}' \xrightarrow{(G'_R, P'_R)} \mathbf{Y}$$

are such spans.

The functor $\Phi : \mathbf{S} \longrightarrow \mathbf{S}'$ is said to satisfy conditions (E) if:

(1) $G'_L \Phi = G_L$ and $G'_R \Phi = G_R$

(2) Φ is surjective on objects

(3) whenever $\Phi S = S'$, we have both $P'_L(S', G'_L S' \xrightarrow{\alpha} X) = \Phi P_L(S, G_L S \xrightarrow{\alpha} X)$ and

$$P'_R(S', G'_R S' \xrightarrow{\beta} Y) = \Phi P_R(S, G_R S \xrightarrow{\beta} Y).$$

By (1) any such Φ is a "2-cell" in spans of categories between the two $\mathbf{X}$ to $\mathbf{Y}$ spans G_L, G_R and G'_L, G'_R. Moreover, Φ is required both to be surjective on objects and also to satisfy (3), a condition which expresses a compatibility with the Puts. Notice that the identity functor on $\mathbf{S}$ satisfies conditions (E).

Definition 10 *Let $\equiv_{Sp}$ be the equivalence relation on spans of d-lenses from $\mathbf{X}$ to $\mathbf{Y}$ which is generated by functors Φ satisfying (E).*

To simplify describing $\equiv_{Sp}$ we now prove some properties of functors satisfying conditions (E).

Lemma 11 *A composite of d-lens span morphisms satisfying (E) also satisfies (E).*

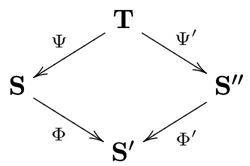
Proof. Suppose that we have spans of d-lenses $(G_L, P_L), (G_R, P_R)$ and $(G'_L, P'_L), (G'_R, P'_R)$ as above, and a third such span is:

$$\mathbf{X} \xleftarrow{(G''_L, P''_L)} \mathbf{S}'' \xrightarrow{(G''_R, P''_R)} \mathbf{Y}$$

Suppose $\Phi : \mathbf{S} \longrightarrow \mathbf{S}'$ and $\Phi' : \mathbf{S}' \longrightarrow \mathbf{S}''$ satisfy (E). Properties (1) and (2) for $\Phi'\Phi$ are immediate. We show the P'_L part of property (3) for $\Phi'\Phi$. Suppose $\Phi'\Phi S = S''$ and consider $P''_L(S'', G''_L S' \xrightarrow{\alpha} X)$. By (E) for Φ and Φ', since $\Phi'(\Phi(S)) = S''$, we have $P''_L(S'', G''_L S'' \xrightarrow{\alpha} X) = \Phi' P'_L(\Phi(S)$, and $G'_L \Phi(S) \xrightarrow{\alpha} X) = \Phi'\Phi P_L(S, G_L S \xrightarrow{\alpha} X)$ as required. ∎

Suppose that Φ satisfies (E). When $\Phi S = S'$ it follows that $G_L S = G'_L \Phi S = G'_L S'$, which we will use below. Note that if Φ were the Get of a d-lens (although it need not be) then it would be surjective on arrows by the *d-PutGet* equation, but not necessarily surjective on hom sets.

Lemma 12 *Suppose once again that $(G_L, P_L), (G_R, P_R)$, $(G'_L, P'_L), (G'_R, P'_R)$ and $(G''_L, P''_L), (G''_R, P''_R)$ are spans of d-lenses as above. Let $\Phi : \mathbf{S} \longrightarrow \mathbf{S}' \longleftarrow \mathbf{S}'' : \Phi'$ be the functors in a cospan of span morphisms satisfying (E). Let*



be a pullback in **cat**. *Then there is a span of d-lenses $\mathbf{X} \xleftarrow{(G^T_L, P^T_L)} \mathbf{T} \xrightarrow{(G^T_R, P^T_R)} \mathbf{Y}$ defined by $G^T_L = G_L \Psi$ and*

$$P^T_L((S, S''), G^T_L(S, S'') \xrightarrow{\alpha} X) = (S \xrightarrow{P_L(S,\alpha)} W, S'' \xrightarrow{P''_L(S'',\alpha)} W'')$$

and similarly for (G^T_R, P^T_R). Moreover, Ψ and Ψ' satisfy (E).

Proof. The first point is that (G^T_L, P^T_L) and (G^T_R, P^T_R) actually are d-lenses. We need to know that P^T_L is well-defined. Since (S, S'') is an object of the pullback $\mathbf{T}$, we know that $\Phi(S) = \Phi'(S'') = S'$, say. We want $P^T_L((S, S''), G^T_L(S, S'') \xrightarrow{\alpha} X)$ to be an arrow of $\mathbf{T}$, so we need to show that $\Phi(S \xrightarrow{P_L(S,\alpha)} W)$ is equal to $\Phi'(S'' \xrightarrow{P''_L(S'',\alpha)} W'')$. However both are equal to $P'_L(S', \alpha)$ since both Φ and Φ' satisfy (E). Thus furthermore, (W, W'') is an object of $\mathbf{T}$ and using this for *d-PutPut* each of the required d-lens equations is easy to establish.

Next, we show that Ψ and Ψ' satisfy (E). First of all, the Gets commute by definition. Moreover, both Ψ and Ψ' are surjective on objects because Φ and Φ' are so.

It remains to check property (3) for Ψ and Ψ'. We need to show that whenever $\Psi(S, S'') = S$, we have

$$P_L(S, G_L S \xrightarrow{\alpha} U) = \Psi(P_L^T((S, S''), G_L^T(S, S'') \xrightarrow{\alpha} U))$$

and this follows immediately from the definitions of Ψ and P_L^T. (Notice that for $S = \Psi(S, S'')$ we have $G_L S = G'_L \Phi S = G'_L \Phi'(S'') = G''_L(S'')$ and thus $P''_L(S'', G''_L S'' \xrightarrow{\alpha} U)$ is well-defined.) Similarly Ψ' satisfies (3). ∎

Corollary 13 *Zig-zags of span morphisms satisfying (E) reduce to spans of span morphisms satisfying (E).* ∎

A zig-zag is any string of arrows (ignoring the direction of the indidual arrows so that neighbouring arrows might be connected head to head or tail to tail as well as tail to head). It follows that any proof that two spans of d-lenses are $\equiv_{Sp}$ equivalent can be reduced to a single span Ψ, Ψ' of span morphisms satisfying (E).

The second equivalence relation we introduce is on the set of fb-lenses from $\mathbf{X}$ to $\mathbf{Y}$. Recall that Diskin et al [3] defined symmetric delta lenses (our fb-lenses), but they did not consider composing them. Like Hoffman et al [4] they would find that they need to consider equivalence classes of their symmetric delta lenses in order for the appropriate composition to be associative. Also like Hoffman et al, there is a need for an equivalence among their lenses to eliminate artificial differences. In fact, defining an equivalence to restore associativity is easy. Choosing the correct equivalence to eliminate the artificial differences is more delicate. And what do we mean by "atificial differences"? Symmetric lenses of various kinds include hidden data — the complements of Hoffmann et al and the corrs of Diskin et al are examples. The hidden data is important for checking and maintaining consistency, but different arrangements of hidden data with the same overall effect should not be counted as different symmetric lenses.

We now introduce such a relation on the set of fb-lenses from $\mathbf{X}$ to $\mathbf{Y}$.

Definition 14 *Let $L = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathsf{f}, \mathsf{b})$ and $L' = (\delta'_{\mathbf{X}}, \delta'_{\mathbf{Y}}, \mathsf{f}', \mathsf{b}')$ be two fb-lenses (from $\mathbf{X}$ to $\mathbf{Y}$) with corrs $R_{\mathbf{XY}}, R'_{\mathbf{XY}}$. We say $L \equiv_{\mathsf{fb}} L'$ iff there is a relation σ from $R_{\mathbf{XY}}$ to $R'_{\mathbf{XY}}$ with the following properties:*

1. *σ is compatible with the δ's, i.e. $r\sigma r'$ implies $\delta_{\mathbf{X}} r = \delta'_{\mathbf{X}} r'$ and $\delta_{\mathbf{Y}} r = \delta'_{\mathbf{Y}} r'$*
2. *σ is total in both directions, i.e. for all r in $R_{\mathbf{XY}}$, there is r' in $R'_{\mathbf{XY}}$ with $r\sigma r'$ and conversely.*
3. *for all r, r', x an arrow of $\mathbf{X}$, if $r\sigma r'$ and $\delta_{\mathbf{X}} r$ is the domain of x then the first components of $\mathsf{f}(x, r)$ and $\mathsf{f}'(x, r')$ are equal and the second components are σ related, i.e. $\pi_0 \mathsf{f}(x, r) = \pi_0 \mathsf{f}'(x, r')$ and $\pi_1 \mathsf{f}(x, r) \sigma \pi_1 \mathsf{f}'(x, r')$*
4. *the corresponding condition for b, i.e. for all r, r', y an arrow of $\mathbf{Y}$, if $r\sigma r'$ and $\delta_{\mathbf{X}} r$ is the domain of x then $\pi_0 \mathsf{b}(y, r) = \pi_0 \mathsf{b}'(y, r')$ and $\pi_1 \mathsf{b}(y, r) \sigma \pi_1 \mathsf{b}'(y, r')$*

Lemma 15 *The relation $\equiv_{\mathsf{fb}}$ is an equivalence relation.*

Proof. For reflexivity take σ to be the identity relation; for symmetry take the opposite relation for σ; for transitivity, the composite relation is easily seen to satisfy conditions 1. to 4. ∎

Lemma 16 *For L, L' as in the definition above, consider a surjection $\varphi : R_{\mathbf{XY}} \longrightarrow R'_{\mathbf{XY}}$ compatible with the δ's and satisfying the following conditions:*

- *for corrs r, r', and an arrow x of $\mathbf{X}$, if $\varphi(r) = r'$ and $\delta_{\mathbf{X}} r$ is the domain of x then the first components of $\mathsf{f}(x, r)$ and $\mathsf{f}'(x, r')$ are equal and the second components are φ related, i.e. $\pi_0 \mathsf{f}(x, r) = \pi_0 \mathsf{f}'(x, r')$ and $\varphi(\pi_1 \mathsf{f}(x, r)) = \pi_1 \mathsf{f}'(x, r')$*
- *the corresponding condition for b, i.e. for all r, r', y an arrow of $\mathbf{Y}$, if $\varphi(r) = r'$ and $\delta_{\mathbf{X}} r$ is the domain of x then $\pi_0 \mathsf{b}(y, r) = \pi_0 \mathsf{b}'(y, r')$ and $\varphi(\pi_1 \mathsf{b}(y, r)) = \pi_1 \mathsf{b}'(y, r')$*

The collection of such surjections (viewed as relations) generates the equivalence relation $\equiv_{\mathsf{fb}}$.

Proof. To see that $\equiv_{\mathsf{fb}}$ is generated by such surjections, consider the span tabulating the relation σ in $\equiv_{\mathsf{fb}}$ viz.

$$R_{\mathbf{XY}} \longleftarrow \sigma \longrightarrow R'_{\mathbf{XY}}.$$

Notice that each leg is a surjection satisfying the conditions of the lemma. Conversely, any zig-zag of such surjections defines a relation which is easily seen to satisfy the conditions of Definition 14. ∎

Remark 17 Although $\equiv_{\mathsf{fb}}$ is generated by zig-zags of certain surjections, we have just seen that any such zig-zag can be reduced to a single span of such surjections between sets of corrs.

Proposition 18 *Suppose that* $M \equiv_{\mathsf{fb}} M'$ *are fb-lenses from* $\mathbf{X}$ *to* $\mathbf{Y}$ *equivalent by a generator for* $\equiv_{\mathsf{fb}}$*, i.e. a surjection* $\varphi : R_{\mathbf{XY}} \longrightarrow R'_{\mathbf{XY}}$*. Then* $(L_M, K_M) \equiv_{Sp} (L_{M'}, K_{M'})$ *as spans of d-lenses from* $\mathbf{X}$ *to* $\mathbf{Y}$*.*

Proof. We first define $\Phi : \mathbf{S} \longrightarrow \mathbf{S}'$ on objects by φ. Notice that Φ is surjective on objects since φ is a surjection. To define Φ on arrows of $\mathbf{S}$, consider an arrow

$$\begin{array}{ccc} X & \stackrel{r}{\longleftrightarrow} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \underset{r'}{\longleftrightarrow} & Y' \end{array}$$

of $\mathbf{S}$. Its image under Φ is defined to be the arrow

$$\begin{array}{ccc} X & \stackrel{\varphi(r)}{\longleftrightarrow} & Y \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \underset{\varphi(r')}{\longleftrightarrow} & Y' \end{array}$$

which is an arrow of $\mathbf{S}'$ since φ is compatible with the δs. This Φ is evidently functorial and commutes with the Gets. It remains to show that Φ satisfies condition (3) of (E), that is, whenever $\Phi(r) = r'$, $P'_L(r', G'_L r' \xrightarrow{x} X') = \Phi P_L(r, G_L r \xrightarrow{x} X')$ (with the similar equation holding for P'_K).

Now, when $r' = \Phi(r)$, we have

$$\begin{aligned} P'_L(r', x) &= (x, \pi_0 \mathsf{f}'(x, r')) \\ &= (x, \pi_0 \mathsf{f}(x, r)) && \text{see Lemma 16} \\ &= \Phi(x, \pi_0 \mathsf{f}(x, r)) && \text{definition of } \Phi \\ &= \Phi P_L(r, x) \end{aligned}$$

as required. Similarly for P'_K. ∎

Proposition 19 *Suppose that* $(L, K) \equiv_{Sp} (L', K')$ *as spans of d-lenses from* $\mathbf{X}$ *to* $\mathbf{Y}$ *are made equivalent by a generator for* $\equiv_{Sp}$*, i.e. a functor* $\Phi : \mathbf{S} \longrightarrow \mathbf{S}'$ *satisfying conditions (E). Then* $M_{L,K} \equiv_{\mathsf{fb}} M_{L',K'}$ *as fb-lenses from* $\mathbf{X}$ *to* $\mathbf{Y}$*.*

Proof. Let φ be the object function of Φ. Since Φ commutes with the Gets, φ is compatible with the δ's.

We need to show that φ satisfies the conditions in Lemma 16. Suppose $\Phi(S) = S'$ and $G_L(S)$ is the domain of x. Then

$$\begin{aligned} \pi_0 \mathsf{f}(x, S) &= G_K P_L(x, S) \\ &= G'_K \Phi P_L(x, S) \\ &= G'_K P'_L(x, S') \\ &= \pi_0 \mathsf{f}'(x, S') \end{aligned}$$

and

$$
\begin{aligned}
\varphi\pi_1\mathsf{f}(x,S) &= \varphi d_1 P_L(x,S) \\
&= d_1 \Phi P_L(x,S) \\
&= d_1 P'_L(x,S') \\
&= \pi_1\mathsf{f}'(x,S')
\end{aligned}
$$

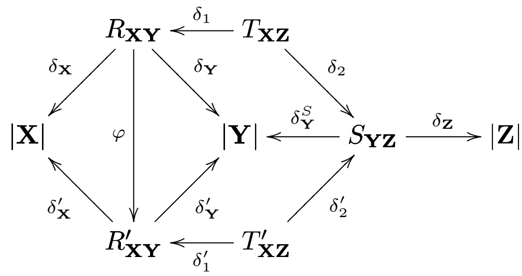
as required. Similarly for the equations involving b. ∎

5 Two categories of lenses

The collections of lenses so far discussed do not form categories since their composition is not associative. We are going to use the equivalence relations of the previous section to resolve this, but first we show that the equivalence relations respect the composites defined above, that is they are "congruences".

Proposition 20 *Suppose that* $M = (\delta_{\mathbf{X}}, \delta_{\mathbf{Y}}, \mathsf{f}^R, \mathsf{b}^R)$*,* $M' = (\delta'_{\mathbf{X}}, \delta'_{\mathbf{Y}}, \mathsf{f}^{R'}, \mathsf{b}^{R'})$ *and* $N = (\delta^S_{\mathbf{Y}}, \delta_{\mathbf{Z}}, \mathsf{f}^S, \mathsf{b}^S)$ *are fb-lenses (see the diagram below in which* $R_{\mathbf{XY}}$*,* $R'_{\mathbf{XY}}$ *and* $S_{\mathbf{YZ}}$ *are the corresponding corrs). Further, suppose* $\varphi : R_{\mathbf{XY}} \longrightarrow R'_{\mathbf{XY}}$ *is a generator of* $\equiv_{\mathsf{fb}}$*. Thus* $M \equiv_{\mathsf{fb}} M'$*. Then* $NM \equiv_{\mathsf{fb}} NM'$*.*

Proof. The composite NM has as corrs the pullback $T_{\mathbf{XZ}}$ as in Definition 5, and similarly NM' has corrs $T'_{\mathbf{XZ}}$.



In order to show that $NM \equiv_{\mathsf{fb}} NM'$, we construct $\varphi' : T_{\mathbf{XZ}} \longrightarrow T'_{\mathbf{XZ}}$. This is straightforward using the universal property of the pullback $T'_{\mathbf{XZ}}$, since $\delta'_{\mathbf{Y}}\varphi\delta_1 = \delta^S_{\mathbf{Y}}\delta_2$.

To finish, we need to check that φ' satisfies the four requirements of Definition 14.

Compatibility with δs is easy when we note that $\varphi\delta_1 = \delta'_1\varphi'$ and $\delta'_2\varphi' = \delta_2$.

The function φ' is a total relation in both directions since it is surjective. To see that φ' is surjective, note that any element of $T'_{\mathbf{XZ}}$ can be thought of as a pair (r', s) compatible over $\mathbf{Y}$, and since φ is surjective, there exists an r in $R_{\mathbf{XY}}$, necessarily compatible with s, such that $\varphi'(r,s) = (\varphi(r), s) = (r', s)$.

Let f be the forward propagation of the composite NM, as defined in Definition 5, and let f' be the forward propagation of the composite NM'. Suppose $r' = \varphi(r)$ and thus $(r', s) = \varphi'(r, s)$. We need to show that the first components of $\mathsf{f}(x,(r,s))$ and $\mathsf{f}'(x,(r',s))$ are equal and that φ' takes the second component of $\mathsf{f}(x,(r,s))$ to the second component of $\mathsf{f}'(x,(r',s))$.

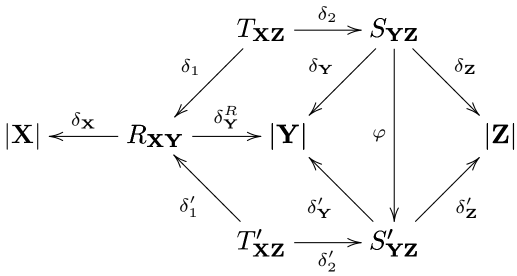
The first component of $\mathsf{f}(x,(r,s))$ is $\pi_0\mathsf{f}^S(\pi_0\mathsf{f}^R(x,r),s)$, while the first component of $\mathsf{f}'(x,(r',s))$ is $\pi_0\mathsf{f}^S(\pi_0\mathsf{f}^{R'}(x,r'),s)$, and these are equal since φ is a generator of $\equiv_{\mathsf{fb}}$ implies $\pi_0\mathsf{f}^R(x,r) = \pi_0\mathsf{f}^{R'}(x,\varphi(r))$.

The second component of $\mathsf{f}(x,(r,s))$ is $(\pi_1\mathsf{f}^R(x,r), \pi_1\mathsf{f}^S(\pi_0 f^R(x,r),s))$, while the second component of $\mathsf{f}'(x,(r',s))$ is $(\pi_1\mathsf{f}^{R'}(x,r'), \pi_1\mathsf{f}^S(\pi_0 f^{R'}(x,r'),s))$, and φ' of the first equals the second since, as before, $\pi_0\mathsf{f}^R(x,r) = \pi_0\mathsf{f}^{R'}(x,\varphi(r))$.

The same arguments work for b and b'. ∎

Proposition 21 *Suppose that* $M = (\delta_{\mathbf{X}}, \delta^R_{\mathbf{Y}}, \mathsf{f}^R, \mathsf{b}^R)$*,* $N = (\delta_{\mathbf{Y}}, \delta_{\mathbf{Z}}, \mathsf{f}^S, \mathsf{b}^S)$ *and* $N' = (\delta'_{\mathbf{Y}}, \delta'_{\mathbf{Z}}, \mathsf{f}^{S'}, \mathsf{b}^{S'})$ *are fb-lenses (see the diagram below in which* $R_{\mathbf{XY}}$*,* $S_{\mathbf{YZ}}$ *and* $S'_{\mathbf{YZ}}$ *are the corresponding corrs). Further, suppose* $\varphi : S_{\mathbf{YZ}} \longrightarrow S'_{\mathbf{YZ}}$ *is a generator of* $\equiv_{\mathsf{fb}}$*. Thus* $N \equiv_{\mathsf{fb}} N'$*. Then* $NM \equiv_{\mathsf{fb}} N'M$*.*

Proof. The composite NM has as corrs the pullback $T_{\mathbf{XZ}}$ as in Definition 5, and similarly $N'M$ has corrs $T'_{\mathbf{XZ}}$.



The proof follows the same argument as in the previous proposition. ∎

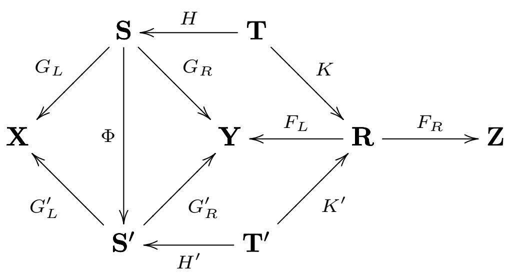
Theorem 22 *Equivalence classes for* $\equiv_{\mathsf{fb}}$ *are the arrows of a category, denoted* $\mathsf{fb}DLens$.

Proof. We first note that Propositions 20 and 21 ensure that composition is well-defined independently of choice of representative. There is an identity fb-lens with obvious structure which acts as an identity for the composition. It remains only to note that associativity follows by standard re-bracketing of iterated pullbacks. The re-bracketing function is the φ for an $\equiv_{\mathsf{fb}}$ equivalence. ∎

Proposition 23 *Suppose that* (G_L, P_L), (G_R, P_R), (G'_L, P'_L), (G'_R, P'_R), (F_L, Q_L), *and* (F_R, Q_R) *are d-lenses whose Gets are shown in the diagram below. Further, suppose* $\Phi : \mathbf{S} \longrightarrow \mathbf{S}'$ *is a functor satisfying properties (E). Thus the span* (G_L, P_L), (G_R, P_R) *is* $\equiv_{Sp}$ *to the span* (G'_L, P'_L), (G'_R, P'_R). *Then the two possible span composites are equivalent, that is*

$$((G_L, P_L), (G_R, P_R)) \circ ((F_L, Q_L), (F_R, Q_R)) \equiv_{Sp} ((G'_L, P'_L), (G'_R, P'_R)) \circ ((F_L, Q_L), (F_R, Q_R)).$$

Proof. The top composite span of d-lenses in the diagram below has head $\mathbf{T}$, the pullback of G_R and F_L (see Definition 3), similarly $\mathbf{T}'$ is the head of the bottom span composite.



In order to show the claimed equivalence, we construct a functor $\Phi' : \mathbf{T} \longrightarrow \mathbf{T}'$. Since $G'_R \Phi H = F_L K$, Φ' is defined by applying the universal property of the pullback $\mathbf{T}'$.

Since $\mathbf{T}$ and $\mathbf{T}'$ are pullbacks of functors, their objects can be taken to be pairs of objects from $\mathbf{S}$ and $\mathbf{R}$, respectively $\mathbf{S}'$ and $\mathbf{R}$. Similarly, their arrows can be taken to be pairs. Also H and K, respectively H' and K' can be taken to be projections. We can now explicitly describe the action of Φ' on an arrow of $\mathbf{T}$ as $\Phi'(t_0, t_1) = (\Phi t_0, t_1)$.

As in Definition 3, we denote the Puts of the lenses whose Gets are H and K by P_H and P_K. Similarly for H' and K'. Denote the composite lens $(G_L, P_L)(H, P_H)$ by (G, P) and similarly $(G', P') = (G'_L, P'_L)(H', P_{H'})$.

We need to show that Φ' satisfies the conditions (E). By its construction Φ' commutes with the Gets, and is surjective on objects.

It remains to show that whenever $\Phi(S, R) = (S', R')$ (which implies that $R = R'$ and $\Phi(S) = S'$) we have

$$P'((S', R'), G'_L H'(S', R') \xrightarrow{\alpha} X') = \Phi' P((S, R), G_L H(S, R) \xrightarrow{\alpha} X')$$

and

$$Q'((S',R'),F_RK'(S',R')\xrightarrow{\gamma}Z') = \Phi'Q((S,R),F_RK(S,R)\xrightarrow{\gamma}Z')$$

We begin by proving the first equation immediately above. We know that whenever $\Phi(S) = S'$, $P'_L(S',G'L(S')\xrightarrow{\alpha}X') = \Phi P_L(S,G_L(S)\xrightarrow{\alpha}X')$. We calculate

$$\begin{aligned}
P'((S',R'),G'_LH'(S',R')\xrightarrow{\alpha}X') &= P'((S',R'),G'_L(S')\xrightarrow{\alpha}X') \\
&= P_{H'}((S',R'),P'_L(S',G'_L(S')\xrightarrow{\alpha}X')) \\
&= (P'_L(S',G'_L(S')\xrightarrow{\alpha}X'),Q_L(R',G'_RP'_L(S',G'_L(S')\xrightarrow{\alpha}X'))) \\
&= (\Phi P_L(S,G_L(S)\xrightarrow{\alpha}X'),Q_L(R,G'_R\Phi P_L(S,G_L(S)\xrightarrow{\alpha}X'))) \\
&= \Phi'(P_L(S,G_L(S)\xrightarrow{\alpha}X'),Q_L(R,G_RP_L(S,G_L(S)\xrightarrow{\alpha}X'))) \\
&= \Phi'(P_H((S,R),P_L(S,G_L(S)\xrightarrow{\alpha}X'))) \\
&= \Phi'P((S,R),G_LH(S,R)\xrightarrow{\alpha}X')
\end{aligned}$$

The first step is merely that H' is a projection; the second is the definition of P' as the Put of the composite lens whose Get is G'_LH'; the third is the definition of $P_{H'}$ (see Proposition 2); the fourth uses $R' = R$ and the hypothesis stated just before the equations; the fifth follows since Φ commutes with G_R and G'_R and the definition of Φ'; the sixth is the definition of P_H (see Proposition 2); the last is the definition of P as the Put of the composite lens whose Get is G_LH.

To establish the second equation, suppose $\Phi'(S,R) = (S',R')$, whence $R = R'$ and $\Phi(S) = S'$, and so because Φ satisfies conditions (E), we have

$$P'_R(S',G'_R(S')\xrightarrow{\beta}Y') = \Phi P_R(S,G_R(S)\xrightarrow{\beta}Y')$$

and since $G_R(S) = G'_R\Phi(S) = G'_R(S')$, the right hand side can be written as $\Phi P_R(S,G'_R(S')\xrightarrow{\beta}Y')$. We calculate

$$\begin{aligned}
Q'((S',R'),F_RK'(S',R')\xrightarrow{\gamma}Z') &= P_{K'}((S',R'),Q_R(R',F_R(R')\xrightarrow{\gamma}Z')) \\
&= (P'_R(S',F_LQ_R(R',F_R(R')\xrightarrow{\gamma}Z')),Q_R(R',F_R(R')\xrightarrow{\gamma}Z'))
\end{aligned}$$

Before continuing the calculation, we simplify by defining β by $(G'_R(S')\xrightarrow{\beta}Y') = F_LQ_R(R',F_R(R')\xrightarrow{\gamma}Z') = F_LQ_R(R,F_R(R)\xrightarrow{\gamma}Z')$ after noting that $G'_R(S')$ is the domain of $F_LQ_R(R',F_R(R')\gamma Z')$ since the T' pullback square commutes. Now, continuing the calculation above:

$$\begin{aligned}
&= (P'_R(S',G'_R(S')\xrightarrow{\beta}Y',Q_R(R',F_R(R')\xrightarrow{\gamma}Z'))) \\
&= (\Phi P_R(S,G_R(S)\xrightarrow{\beta}Y',Q_R(R',F_R(R')\xrightarrow{\gamma}Z'))) \\
&= (\Phi P_R(S,G'_R(S')\xrightarrow{\beta}Y',Q_R(R,F_R(R)\xrightarrow{\gamma}Z'))) \\
&= \Phi'(P_R(S,F_LQ_R(R,F_R(R)\xrightarrow{\gamma}Z'),Q_R(R,F_R(R)\xrightarrow{\gamma}Z'))) \\
&= \Phi'P_K((S,R),Q_R(R,F_R(R)\xrightarrow{\gamma}Z'),Q_R(R,F_R(R)\xrightarrow{\gamma}Z')) \\
&= \Phi'Q((S,R),F_RK(S,R)\xrightarrow{\gamma}Z')
\end{aligned}$$

The first step uses that K' is a projection and the definition of Q' as the Put of the composite lens whose Get is F'_RK' ; the second is the definition of $P_{K'}$ (see Proposition 2); the third is the definition of β above; the fourth uses the hypothesis stated before the equations; the fifth uses $R = R'$ and the note just before the equations; the sixth uses the definitions of Φ' and β; the seventh is the definition of P_K (see Proposition 2); the last is the definition of Q as the Put of the composite lens whose Get is F_RK. ∎

Like Propositions 20 and 21, there is a reflected version of Proposition 23, showing that equivalent spans of d-lenses when composed on the left with another span of d-lenses are equivalent.

Proposition 24 *In notation analogous to Proposition 23,*

$$((G_L, P_L), (G_R, P_R)) \circ ((F_L, Q_L), (F_R, Q_R)) \equiv_{Sp} ((G_L, P_L), (G_R, P_R)) \circ ((F'_L, Q'_L), (F'_R, Q'_R)).$$

Theorem 25 *Equivalence classes for $\equiv_{Sp}$ are the arrows of a category, denoted SpDLens.*

Proof. We first note that Proposition 23 and Proposition 24 ensure that composition is well-defined independently of choice of representative. There is a span of identity d-lenses which acts as the identity for the composition. Again, associativity follows by standard re-bracketing of iterated pullbacks of categories. The re-bracketing functor is the Φ for an $\equiv_{Sp}$ equivalence. ∎

6 An isomorphism of categories of lenses

Now that we have the categories $\mathsf{fb}DLens$ and $SpDLens$, we can extend the constructions of Section 3 to functors on them.

Definition 26 *For the $\equiv_{\mathsf{fb}}$ equivalence class $[M]$ of an fb-lens M, let $\mathcal{A}([M])$ be the $\equiv_{Sp}$ equivalence class of, in the notation of Lemma 8, the span L_M, K_M.*

Proposition 27 *$\mathcal{A}$ is the arrow function of a functor, also denoted $\mathcal{A}$, from $\mathsf{fb}DLens$ to SpDLens.*

Proof. We need to show that $\mathcal{A}$ preserves identities and composition.

For the former denote by $M_{\mathbf{X}}$ the identity fb-lens on a category $\mathbf{X}$. We begin by noticing that the category $\mathbf{X}_p$ at the head of the span of d-lenses constructed from $M_{\mathbf{X}}$ has as its objects exactly those of $\mathbf{X}$. Its arrows from X to X' are arbitrary pairs of $\mathbf{X}$ arrows, both of which are from X to X'. Define the functor Φ from the head $\mathbf{X}$ of the identity span on $\mathbf{X}$ to $\mathbf{X}_p$ by sending an arrow x of $\mathbf{X}$ to the pair of arrows (x, x). This functor Φ satisfies conditions (E), and so $\mathcal{A}([M_{\mathbf{X}}]) = [\mathbf{X}]$ as required.

Let M and M' be a composable pair of fb-lenses from $\mathbf{X}$ to $\mathbf{Y}$ and $\mathbf{Y}$ to $\mathbf{Z}$ respectively. The composite fb-lens $M'M$ has as corrs compatible pairs of corrs, one from M and one from M' (see Definition 5). The head $\mathbf{S}$ of the span of d-lenses constructed from $M'M$ has objects compatible pairs of corrs and as arrows from compatible corrs (r_1, r_2) to compatible corrs (r'_1, r'_2), pairs of arrows, one from $\mathbf{X}$ and one from $\mathbf{Z}$ as shown

$$\begin{array}{ccccc} X & \overset{r_1}{\longleftrightarrow} & Y & \overset{r_2}{\longleftrightarrow} & Z \\ {\scriptstyle x}\downarrow & & & & \downarrow{\scriptstyle z} \\ X' & \underset{r'_1}{\longleftrightarrow} & Y' & \underset{r'_2}{\longleftrightarrow} & Z' \end{array}$$

On the other hand, the span composite of the spans constructed from M and M' has as head a category $\mathbf{T}$ whose objects are pairs of compatible corrs from M and M' respectively. The arrows of $\mathbf{T}$ are triples of arrows (x, y, z) as shown

$$\begin{array}{ccccc} X & \overset{r_1}{\longleftrightarrow} & Y & \overset{r_2}{\longleftrightarrow} & Z \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} \\ X' & \underset{r'_1}{\longleftrightarrow} & Y' & \underset{r'_2}{\longleftrightarrow} & Z' \end{array}$$

Define the functor Φ from $\mathbf{T}$ to $\mathbf{S}$ by sending the triple of arrows (x, y, z) to the pair of arrows (x, z). This functor Φ satisfies conditions (E), and so $\mathcal{A}([M'M]) = \mathcal{A}([M'])\mathcal{A}([M])$ as required. ∎

Definition 28 *For the $\equiv_{Sp}$ equivalence class $[L, K]$ of a span of d-lenses L, K, let $\mathcal{S}([L, K])$ be the $\equiv_{\mathsf{fb}}$ equivalence class of, in the notation of Lemma 7, the fb-lens $M_{L,K}$.*

Proposition 29 *$\mathcal{S}$ is the arrow function of a functor, also denoted $\mathcal{S}$, from SpDLens to $\mathsf{fb}DLens$.*

Proof. We need to show that $\mathcal{S}$ preserves identities and composition.

Unlike the previous proof, the preservation of identities and composition is "on the nose". That is, the construction applied to the identity gives precisely the identity fb-lens. Moreover, with judicious choice of pullbacks, the construction applied to the composite of two composable spans of d-lenses *is* the composite of the fb-lenses constructed from each of the spans.

Thus $\mathcal{S}$ preserves the equivalence class of the identity span and $\mathcal{S}([L_1, K_1][L_2, K_2]) = \mathcal{S}([L_1, K_1])\mathcal{S}([L_2, K_2])$. ∎

Theorem 30 *The functors $\mathcal{A}$ and $\mathcal{S}$ are an isomorphism of categories $SpDLens \cong \mathsf{fb}DLens$.*

Proof. We need to show that the composites $\mathcal{AS}$ and $\mathcal{SA}$ are identities. Recall first that both $\mathcal{A}$ and $\mathcal{S}$ have identity functions as object functions. Considering the arrows, Proposition 9 shows that $\mathcal{SA}$ is the identity functor. We now consider $\mathcal{AS}$.

For a span L, K of d-lenses between $\mathbf{X}$ and $\mathbf{Y}$, using the notation of Lemmas 7 and 8, $\mathcal{AS}([L, K]) = [L_{M_{L,K}}, K_{M_{L,K}}]$, so we consider the span $L_{M_{L,K}}, K_{M_{L,K}}$ of d-lenses whose Gets and Puts we denote by F_L, Q_L and F_K, Q_K respectively. The head of the span is a category we denote $\mathbf{S}_{L,K}$ whose objects are the same as the objects of $\mathbf{S}$, the head of the span L, K. We define an identity on objects functor $\Phi : \mathbf{S} \longrightarrow \mathbf{S}_{L,K}$ on arrows by $\Phi(s) = (G_L(s), G_K(s))$ (recalling that arrows of $\mathbf{S}_{L,K}$ are pairs of arrows from $\mathbf{X}$ and $\mathbf{Y}$, respectively). We finish by showing that Φ satisfies conditions (E), and so witnesses $\mathcal{AS}([L, K]) = [L, K]$.

It remains to show that Φ satisfies conditions (E). Being identity on objects, Φ is certainly surjective on objects, and it commutes with the Gets by its construction. For condition (3), given an object S' of $\mathbf{S}_{L,K}$, an object S of $\mathbf{S}$ such that $\Phi S = S'$, and an arrow $\alpha : G_L(S) = F_L(S') \longrightarrow X'$ in $\mathbf{X}$, we have $P_L(S, \alpha)$ an arrow of $\mathbf{S}$. We need to show that $\Phi P_L(S, \alpha) = Q_L(S', \alpha)$. Since $\Phi S = S'$ we have $S = S'$. Now $Q_L(S', \alpha) = Q_L(S, \alpha) = (\alpha, \pi_0 \mathsf{f}(\alpha, S))$, for the forward propagation f of $M_{L,K}$ constructed as in Lemma 7. By that construction $\pi_0 \mathsf{f}(\alpha, S) = G_K(P_L(S, \alpha))$. But $\Phi P_L(S, \alpha) = (G_L(P_L(S, \alpha)), G_K(P_L(S, \alpha))) = (\alpha, \pi_0 \mathsf{f}(\alpha, S)) = Q_L(S', \alpha)$.

Thus, since $\mathcal{AS}([L, K]) = [L, K]$, $\mathcal{AS}$ is the identity. ∎

7 Conclusions

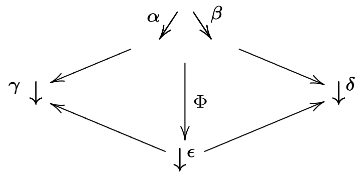
Because asymmetric delta lenses and symmetric delta lenses are so useful in applications, it is important that we understand them well and provide a firm mathematical foundation. This paper provides such a foundation by formalizing fb-lenses and their composition, including determining the equivalence required on fb-lenses for the composition to be well-defined and associative. Furthermore the resultant category $\mathsf{fb}DLens$ of fb-lenses is equivalent, indeed isomorphic, to the category $SpDLens$ of equivalence classes of spans of d-lenses.

This last result, the isomorphism between $\mathsf{fb}DLens$ and $SpDLens$, furthers the program to unify the treatment of symmetric lenses of type X as equivalence classes of spans of asymmetric lenses of type X, carrying that program for the first time into category-based lenses. (And that extension came with a surprise — see below.)

Naturally a unified treatment needs to be tested extensively on a wide range of lens types, and more work remains. The present paper is an important step in the program, and provides a reason to be hopeful that the unification is close at hand. Indeed, with this work the program encounters the important category-based lenses for the first time and that substantially widens the base of unified examples.

We end with a distilled example. It shows in a simplified way why the equivalence used here, based on conditions (E), needs to be coarser than an equivalence generated by lenses commuting with the spans though it remains compatible with the earlier work. Thus it is also a coarser equivalence relation than might have been expected based on [6].

The figure below presents two spans of d-lenses. The categories at the head and feet of the spans have been shown explicitly. In three cases the category has a single non-identify morphism called variously γ, δ and ϵ while in the fourth case the category has two distinct nonidentity morphisms denoted α and β. In all cases objects and identity morphisms have not been shown. In three cases there are just two objects, while in the fourth case there are three, with a single object serving as the domain of both α and β.



The arrows displayed in both spans represent (the Gets of) d-lenses. In the lower span the d-lenses are simply identity d-lenses (the Gets are isomorphisms sending ϵ to γ in the left hand leg, and to δ in the right hand leg). Both of the Puts are then determined. The upper span is made up of two non-identity d-lenses. In both cases the Gets send both α and β to the one non-identity morphism (γ in the left hand leg and δ in the right hand leg). We specify the Puts for the upper span (eliding reference to objects in the Puts' parameters since they can be easily deduced): $P_L(\gamma) = \alpha$ and $P_R(\delta) = \beta$ for the left and right Puts respectively.

Notice that Φ, the functor that sends both α and β to ϵ, satisfies conditions (E) showing, as expected, that the two spans are equivalent. After all, if one traces through the forward and backward behaviours across the two spans the results at the extremities are in all cases the same, though the intermediate results at the heads of the spans differ. However, Φ cannot be the Get of a lens which commutes with the other four d-lenses. Indeed, to commute with the left hand lenses would require $P_\Phi(\epsilon) = \alpha$ while to commute with the right hand lenses would require $P_\Phi(\epsilon) = \beta$, but $\alpha \neq \beta$.

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