Between Belief Bases and Belief Sets: Partial Meet Contraction

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Abstract

In belief revision, the result of a contraction should be a logically closed set (closure) such that the input is not inferred (success). Traditionally, these two postulates depend on the same consequence operator.

The present work investigates the consequences of using two different consequence operators for each of these desiderata: the classic Cn to compute success and a second weaker operator to compute the closure. We prove a representation theorem and some other properties for the new contraction.

1 Introduction

Belief revision is the branch of knowledge representation that deals with the dynamics of epistemic states. In their seminal paper [Alchourrón *et al.*, 1985], Alchourrón, Gärdenfors and Makinson proposed to represent the epistemic state of an agent as a logically closed set of propositions called *belief sets*. The authors focus on three main epistemic operations over belief sets: expansion, contraction and revision. Expansion is the simple addition of a proposition to the epistemic state, contraction is the removal of a proposition and revision is the *consistent* addition of a proposition. Each of this operations is characterized by a set of rationality postulates.

Consider, for example, the following postulates for contraction:

(closure) $K - \alpha = Cn(K - \alpha)$

(success) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(K - \alpha)$

Closure guarantees the resulting epistemic state to be represented as a logically closed set of propositions, i.e., the result of contracting a belief set is also a belief set. **Success** guarantees that the removed proposition α is no longer implied by the new belief set, unless it is a tautology.

Hansson in [Hansson, 1992a] suggests a generalization of the AGM framework where the epistemic state is not necessarily closed under logical consequences. Although in his work epistemic states are represented as arbitrary sets of sentences called belief bases (i.e. **closure** is not necessarily satisfied), the removal of a sentence in contraction is still evaluated against the closure of the belief base (i.e. **success** is satisfied). Hence, in belief base contraction there is a distinction between the sentences in the base and the sentences inferred from the base. Especially in computer science, belief base revision is very useful since computing the logical closure of a set may be hard if possible at all.

Operations in belief bases, however, have some undesirable properties due to the fact that equivalent sentences may be treated differently. In the present work we explore the consequences of keeping both closure and success where the consequence operations in these two postulates do not necessarily coincide. More precisely, we assume that closure is guaranteed only for a weaker consequence operation. This weaker notion of consequence may be easy to compute and at the same time useful to avoid some undesirable consequences of belief base operations.

Throughout this paper, we consider the language of classical propositional logic, closed under the usual boolean connectives.

We call consequence operator a function that takes sets of formulas into sets of formulas. A consequence operator C is *Tarskian* if and only if it satisfies:

(inclusion) $A \subseteq C(A)$

(idempotence) $C(A) = C(C(A))$

(monotonicity) If $A \subseteq B$ then $C(A) \subseteq C(B)$

Cn denotes the classical consequence operator and $\vdash$ denotes the associated relation: $A \vdash \alpha$ stands for $\alpha \in Cn(A)$.

Lowercase Latin letters (p, q, r) stand for atoms, lowercase Greek letters (α, β) stand for formulas, uppercase Latin letters (A, B, K) stand for sets of formulas.

The rest of the paper is structured as follows. A brief overview of classical belief revision is given in Section 2. Section 3 presents pseudo-contractions, the kind of operation we are exploring in this paper. In Section 4, we describe our proposal, depicting it with examples in Section 5. Section 6 highlights some previous works that are related to ours. We delineate our conclusions in Section 7. Proofs for theorems and propositions are found in the Appendix.

2 AGM Paradigm

In the AGM paradigm [Alchourrón *et al.*, 1985], belief states are typically represented by sets of sentences closed under logical consequence, the so-called *belief sets*. Three change operations were initially defined: expansion, contraction and revision. Expansion is the simple addition of a new sentence, followed by logically closing the resulting set, i.e, $K + \alpha = Cn(K \cup \{\alpha\})$. In the case of contraction, we have a class of operations, all delimited by *rationality postulates* that they should satisfy. Let K be a belief set and α and β be formulas. The original AGM basic postulates for contraction are:

(closure) $K - \alpha = Cn(K - \alpha)$

(success) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin K - \alpha$

(inclusion) $K - \alpha \subseteq K$

(vacuity) If $\alpha \notin K$, then $K - \alpha = K$

(recovery) $K \subseteq (K - \alpha) + \alpha$

(extensionality) If $Cn(\alpha) = Cn(\beta)$, then $K - \alpha = K - \beta$

Note that in the presence of **closure**, we can use $\alpha \notin K - \alpha$ or $\alpha \notin Cn(K - \alpha)$ interchangeably in the **success** postulate.

AGM revision is usually defined from contraction and expansion by means of the Levi identity:

$$K * \alpha = (K - \neg\alpha) + \alpha$$

Therefore, in this paper we will focus on contraction.

Besides defining rationality postulates for contraction, Alchourrón et al.[1985] have also proposed a construction for a contraction operation. *Partial meet contraction* is based on the notion of a *remainder set*, the set of all maximal subsets that do not imply the element that is to be contracted. Formally:

Definition 1 *Let B be a set and α a formula. The remainder set $B \perp \alpha$ is such that $X \in B \perp \alpha$ if and only if:*

- $X \subseteq B$
- $X \nvdash \alpha$
- *For all sets Y, if $X \subset Y \subseteq B$, then $Y \vdash \alpha$*

Definition 2 *A function γ is a selection function for the set B if and only if:*

- *If $B \perp \alpha \neq \emptyset$ then $\emptyset \neq \gamma(B \perp \alpha) \subseteq B \perp \alpha$*
- *Otherwise, $\gamma(B \perp \alpha) = \{B\}$*

Definition 3 *Let γ be a selection function for a set of sentences B. The partial meet contraction of B by a sentence α is given by $B - \alpha = \bigcap \gamma(B \perp \alpha)$.*

The partial meet operation previously defined can be applied directly over belief bases, in such a way that it satisfies the following postulates:

(success) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(B - \alpha)$

(inclusion) $B - \alpha \subseteq B$

(relevance) If $\beta \in B \setminus (B - \alpha)$, then there is a B' such that $B - \alpha \subseteq B' \subseteq B$, $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$

(uniformity) If for all $B' \subseteq B$, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $B - \alpha = B - \beta$

Furthermore, we have the following representation theorem:

Theorem 4 *[Hansson, 1992b] An operation is a partial meet contraction over belief bases if and only if it satisfies the postulates of success, inclusion, relevance and uniformity.*

For logically closed sets, both characterizations are equivalent (this is true for classical logics, for other cases, cf. [Ribeiro *et al.*, 2013]) in the sense that all initial AGM postulates are consequence of the postulates for bases [Hansson, 1999].

3 Pseudo-Contractions

Contraction operators on belief sets can be generated from operations on belief bases. As an example, one can define a contraction operator for belief sets as $K \div \alpha = Cn(B - \alpha)$, where $K = Cn(B)$ and $-$ is a base contraction operator. If $-$ is a partial meet base contraction, $\div$ satisfies five of the six AGM postulates, but not **recovery**.

In [Hansson, 1989] a weakening of the inclusion postulate was proposed, called **logical inclusion**.

(logical inclusion) $Cn(B - \alpha) \subseteq Cn(B)$

Hansson has suggested to call operations that satisfy **success** and **logical inclusion** *pseudo-contractions*.

Nebel has proposed a pseudo-contraction for bases that generates a contraction that satisfies all the six AGM postulates [Nebel, 1989].

Definition 5 *Let $\bigwedge B$ be the conjunction of all elements of B. Nebel's pseudo-contraction for the set B is the operator $-$ such that for all sentences α:*

$$B - \alpha = \begin{cases} B & \text{if } \alpha \in Cn(\emptyset) \\ \bigcap \gamma(B \perp \alpha) \cup \{\alpha \rightarrow \bigwedge B\} & \text{otherwise} \end{cases}$$

Although the belief set operation generated from Nebel's pseudo-contraction satisfies all the AGM postulates, it adds unnecessary information to the base. As shown in [Ribeiro and Wassermann, 2008], it suffices to add $\{\alpha \rightarrow \bigwedge B'\}$, where $B' = B \setminus \bigcap \gamma(B \perp \alpha)$. As already noted in [Ribeiro and Wassermann, 2008], there is no other intuition behind Nebel's operation than maintaining **recovery**, a postulate which has been deemed as polemic already since the 80's [Makinson, 1987].

In this work we want to further explore the possibility of working with belief bases with **logical inclusion**, allowing for some syntax independence without having to resort to belief sets.

4 Between Belief Sets and Belief Bases

As stated earlier, the direct application of partial meet contraction over closed belief sets and over belief bases creates problems of practical (computational infeasibility) and theoretical (syntax dependence) nature, respectively. One of the aims of this study is to assess the effects of doing the traditional partial meet contraction on belief bases closed by a consequence operation that is between the classical consequence operator and the identity (i.e., the base itself). Hence, we will assume that this operator (here called Cn^*) is Tarskian.

We will study the properties of the application of the partial meet contraction over a set closed under Cn^*, i.e., the operator defined as:

Definition 6 *Let B be a set of sentences, Cn^* a function from sets of sentences to sets of sentences and γ a selection function for $Cn^*(B)$. The operator $-_*$ is such that, for all sentences α:*

$$B -_* \alpha = \bigcap \gamma(Cn^*(B) \bot \alpha)$$

Notice that $B -_* \alpha = \bigcap \gamma(Cn^*(B)\bot\alpha) = Cn^*(B) -_\gamma \alpha$, where $-_\gamma$ is the partial meet contraction. Since $-_\gamma$ satisfies the postulates of **success**, **inclusion**, **relevance** and **uniformity**, it follows directly (details in the Appendix) that $-_*$ satisfies **success** and:

(inclusion*) $B - \alpha \subseteq Cn^*(B)$

(relevance*) If $\beta \in Cn^*(B) \setminus (B - \alpha)$, then there is a B' such that $B - \alpha \subseteq B' \subseteq Cn^*(B)$, $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$

(uniformity*) If for all $B' \subseteq Cn^*(B)$, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $B - \alpha = B - \beta$

For several applications it is important that the construction satisfies the original **success** postulate, and not only a starred version of it:

(success*) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn^*(B - \alpha)$

We want that the sentence to be contracted ceases to be logically (classically) implied by the resulting set after the contraction. In this case, the role of the logic Cn^* is just to give a degree of syntactic independence to the operation.

As our purpose here is to make the contraction on a set closed by a Cn^* that does not generate as many consequences as the classic Cn, the following property is desirable:

(subclassicality) $Cn^*(A) \subseteq Cn(A)$

Clearly, if $Cn^*(A) = A$ (identity), we have that $-_*$ is the usual operation of partial meet contraction on bases. Similarly, for all Tarskian Cn^* that also satisfies subclassicality, applying $-_*$ to belief sets (i.e., $K = Cn^*(K) = Cn(K)$) yields the usual AGM partial meet contraction on belief sets.

Following the same idea as **logical inclusion**, in [Ribeiro and Wassermann, 2008] we have a weakening of **relevance**:

(logical relevance) If $\beta \in B \setminus (B - \alpha)$, then there is a B' such that $B - \alpha \subseteq B' \subseteq Cn(B)$, $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$

The next corollary follows:

Corollary 7 *If Cn^* is Tarskian and satisfies subclassicality, then an operation that satisfies success, inclusion*, relevance* and uniformity* also satisfies logical inclusion, logical relevance and uniformity.*

With a proof that is very similar to that of the representation theorem for partial meet contraction on bases (which can be found in [Hansson, 1999]), we can prove the following representation theorem:

Theorem 8 *Provided that Cn^* is Tarskian, subclassical and compact, an operation is a $-_*$ operator if and only if it satisfies success, inclusion*, relevance* and uniformity*.*

From this theorem and the previous corollary, it also follows:

Corollary 9 *If Cn^* is Tarskian and satisfies subclassicality, then $-_*$ satisfies success, logical inclusion, logical relevance and uniformity.*

It is interesting that we have here a set of postulates that are independent from Cn^*. Nonetheless, these postulates do not characterize the operation, and are in general weaker than the postulates with *. Hansson's **logical inclusion** postulate is quite reasonable for base operations, as it brings syntactic independence, although with **inclusion*** we already have a degree of independence, and with better preservation of the original set (since Cn^* is subclassical), and, depending on the chosen Cn^*, we avoid the complexity problem we have with the closure of Cn.

Another desirable property in rational contraction operations is **relative closure** [Hansson, 1991].

(relative closure) $B \cap Cn(B - \alpha) \subseteq B - \alpha$

This property is a consequence of the postulate of **relevance**, which $-_*$ does not satisfy. Nevertheless, **relative closure** is satisfied, given the condition that Cn^* is Tarskian.

Proposition 10 *If Cn^* is Tarskian, the $-_*$ operator satisfies relative closure.*

Kernel contraction [Hansson, 1994] is an alternative construction for contraction, which instead of considering maximal sets not implying a given formula as in partial meet contraction, considers minimal sets implying it. It works by finding the minimal sets (called α-kernels) that imply the element α being contracted and then selecting at least one element of each α-kernel to remove from the belief base. The operation is characterized by the same postulates as partial meet contraction on bases, except for **relevance**, which is weakened to **core-retainment**:

(core-retainment) If $\beta \in B \setminus (B - \alpha)$, then there is a B' such that $B' \subseteq B$, $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$

Kernel contraction may have erratic behaviour due to its non-satisfaction of **relevance** [Hansson, 1999]. For instance, consider the logically independent sentences p and q, and let $A = \{p, p \vee q, p \leftrightarrow q\}$. The kernel contraction $A - (p \wedge q) = \{p\}$ is possible, whereas partial meet contraction cannot have this outcome. As observed by Hansson, it is not sensible to give up $p \vee q$, since p was kept.

Would the weakening of **relevance** to **relevance*** or **logical relevance** be enough so as to make these behaviours show up in the $-_*$ operator? Kernel contraction does not satisfy any of these last two postulates. Furthermore, Hansson had already noticed that the lack of **relative closure** also contributes to these unnecessary removals in contraction. The operator $-_*$, as shown above, satisfies **relative closure**.

A property of our pseudo-contraction is **enforced closure***.

(enforced closure*) $B - \alpha = Cn^*(B - \alpha)$

Proposition 11 *If Cn^* is Tarskian and satisfies subclassicality, an operator that satisfies inclusion* and relevance* also satisfies enforced closure*.*

Whenever $A \subset Cn^*(A)$, this postulate will imply that **vacuity** is not satisfied, which is an essential postulate in the point of view of rational contractions (that respect the principle of minimal change). It has as effect that the belief base will always end up closed by Cn^* after the contraction, even though the original base was not closed.

(vacuity) If $\alpha \notin Cn(B)$, then $B - \alpha = B$

Nevertheless, our construction does satisfy a weaker form of **vacuity**:

(vacuity*) If $\alpha \notin Cn(B)$, then $B - \alpha = Cn^*(B)$

Proposition 12 *If Cn^* is Tarskian and satisfies subclassicality, an operator that satisfies inclusion* and relevance* also satisfies vacuity*.*

Corollary 13 *If Cn^* is Tarskian and satisfies subclassicality, then the operator $-_*$ satisfies enforced closure* and vacuity*.*

A simple way to restore **vacuity** is to redefine the $-_*$ operator in the following manner:

Definition 14 *Let B be a set of sentences, Cn^* a function from sets of sentences to sets of sentences and γ a selection function for $Cn^*(B)$. The operator $-'_*$ is such that, for all sentence α:*

$$B -'_* \alpha = \begin{cases} B & \text{if } \alpha \notin Cn(B) \\ \bigcap \gamma(Cn^*(B) \perp \alpha) & \text{otherwise} \end{cases}$$

Observation 15 *The $-'_*$ operator satisfies success, inclusion*, uniformity* and vacuity. If Cn^* is Tarskian and subclassical, $-'_*$ also satisfies logical inclusion, uniformity and relative closure.*

The proof of the observation above is not given, but can be trivially obtained from theorem 8, corollary 7 and proposition 10. Note that **relevance*** and **logical relevance** were lost.

Although we have attained **vacuity**, we have only partly gotten rid of **enforced closure*** (just when $\alpha \notin Cn(B)$).

If on one hand **logical inclusion** seems to make more sense than **inclusion** for base contractions, by allowing some syntactic independence, effects such as **enforced closure*** illustrate the need to refrain from careless additions of sentences in the contraction. Here we should mention the postulate of **core-addition** [Ribeiro and Wassermann, 2008]:

(core-addition) If $\beta \in (B - \alpha) \setminus B$, then there is a $\beta' \in B \setminus (B - \alpha)$ and a $B' \subseteq B - \alpha$ such that $\alpha \to \beta' \notin Cn(B')$ but $\alpha \to \beta' \in Cn(B' \cup \{\beta\})$.

Any operator satisfying **inclusion** will satisfy this postulate trivially. If we break $\{\alpha \to \bigwedge B\}$ into the set of sentences $\{\alpha \to \beta \mid \beta \in B\}$, Nebel's pseudo-contraction will not satisfy **core-addition**. Clearly the $-_*$ operator does not satisfy it also (neither does $-'_*$), and it does not satisfy **vacuity** as well. In the effort to fix these two problems, the operations of general partial-meet pseudo-contraction and Δ-partial-meet pseudo-contraction, proposed in [Ribeiro and Wassermann, 2008], seem to be viable solutions.

5 Examples

In this section we are going to show some concrete examples of Cn^* functions that can be useful in the solution of practical problems. The first example we mention is the Cleopatra example, adapted from [Ribeiro and Wassermann, 2008].

Example 1 *Consider a language with three propositional letters, p, q and r and a belief base $B = \{p \wedge q\}$, where p stands for* Cleopatra had a son *and q,* Cleopatra had a daughter. *If we want to contract by p, applying a partial meet contraction produces $B - p = \emptyset$. This is not always the expected result, because the loss of faith in the belief that Cleopatra had a son also made us lose faith in the belief that she had a daughter.*

With the classic partial meet construction for bases we would have $B \perp p = \{\emptyset\}$, so the selection function needs to choose $\{\emptyset\}$, causing the overall contraction process to produce $\emptyset$ as final result.

On the other hand, if we take B to represent the belief set $K = Cn(B)$, then K contains both p and q and the belief that Cleopatra had a daughter (q) may survive the contraction, i.e., we may have $q \in K - p$. But then we would also have $q \vee r$, $r \to q$ and many other irrelevant formulas in the resulting set, since it is closed under Cn.

Let us consider an intermediate consequence operator:

$$Cn_1^*(A) = \{\alpha \mid \alpha \in A \text{ or for any formulas } \beta, \delta, \\ \alpha \wedge \beta \in A \text{ or } \beta \wedge \alpha \in A \text{ or } \beta \wedge \alpha \wedge \delta \in A\}$$

We can use Cn_1^* with the $-_*$ operator to solve the problem of the preceding example. In this case, we have $Cn_1^*(B) = \{p \wedge q, p, q\}$ and $Cn_1^*(B) \perp p = \{\{q\}\}$, hence the selection function would choose the whole remainder set, $\{\{q\}\}$, and, accordingly, $B -_* p = \{q\}$.

Although the usefulness of this consequence operation is dubious, its use already brings better results than the typical base contraction in some cases, as in the former example.

Example 2 *Suppose I believe that* the town of Juazeiro do Norte is located in the state of Pernambuco *($j \to p$) and that* the state of Pernambuco is located in Brazil *($p \to b$). Speaking with a colleague, I found that this town is not located in his state (Pernambuco), that is, I contract $j \to p$ from my base. The outcome is $B - (j \to p) = \{p \to b\}$. So, I no longer know whether Juazeiro do Norte is located in Brazil.*

In this example, as well as in the previous one, one can blame the poor codification of the belief base for the problems. The knowledge that Juazeiro do Norte is located in Brazil, if obvious, perhaps would be individually justified, and so it would deserve to be explicitly in the base. At this point the syntactic independence dilemma reappears. In some cases we want to have it, but without having to generate infinitely many derivative sentences with little utility.

However, when working with ontologies, for instance, it is possible that the user does not want to make explicit every possible relationship, trusting the transitivity of some properties (i.e., he would be more concerned with his ontology on the knowledge level than on the syntactic level). One may also want a knowledge base with little redundancy.

Again, neither the belief base nor the belief set approach would give us the desired result.

Returning to the foregoing example, we could use a Cn_2^* that adds to the base the transitive closure of $\rightarrow$:

$$Cn_2^*(A) = A \cup \{\alpha_1 \rightarrow \alpha_2 | \text{ for some } \beta, \\ \alpha_1 \rightarrow \beta, \beta \rightarrow \alpha_2 \in A\}.$$

In that case, we would have $Cn_2^*(B) = B \cup \{j \rightarrow b\}$, what results in $Cn_2^*(B) \perp (j \rightarrow p) = \{\{p \rightarrow b, j \rightarrow b\}\}$. As in the last example, the selection function must choose the only member of the remainder set, therefore $B -_* (j \rightarrow p) = \{p \rightarrow b, j \rightarrow b\}$. It is interesting to note that we could also use here $Cn_{2'}^*(B)$ to be the set of all Horn consequences of B.

The following example was adapted from [Hansson, 1993].

Example 3 *Suppose I believe, for good and independent reasons, that* Andy is son of the mayor *(a) and* Bob is son of the mayor *(b). Then I hear the mayor say: "I certainly have nothing against our youth studying abroad. My only son did it for three years". I then have to retract* $a \wedge b$ *from my base* $B = \{a, b\}$. *But it is reasonable to retain a belief that either Andy or Bob is the son of the mayor, i.e., the result of the contraction should be* $\{a \vee b\}$.

The remainder set for the operation above is $B \perp (a \wedge b) = \{\{a\}, \{b\}\}$. So, the resulting partial meet contraction is either $\{a\}$, $\{b\}$ or $\emptyset$, the first two being odd since we do not seem to have reasons to prefer a over b or vice-versa.

In the same paper where he presented the example above, Hansson has done an extensive study of partial meet contraction on disjunctively closed bases. If we define $Cn_3^*(A)$ as the disjunctive closure of A, as defined by Hansson, that is, $Cn_3^*(A)$ is the set of sentences that are either elements of A or disjunctions of elements of A, we can manage to get the desired result.

$$Cn_3^*(A) = A \cup \{\bigvee \alpha_i | \alpha_i \in A\}$$

We have $Cn_3^*(B) = \{a, b, a \vee b\}$. Then, the remainder set is $Cn_3^*(B) \perp (a \wedge b) = \{\{a, a \vee b\}, \{b, a \vee b\}\}$, and so the selection function may choose both sets, producing the expected result in the lack of evidence for a or b: $B -_* (a \wedge b) = \{a \vee b\}$.

Consider now the case where the language has three propositional letters (a, b, and c). If we take the belief set $K = Cn(B)$, we have that K contains $a \vee c$ and $b \vee c$. It is not hard to see that there are two remainder sets containing $\{a, a \vee b, a \vee c, b \vee c\}$, $\{b, a \vee b, a \vee c, b \vee c\}$ and hence, these two formulas may survive contraction, even if the original set did not mention c.

6 Related Work

There have been several attempts in the literature to study AGM-like contraction operations based on different consequence operations.

Concerning belief bases, Hansson and Wassermann 2002 have shown that the original representation results for characterizing partial meet contraction only depended on the underlying logic being compact and monotonic. The main difference from what we are proposing in this paper is that the underlying consequence operator C is used for computing the remainder sets and everywhere in the postulates. So, for example, if we take as underlying logic one for approximate reasoning, as suggested in [Chopra *et al.*, 2001], the remainders will be different than the ones we use for our $-_*$ operation. If the approximation is done from below (cf. [Schaerf and Cadoli, 1995]), then each element of the approximate remainder will contain an element of the classical remainder. **Success** will also be computed in terms of the chosen underlying consequence operation, so the outcome of the contraction by α may still classically imply α.

More recently, there has been a series of papers considering Horn Contraction [Delgrande, 2008; Booth *et al.*, 2011; Delgrande and Wassermann, 2013], which again, replace all classical reasoning by Horn reasoning. Contraction of belief sets has also been studied for non-classical logics [Ribeiro *et al.*, 2013; Ribeiro, 2013] along the same lines of the work on belief bases.

The approach of [Meyer, 2001] is similar to ours in the fact that it "weakens" the formulas that must be removed from the base, thus it is also a pseudo-contraction. However, their purpose is different since they do not take computational costs into account (they define base contraction from theory contraction, using logical closure), whereas it is one of our main concerns in this paper.

The closest proposal to the one described in this paper is the idea of disjunctively closed belief bases, proposed by Hansson [1993], where a single closure is studied, the one we used in Example 3.

To the best of our knowledge, [Ribeiro and Wassermann, 2008] was the first proposal where general alternative consequence operations are used only to extend the set of formulas being considered on contraction. The present work follows the line of [Ribeiro and Wassermann, 2008] (even borrowing the Cn^* notation), but with a different focus. While the idea there was to study forms of **recovery**, here we are interested in characterizing partial meet operations where the initial belief state is closed under some subclassical consequence operator.

7 Conclusion

In this paper, we have proposed an operation of pseudo-contraction, denoted by $-_*$. We have provided a characterization of this operation in terms of postulates which were adapted from the classical ones to use subclassical operators of consequence, which we denote by Cn^*.

One of the drawbacks of the construction of $-_*$ is that it satisfies **enforced closure***, i.e., the result of contraction is always closed under Cn^*, causing $-_*$ to violate **vacuity**.

We have found a way to partially circumvent the problem by defining the $-'_*$ operator. We have also noted that it would be worthwhile that the operator could satisfy the postulate of core-addition, that avoid needless inclusions. Finally, we conclude that these two flaws would be possibly amended by the use of the pseudo-contractions proposed in [Ribeiro and Wassermann, 2008].

Despite these problems, we have shown some practical examples of use of this operator in which it does better than the direct application of partial meet contraction for bases. The practical usefulness of this operator is limited on isolation, but the importance of the study of its properties can be better understood in the context of the pseudo-contraction operators suggested in [Ribeiro and Wassermann, 2008].

Future work involves exploring the use of different Cn^* to model real problems encountered in reasoning with ontologies and with Horn theories.

Appendix

Proof of Theorem 8:

Construction-to-postulates: We know that $A -_* \alpha = \bigcap \gamma(Cn^*(A) \perp \alpha) = Cn^*(A) -_\gamma \alpha$, where $-_\gamma$ is the partial meet contraction. We also know that $-_\gamma$ satisfies success, inclusion, relevance and uniformity. So, we have:

- If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(Cn^*(A) -_\gamma \alpha)$
- $Cn^*(A) -_\gamma \alpha \subseteq Cn^*(A)$
- If $\beta \in Cn^*(A) \setminus (Cn^*(A) -_\gamma \alpha)$, then there is a B' such that $Cn^*(A) -_\gamma \alpha \subseteq B' \subseteq Cn^*(A)$, $\alpha \notin Cn(B')$, but $\alpha \in Cn(B' \cup \{\beta\})$.
- If for all $B' \subseteq Cn^*(A)$, $\alpha \in Cn(B')$ if and only if $\beta \in Cn(B')$, then $Cn^*(A) -_\gamma \alpha = Cn^*(A) -_\gamma \beta$.

Since $Cn^*(A) -_\gamma \alpha = A -_* \alpha$, we are done.

Postulates-to-construction: This part is is almost trivially obtained from the proof of the representation theorem for partial meet contraction for bases, which can be found in [Hansson, 1999].

Let $-_*$ be an operation for A that satisfies success, inclusion*, relevance* and uniformity*. From the last proof and corollary 7 we conclude that $-_*$ also satisfies logical relevance and uniformity. Let γ be a function such that:

- If $Cn^*(A) \perp \alpha = \emptyset$, then $\gamma(Cn^*(A) \perp \alpha) = \{Cn^*(A)\}$.
- Otherwise $\gamma(Cn^*(A) \perp \alpha) = \{X \in Cn^*(A) \perp \alpha \mid A -_* \alpha \subseteq X\}$

We need to show that (1) γ is a well-defined function, (2) γ is a selection function and (3) $\bigcap \gamma(Cn^*(A) \perp \alpha) = A -_* \alpha$ for all α.

Part 1: For γ to be a well-defined function, for all α and β, if $Cn^*(A) \perp \alpha = Cn^*(A) \perp \beta$, we must have $\bigcap \gamma(Cn^*(A) \perp \alpha) = \bigcap \gamma(Cn^*(A) \perp \beta)$. Suppose that $Cn^*(A) \perp \alpha = Cn^*(A) \perp \beta$. It follows from observation 1.39 in [Hansson, 1999] that any subset of $Cn^*(A)$ implies α if and only if it implies β. By uniformity, $Cn^*(A) -_* \alpha = Cn^*(A) -_* \beta$. By the definition of γ we have $\gamma(Cn^*(Cn^*(A)) \perp \alpha) = \gamma(Cn^*(Cn^*(A)) \perp \beta)$. Since Cn^* is Tarskian, by idempotence, the result follows.

Part 2: For γ to be a selection function it remains to be proven that if $Cn^*(A) \perp \alpha$ is not empty, then $\gamma(Cn^*(A) \perp \alpha)$ is not empty as well. Then, assuming $Cn^*(A) \perp \alpha \neq \emptyset$, we know that there is at least one $X \in Cn^*(A) \perp \alpha$, and we must show that at least one of these X contains $A -_* \alpha$. Since $Cn^*(A) \perp \alpha$ is not empty, $\alpha \notin Cn(\emptyset)$, and by success, $\alpha \notin Cn(A -_* \alpha)$. By inclusion*, $A -_* \alpha \subseteq Cn^*(A)$, then, by the upper bound property [Alchourrón and Makinson, 1981], there is an A' such that $A -_* \alpha \subseteq A'$ and $A' \in Cn^*(A) \perp \alpha$. By the construction of γ, $\gamma(Cn^*(A) \perp \alpha)$ is non-empty.

Part 3: Case 1, $\alpha \in Cn(\emptyset)$. Then, by logical relevance, since there is no A' such that $\alpha \notin Cn(A')$, no element is in $A \setminus A -_* \alpha$, then, using inclusion*, $A \subseteq A -_* \alpha \subseteq Cn^*(A)$. We know that $Cn^*(A) \perp \alpha = \emptyset$, then $\bigcap \gamma(Cn^*(A) \perp \alpha) = Cn^*(A)$. We need to show that $Cn^*(A) \subseteq A -_* \alpha$. By relevance*, we know that $Cn^*(A) \setminus A -_* \alpha = \emptyset$, then $Cn^*(A) \subseteq A -_* \alpha$.

Case 2, $\alpha \notin Cn(\emptyset)$. $Cn^*(A) \perp \alpha$ is non-empty and by part 2, $\gamma(Cn^*(A) \perp \alpha)$ is non-empty as well. Since $A -_* \alpha$ is a subset of all elements of $\gamma(Cn^*(A) \perp \alpha)$, $A -_* \alpha \subseteq \bigcap \gamma(Cn^*(A) \perp \alpha)$. We need to show that $\bigcap \gamma(Cn^*(A) \perp \alpha) \subseteq A -_* \alpha$.

Take $\varepsilon \notin A -_* \alpha$. If $\varepsilon \notin Cn^*(A)$, obviously $\varepsilon \notin \bigcap \gamma(Cn^*(A) \perp \alpha)$. If $\varepsilon \in Cn^*(A) \setminus A -_* \alpha$, then by relevance* there is an A' such that $A -_* \alpha \subseteq A' \subseteq Cn^*(A)$, $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\varepsilon\})$. It follows from the upper bound property that there is an A'' such that $A \subseteq A''$ and $A'' \in Cn^*(A) \perp \alpha$. From $A \subseteq A''$, $\alpha \in Cn(A' \cup \varepsilon)$ and $\varepsilon \in A''$ we conclude that $\alpha \in Cn(A'')$, so we must have $\varepsilon \notin A''$. By our definition of γ, $A'' \in \gamma(Cn^*(A) \perp \alpha)$, and since $\varepsilon \notin A''$, we conclude that $\varepsilon \notin \bigcap \gamma(Cn^*(A) \perp \alpha)$. ■

Proof of Proposition 10: We know that $A -_* \alpha = Cn^*(A) -_\gamma \alpha$, where $-_\gamma$ is the partial meet contraction. Since partial meet satisfies relative closure [Hansson, 1999], $Cn^*(A) \cap Cn(Cn^*(A) -_\gamma \alpha) \subseteq Cn^*(A) -_\gamma \alpha$ is valid. From this we have $Cn^*(A) \cap Cn(A -_* \alpha) \subseteq A -_* \alpha$. By the inclusion property of Cn^* (which is Tarskian) and set theory we get $A \cap Cn(A -_* \alpha) \subseteq Cn^*(A) \cap Cn(A -_* \alpha)$. ■

Proof of Proposition 11: Since Cn^* is Tarskian, by inclusion, $A - \alpha \subseteq Cn^*(A - \alpha)$. We want to show that $Cn^*(A - \alpha) \subseteq A - \alpha$. Suppose by contradiction that $\beta \in Cn^*(A - \alpha) \setminus (A - \alpha)$. From inclusion*, monotonicity and idempotence of Cn^* we obtain $\beta \in Cn^*(A) \setminus A - \alpha$. Relevance* guarantees that there is an A' such that $A - \alpha \subseteq A' \subseteq Cn^*(A)$, $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$. By subclassicality of Cn^* and $\beta \in Cn^*(A - \alpha)$ we have $\beta \in Cn(A - \alpha)$. By $A - \alpha \subseteq A'$ and by the inclusion property of Cn, $\beta \in Cn(A')$. So, we have $Cn(A') = Cn(A' \cup \{\beta\})$, which is a contradiction. ■

Proof of Proposition 12: Assume $\alpha \notin Cn(B)$.

By inclusion* we already have $B - \alpha \subseteq Cn^*(B)$. To finish the proof it is sufficient to prove that $Cn^*(B) \setminus (B - \alpha) = \emptyset$.

By relevance*, if $\beta \in Cn^*(B) \setminus (B - \alpha)$, then there is a B' such that $B - \alpha \subseteq B' \subseteq Cn^*(B)$ and $\alpha \in Cn(B' \cup \{\beta\})$. From this and subclassicality of Cn^* we get $B' \cup \{\beta\} \subseteq Cn^*(B) \subseteq Cn(B)$ and then by monotony and idempotence of Cn we have $Cn(B' \cup \{\beta\}) \subseteq Cn(B)$. Since $\alpha \notin Cn(B)$ by assumption, we cannot have such α. ■

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