Type-2 Fuzzy Uncertainty in Goal Programming

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Abstract

This paper presents a goal programming model for problems where resources are defined by the opinion of multiple experts. Through the use of Type-2 fuzzy sets, we propose a model that includes human being like information in order to define the parameters of a goal programming problem, and then solve it using a constructive approach that uses LP models due to its efficiency. An application example is provided and explained, and some concluding remarks are provided.

1 Introduction

Decision making in practical applications has to face human being interaction and social aspects. Some situations have to solve multiple goals involving multiple people that try to solve the same problem with different objectives. To solve those problems, goal programming offers an efficient tool to find a solution.

A common situation in applied goal programming includes multiple experts and uncertainty around the exact value of a desired goal, where fuzzy sets appear as a useful tool for handling uncertainty coming from different people. Classical fuzzy goal programming has been proposed by Narasimhan [Narasimhan, 1980], and later developed by Yang [Yang *et al.*, 1991], Turgay & Taşkın [Safiye and Harun, 2014], Li & Gang [Li, 2012],Hu, Zhang & Wang [Hu *et al.*, 2014], Khalili-Damghani & Sadi-Nezhad [Khalili-Damghani *et al.*, 2013], in both theoretical and practical situations.

Using the results of Narasimhan [Narasimhan, 1980], Yang [Yang *et al.*, 1991] has designed a smaller model (in terms of amount of variables) that leads to the same solution. In this paper we propose to extend the classical goal programming problem to a case where multiple experts deal with multiple goals by using Type-2 fuzzy sets and α-cuts to handle linguistic/numerical uncertainty coming from experts and Linear Programming (LP) methods for handling goal programming.

The paper is organized as follows: Section 1 introduces the main problem. Section 2 presents some basics on fuzzy sets. In Section 3, goal programming LP model is referred. Section 4 presents the Yang [Yang *et al.*, 1991] proposal for fuzzy goal programming. Section 5 contains the proposal; Section 6 shows an application example; and finally Section 7 presents the concluding remarks of the study.

2 Basic on Fuzzy sets

According to Klir and Yuan [Klir and Yuan, 1995], a fuzzy set is a function $A : X \to [0, 1]$. The notation μ_A is equivalent to describe the membership function μ that describes A, this is $\mu_A : X \to [0, 1]$ where $x \in X$ is the universe of discourse over A is defined, as follows:

$$\begin{aligned} A : X &\to [0,1] \\ A = (x, \mu_A(x)) &: x \in X \end{aligned} \tag{1}$$

2.1 Type-2 Fuzzy Sets

A Type-2 Fuzzy set, Mendel [Mendel, 2001] is an ordered pair $\{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where A is a linguistic label $\tilde{A}$ represents the uncertainty about the word A. And its mathematical definition is:

$$\begin{aligned} \tilde{A} : X &\to \mathcal{F}[0,1] \\ \tilde{A} = (x, \mu_{\tilde{A}}(x)) &: x \in X \\ \tilde{A} = \int_{x\in X} \int_{u \in J_x} f_x(u)/(x,u),\ & J_x \subseteq [0,1] \end{aligned} \tag{2}$$

where $f_x(u)/u$ is a secondary membership function of $\tilde{A}$ on $x \in X$ and u is the domain of uncertainty.

Why Fuzzy Sets? Fuzzy sets has the property of handling uncertainty coming from human knowledge, which commonly appear in decision making. In the case of numerical uncertainty, fuzzy sets handle imprecision about X that appears in cease where no historical/statistical data is available, so the only way to estimate parameters and/or variables is by using approximate information coming from the experts of the problem that can be represented through fuzzy numbers.

2.2 α-cuts

One of the most used ways to decompose A is through α-cuts. The α-cut of a A, namely ${}^{\alpha}A$, is defined as:

$${}^{\alpha}A = \{x \in X : \mu_A(x) \geqslant \alpha\}, \tag{3}$$

Thus, a fuzzy set A is the union of its α-cuts, $\bigcup_{\alpha\in[0,1]} \alpha \cdot {}^{\alpha}A$, where $\cup$ denotes union [Klir and Yuan, 1995]. Now, the extension of α-cut of A to the α-cut of $\tilde{A}$ (see [Figueroa-García *et al.*, 2015]) allows us to say that the primary α-cut of an Interval Type-2 fuzzy set ${}^{\alpha}\tilde{A}$ is the union of all $x \in X$ whose primary memberships J_x are greater than α, $J_x \geqslant \alpha$, this is:

$$
{}^{\alpha}\tilde{A} = \{x \in X : \mu_{\tilde{A}}(x, u) \geqslant \alpha;\ u \in J_x \subseteq [0, 1]\}, \tag{4}
$$

3 Goal programming

Charnes, Cooper & Wagner [Charnes and Cooper, 1961; 1977] has proposed an LP model that tries to minimize deviations from different goals (desired objectives) through minimizing the absolute deviations d_k of the constraints of the problem $A_k x$ regarding its desired value B_k e g. $\min_k\{D = \sum_{k=1}^{n} |A_k x - B_k|\}$. This model is equivalent to the following LP model (see Charnes, Cooper & Wagner [Charnes and Cooper, 1961; 1977]):

$$
\begin{gathered}
\min_k \sum_{k=1}^{n} d_{k1} + d_{k2} \\
s.t. \\
A_k x + d_{k1} - d_{k2} = B_k, \\
A'_k x \leq B'_k \\
x, d_{k1}, d_{k2} \geq 0; \forall\, k,
\end{gathered} \tag{5}
$$

where $B_k \in \mathbb{R}$ is the aspiration level, $d_{k1}, d_{k2} \in \mathbb{R}$ are negative and positive deviations from the goal B_k, A_k is the set of n constraints related to goals, A'_k is a set of crisp constraints of the problem, B'_k is its set of boundaries, and $x \in \mathbb{R}^m$ is the set of decision variables of the problem. A negative deviation quantifies a lack of satisfaction of the desired aspiration level, and a positive deviation quantifies an excess over the desired aspiration level.

4 Fuzzy Goal Programming

Fuzzy goal programming has been proposed by Narasimhan [Narasimhan, 1980], Narasimhan & Hanna [Hannan, 1981], and Yang [Yang *et al.*, 1991] has proposed a smaller model that obtains an equivalent solution that the presented by [Narasimhan, 1980; Hannan, 1981]. Yang's proposal defines the membership function of the k_{th} fuzzy goal B_k namely μ_{B_k}, as follows:

$$
\mu_{B_k} = \begin{cases}
0 & \textit{if}\ G_k(x) \leq b_k + b_{k2}, \\
1 - \dfrac{G_k(x) - b_k}{b_{k2}}, & \textit{if}\ b_k \leq G_k(x) \leq b_k + b_{k2}, \\
1 & \textit{if}\ G_k(x) = b_k, \\
1 - \dfrac{b_k - G_k(x)}{b_{k1}}, & \textit{if}\ b_k - b_{k1} \leq G_k(x) \leq b_k, \\
0 & \textit{otherwise},
\end{cases} \tag{6}
$$

where $k \in n$ denotes the k_{th} goal, $G_k(x)$ is the k_{th} constraint to be fulfilled, $b_k \in \mathbb{R}$ is the aspiration level of the k_{th} goal, and d_{k1} and d_{k2} are the maximum negative and positive deviations from b_k, respectively. Then the resulting LP model is

$$
\begin{gathered}
\min_k \sum_{k=1}^{n} d_{k1} + d_{k2} \\
s.t. \\
A_k x + d_{k1} - d_{k2} \cong \tilde{B}_k, \\
A'_k x \leq B'_k \\
x, d_{k1}, d_{k2} \geq 0; \forall\, k,
\end{gathered} \tag{7}
$$

where $\tilde{B}_k \in \mathcal{F}_1$ the fuzzy aspiration level, $d_{k1}, d_{k2} \in \mathbb{R}$ are negative and positive deviations from the goal b_k, A_k is the set of n constraints related to fuzzy goals, A'_k is a set of crisp constraints of the problem, B'_k is its set of boundaries, and $x \in \mathbb{R}^m$ is the set of decision variables of the problem.

Every Type-2 fuzzy goal is defined by its LMF and UMF, as shown as follows:

$$
\overline{\mu}_{\tilde{b}_k} = \begin{cases}
0 & \textit{if}\ G_k(x) \leq b_k + \overline{b}_{k2}, \\
1 - \dfrac{G_k(x) - b_k}{\overline{b}_{k2}}, & \textit{if}\ \overline{b}_k \leq G_k(x) \leq b_k + \overline{b}_{k2}, \\
1 & \textit{if}\ G_k(x) = b_k, \\
1 - \dfrac{b_k - G_k(x)}{\overline{b}_{k1}}, & \textit{if}\ b_k - \overline{b}_{k1} \leq G_k(x) \leq b_k, \\
0 & \textit{otherwise},
\end{cases} \tag{8}
$$

$$
\underline{\mu}_{\tilde{b}_k} = \begin{cases}
0 & \textit{if}\ G_k(x) \leq \underline{b}_k + \underline{b}_{k2}, \\
1 - \dfrac{G_k(x) - \underline{b}_k}{\underline{b}_{k2}}, & \textit{if}\ \underline{b}_k \leq G_k(x) \leq \underline{b}_k + \underline{b}_{k2}, \\
1 & \textit{if}\ G_k(x) = \underline{b}_k, \\
1 - \dfrac{\underline{b}_k - G_k(x)}{\underline{b}_{k1}}, & \textit{if}\ \underline{b}_k - \underline{b}_{k1} \leq G_k(x) \leq \underline{b}_k, \\
0 & \textit{otherwise},
\end{cases} \tag{9}
$$

where $\overline{\mu}$ defines the LMF of the k_{th} goal, and $\underline{\mu}$ defines the UMF of the k_{th} goal. A graphical display of a Type-2 fuzzy goal is shown in Figure 1.

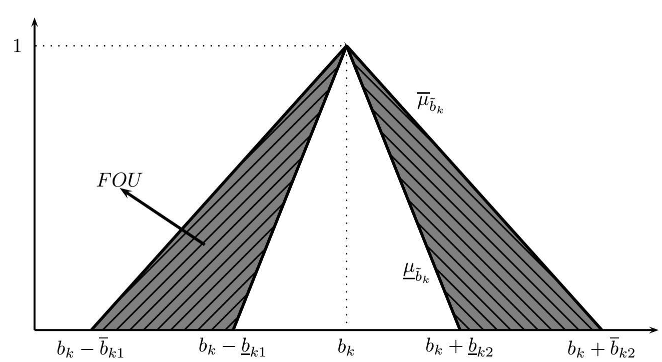

Figure 1: Interval Type-2 fuzzy goal $\tilde{B}_k$

4.1 The Proposal

Our proposal extends the classical fuzzy goal programming model to a Type-2 fuzzy environment, as follows:

$$\begin{aligned} \min_{k} \ & \sum_{k=1}^{n} d_{k1} + d_{k2} \\ & s.t. \\ A_k x + d_{k1} - d_{k2} & \approx \tilde{B}_k, \qquad (10) \\ A'_k x & \leq B'_k \\ x, d_{k1}, d_{k2} & \geq 0; \forall\, k, \end{aligned}$$

where $\tilde{B}_k \in \mathbb{R}$ is a Type-2 fuzzy aspiration level, $d_{k1}, d_{k2} \in \mathbb{R}$ are negative and positive deviations from the goal $\tilde{B}_k$, A_k is the set of n constraints related to goals, A'_k is a set of crisp constraints of the problem, B'_k is its set of boundaries, and $x \in \mathbb{R}^m$ is the set of decision variables of the problem.

The proposed approach to find a solution of the problem is by using a constructive method based on α-cuts which basically decomposes $\tilde{B}_k$ into α-cuts and find a crisp solution for every of the 4 boundaries of every α-cut. The method is described as follows.

5 α-cuts and deviations in Fuzzy Goal Programming

There is a relationship between satisfaction levels, α-cuts, and the goal value. It is clear that there exists a set X that satisfies every α-cut which leads to two intervals, one for the left side $[{}^{\alpha}\hat{B}_{k,l}, {}^{\alpha}\check{B}_{k,l}]$ and one for the right side $[{}^{\alpha}\check{B}_{k,r}, {}^{\alpha}\hat{B}_{k,r}]$ which are computed using Eq. (4) and shown as follows:

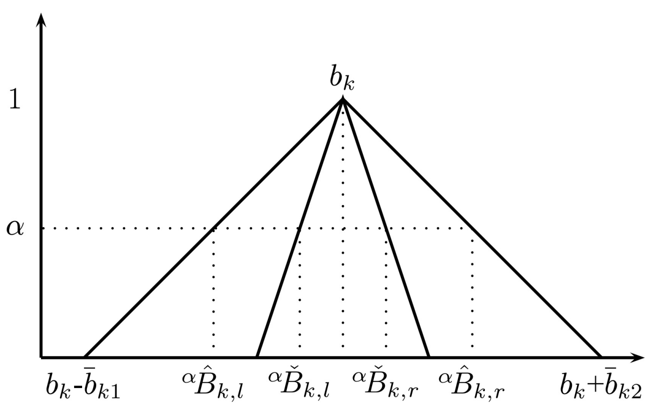

Figure 2: FGP

where ${}^{\alpha}\hat{B}_{k,l}, {}^{\alpha}\check{B}_{k,l}$ are the left values of the cut for its UMF and LMF respectively, and ${}^{\alpha}\check{B}_{k,r}, {}^{\alpha}\hat{B}_{k,r}$ are the right values of the cut for its LMF and UMF respectively. To do so, all crisp boundaries of $\tilde{B}_{k,r}$ are computed as follows:

$${}^{\alpha}\hat{B}_{k,l} = (b_k - \overline{b}_{k1}) + \alpha(b_k - (b_k - \overline{b}_{k1})), \qquad (11)$$

$${}^{\alpha}\check{B}_{k,l} = (b_k - \underline{b}_{k1}) + \alpha(b_k - (b_k - \underline{b}_{k1})), \qquad (12)$$

$${}^{\alpha}\hat{B}_{k,r} = (b_k + \overline{b}_{k2}) - \alpha((b_k + \overline{b}_{k2}) - b_k), \qquad (13)$$

$${}^{\alpha}\check{B}_{k,r} = (b_k + \overline{b}_{k2}) - \alpha((b_k + \overline{b}_{k2}) - b_k), \qquad (14)$$

Then from the k goal values the value of the deviations in the linear goal programming problem (7) are computed, as a four-step LP method which finds the following crisp solutions:

$${}^{\alpha}\check{B}_{k,l} \rightarrow {}^{\alpha}\check{z}_l \qquad (15)$$

$${}^{\alpha}\hat{B}_{k,l} \rightarrow {}^{\alpha}\hat{z}_l \qquad (16)$$

$${}^{\alpha}\check{B}_{k,r} \rightarrow {}^{\alpha}\check{z}_r \qquad (17)$$

$${}^{\alpha}\hat{B}_{k,r} \rightarrow {}^{\alpha}\hat{z}_r \qquad (18)$$

Now, every set of goals ${}^{\alpha}\check{B}_{k,l}, {}^{\alpha}\hat{B}_{k,l}, {}^{\alpha}\check{B}_{k,r}, {}^{\alpha}\hat{B}_{k,r}$ has to be solved using (7). This way, the set of Type-2 fuzzy goals $\tilde{B}$ leads to a set of optimal solutions $\tilde{z}$, as follows:

$$\tilde{B} \xrightarrow{f} \tilde{z} \qquad (19)$$

where f is a function, in this case an LP method.

6 Experimentation and results

As application example we use the proposed by [Narasimhan, 1980] and extended by [Chen and Tsai, 2001] which is composed by three fuzzy goals, as shown as follows:

$$\begin{aligned} G_1 &: 80x_1 + 40x_2 \cong 630, \\ G_2 &: x_1 \cong 7, \qquad (20) \\ G_3 &: x_2 \cong 4, \end{aligned}$$

where x_1 and x_2 are the manufacturing quantities of two products which regard to three goals: G_1 is a profit goal, and $G_2 - G_3$ are the expected selling quantities per product. The maximum deviations from $G_k = \{630, 7, 4\}$ and modifying them to get a Type-2 fuzzy goal programming which can be symmetrically handled where $\underline{b}_{k1} = \underline{b}_{k2} = \{10, 2, 2\}$ and $\overline{b}_{k1} = \overline{b}_{k2} = \{15, 3, 3\}$.

Using Eq. (7) we can obtain the values of the goals $G1, G2$ and $G3$ for every α-cut. The idea is then to minimize the deviations from the goals through Eqs. (7), so we obtain four crisp points that compose ${}^{\alpha}\tilde{z}$ and therefore $\tilde{z}$ as stated in Eq. (19).

α-cut	d_{11}	d_{12}	d_{21}	d_{22}	d_{31}	d_{32}
0.1	0	0	0	1.46	0	0
0.2	0	0	0	1.18	0	0
0.3	0	0	0	0	0	1.78
0.4	0	0	0	0	0	1.20
0.5	0	0	0	1.00	1.38	0
0.6	2.00	0	0	0	0	0
0.7	0	0	1.00	0	0	1.48
0.8	0	0	0	0	1.10	0
0.9	0	0	0	0	1.68	0
1	0	0	1.13	0	0	0

Table 1: Optimal deviations for the left side UMF, LMF

As seen in Table 6, goal $G2$ was the only goal which obtained its desired value on its left side while its right side has a linear behavior (see Table 6). There is a nonlinear behavior on all deviations from goals even when all goals were accomplished, this is, there is no direct relationship between the

α-cut	X_1	X_2	OF
0.1	6.66	2.20	1.46
0.2	6.58	2.40	1.18
0.3	5.60	4.38	1.78
0.4	5.80	4.00	1.20
0.5	7.00	1.63	2.38
0.6	6.20	3.20	2.00
0.7	5.40	4.88	2.48
0.8	6.60	2.50	1.10
0.9	6.80	2.13	1.68
1	5.88	4.00	1.13

Table 2: Optimal variables X_1, X_2 for the left side UMF, LMF

α-cut	d_{11}	d_{12}	d_{21}	d_{22}	d_{31}	d_{32}
0.1	0	0	3.71	0	0	0
0.2	0	0	3.43	0	0	0
0.3	0	0	3.14	0	0	0
0.4	0	0	2.85	0	0	0
0.5	0	0	2.56	0	0	0
0.6	0	0	2.28	0	0	0
0.7	0	0	1.99	0	0	0
0.8	0	0	1.70	0	0	0
0.9	0	0	1.41	0	0	0
1	0	0	1.13	0	0	0

Table 3: Optimal deviations for the right side UMF, LMF

objective function of the LP and the α-cuts, although the results of the right side (for both UMF and LMF) as a function of the α-cuts fit the shape of the goal. Roughly speaking, the behavior of the deviations is not a function of α.

Even when all goals were defined by linear UMFs and LMFs, the results of every α-cut have shown that the optimal solution (in terms of deviations from goals) are not linear, so GP problems seem to be nonlinearly shaped which confirms that fuzzy sets can efficiently represent nonlinear systems.

Also note that every goal is fulfilled for every α-cut with some deviations, so the real behavior of the problem is given by their deviations. In our example those deviations have shown a nonlinear behavior (chaotic in some sense) which provides some information to us: it seems that GP problems has no a predictable behavior. This happens because every α-cut operates as a single GP problem whose optimal deviations has no a linear relationship between α-cuts.

7 Conclusions and recommendation

There is not a direct relationship among α and the objective value given by the LP (7), this is because no matter what is the value of α is, the model tries to minimize their deviations, turning out decision variables in a nonlinear way.

The example shows an interesting behavior: when deviations d_{21} always are zero, the expected shapes of the goals are accomplished, in this case its right shape. For the left side, the expected shape is not reached due to the deviations have a nonlinear behavior.

Our recommendation is to analyze every α-cut as a single problem. We can see an α-cut as a fuzzy aspiration level

α-cut	X_1	X_2	OF
0.1	5.09	5.80	3.71
0.2	5.18	5.60	3.43
0.3	5.26	5.40	3.14
0.4	5.35	5.20	2.85
0.5	5.44	5.00	2.56
0.6	5.53	4.80	2.28
0.7	5.61	4.60	1.99
0.8	5.70	4.40	1.70
0.9	5.79	4.20	1.41
1	5.88	4.00	1.13

Table 4: Optimal variables X_1, X_2 for the left side UMF, LMF

of every goal $\tilde{B}_k$ that comes from the opinion of multiple experts, so its optimal solution should be interpreted apart from other α-cuts. A practical way to find a crisp solution is by selecting an α-cut and then solve the problem keeping in mind its results.

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