Algebraic Semantics for Graded Propositions

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Abstract. We present $Log_A\mathbf{G}$, an algebraic language for reasoning about graded propositions. $Log_A\mathbf{G}$ is algebraic in that it is a language of only terms, some of which denote propositions. Both propositions and their grades are taken as individuals in the $Log_A\mathbf{G}$ ontology. Thus, the language includes terms denoting graded propositions, grades of propositions, grading propositions, and graded grading propositions in an arbitrary compositional structure. In this paper, we present the syntax and semantics of $Log_A\mathbf{G}$, defining an infinite sequence of graded logical consequence relations, each corresponding to accepting graded propositions at some nesting depth. We show the utility of $Log_A\mathbf{G}$ in default reasoning, reasoning about information provided by a chain of sources with varying degrees of trust, and representing the dilemma one is in when facing paradoxical liar-like sentences.

1 Introduction

Graded, or weighted, logics have witnessed increased attention and interest over the years, which may be attested by the sheer length of the bibliography of a recent comprehensive survey [1]. Whether the interest is in modelling uncertain beliefs [2, 3, for instance], reasoning with vague predicates [4, 5, for instance], revising a logical theory [6], or jumping to default conclusions [7], weighted logics are always an obvious resort. To demonstrate the variety of phenomena falling under the rubric of weighted logics, we present three problems (two of which are classical) that we shall carefully revisit in Section 4.

The Case of Opus and Tweety. Tweety is a bird and Opus is a penguin. You believe that penguins are birds. In the absence of other information, you would like to jump to the conclusion that Tweety flies and Opus does not. How do you do so gracefully and without succumbing to absurdity?

The Case of Superman. You open the Daily Planet and read a report by Lois Lane claiming that Superman was seen in downtown Metropolis at noon. You happen to have seen Clark Kent at his office at noon, and you have always had a feeling that Superman is Clark Kent. What should you believe about the whereabouts of Superman if you trust your perception very much, you trust Lois Lane's honesty, you only mildly trust the Daily Planet, and you still have your doubts about whether Superman is indeed Clark Kent?

The Case of the Liar. The first sentence of the paragraph titled "The Case of the Liar" in this paper is not true. Having read the previous sentence, should you believe it or not? (cf. [8].)

Weighted logics have something to say about each of the above cases (or, at least, so we claim). In this report, we present the syntax and semantics of a family of logical languages, $Log_A\mathbf{G}$, for reasoning about graded propositions and, in Section 4, we demonstrate how the above cases are treated within $Log_A\mathbf{G}$ theories. While most of the weighted logics we are aware of employ some form of non-classical, possible-worlds, semantics, basically assigning some notion of grade to possible worlds or truth values, $Log_A\mathbf{G}$ is a non-modal logic, with classical notions of worlds and truth values. This is not to say that $Log_A\mathbf{G}$ is a common classical logic—it surely is not—but it is closer in spirit to classical non-monotonic logics in artificial intelligence [9, 7, for example].[3] In such formalisms, as in $Log_A\mathbf{G}$, there is a classical logical consequence relation on top of which we define a non-classical relation which is more restrictive, selecting only a subset of the classical models. We achieve this by taking the algebraic, rather than the modal, route.

$Log_A\mathbf{G}$ is algebraic in the sense that it only contains terms, algebraically constructed from function symbols. No sentences are included in a $Log_A\mathbf{G}$ language; instead, there are terms of a distinguished syntactic type that are taken to denote propositions. $Log_A\mathbf{G}$ is a variant of $Log_A\mathbf{B}$ [10] and $Log_A\mathbf{S}$ [11], which are algebraic languages for reasoning about, respectively, beliefs and temporal phenomena. The inclusion of propositions in the ontology, though non-standard, has been suggested by several authors [12–15, for example]. (See [10] and [15] for a thorough defense of this position.) In the $Log_A\mathbf{G}$ ontology, propositions are structured in a Boolean algebra, giving us, almost for free, all standard truth conditions and standard notions of consequence and validity. In addition, we also admit *grades* as first-class individuals in the ontology. Thus, we combine propositions and grades to construct propositions *about* graded propositions, which, recursively, are themselves gradable. This yields a language that is on one hand quite expressive and, on the other hand, intuitive and very similar in syntax to first-order logic.[4]

2 Log_AG Languages

$Log_A\mathbf{G}$ is a class of many-sorted languages that share a common core of logical symbols and differ in a signature of non-logical symbols. In what follows, we identify a sort σ with the set of symbols of sort σ. A $Log_A\mathbf{G}$ language is a set of terms partitioned into three base syntactic sorts, σ_P, σ_D and σ_I. Intuitively, σ_P

[3] But it is neither second-order like circumscriptive theories [9] nor dependent on special default rules like default logic [7].

[4] While multi-modal logics such as those presented in [16] and [17] may be used to express graded *grading* propositions, the grades themselves are embedded in the modal operators and are not amenable to reasoning and quantification.

is the set of terms denoting propositions, σ_D is the set of terms denoting grades of propositions, and σ_I is the set of terms denoting anything else.

As is customary in many-sorted languages, an alphabet of *Log*$_A$**G** is made up of a set of syncategorematic punctuation symbols and a set of denoting symbols each from a set $\sigma = \{\sigma_P, \sigma_D, \sigma_I\} \cup \{\tau_1 \longrightarrow \tau_2 | \tau_1 \in \{\sigma_P, \sigma_D, \sigma_I\}$ and $\tau_2 \in \sigma\}$ of syntactic sorts. Intuitively, $\tau_1 \longrightarrow \tau_2$ is the syntactic sort of function symbols that take a single argument of sort σ_P, σ_D, or σ_I and produce a functional term of sort τ_2. Given the restriction of the first argument of function symbols to base sorts, *Log*$_A$**G** is, in a sense, a first-order language.

A *Log*$_A$**G** alphabet is a union of four disjoint sets: $\Omega \cup \Xi \cup \Sigma \cup \Lambda$. The set Ω, the *signature* of the language, is a non-empty, countable set of constant and function symbols. Each symbol in the signature has a designated syntactic type from σ. The set $\Xi = \{x_i, d_i, p_i\}_{i \in \mathbb{N}}$ is a countably infinite set of variables, where $x_i \in \sigma_I$, $d_i \in \sigma_D$, and $p_i \in \sigma_P$, for $i \in \mathbb{N}$. Σ is a set of syncategorematic symbols, including the comma, various matching pairs of brackets and parentheses, and the symbol $\forall$. The set Λ is the set of logical symbols of *Log*$_A$**G**, defined as the union of the following sets.

1. $\{\neg\} \subseteq \sigma_P \longrightarrow \sigma_P$
2. $\{\wedge, \vee\} \subseteq \sigma_P \longrightarrow \sigma_P \longrightarrow \sigma_P$
3. $\{\prec, \doteq\} \subseteq \sigma_D \longrightarrow \sigma_D \longrightarrow \sigma_P$
4. $\{\mathbf{G}\} \subseteq \sigma_P \longrightarrow \sigma_D \longrightarrow \sigma_P$

A *Log*$_A$**G** language with signature Ω is denoted by L_Ω. It is the smallest set of terms formed according to the following rules; as usual, terms involving $\Rightarrow$, $\Leftrightarrow$, and $\exists$ may be introduced as abbreviations in the standard way.

– $\Xi \subset L_\Omega$
– $c \in L_\Omega$, where $c \in \Omega$ is a constant symbol.
– $f(t_1, \ldots, t_n) \in L_\Omega$, where $f \in \Omega$ is of sort $\tau_1 \longrightarrow \ldots \longrightarrow \tau_n \longrightarrow \tau$ $(n > 0)$ and t_i is of sort τ_i.
– $\{\neg t_1, (t_1 \wedge t_2), (t_1 \vee t_2), \forall x(t_1), \mathbf{G}(t_1, t_2), t_3 \prec t_4, t_3 \doteq t_4\} \subset L_\Omega$; where $t_1, t_2 \in \sigma_P$; $t_3, t_4 \in \sigma_D$; and $x \in \Xi$.

The basic ingredient of the *Log*$_A$**G** semantic apparatus is the notion of a *Log*$_A$**G** structure.

Definition 1. *A Log*$_A$**G** structure *is a quintuple* $\mathfrak{S} = \langle \mathcal{D}, \mathfrak{A}, \mathfrak{g}, <, \mathfrak{e} \rangle$*, where*

– *$\mathcal{D}$, the domain of discourse, is a set with two disjoint, non-empty, countable subsets $\mathcal{P}$ and $\mathcal{G}$.*
– *$\mathfrak{A} = \langle \mathcal{P}, +, \cdot, -, \perp, \top \rangle$ is a complete (closed under arbitrary products and sums), non-degenerate ($\top \neq \perp$) Boolean algebra [18].*
– *$\mathfrak{g} : \mathcal{P} \times \mathcal{G} \longrightarrow \mathcal{P}$.*
– *$<: \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{P}$ satisfies the following properties for every distinct $g_1, g_2, g_3 \in \mathcal{G}$:*
  O1. $g_1 < g_2 = -(g_2 < g_1)$.
  O2. $[(g_1 < g_2) \cdot (g_2 < g_3)] + g_1 < g_3 = g_1 < g_3$.

O3. $g_1 < g_1 = \bot$.
O4. $\sum_{g \in \mathcal{G}} g_1 < g = \sum_{g \in \mathcal{G}} g < g_1 = \top$.

– $\mathfrak{e} : \mathcal{G} \times \mathcal{G} \longrightarrow \{\bot, \top\}$, *where, for every* $g_1, g_2 \in \mathcal{G}$, $\mathfrak{e}(g_1, g_2) = \top$, *if* $g_1 = g_2$, *and* $\mathfrak{e}(g_1, g_2) = \bot$, *otherwise.*

Intuitively, the domain $\mathcal{D}$ is partitioned into three cells: (i) a set of propositions $\mathcal{P}$, structured as a Boolean algebra; (ii) a set of grades $\mathcal{G}$, and (iii) a set of individuals $\overline{\mathcal{P} \cup \mathcal{G}}$. These stand in correspondence to the syntactic sorts of $Log_A\mathbf{G}$. In what follows, we let $\mathcal{D}_{\sigma_P} = \mathcal{P}$, $\mathcal{D}_{\sigma_D} = \mathcal{G}$, and $\mathcal{D}_{\sigma_I} = \overline{\mathcal{P} \cup \mathcal{G}}$. $\mathfrak{g}$ is a function which maps a proposition p and a grade g to the proposition that p is a proposition of grade g. By refraining from imposing any constraints on $\mathfrak{g}$ (other than functionality), we are admitting virtually any intuitive interpretation of grading. Properties **O1**–**O4** require propositions in the range of $<$ to give rise to an irreflexive linear order on $\mathcal{G}$ which is serial in both directions. Similarly, the rigid definition of $\mathfrak{e}$ gives rise to the identity relation on $\mathcal{G}$.

Definition 2. *A* valuation $\mathcal{V}$ *of a* $Log_A\mathbf{G}$ *language* L_Ω *is a triple* $\langle \mathfrak{S}, \mathcal{V}_\Omega, \mathcal{V}_\Xi \rangle$, *where*

– $\mathfrak{S} = \langle \mathcal{D}, \mathfrak{A}, \mathfrak{g}, <, \mathfrak{e} \rangle$ *is a* $Log_A\mathbf{G}$ *structure;*
– $\mathcal{V}_\Omega$ *is a function that assigns to each constant of sort* τ *in* Ω *an element of* $\mathcal{D}_\tau$, *and to each function symbol* $f \in \Omega$ *of sort* $\tau_1 \longrightarrow \ldots \longrightarrow \tau_n \longrightarrow \tau$ *an* n*-adic function* $\mathcal{V}_\Omega(f) : \bigtimes_{i=1}^{n} \mathcal{D}_{\tau_i} \longrightarrow \mathcal{D}_\tau$; *and*
– $\mathcal{V}_\Xi : \Xi \longrightarrow \mathcal{D}$ *is a variable assignment, where, for every* $i \in \mathbb{N}$, $v_\Xi(p_i) \in \mathcal{D}_{\sigma_P}$, $v_\Xi(d_i) \in \mathcal{D}_{\sigma_D}$, *and* $v_\Xi(x_i) \in \mathcal{D}_{\sigma_I}$.

In what follows, for a valuation $\mathcal{V} = \langle \mathfrak{S}, \mathcal{V}_\Omega, \mathcal{V}_\Xi \rangle$ with $x \in \Xi$ of sort τ and $a \in \mathcal{D}_\tau$, $\mathcal{V}[a/x] = \langle \mathfrak{S}, \mathcal{V}_\Omega, \mathcal{V}_\Xi[a/x] \rangle$, where $\mathcal{V}_\Xi[a/x](x) = a$, and $\mathcal{V}_\Xi[a/x](y) = \mathcal{V}_\Xi(y)$ for every $y \neq x$.

Definition 3. *Let* L_Ω *be a* $Log_A\mathbf{G}$ *language and let* $\mathcal{V}$ *be a valuation of* L_Ω. *An* interpretation *of the terms of* L_Ω *is given by a function* $[\![\cdot]\!]^\mathcal{V}$:

– $[\![x]\!]^\mathcal{V} = \mathcal{V}_\Xi(x)$, *for* $x \in \Xi$
– $[\![c]\!]^\mathcal{V} = \mathcal{V}_\Omega(c)$, *for a constant* $c \in \Omega$
– $[\![f(t_1, \ldots, t_n)]\!]^\mathcal{V} = \mathcal{V}_\Omega(f)([\![t_1]\!]^\mathcal{V}, \ldots, [\![t_n]\!]^\mathcal{V})$, *for an* n*-adic* $(n \geq 1)$ *function symbol* $f \in \Omega$
– $[\![(t_1 \wedge t_2)]\!]^\mathcal{V} = [\![t_1]\!]^\mathcal{V} \cdot [\![t_2]\!]^\mathcal{V}$
– $[\![(t_1 \vee t_2)]\!]^\mathcal{V} = [\![t_1]\!]^\mathcal{V} + [\![t_2]\!]^\mathcal{V}$
– $[\![\neg t]\!]^\mathcal{V} = -[\![t]\!]^\mathcal{V}$
– $[\![\forall x(t)]\!]^\mathcal{V} = \prod_{a \in \mathcal{D}_\tau} [\![t]\!]^{\mathcal{V}[a/x]}$
– $[\![\mathbf{G}(t_1, t_2)]\!]^\mathcal{V} = \mathfrak{g}([\![t_1]\!]^\mathcal{V}, [\![t_2]\!]^\mathcal{V})$
– $[\![t_1 \prec t_2]\!]^\mathcal{V} = [\![t_1]\!]^\mathcal{V} < [\![t_2]\!]^\mathcal{V}$
– $[\![t_1 \doteq t_2]\!]^\mathcal{V} = \mathfrak{e}([\![t_1]\!]^\mathcal{V}, [\![t_2]\!]^\mathcal{V})$

In *Log*$_A$**G**, logical consequence is defined in pure algebraic terms without alluding to the notion of truth. This is achieved using the natural partial order $\leq$ associated with $\mathfrak{A}$ [18], where, for $p_1, p_2 \in \mathcal{P}$, $p_1 \leq p_2 =_{\text{def}} p_1 \cdot p_2 = p_1$.

Definition 4. *Let L_Ω be a Log$_A$**G** language. For every $\phi \in \sigma_P$ and $\Gamma \subseteq \sigma_P$, ϕ is a logical consequence of Γ, denoted $\Gamma \models \phi$, if, for every L_Ω valuation $\mathcal{V}$, $\prod_{\gamma \in \Gamma} [\![\gamma]\!]^{\mathcal{V}} \leq [\![\phi]\!]^{\mathcal{V}}$.*

In [10], it is shown that $\models$ has the distinctive properties of classical Tarskian logical consequence and that it satisfies a counterpart of the deduction theorem.

3 Graded Filters

In what follows, for every $p \in \mathcal{P}$ and $g \in \mathcal{G}$, we say that $\mathfrak{g}(p, g)$ *grades* p and that $\mathfrak{g}(p, g)$ is a *grading proposition*. Moreover, if $\mathfrak{g}(p, g) \in \mathcal{Q} \subseteq \mathcal{P}$, we say that p is *graded* in $\mathcal{Q}$. We define the set of *p graders in $\mathcal{Q}$* to be the set $G(p, \mathcal{Q}) = \{q | q \in \mathcal{Q}$ and q grades $p\}$. Throughout, we assume a *Log*$_A$**G** structure $\mathfrak{S} = \langle \mathcal{D}, \mathfrak{A}, \mathfrak{g}, <, \mathfrak{e} \rangle$.

According to Definition 4, the set of logical consequences of a set Γ of σ_P-terms corresponds to the filter $F([\![\Gamma]\!])$ generated by the set $[\![\Gamma]\!]$ of denotations of members of Γ [18]. In order to accommodate a richer, non-classical set of consequences which includes some *acceptable* propositions graded in $[\![\Gamma]\!]$, we need a more liberal notion of *graded filters*.

3.1 Embedding and Chains

In order to develop the notion of a graded filter, we need to sharpen our intuitions about the nesting structure of propositions graded in a given set.

Definition 5. *A proposition $p \in \mathcal{P}$ is embedded in $\mathcal{Q} \subseteq \mathcal{P}$ if (i) $p \in \mathcal{Q}$ or (ii) for some $g \in \mathcal{G}$, $\mathfrak{g}(p, g)$ is embedded in $\mathcal{Q}$. Henceforth, let $E(\mathcal{Q}) = \{p | p$ is embedded in $\mathcal{Q}\}$.*

Definition 6. *For $\mathcal{Q} \subseteq \mathcal{P}$, let $\delta_\mathcal{Q} : E(\mathcal{Q}) \longrightarrow \mathbb{N}$, where*

1. *if $p \in \mathcal{Q}$, then $\delta_\mathcal{Q}(p) = 0$; and*
2. *if $p \notin \mathcal{Q}$, then $\delta_\mathcal{Q}(p) = e + 1$, where $e = \min_{q \in G(p, E(\mathcal{Q}))}\{\delta_\mathcal{Q}(q)\}$.*

$\delta_\mathcal{Q}(p)$ is referred to as the degree of embedding of p in $\mathcal{Q}$.

In the sequel, we let $E^n(\mathcal{Q}) = \{p \in E(\mathcal{Q}) \mid \delta_\mathcal{Q}(p) \leq n\}$, for every $n \in \mathbb{N}$.

Definition 7. *A grading chain of $p \in \mathcal{P}$ is a finite sequence $\langle q_0, q_1, \ldots, q_n \rangle$ of grading propositions such that q_n grades p and q_i grades q_{i+1}, for $0 \leq i < n$. $\langle q_0, q_1, \ldots, q_n \rangle$ is a grading chain if it is a grading chain of some $p \in \mathcal{P}$.*

A grading chain $C_2 = \langle q_0, q_1, \ldots, q_n \rangle$ *extends* a grading chain C_1 if C_1 is a grading chain of q_0. The grading chain $C_1 \odot C_2$ is said to be an extension of C_1 (where $\odot$ denotes sequence concatenation).

We say that C is a grading chain in $\mathcal{Q}$, if every proposition in C is in $\mathcal{Q}$. Given that we impose no special restrictions on the function $\mathfrak{g}$ and the proposition algebra, grading chains in a set $\mathcal{Q}$ may, in general, be quite counter-intuitive. Hence, we need to introduce some especially interesting sets of propositions.

Definition 8. *Let $\mathcal{Q} \subseteq \mathcal{P}$.*

1. *A grading chain is well-founded if all its extensions are well-founded. $\mathcal{Q}$ is well-founded if every grading chain in $\mathcal{Q}$ is well-founded.*
2. *A grading chain $\langle q_0, q_1, \ldots, q_n \rangle$ is acyclic if, for every $0 \leq i, j \leq n$, $q_i = q_j$ only if $i = j$. $\mathcal{Q}$ is acyclic if every grading chain in $\mathcal{Q}$ is acyclic.*
3. *$\mathcal{Q}$ is depth-bounded if there is some $d \in \mathbb{N}$ such that every grading chain in $\mathcal{Q}$ has at most d distinct grading propositions.*
4. *$\mathcal{Q}$ is fan-out-bounded if there is some $f_{\text{out}} \in \mathbb{N}$ such that every grading proposition in $\mathcal{Q}$ grades at most f_{out} propositions.*
5. *$\mathcal{Q}$ is fan-in bounded if there is some $f_{\text{in}} \in \mathbb{N}$ where $|G(p, \mathcal{Q})| \leq f_{\text{in}}$, for every $p \in \mathcal{Q}$.*
6. *$\mathcal{Q}$ is non-explosive if for every $\mathcal{R} \subseteq \mathcal{Q}$, if $\mathcal{R}$ has finitely-many grading propositions, then so does $F(\mathcal{R})$.*

Given the above notions, if $p \in E(\mathcal{Q})$, then a grading chain C of p in $\mathcal{Q}$ is a *longest grading chain* of p in $\mathcal{Q}$ if

1. C is acyclic; and
2. if C extends a grading chain C', then $C' \odot C$ is not acyclic.

Proposition 1. [5] *Let $\mathcal{Q} \subseteq \mathcal{P}$.*

1. *If $\mathcal{P}$ is depth-bounded, then it is not acyclic.*
2. *If $\mathcal{Q}$ is depth-bounded and acyclic, then $\mathcal{Q}$ is well-founded.*
3. *If $E(\mathcal{Q})$ is depth-bounded, then there is some $n \in \mathbb{N}$ such that $\delta_{\mathcal{Q}}(p) \leq n$, for every $p \in E(\mathcal{Q})$.*
4. *If $\mathcal{P}$ is fan-out-bounded and $\mathcal{Q}$ is non-explosive with finitely-many grading propositions, then $E^n(F(\mathcal{Q}))$ has finitely-many grading propositions, for every $n \in \mathbb{N}$. Further, if $E(F(\mathcal{Q}))$ is depth-bounded, then it has finitely-many grading propositions.*
5. *If $\mathcal{Q}$ is depth-bounded, then every graded $p \in \mathcal{Q}$ has a longest grading chain in $\mathcal{Q}$. Further, if $\mathcal{Q}$ is fan-in-bounded, then every graded $p \in \mathcal{Q}$ has finitely-many longest grading chains in $\mathcal{Q}$.*

Note that, for the existence of longest grading chains, it suffices to have a single, bounded grading chain of p.

[5] Proofs of observations and propositions are omitted for space limitations. A longer version of the paper includes all the proofs.

3.2 Telescoping

The key to defining graded filters is the intuition that the set of consequences of a proposition set $\mathcal{Q}$ may be further enriched by *telescoping* $\mathcal{Q}$ and accepting some of the propositions embedded therein. For this, we need to define (i) the process of telescoping, which is a step-wise process that considers propositions at increasing degrees of embedding, and (ii) a criterion for accepting embedded propositions which, as should be expected, depends on the grades of said propositions.

Definition 9. *Let $\mathfrak{S}$ be a $Log_A\mathbf{G}$ structure with a depth- and fan-out-bounded $\mathcal{P}$. A telescoping structure for $\mathfrak{S}$ is a quadruple $\mathfrak{T} = \langle \mathcal{T}, \mathfrak{O}, \otimes, \oplus \rangle$, where*

- *$\mathcal{T} \subseteq \mathcal{P}$;*
- *$\mathfrak{O}$ is an ultrafilter of the subalgebra induced by Range($<$) (see [18]);*
- *$\otimes : \bigcup_{i=1}^{\infty} \mathcal{G}^i \longrightarrow \mathcal{G}$; and*
- *$\oplus : \bigcup_{i=1}^{\infty} \mathcal{G}^i \longrightarrow \mathcal{G}$ is commutative in the sense that $\oplus(t) = \oplus(\pi(t))$, where $\pi(t)$ is any permutation of the tuple t.*

Definition 10. *Let $\otimes : \bigcup_{i=1}^{\infty} \mathcal{G}^i \longrightarrow \mathcal{G}$, and let $C = \langle q_0, q_1, \ldots, q_n \rangle$ be a grading chain of $p \in \mathcal{P}$. The fused $\otimes$-grade of p with respect to C is the grade $\mathfrak{f}_{\otimes}(p, C) = \otimes(\langle g_0, \ldots, g_n \rangle)$, where $q_i = \mathfrak{g}(q_{i+1}, g_i)$, for $0 \leq i < n$, and $q_n = \mathfrak{g}(p, g_n)$.*

Definition 11. *Let $\mathfrak{T}$ be a telescoping structure. If $p \in \mathcal{Q}$, for a fan-in-bounded $\mathcal{Q} \subset \mathcal{P}$, then the $\mathfrak{T}$-fused grade of p in $\mathcal{Q}$ is defined as*

$$\mathfrak{f}_{\mathfrak{T}}(p, \mathcal{Q}) = \bigoplus \langle \mathfrak{f}_{\otimes}(p, C_k) \rangle_{k=1}^{n}$$

where $\langle C_k \rangle_{k=1}^{n}$ is a permutation of the set of longest grading chains of p in $\mathcal{Q}$.[6]

Recasting the familiar notion of a *kernel* of a belief base [19] into the context of $Log_A\mathbf{G}$ structures, we say that a kernel of $\mathcal{Q} \subseteq \mathcal{P}$ is a subset-minimal $\mathcal{X} \subseteq \mathcal{Q}$ such that $F(\mathcal{X})$ is improper ($\neq \mathcal{P}$). Let $\mathcal{Q}^{\perp\!\!\perp}\bot$ be the set of $\mathcal{Q}$ kernels.

Definition 12. *For a telescoping structure $\mathfrak{T} = \langle \mathcal{T}, \mathfrak{O}, \otimes, \oplus \rangle$ and a fan-in-bounded $\mathcal{Q} \subseteq \mathcal{P}$, if $\mathcal{X} \subseteq \mathcal{Q}$, then $p \in \mathcal{X}$* survives $\mathcal{X}$ in $\mathfrak{T}$ *if*

1. *$p \in \mathcal{T}$; or*
2. *$G(p, \mathcal{Q}) \neq \varnothing$ and there is some $q \in \mathcal{X}$, with $G(q, \mathcal{Q}) \neq \varnothing$, such that $(\mathfrak{f}_{\mathfrak{T}}(q, \mathcal{Q}) < \mathfrak{f}_{\mathfrak{T}}(p, \mathcal{Q})) \in \mathfrak{O}$.*

The set of kernel survivors of $\mathcal{Q}$ in $\mathfrak{T}$ is the set

$$\kappa(\mathcal{Q}, \mathfrak{T}) = \{p \in \mathcal{Q} | \text{ if } p \in \mathcal{X} \in \mathcal{Q}^{\perp\!\!\perp}\bot \text{ then } p \text{ survives } \mathcal{X} \text{ given } \mathfrak{T}\}.$$

Observation 1 *If $F(\mathcal{T})$ is proper, then $F(\kappa(\mathcal{Q}, \mathfrak{T}))$ is proper.*

Definition 13. *Let $\mathcal{Q}, \mathcal{T} \subseteq \mathcal{P}$. We say that p is supported in $\mathcal{Q}$ given $\mathcal{T}$ if*

[6] Note that $\mathfrak{T}$-fusion is well-defined given the fan-in-boundedness of $\mathcal{Q}$ and the final clause of Proposition 1.

1. *$p \in F(\mathcal{T})$; or*
2. *there is a grading chain $\langle q_0, q_1, \ldots, q_n\rangle$ of p in $\mathcal{Q}$ with $q_0 \in F(\mathcal{R})$ where every member of $\mathcal{R}$ is supported in $\mathcal{Q}$.*

The set of propositions supported in $\mathcal{Q}$ given $\mathcal{T}$ is denoted by $\varsigma(\mathcal{Q}, \mathcal{T})$.

The following simple observation will prove useful later.

Observation 2 *$\varsigma(\mathcal{Q}, \mathcal{T}) = F(\mathcal{T}) \cup G$, for some set G of propositions graded in $\mathcal{Q}$.*

Definition 14. *Let $\mathfrak{T}$ be a telescoping structure for $\mathfrak{S}$. If $\mathcal{Q} \subset \mathcal{P}$ such that $E^1(F(\mathcal{Q}))$ is fan-in-bounded, then the $\mathfrak{T}$-induced telescoping of $\mathcal{Q}$ is given by $\tau_{\mathfrak{T}}(\mathcal{Q}) = \varsigma(\kappa(E^1(F(\mathcal{Q})), \mathfrak{T}), \mathcal{T})$*

Proposition 2. *For a telescoping structure $\mathfrak{T}$, $\tau_{\mathfrak{T}}$ is a function from fan-in-bounded sets in $2^{\mathcal{P}}$ to sets in $2^{\mathcal{P}}$.*

It may be shown that if $\mathcal{Q}$ is non-explosive with finitely-many grading propositions, then $\tau_{\mathfrak{T}}(\mathcal{Q})$ is defined, for every telescoping structure $\mathfrak{T}$. On the other hand, if $F(\mathcal{Q})$ is improper, then $\tau_{\mathfrak{T}}(\mathcal{Q})$ is undefined. In what follows, provided that the right-hand side is defined, let

$$\tau^n_{\mathfrak{T}}(\mathcal{Q}) = \begin{cases} \mathcal{Q} & \text{if } n = 0 \\ \tau_{\mathfrak{T}}(\tau^{n-1}_{\mathfrak{T}}(\mathcal{Q})) & \text{otherwise} \end{cases}$$

Definition 15. *Let $\mathfrak{T}$ be a telescoping structure. We refer to $F(\tau^n_{\mathfrak{T}}(\mathcal{T}))$ as a degree $n (\in \mathbb{N})$ graded filter of $\mathfrak{T}$, denoted $\mathfrak{F}^n(\mathfrak{T})$.*

Unfortunately, even with a finite and fan-in-bounded $\mathcal{T}$, the existence of a fixed-point for graded filters is not secured. (Check Example 3 in Section 4.) We can only prove a weaker property for a special class of telescoping structures.

Theorem 1. *Let $\mathfrak{T}$ be a telescoping structure where $\mathcal{T}$ is finite. There is some $n \in \mathbb{N}$ such that if $\mathfrak{F}^i(\mathfrak{T})$ is defined and $\mathfrak{F}^i(\mathfrak{T}) \cap Range(\mathfrak{g}) = \tau^i(\mathfrak{T}) \cap Range(\mathfrak{g})$ for every $i \leq n$, then for every $j \in \mathbb{N}$, there is some $k \leq n$ such that $\mathfrak{F}^{n+j}(\mathfrak{T}) = \mathfrak{F}^k(\mathfrak{T})$.*

A fixed-point is guaranteed if, under the same conditions in Theorem 1, we happen to stumble upon a *maximal* graded filter in the following sense.

Corollary 1. *If, in Theorem 1, $\mathfrak{F}^n(\mathfrak{T}) = F(E(\mathcal{T}))$ for some $n < 2^{|E(\mathcal{T})|} + 1$, then $\mathfrak{F}^{n+k}(\mathfrak{T}) = \mathfrak{F}^n(\mathfrak{T})$, for every $k \in \mathbb{N}$.*

Telescoping can never generate an inconsistent theory if the top theory is consistent.

Theorem 2. *If $\mathfrak{T}$ is a telescoping structure where $F(\mathcal{T})$ is proper, then, if defined, $\mathfrak{F}^n(\mathfrak{T})$ is proper, for every $n \in \mathbb{N}$.*

4 *Log$_A$*G Theories

Given the definition of a *Log$_A$***G** structure, we impose some reasonable constraints on which sets of *Log$_A$***G** terms qualify as *Log$_A$***G** *theories.* A *Log$_A$***G** theory is a finite set $\mathbb{T} \subseteq \sigma_P$ such that $\mathbb{E} \cup \mathbb{O} \subseteq \mathbb{T}$, where

- $\mathbb{E}$ is the smallest set containing the following terms:
  1. $\forall d[d \doteq d]$
  2. $\forall d_1, d_2[d_1 \doteq d_2 \Rightarrow d_2 \doteq d_1]$
  3. $\forall d_1, d_2, d_3[(d_1 \doteq d_2 \wedge d_2 \doteq d_3) \Rightarrow d_1 \doteq d_3]$
  4. $\forall p, d_1, d_2[(d_1 \doteq d_2 \wedge \mathbf{G}(p, d_1)) \Rightarrow \mathbf{G}(p, d_2)]$

  and
- $\mathbb{O}$ is the smallest set containing the following terms:
  1. $\forall d_1, d_2[\neg(d_1 \prec d_2) \Leftrightarrow (d_2 \prec d_1 \vee d_2 \doteq d_1)]$
  2. $\forall d_1, d_2, d_3[(d_1 \prec d_2 \wedge d_2 \prec d_3) \Rightarrow d_1 \prec d_3]$

Given a *Log$_A$***G** theory $\mathbb{T}$ and a valuation $\mathcal{V} = \langle \mathfrak{S}, \mathcal{V}_\Omega, \mathcal{V}_\Xi \rangle$, let $\mathcal{V}(\mathbb{T}) = \{[\![\phi]\!]^\mathcal{V} \mid \phi \in \mathbb{T}\})$. Further, for a *Log$_A$***G** structure $\mathfrak{S}$, an $\mathfrak{S}$ grading canon is a triple $\mathcal{C} = \langle \otimes, \oplus, n \rangle$ where $n \in \mathbb{N}$ and $\otimes$ and $\oplus$ are as indicated in Definition 9.

Definition 16. *Let $\mathbb{T}$ be a Log$_A$***G** *theory and $\mathcal{V} = \langle \mathfrak{S}, \mathcal{V}_\Omega, \mathcal{V}_\Xi \rangle$ a valuation, where $\mathfrak{S}$ has a set $\mathcal{P}$ which is depth- and fan-out-bounded, for some Log$_A$***G** *language L_Ω. For every $\phi \in \sigma_P$ and $\mathfrak{S}$ grading canon $\mathcal{C} = \langle \otimes, \oplus, n \rangle$, ϕ is a graded consequence of $\mathbb{T}$ with respect to $\mathcal{C}$, denoted $\mathbb{T} \models^{\mathcal{C}} \phi$, if $\mathfrak{F}^n(\mathfrak{T})$ is defined and $[\![\phi]\!]^\mathcal{V} \in \mathfrak{F}^n(\mathfrak{T})$, for every telescoping structure $\mathfrak{T} = \langle \mathcal{V}(\mathbb{T}), \mathfrak{O}, \otimes, \oplus \rangle$ for $\mathfrak{S}$, where $\mathfrak{O}$ extends $F(\mathcal{V}(\mathbb{T}) \cap Range(<))$.*[7]

It should be clear that $\models^{\mathcal{C}}$, where $\mathcal{C} = \langle \otimes, \oplus, n \rangle$, reduces to $\models$ if $n = 0$ or if $F(E(\mathcal{V}(\mathbb{T})))$ does not contain any grading propositions. Further, for $n > 0$, no ϕ is a graded consequence of $\mathbb{T}$ with respect to $\mathcal{C}$ if $F(\mathcal{V}(\mathbb{T}))$ is not proper. In what follows, let $\mathbb{T}^\mathcal{C} = \{\phi \mid \mathbb{T} \models^{\mathcal{C}} \phi\}$. When we are considering a set of canons which only differ in the value of n, we write $\mathbb{T}^n$ instead of $\mathbb{T}^\mathcal{C}$.

Unlike $\models$, $\models^{\mathcal{C}}$ is, in general, non-monotonic. (In the sequel, we interpret grades by the rational numbers, with their natural order remaining implicit.)

Example 1 (Opus and Tweety). We can represent the case of Opus and Tweety from Section 1 using a *Log$_A$***G** theory $\mathbb{T}_{\mathrm{OT1}} = \mathbb{E} \cup \mathbb{O} \cup \Gamma_{\mathrm{OT1}}$, where Γ_{OT1} is made up of the following terms.

1. $\forall x[Bird(x) \Rightarrow \mathbf{G}(Flies(x), 5)]$
2. $\forall x[Penguin(x) \Rightarrow \mathbf{G}(\neg Flies(x), 10)]$
3. $\forall x[Penguin(x) \Rightarrow Bird(x)]$
4. $Penguin(Opus)$
5. $Bird(Tweety)$

Figure 1 displays the relevant graded consequences of $\mathbb{T}_{\mathrm{OT1}}$ with respect to a series of canons, with $0 \leq n \leq 2$. Upon telescoping to $n = 1$, we believe

[7] An ultrafilter U extends a filter F, if $F \subseteq U$.

$n = 0$	0.1. $\mathbb{T}_{\mathrm{OT1}}$ 0.2. $Bird(Opus)$ 0.3. $\mathbf{G}(Flies(Tweety), 5)$ 0.4. $\mathbf{G}(Flies(Opus), 5)$ 0.5. $\mathbf{G}(\neg Flies(Opus), 10)$
$n = 1$	1.1. $\mathbb{T}^0_{\mathrm{OT1}}$ 1.2. $Flies(Tweety)$ 1.3. $\neg Flies(Opus)$

Fig. 1. Graded consequences of the $Log_A\mathbf{G}$ theory $\mathbb{T}_{\mathrm{OT1}}$ from Example 1

that Tweety flies and Opus does not fly. The embedded proposition that Opus flies does not survive telescoping since we trust that Opus does not fly, being a penguin, more than we trust that it flies, being a bird. $\mathbb{T}^1_{\mathrm{OT1}}$ is a fixed point. Now, consider the theory $\mathbb{T}_{\mathrm{OT2}} = \mathbb{E} \cup \mathbb{O} \cup \Gamma_{\mathrm{OT2}}$, where Γ_{OT2} is similar to Γ_{OT1}, but with propositions (1) and (2) replaced by "$\mathbf{G}(\forall x[Bird(x) \Rightarrow Flies(x), 5)$" and "$\mathbf{G}(\forall x[Penguin(x) \Rightarrow \neg Flies(x), 10)$", respectively. Thus, we trade the "*de re*" representation of $\mathbb{T}_{\mathrm{OT1}}$ for the "*de dicto*" representation in $\mathbb{T}_{\mathrm{OT2}}$. This change results in a change in the fixed point that we reach. In $\mathbb{T}^1_{\mathrm{OT2}}$, as in $\mathbb{T}^1_{\mathrm{OT1}}$, we end up believing that Opus does not fly. Unlike $\mathbb{T}^1_{\mathrm{OT1}}$ however, we give up our belief in the proposition that birds fly and, hence, cannot conclude that Tweety flies. □

Example 1 showcases the use of graded propositions in default reasoning. The following example illustrates the utility of nested grading.

Example 2 (Superman). Recalling the case of Superman from Section 1, we can describe the situation using $Log_A\mathbf{G}$ in at least two theories. Consider the theory $\mathbb{T}_{\mathrm{SM1}} = \mathbb{E} \cup \mathbb{O} \cup \Gamma_{\mathrm{SM1}}$, where Γ_{SM1} is made up of the following terms.

1. $\forall p[Source(p, LL) \Rightarrow \mathbf{G}(p, 11)]$
2. $\forall p[Source(p, DP) \Rightarrow \mathbf{G}(p, 4)]$
3. $\forall p[Perceive(p) \Rightarrow \mathbf{G}(p, 15)]$
4. $\forall l, t[\mathbf{G}(At(KC, l, t) \Leftrightarrow At(SM, l, t), 10.5)]$
5. $\forall l_1, l_2, t, x[(Disjoint(l_1, l_2) \wedge At(x, l_1, t)) \Rightarrow \neg At(x, l_2, t)]$
6. $Perceive(Source(Source(At(SM, DT, 12:00), LL), DP))$
7. $Perceive(At(KC, Office, 12:00))$
8. $Disjoint(Office, DT)$

Figure 2 displays relevant members of $\mathbb{T}^n_{\mathrm{SM1}}$ with respect to a series of canons, with $\otimes = $ mean and $\oplus = $ max, and with $0 \leq n \leq 3$. Note that, for $1 \leq n \leq 2$, we trust the proposition that Superman was at the office at noon (1.6). However, upon telescoping to $n = 3$, we lose our trust in said proposition, since we trust what Lois Lane says more than we trust our belief in the identity of Superman and Clark Kent (1.3).

Alternatively, consider the theory $\mathbb{T}_{\mathrm{SM2}} = \mathbb{E} \cup \mathbb{O} \cup \Gamma_{\mathrm{SM2}}$, where Γ_{SM2} is made up of the following terms.

1. $\mathbf{G}(\mathbf{G}(\mathbf{G}(At(SM, DT, 12:00), 11), 4), 15)$

$n = 0$	0.1. $\mathbb{T}_{\mathrm{SM1}}$ 0.2. $\mathbf{G}(Source(Source(At(SM, MDT, 12:00), LL), DP), 15)$ 0.3. $\mathbf{G}(At(KC, Office, 12:00) \Leftrightarrow At(SM, Office, 12:00), 10.5)$ 0.4. $\mathbf{G}(At(KC, Office, 12:00), 15)$
$n = 1$	1.1. $\mathbb{T}^0_{\mathrm{SM1}}$ 1.2. $Source(Source(At(SM, DT, 12:00), LL), DP)$ 1.3. $At(KC, Office, 12:00) \Leftrightarrow At(SM, Office, 12:00)$ 1.4. $At(KC, Office, 12:00)$ 1.5. $\mathbf{G}(Source(At(SM, DT, 12:00), LL), 4)$ 1.6. $At(SM, Office, 12:00)$
$n = 2$	2.1. $\mathbb{T}^1_{\mathrm{SM1}}$ 2.2. $Source(At(SM, DT, 12:00), LL)$ 2.3. $\mathbf{G}(At(SM, DT, 12:00), 11)$
$n = 3$	3.1. Everything at $n = 2$ except 1.3, 1.6, and anything they support 3.2. $At(SM, DT, 12:00)$

Fig. 2. Graded consequences of the $Log_A\mathbf{G}$ theory $\mathbb{T}_{\mathrm{SM1}}$ from Example 2

2. $\mathbf{G}(At(KC, Office, 12:00), 15)$
3. $\forall l, t[\mathbf{G}(At(KC, l, t) \Leftrightarrow At(SM, l, t), 10.5)]$
4. $\forall l_1, l_2, t, x[(Disjoint(l_1, l_2) \wedge At(x, l_1, t)) \Rightarrow \neg At(x, l_2, t)]$
5. $Disjoint(Office, DT)$

Figure 3 displays the different consequences with respect to the same canons employed with $\mathbb{T}_{\mathrm{SM1}}$. In this case, we get a different fixed point; we end up (at $n = 2$) believing that Superman was at the office at noon, contrary to what has been reported by Lois Lane in the Daily Planet. The reason is that, due to the nesting of grading propositions, the fused grade of "$AT(SM, DT, 12:00)$" is now only 10 (which is less than 10.5, the grade of (1.4)), being pulled down by the low grade attributed to the Daily Planet. □

$n = 0$	0.1. $\mathbb{T}_{\mathrm{SM2}}$ 0.2. $\mathbf{G}(At(KC, Office, 12:00) \Leftrightarrow At(SM, Office, 12:00), 10.5)$
$n = 1$	1.1. $\mathbb{T}^0_{\mathrm{SM2}}$ 1.2. $\mathbf{G}(\mathbf{G}(At(SM, DT, 12:00), 11), 4)$ 1.3. $At(KC, Office, 12:00)$ 1.4. $At(KC, Office, 12:00) \Leftrightarrow At(SM, Office, 12:00)$ 1.5. $At(SM, Office, 12:00)$ 1.6. $\neg At(SM, DT, 12:00)$
$n = 2$	2.1. $\mathbb{T}^1_{\mathrm{SM2}}$ 2.2. $\mathbf{G}(At(SM, DT, 12:00), 11)$
$n = 3$	3.1. $\mathbb{T}^2_{\mathrm{SM2}}$

Fig. 3. Graded consequences of the $Log_A\mathbf{G}$ theory $\mathbb{T}_{\mathrm{SM2}}$ from Example 2

$n = 0$	0.1. $\mathbb{T}_{\mathrm{L}}$
$n = 1$	1.1. $\mathbb{T}^{0}_{\mathrm{L}}$ 1.2. ϕ 1.3. $\mathbf{G}(\neg\phi, 5)$
$n = 2$	2.1. $\mathbb{T}^{0}_{\mathrm{L}}$

Fig. 4. Graded consequences of the $\mathit{Log}_A\mathbf{G}$ theory $\mathbb{T}_{\mathrm{L}}$ from Example 3

Finally, we revisit the pathological case of the liar.

Example 3 (The Liar). Consider the theory $\mathbb{T}_{\mathrm{L}} = \mathbb{E} \cup \mathbb{O} \cup \{\mathbf{G}(\phi, 5), \phi \Leftrightarrow \mathbf{G}(\neg\phi, 5)\}$. This is the closest we can get, within $\mathit{Log}_A\mathbf{G}$, to the case of the liar from Section 1; given the non-degeneracy of the Boolean algebra, we can never have the situation where $[\![\phi]\!]^{\mathcal{V}} = -[\![\phi]\!]^{\mathcal{V}}$ (cf. [10]). Figure 4 shows what happens as n increases, given any grading canon. In such a problematic situation, we never reach a fixed point, indefinitely iterating through $\mathbb{T}^{0}_{\mathrm{L}}$ and $\mathbb{T}^{1}_{\mathrm{L}}$. But this is just as well, for it fairly captures the dilemma one is in when encountering liar-like sentences. While this is similar in spirit, but not identical, to the treatment of the liar paradox within the revision theory of truth (RTT) [20], RTT tackles the liar paradox head-on, not via the more tame version expressible in $\mathit{Log}_A\mathbf{G}$. (Also see [21] for a treatment of the liar paradox within a fuzzy logical framework.) □

5 Conclusion

Notwithstanding the abundance of weighted logics in the literature, it is our conviction that $\mathit{Log}_A\mathbf{G}$ provides an interesting alternative. While it has a non-classical semantics, $\mathit{Log}_A\mathbf{G}$ is arguably intuitive, expressive, and quite similar in syntax to first-order logic. We hope to have demonstrated the utility of $\mathit{Log}_A\mathbf{G}$ in default reasoning, reasoning with information reported through a chain of sources, and even reasoning with paradoxical propositions. A careful examination of how $\mathit{Log}_A\mathbf{G}$ relates to other graded logics and non-monotonic formalisms is called for. On a first pass, we believe that $\mathit{Log}_A\mathbf{G}$ subsumes possibilistic logic [2], circumscription [9], and default theories with at most one justification per default (which includes normal defaults) [7]. We are currently working on the implementation of a proof theory for $\mathit{Log}_A\mathbf{G}$ based on a reason maintenance system (cf. [22]). The primary objective we have in mind is prioritized belief revision based on graded propositions.

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A Proofs

A.1 Proof of Theorem 1

To prove Theorem 1, we need the following result.

Lemma 1. *If $\mathfrak{T}$ is a telescoping structure then, for every $n \in \mathbb{N}$, if $\mathfrak{F}^i(\mathfrak{T})$ is defined and $\mathfrak{F}^i(\mathfrak{T}) \cap Range(\mathfrak{g}) = \tau^i(\mathfrak{T}) \cap Range(\mathfrak{g})$ for every $i \leq n$, then $\mathfrak{F}^n(\mathfrak{T}) = F(\mathcal{Q})$, for some $\mathcal{Q} \subseteq E(\mathcal{T})$ and $\mathfrak{F}^n(\mathfrak{T}) \cap Range(\mathfrak{g}) \subseteq E^n(\mathcal{T})$.*

Proof. We prove the lemma by induction on n. For $n = 0$, $\mathfrak{F}^0(\mathcal{T}) = F(\mathcal{T})$, where $\mathcal{T} \subseteq E(\mathcal{T})$. Further, $\mathfrak{F}^0(\mathcal{T}) \cap Range(\mathfrak{g}) = \tau^0(\mathfrak{T}) \cap Range((g)) = \mathcal{T} \cap Range(\mathfrak{g}) \subseteq E^0(\mathcal{T})$. Now assume that the statement holds for some $k \in \mathbb{N}$. By the induction hypothesis, $\mathfrak{F}^k(\mathfrak{T}) \cap Range(\mathfrak{g}) \subseteq E^k(\mathcal{T})$. Therefore, all propositions graded in $E^1(\mathfrak{F}^k(\mathfrak{T}))$ are in $E^{k+1}(\mathcal{T})$. Moreover, $E^1(\mathfrak{F}^k(\mathfrak{T})) \cap Range(\mathfrak{g}) \subseteq E^{k+1}(\mathcal{T})$. By Observation 2, $\tau_{\mathfrak{T}}^{k+1}(\mathcal{T}) = \tau_{\mathfrak{T}}(\mathfrak{F}^k(\mathfrak{T})) = \varsigma(\kappa(E^1(\mathfrak{F}^k(\mathfrak{T}))), \mathfrak{T}), \mathcal{T}) = F(\mathcal{T}) \cup G$, where G is a set of propositions graded in $\kappa(E^1(\mathfrak{F}^k(\mathfrak{T})))$. By Definition 12, $\kappa(E^1(\mathfrak{F}^k(\mathfrak{T}))) \subseteq E^1(\mathfrak{F}^k(\mathfrak{T}))$. Hence, $G \subseteq E^{k+1}(\mathcal{T})$. Thus, $\mathfrak{F}^{k+1}(\mathfrak{T}) = F([F(\mathcal{T}) \cup G]) = F(\mathcal{T} \cup G)$, where $\mathcal{T} \cup G \subseteq E(\mathcal{T})$. Moreover, $\mathfrak{F}^{k+1}(\mathfrak{T}) \cap Range(\mathfrak{g}) = \tau^{k+1}(\mathfrak{T}) \cap Range(\mathfrak{g}) \subseteq E^{k+1}(\mathcal{T})$. □

We now proceed to proving the theorem. By Definition 9 and Clause 4 of Proposition 1, $E(\mathcal{T})$ has finitely-many grading propositions. In fact, since $\mathcal{T}$ is finite, then $E(\mathcal{T})$ is finite. Let $b = |E(\mathcal{T})|$. Now, taking $n = 2^b + 1$, suppose that, for every $i \leq 2^b + 1$, $\mathfrak{F}^i(\mathfrak{T})$ is defined and $\mathfrak{F}^i(\mathfrak{T}) \cap Range(\mathfrak{g}) = \tau^i(\mathfrak{T}) \cap Range(\mathfrak{g})$. By Lemma 1, $\mathfrak{F}^i(\mathfrak{T}) = F(\mathcal{Q}_i)$, for some $\mathcal{Q}_i \subseteq E(\mathcal{T})$. Since there are only 2^b subsets of $E(\mathcal{T})$, then $\mathfrak{F}^n(\mathfrak{T}) = \mathfrak{F}^j(\mathfrak{T})$, for some $j \leq 2^b$. Further, by Proposition 2, for every $j \in \mathbb{N}$, there is some $k \leq n$ such that $\mathfrak{F}^{n+j}(\mathfrak{T}) = \mathfrak{F}^k(\mathfrak{T})$. □

A.2 Proof of Corollary 1

We prove the case of $k = 1$ and the result follows by the same argument for all $k \in \mathbb{N}$.

$\mathfrak{F}^{n+1}(\mathfrak{T})$	
$= F(\varsigma(\kappa(E^1(\mathfrak{F}^n(\mathfrak{T})), \mathfrak{T})), \mathcal{T})$	(Definitions 14 and 15)
$= F(\varsigma(\kappa(E^1(F(E(\mathcal{T}))))))$	
$= F(\varsigma(\kappa(F(E(\mathcal{T})))))$	(Definition of E and the assumption of Theorem 1)
$= F(\varsigma(F(E(\mathcal{T}))))$	(Since $F(E(\mathcal{T}))$ is proper given that $\mathfrak{F}^{n+1}(\mathfrak{T})$ is defined)
$= F(F(E(\mathcal{T})))$	(Since every $p \in F(E(\mathcal{T}))$ is supported given that $F(E(\mathcal{T})) = \mathfrak{F}^n(\mathfrak{T})$ and Definitions 14 and 15)
$= F(E(\mathcal{T}))$	

□

A.3 Proof of Theorem 2

For $n = 0$, the statement is trivial, since $\mathfrak{F}^0(\mathfrak{T}) = F(\mathcal{T})$. Otherwise, the statement follows directly from Observation 1 since, by Definition 15, $\mathfrak{F}^{k+1}(\mathfrak{T}) = F(K)$, for some $K \subseteq \kappa(E^1(\mathfrak{F}^k(\mathfrak{T})), \mathfrak{T})$. □