Importance Analysis of *k*-out-of-*n* Multi-State Systems based on Direct Partial Logic Derivatives

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Abstract. In this paper, analysis of *k*-out-of-*n* Multi-State Systems (MSSs) is considered. This type of systems consists of *n* components, and it can be in state *j* if and only if at least *k* components are in state *j* or greater. Investigation of such systems has been considered in several works. However, most of them have dealt only with an efficient computation of several global characteristics, such as system state probability or system availability. In this paper, we deal with importance analysis for such systems. Particularly, we focus on two commonly used importance measures – structural importance and Birnbaum's importance. Using logical differential calculus, we propose an efficient way of how to calculate these measures for a *k*-out-of-*n* MSS. The obtained results are then used to analyze an oil supply system.

Keywords. Multi-State System, Structure Function, Structural Importance, Birnbaum's Importance, Logical Differential Calculus

Key Terms. Reliability, Model, Approach, Methodology, Scientific Field

1 Introduction

Reliability has been considered as an important characteristic of many systems [1], [2], [3], [4]. Most of the systems, whose reliability has to be investigated, are composed of more than one element (component) and, therefore, one of the principal tasks of reliability analysis is investigation of influence of individual system components on the proper work of the system [4]. Such investigation requires creation of a mathematical model of the system. As a rule, two approaches are used in reliability analysis. The first one is based on the assumption that the system and all its components (system elements that are assumed to be indivisible into smaller parts) can be in one of only two possible states – functioning (represented by number 1) and failed (presented as number 0). These systems are known as Binary-State Systems (BSSs) [1], [4]. Models based on this approach are suitable for the analysis of consequences of system failure, but they are not very appropriate for the investigation of processes that

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result in system failure. For this purpose, the approach based on the idea that the system and all its components can be in one of more than two states is more suitable. In this case, we say about Multi-State Systems (MSSs) [2], [3], [4].

One of the current issues of reliability engineering is evaluation of complex systems. Such systems are composed of many components with very various natures [5]. Typical instances of such systems are healthcare systems [6] containing hardware and software components, human factor, and organizational elements, or complex distribution networks composing of many different hardware elements [7]. This heterogeneity indicates that it can be quite difficult to model such systems as BSSs and, therefore, MSSs are more appropriate.

Reliability analysis of MSSs is a complex problem that includes a lot of tasks. This paper focuses on two specific tasks: identification of situations in which component or its state is critical for system activity, i.e. situations in which a degradation of a component results in system degradation, and quantification of importance of individual system components, i.e. finding components with the greatest influence on system activity. One of the possible ways of how to perform this analysis is application of logical differential calculus.

Logical differential calculus has originally been developed for analysis of dynamic properties of Multiple-Valued Logic (MVL) functions [8]. This tool can also be applied in reliability analysis to identify circumstances under which a change of a state of a system component results in a change of system state. So, it allows us to find situations in which a degradation of a given component or its state is critical for system degradation [9], [10]. In this paper, we consider its application in importance analysis of k-out-of-n MSSs.

A k-out-of-n system is composed of n components. Based on [11], behavior of this system can be described as follows:

- if at least k components are in state m -1, then the system is in state m -1,
- else if at least k components are in state m -2 or better, then it is in state m -2,

  …
- else if at least k components are in state 1 or better, then it is in state 1,
- else the system is in state 0.

Efficient ways of how to calculate some global characteristics, e.g. system state probability or system availability, for this kind of systems have been considered, for example, in [12], [13]. However, those papers have not considered investigation of importance of individual system components (or their states) on system activity. This problem is taken into account on the next pages.

2 Reliability Analysis of Multi-State Systems

A MSS is a mathematical representation of a system under consideration. It allows us to define m levels at which the system or its components can operate. These levels are known as states of the system/component and they take values from the set $\{0,1,\ldots, m-1\}$. State 0 means that the system/component is completely failed, while state m -1 implies that it is perfectly functioning. A mapping that defines the dependency of system state on the states of its components is known as structure function. For a MSS composed of n components, this function has the following form [3], [9]:

$$\phi(\boldsymbol{x}): \{0,1,\ldots, m-1\}^n \rightarrow \{0,1,\ldots, m-1\} , \tag{1}$$

where x_i is a variable defining state of the i-th system component for $i = 1,2,\ldots, n$, and $\boldsymbol{x} = (x_1, x_2,\ldots, x_n)$ is a vector of components states (state vector). Specially, if $m = 2$, then definition (1) agrees with the structure function of a BSS. Please note that the structure function of a MSS can also be viewed as a MVL function. In this case, x_i is known as a MVL variable and vector $\boldsymbol{x}$ can be named as a MVL vector.

Based on the properties of the structure function, two classes of MSSs can be defined – coherent and incoherent. A MSS is coherent if its structure function is non-decreasing in all its arguments, i.e. there exist no circumstances under which degradation (improvement) of a system component can result in improvement (deterioration) of system state. In what follows, only coherent systems are considered.

The structure function defines system topology. However, if we want to investigate not only system topology but also some others characteristics, such as system state probability, system availability, or importance of individual system components, the state probabilities of the system components have to be known. For the i-th system component, they will be denoted as follows:

$$p_{i,s} = \Pr\{x_i = s\},\ s = 0,1,\ldots, m-1 . \tag{2}$$

Using these probabilities and the system structure function, we can compute two basic characteristics of MSSs – system state probability [3], [9]:

$$\Pr\{\phi(\boldsymbol{x}) = j\}, \text{ for } j \in \{0,1,\ldots, m-1\} , \tag{3}$$

and system availability/unavailability with respect to state j of the system [3], [9]:

$$A^{\geq j} = \Pr\{\phi(\boldsymbol{x}) \geq j\},\ U^{\geq j} = \Pr\{\phi(\boldsymbol{x}) < j\},\ \text{for } j \in \{1,2,\ldots,m-1\}. \tag{4}$$

This definition implies that system availability (unavailability) for system state j agrees with the probability that the system is in such state that its performance can (cannot) satisfy a demand corresponding to state j. For illustration, let us consider a power supply unit that can generate 30 MW, 10 MW, or 0 MW of electricity. Clearly, the system has 3 performance levels from which level 30 MW corresponds to state 2, level 10 MW to state 1 and level 0 MW to state 0. If there is a demand of at least 5 MW of electricity, then the unit is working if it is at least in state 1. This implies that it is available if it is at least in state 1 and, therefore, its availability (unavailability) should be computed with respect to state 1 for this situation.

2.1 Importance Analysis of Multi-State Systems

System state probability and availability are important characteristics of a system. They give us a global view on the system. On the other hand, they carry no information about the system structure, i.e. they do not allow investigating influence of individual system components or their states on the system. For this purpose, other indices are used. These indices are known as Importance Measures (IMs), and some of the most commonly known are Structural Importance (SI) and Birnbaum's Importance (BI). The SI investigates only system topology while the BI takes into account also state probabilities of the system components. These two indices play a key role in importance analysis because a lot of other measures are defined based on them [4].

Importance analysis of MSSs based on SI and BI has been considered in several papers, e.g., [9], [14], [15], [16]. In those papers, several versions of these measures have been proposed depending on whether we want to:

- investigate influence of a given component state on a given system state/ availability level [9], [14],
- analyze the total influence of a given component state on the system (not only on a specific system state) [15],
- inspect the total importance of a given component [16].

The approaches presented in the aforementioned works have been combined into one complex framework in [10]. According to that work, the SI agrees with a relative number of situations in which a given component (state) is critical for degradation of (a given state/availability level of) the system, while the BI corresponds to the probability that such situation occurs. (The criticality means that degradation of a given component results in system degradation.) These definitions indicate that identification of situations in which a given component (state) is critical for degradation of (a given state/availability level of) the system represents the main issue in the computation of these measures. In the considered paper, this task has been solved using a special tool of MVL that is known as logical differential calculus [8].

Logical differential calculus has been developed for analysis of dynamic properties of MVL functions. Logic derivative is a key term of this tool. There exist several types of logic derivatives but, for the purpose of this paper, Direct Partial Logic Derivatives (DPLDs) are the most important.

A DPLD reveals circumstances under which a considered change of a MVL variable results in the studied change of the analyzed MVL function. Since the formal definition of the structure function of a MSS agrees with the definition of a MVL function, this derivative can also be used in the analysis of MSSs. In this case, it allows us to detect situations in which a given change of a given component state results in the studied change of the system state. More formally, a DPLD with respect to variable x_i is defined as follows [8], [9]:

$$\frac{\partial\phi(j \to h)}{\partial x_i(s \to r)} = \begin{cases} 1, & \text{if } \phi(s_i, \boldsymbol{x}) = j \text{ and } \phi(r_i, \boldsymbol{x}) = h \\ 0, & \text{otherwise} \end{cases}, \quad \text{for } s,r,j,h \in \{0,1,\ldots,m\text{-}1\}, s \neq r, j \neq h, \tag{5}$$

where $(a_i, \boldsymbol{x}) = (x_1, x_2,\ldots, x_{i-1}, a, x_{i+1},\ldots, x_n)$ for $a \in \{s, r\}$.

Depending on the relations between s and r and j and h in (5), four kinds of DPLDs with different physical meaning can be used in reliability analysis of MSSs:

- if s > r and j > h, then the DPLD identifies situations in which degradation of component i from state s to r results in degradation of system from state j to h,

- if s < r and j < h, then the DPLD detects circumstances under which improvement of component i from state s to r causes improvement of system state from value j to h,
- if s > r and j < h, then the DPLD finds circumstances under which degradation of component i from state s to r results in improvement of system state from value j to h,
- if s < r and j > h, then the DPLD reveals situations in which degradation of the system from state j to h is caused by improvement of component i from state s to r.

Clearly, situations identified by the last two kinds of DPLDs cannot occur in case of a coherent system and, therefore, only the first two DPLDs are meaningful in the analysis of coherent MSSs. Furthermore, in what follows, we will primarily deal with investigation of consequences of component degradation on system activity. This implies that only DPLDs in which $s > r$ and $j > h$ will be taken into account.

Based on the meaning of DPLDs, it is clear that they allow us to find state vectors of the form of $(s_i, \boldsymbol{x})$ at which deterioration of state s of component i to state r results in degradation of system state j to h. These state vectors are known as critical state vectors and, clearly, they describe circumstances under which a given component state is critical for a given degradation of system state j.

One of the assumptions that are often used in importance analysis of MSSs is that the system components degrade gradually state by state. This assumption is not unrealistic because even if a component deteriorates from state m -1 to state 0, we can assume that it stays in every state from set {1,2,…, m -2} for very short time [9]. This implies that only DPLDs of the form of $\partial\phi(j \to h)/\partial x_i(s \to s-1)$ have to be taken into account if we want to investigate importance of individual system components.

DPLDs give us a detailed view on the dependency between component degradation and system degradation. However, they are not very appropriate for importance analysis of a general MSS, i.e. a MSS in which a minor degradation (degradation by one state) of any system component can result in degradation of the system by more than one state. This inadequacy results from the fact that a lot of DPLDs have to be computed in such situations, e.g., if we want to investigate consequences of a minor degradation of state s of component i, then we have to compute DPLDs of the form of $\partial\phi(j \to h)/\partial x_i(s \to s-1)$ for all $h < j$, i.e. j -1 DPLDs. The similar fact can also be observed if we want to use DPLDs to investigate the coincidence between component degradation and decrease in system availability level. To avoid this problem, new types of logic derivatives have been introduced in [10]. These derivatives were named as Integrated Direct Partial Logic Derivatives (IDPLDs) because they combine several types of DPLDs together. Depending on the combined DPLDs, three types of IDPLDs have been defined. In this paper, only IDPLDs of type I and III are used.

An IDPLD of type I is defined as follows:

$$\frac{\partial\phi(j\downarrow)}{\partial x_i(s \to r)} = \bigcup_{h=0}^{j-1} \frac{\partial\phi(j \to h)}{\partial x_i(s \to r)} = \begin{cases} 1, & \text{if } \phi(s_i, \boldsymbol{x}) = j \text{ and } \phi(r_i, \boldsymbol{x}) < j \\ 0, & \text{otherwise} \end{cases}, \qquad (6)$$
$$\text{for } s, r \in \{0,1,\dots,m\text{-}1\}, s \neq r,\ \ j \in \{1,2,\dots,m\text{-}1\},$$

and it allows us to find situations in which a given degradation of state s of system component i results in a deterioration of system state j. Quantification of these situations allows us to estimate influence of the considered component degradation on system state j.

An IDPLD of type III has the following form:

$$\frac{\partial\phi(h_{\geq j} \to h_{<j})}{\partial x_i(s \to r)} = \bigcup_{h_u=j}^{m-1} \bigcup_{h_d=0}^{j-1} \frac{\partial\phi(h_u \to h_d)}{\partial x_i(s \to r)} = \begin{cases} 1, & \text{if } \phi(s_i, \boldsymbol{x}) \geq j \text{ and } \phi(r_i, \boldsymbol{x}) < j \\ 0, & \text{otherwise} \end{cases}, \tag{7}$$
$$\text{for } s, r \in \{0,1,\ldots,m\text{-}1\}, s \neq r, \;\; j \in \{1,2,\ldots,m\text{-}1\},$$

where notation $h_{\geq j}$ ($h_{<j}$) means that all system states that are greater than or equal to (less than) j are taken into account. Please note, this definition implies that IDPLDs of type III can be used to find state vectors at which a degradation of a given component state causes degradation of a given level of system availability and, therefore, they can be used to quantify consequences of a given deterioration of state s of component i on level j of system availability.

It has been mentioned in the previous paragraphs that quantification of situations in which an IDPLD of type I or III takes nonzero value allows us to estimate influence of degradation a given component state on system state/availability level. This quantification can be done in two ways [10]. Firstly, we can compute truth density (a relative count of situations in which a function with a Boolean-valued output takes nonzero value) of the considered IDPLD. Result of this computation agrees with the relative number of situations in which a considered degradation of a given component state results in degradation of a given system state/availability level. If we assume that the system components degrade gradually state by state, then this number corresponds to SI of a given component state for a given system state/availability level [10]. This measure does not take the components states probabilities into account and, therefore, it investigates only topological importance of a given component state.

Another possibility is to calculate the probability that the considered IDPLD is nonzero. This agrees with the probability that the studied degradation of a given component state causes decrease in a given state/availability level of the system. If we assume that only minor degradations of the system components can occur, then this number agrees with BI of a given component state for a given system state/availability level [10]. Unlike the SI, the BI provides more information because it considers not only system topology but also state probabilities of the components.

The previously mentioned versions of SI and BI deals with importance of a given component state for a given state/availability level of the system. It has been shown in [10] that these measures can also be used to investigate:

- the total importance of a given component state,
- the total importance of a component for a given system state/availability level,
- the total importance of a given component.

The SI measures that can be used for these purposes are presented in Table 1. The similar table can be shown for the BI measures, but the only difference will be in replacement of the truth density notation with the probability that the IDPLD takes nonzero value. Please note that complex importance analysis of a MSS can be performed by computation of all types of SI or BI measures. Based on the formulae presented in Table 1, the SI (BI) measures investigating all possible dependencies between component degradation and system deterioration can be expressed in the form of Table 2 (IMs concerning with system state) or Table 3 (IMs focusing on system availability level).

Table 1. Summary of structural importance measures investigating topological properties of the system based on component degradation

Structural importance	Definition	Interpretation
The SI of a given component state and for a given system state	$\mathrm{SI}_{i,s}^{j\downarrow} = \mathrm{TD}\left(\frac{\partial\phi(j\downarrow)}{\partial x_i(s \to s-1)}\right)$	A relative number of situations in which degradation of state *s* of component *i* results in degradation of state *j* of the system.
The SI of a given component state and for a given level of system availability	$\mathrm{SI}_{i,s}^{\geq j} = \mathrm{TD}\left(\frac{\partial\phi(h_{\geq j} \to h_{<j})}{\partial x_i(s \to s-1)}\right)$	A relative number of situations in which degradation of state *s* of component *i* results in degradation of level *j* of system availability.
The SI of a given component state	$\mathrm{SI}_{i,s}^{\downarrow} = \sum_{j=1}^{m-1} \mathrm{SI}_{i,s}^{j\downarrow}$	A relative number of situations in which state *s* of component *i* results in system degradation.
The SI of a given component for a given system state	$\mathrm{SI}_{i}^{j\downarrow} = \frac{1}{m-1}\sum_{s=1}^{m-1} \mathrm{SI}_{i,s}^{j\downarrow}$	A relative number of situations in which degradation of component *i* causes degradation of state *j* of the system.
The SI of a given component for a given system availability level	$\mathrm{SI}_{i}^{\geq j} = \frac{1}{m-1}\sum_{s=1}^{m-1} \mathrm{SI}_{i,s}^{\geq j}$	A relative number of situations in which degradation of component *i* causes degradation of level *j* of system availability.
The total SI of a given component	$\mathrm{SI}_{i}^{\downarrow} = \frac{1}{m-1}\sum_{s=1}^{m-1} \mathrm{SI}_{i,s}^{\downarrow}$	A relative number of situations in which degradation of component *i* results in system degradation.

*note: TD(.) – truth density of the argument interpreted as a function with a Boolean-valued output

3 Importance Analysis of *k*-out-of-*n* Multi-State Systems

Let us consider a *k*-out-of-*n* MSS. According to Table 1 – Table 3, the most important thing in computation of the IMs considered above is efficient identification of nonzero elements of IDPLDs. For example, in case of computing $\mathrm{SI}_{i,s}^{j\downarrow}$, this agrees with finding all state vectors $(._i, \boldsymbol{x}) = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ for which IDPLD $\partial\phi(j\downarrow)/\partial x_i(s \to s-1)$ takes nonzero value. Now, let us find IDPLDs of which form can be nonzero. Firstly, let us assume that $j > s$. Such derivative cannot be nonzero because the *k*-out-of-*n* MSS can be in state *j* if and only if at least *k* components are in state *j* or greater. This implies that degradation of a component that is in a state less than *j* cannot result in degradation of system state *j* because system state is not

Table 2. Structural importance measures investigating topological properties of the system with respect to system state

		Component state				Average
		1	2	…	m -1	
System state	1	$\mathrm{SI}_{i,1}^{1\downarrow}$	$\mathrm{SI}_{i,2}^{1\downarrow}$	…	$\mathrm{SI}_{i,m-1}^{1\downarrow}$	$\mathrm{SI}_{i}^{1\downarrow}$
	2	$\mathrm{SI}_{i,1}^{2\downarrow}$	$\mathrm{SI}_{i,2}^{2\downarrow}$	…	$\mathrm{SI}_{i,m-1}^{2\downarrow}$	$\mathrm{SI}_{i}^{2\downarrow}$
	⋮	⋮	⋮	⋱	⋮	⋮
	m -1	$\mathrm{SI}_{i,1}^{(m-1)\downarrow}$	$\mathrm{SI}_{i,2}^{(m-1)\downarrow}$	…	$\mathrm{SI}_{i,m-1}^{(m-1)\downarrow}$	$\mathrm{SI}_{i}^{(m-1)\downarrow}$
Sum		$\mathrm{SI}_{i,1}^{\downarrow}$	$\mathrm{SI}_{i,2}^{\downarrow}$	…	$\mathrm{SI}_{i,m-1}^{\downarrow}$	$\mathrm{SI}_{i}^{\downarrow}$

Table 3. Structural importance measures investigating topological properties of the system with respect to system availability level

		Component state				Average
		1	2	…	m -1	
System availability level	1	$\mathrm{SI}_{i,1}^{\geq 1}$	$\mathrm{SI}_{i,2}^{\geq 1}$	…	$\mathrm{SI}_{i,m-1}^{\geq 1}$	$\mathrm{SI}_{i}^{\geq 1}$
	2	$\mathrm{SI}_{i,1}^{\geq 2}$	$\mathrm{SI}_{i,2}^{\geq 2}$	…	$\mathrm{SI}_{i,m-1}^{\geq 2}$	$\mathrm{SI}_{i}^{\geq 2}$
	⋮	⋮	⋮	⋱	⋮	⋮
	m -1	$\mathrm{SI}_{i,1}^{\geq(m-1)}$	$\mathrm{SI}_{i,2}^{\geq(m-1)}$	…	$\mathrm{SI}_{i,m-1}^{\geq(m-1)}$	$\mathrm{SI}_{i}^{\geq(m-1)}$

determined by this component. Secondly, let us assume that $j < s$. This derivative cannot also take nonzero values because the system can be in state j if and only if not more than k -1 components are in a state greater than j. It follows that a minor degradation of a component that is in a state greater than system state j does not result in violation of this condition and, therefore, there exist no circumstances under which degradation of this component from state s to state s -1 can result in degradation of the system if the system is in state j such that $j < s$. Finally, let us consider an IDPLD of the form of $\partial\phi(j\downarrow)/\partial x_i(j \to j-1)$. This derivative identifies situations in which a minor degradation of state j of component i results in degradation of system state j. Such situations can occur if and only if component i is in state j and exactly k -1 from the remaining components are in state j or greater than j. These situations correspond to state vectors of the form of $(j_i, r_{u_1}^{\geq j,(1)}, r_{u_2}^{\geq j,(2)}, \ldots, r_{u_{k-1}}^{\geq j,(k-1)}, r_{v_1}^{<j,(1)}, r_{v_2}^{<j,(2)}, \ldots, r_{v_{n-k}}^{<j,(n-k)})$ where u_1, u_2,…, u_{k-1} are components that are in states greater than or equal to j; v_1, v_2,…, v_{n-k} are components that are in states less than j; $r_{u_t}^{\geq j,(t)}$, for t = 1,2,…, k -1, de-

notes state of component u_t (this state is greater than or equal to j); and $r_{v_t}^{<j,(t)}$, for t = 1,2,…, n - k, means that component v_t is in a state less than j. It can be simply shown that $\binom{n-1}{k-1}(m-j)^{k-1}j^{n-k}$ such state vectors exist. It follows that integrated derivatives $\partial\phi(j\downarrow)/\partial x_i(j\to j-1)$ has $\binom{n-1}{k-1}(m-j)^{k-1}j^{n-k}$ nonzero elements. Since every IDPLD and DPLD has m^{n-1} elements [9], [10], the truth density of integrated derivative $\partial\phi(j\downarrow)/\partial x_i(j\to j-1)$ can be computed in the following way:

$$\mathrm{TD}\left(\frac{\partial\phi(j\downarrow)}{\partial x_i(j\to j-1)}\right)=\binom{n-1}{k-1}\frac{(m-j)^{k-1}j^{n-k}}{m^{n-1}}. \tag{8}$$

Based on the results obtained in the previous paragraph and information presented in Table 1, $\mathrm{SI}_{i,s}^{j\downarrow}$ can be computed for a k-out-of-n MSS using the following formula:

$$\mathrm{SI}_{i,s}^{j\downarrow}=\mathrm{TD}\left(\frac{\partial\phi(j\downarrow)}{\partial x_i(s\to s-1)}\right)=\begin{cases}\binom{n-1}{k-1}\frac{(m-j)^{k-1}j^{n-k}}{m^{n-1}}, & \text{if } s=j\\ 0, & \text{otherwise}\end{cases}. \tag{9}$$

Above, we have shown that only IDPLDs of the form of $\partial\phi(j\downarrow)/\partial x_i(j\to j-1)$ are nonzero in case of a k-out-of-n MSS. Since this IDPLD is nonzero if and only if states of the system components are characterized by state vectors of the form of $(j_i, r_{u_1}^{\geq j,(1)}, r_{u_2}^{\geq j,(2)},\ldots, r_{u_{k-1}}^{\geq j,(k-1)}, r_{v_1}^{<j,(1)}, r_{v_2}^{<j,(2)},\ldots, r_{v_{n-k}}^{<j,(n-k)})$, then degradation of component i from state j to j -1 has to result in degradation of system state from j to j -1. This implies that the nonzero elements of this derivative agrees with the nonzero elements of DPLD $\partial\phi(j\to j-1)/\partial x_i(j\to j-1)$. It follows that only such DPLDs are nonzero in case of a k-out-of-n MSS. Using this fact and definitions (6) and (7) of IDPLDs, the next formula can be proved simply for k-out-of-n MSSs:

$$\frac{\partial\phi(j\downarrow)}{\partial x_i(s\to s-1)}=\frac{\partial\phi(h_{\geq j}\to h_{<j})}{\partial x_i(s\to s-1)}=\frac{\partial\phi(j\to j-1)}{\partial x_i(s\to s-1)}=\begin{cases}\frac{\partial\phi(j\to j-1)}{\partial x_i(j\to j-1)}, & \text{if } s=j\\ 0, & \text{otherwise}\end{cases}. \tag{10}$$

This formula implies that the k-out-of-n MSS represents a special type of MSSs in which a minor degradation of a system component can result only in a minor degradation of system state. Furthermore, it follows that the following relationships exist between SI measures presented in Table 1:

$$\mathrm{SI}_{i,s}^{j\downarrow}=\mathrm{SI}_{i,s}^{\geq j}=\begin{cases}\binom{n-1}{k-1}\frac{(m-j)^{k-1}j^{n-k}}{m^{n-1}}, & \text{if } s=j\\ 0, & \text{otherwise}\end{cases},$$

$$\mathrm{SI}_{i}^{j\downarrow}=\mathrm{SI}_{i}^{\geq j}=\frac{1}{m-1}\mathrm{SI}_{i,j}^{j\downarrow}=\binom{n-1}{k-1}\frac{(m-j)^{k-1}j^{n-k}}{(m-1)m^{n-1}}, \tag{11}$$

$$\mathrm{SI}_{i,s}^{\downarrow}=\mathrm{SI}_{i,s}^{s\downarrow}=\binom{n-1}{k-1}\frac{(m-s)^{k-1}s^{n-k}}{m^{n-1}}.$$

Finally, the total topological importance of component i for the activity of the k-out-of-n MSS can be computed using the following formula:

$$\mathrm{SI}_{i}^{\downarrow}=\frac{1}{m-1}\sum_{j=1}^{m-1}\mathrm{SI}_{i,j}^{j\downarrow}=\binom{n-1}{k-1}\frac{\sum_{j=1}^{m-1}(m-j)^{k-1}j^{n-k}}{(m-1)m^{n-1}}. \tag{12}$$

All these formulae imply that there is no sense to distinguish between the SI measures investigating topological properties of the k-out-of-n MSS with respect to system state (Table 2) and with respect to system availability level (Table 3):

Now, let us consider the BI measures. These measures can also be computed using IDPLDs I or III. However, as has been shown in (10), these IDPLDs computed with respect to system state/availability level j agree with $\partial\phi(j\to j-1)/\partial x_i(s\to s-1)$ in case of k-out-of-n MSSs and, therefore, the following relationships will hold between BI measures calculated with respect to system state and with respect to system availability level:

$$\mathrm{BI}_{i,s}^{j\downarrow}=\mathrm{BI}_{i,s}^{\geq j}=\begin{cases}\Pr\left\{\frac{\partial\phi(j\to j-1)}{\partial x_i(j\to j-1)}=1\right\}, & \text{if } s=j\\ 0, & \text{otherwise}\end{cases},$$

$$\mathrm{BI}_{i}^{j\downarrow}=\mathrm{BI}_{i}^{\geq j}=\frac{1}{m-1}\mathrm{BI}_{i,j}^{j\downarrow}, \tag{13}$$

$$\mathrm{BI}_{i,s}^{\downarrow}=\mathrm{BI}_{i,s}^{s\downarrow},$$

$$\mathrm{BI}_{i}^{\downarrow}=\frac{1}{m-1}\sum_{j=1}^{m-1}\mathrm{BI}_{i,j}^{j\downarrow}.$$

This implies that only problem in computation of the BI measures is calculation of the probability that DPLD $\partial\phi(j\to j-1)/\partial x_i(j\to j-1)$ takes nonzero value. According to the results presented in the previous paragraphs, this DPLD takes nonzero values for all state vectors of the structure function that have the form of $(j_i, r_{u_1}^{\geq j,(1)}, r_{u_2}^{\geq j,(2)},\ldots, r_{u_{k-1}}^{\geq j,(k-1)}, r_{v_1}^{<j,(1)}, r_{v_2}^{<j,(2)},\ldots, r_{v_{n-k}}^{<j,(n-k)})$. Since a DPLD computed with respect to variable x_i does not depend on this variable [8], [9], the nonzero elements of the DPLD agree with the state vectors that have the following form: $(\cdot_i, r_{u_1}^{\geq j,(1)}, r_{u_2}^{\geq j,(2)},\ldots, r_{u_{k-1}}^{\geq j,(k-1)}, r_{v_1}^{<j,(1)}, r_{v_2}^{<j,(2)},\ldots, r_{v_{n-k}}^{<j,(n-k)})$. Therefore, the probability that

the DPLD is nonzero can be computed simply as the probability that the system components are in states corresponding to state vectors of this form.

4 Case Study: Oil Supply System

For illustration of our approach, let us consider a modified oil supply system proposed in [12] and considered in [13]. This system is depicted in Fig. 1. There are 4 pipelines that deliver oil from the oil source to 3 oil stations. The system and every pipeline have 4 possible states. The system state is defined by the number of oil stations to which oil can be delivered through the pipelines (Table 4). The state of a pipeline identifies which oil stations can be supplied through the pipeline (Table 4). Next, let us assume that the oil source is perfectly functioning and an oil station is working if at least k pipelines are able to deliver oil to it. This description implies that only relevant components that determine system state are 4 pipelines. So, we obtain a k-out-of-4 MSS in which $m = 4$ and $k \in \{1,2,3,4\}$.

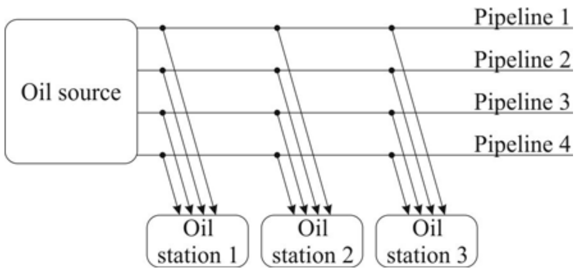

Fig. 1. Oil supply system considered in [12] and [13]

Table 4. Interpretation of system and component states for the oil supply system

State	System	Component (pipeline) *i*
0	No oil station is supplied.	Pipeline *i* is not able to deliver oil to any station.
1	Only oil station 1 is supplied.	Pipeline *i* is able to deliver oil only to station 1.
2	Only oil stations 1 and 2 are supplied.	Pipeline *i* is able to deliver oil only to oil stations 1 and 2.
3	All oil stations are supplied.	Pipeline *i* is able to deliver oil to all oil stations.

Firstly, let us investigate topological properties of this system. This can be done simply using SI measures shown in Table 1 and formulae (11) and (12). For pipeline 1 and individual values of k, these results are presented in Table 5. According to the results presented in the form of formulae (11) and (12), the SI measures computed in these tables investigate topological properties of the system not only with respect to system state but also with respect to system availability level. As we can see (the lower right corner of sub-tables), degradation of component 1 has the greatest influence on system activity if $k \in \{2,3\}$ and the least if $k \in \{1,4\}$. Next, in the bottom parts of the sub-tables, we can see that, in case of $k \in \{1,2\}$, the most important state of the component is state 3, while the least important is state 1. On the other hand, if $k \in \{3,4\}$, then the situation is completely different, i.e. the state with the greatest topological influence on system degradation is state 1, while state 3 has the least influence. Similarly, using the information presented in the right columns of the subtales, we can state if $k \in \{1,2\}$, then pipeline 1 has the greatest influence on system state/availability level 3 and the least on system state/availability level 1; while if $k \in \{3,4\}$, then pipeline 1 has the greatest influence on system state/availability level 1 and the least on system state/availability level 3. Clearly, the same results can be obtained for the remaining pipelines since formulae (11) and (12) imply that all components (or components states) have the same topological influence in case of *k*-out-of-*n* systems and fixed values of *k* and *n*.

Table 5. Structural importance measures investigating pipelines 1,2,3,4 for k = 1, 2, 3, 4

		$k = 1$				$k = 2$			
		Component state			**Average**	**Component state**			**Average**
		1	2	3		1	2	3	
System state	1	0.0156	0	0	0.0052	0.1406	0	0	0.0469
	2	0	0.1250	0	0.0417	0	0.3750	0	0.1250
	3	0	0	0.4219	0.1406	0	0	0.4219	0.1406
Sum		0.0156	0.1250	0.4219	**0.1875**	0.1406	0.3750	0.4219	**0.3125**

		$k = 3$				$k = 4$			
		Component state			**Average**	**Component state**			**Average**
		1	2	3		1	2	3	
System state	1	0.4219	0	0	0.1406	0.4219	0	0	0.1406
	2	0	0.3750	0	0.1250	0	0.1250	0	0.0417
	3	0	0	0.1406	0.0469	0	0	0.0156	0.0052
Sum		0.4219	0.3750	0.1406	**0.3125**	0.4219	0.1250	0.0156	**0.1875**

Now, let us calculate the BI measures for this system. These measures take into account not only system topology but also state probabilities of the pipelines. In this case, we use numbers presented in [13] (Table 6).

The BI measures for the oil supply system can be computed using formulae (13). According to these formulae, the most important part in computation of these measures is calculating the probability that DPLD $\partial\phi(j \to j-1)/\partial x_i(j \to j-1)$, for $j = 1,2,3$ and $i = 1,2,3,4$, takes nonzero value. This can be done simply by identifying the state vectors of the form of $(\cdot_i, r_{u_1}^{\geq j,(1)}, r_{u_2}^{\geq j,(2)}, \ldots, r_{u_{k-1}}^{\geq j,(k-1)}, r_{v_1}^{<j,(1)}, r_{v_2}^{<j,(2)}, \ldots, r_{v_{n-k}}^{<j,(n-k)})$ for specific values of j and k. For example, if we want to compute the probability that DPLD $\partial\phi(1 \to 0)/\partial x_1(1 \to 0)$ is nonzero for $k = 1$, then its nonzero elements agree

Table 6. State probabilities of the pipelines of the oil supply system

Component	Components state			
	0	1	2	3
1	0.0500	0.0950	0.0684	0.7866
2	0.0500	0.0950	0.0684	0.7866
3	0.0300	0.0776	0.0446	0.8478
4	0.0300	0.0776	0.0446	0.8478

with the state vectors of the form of $(.,r^{<1},r^{<1},r^{<1})$. Only one state vector has this form, i.e. state vector (.,0,0,0); therefore, the probability that the DPLD takes nonzero value is computed as follows:

$$\Pr\left\{\frac{\partial\phi(2\to1)}{\partial x_1(2\to1)}=1\right\}=\Pr\{(._1,\boldsymbol{x})=(.,0,0,0)\}=p_{2,0}\,p_{3,0}\,p_{4,0} \\ =0.05*0.03*0.03=0.000045. \tag{14}$$

If we want to compute the probability that DPLD $\partial\phi(2\to1)/\partial x_1(2\to1)$ takes nonzero value for $k=1$, then we should calculate the probability that state vector $(._1,\boldsymbol{x})$ has the form of $(.,r^{<2},r^{<2},r^{<2})$. This condition is met if $(._1,\boldsymbol{x})\le(.,1,1,1)$, where relation "$\le$" between state vectors $(._i,\boldsymbol{x})=(x_1,x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ and $(._i,\boldsymbol{y})=(y_1,y_2,\ldots,y_{i-1},y_{i+1},\ldots,y_n)$ means that $x_k\le y_k$ for $k=1,2,\ldots,i-1,i+1,\ldots,n$. So, the DPLD is nonzero with the following probability:

$$\Pr\left\{\frac{\partial\phi(2\to1)}{\partial x_1(2\to1)}=1\right\}=\Pr\{(._1,\boldsymbol{x})\le(.,1,1,1)\}=(p_{2,0}+p_{2,1})(p_{3,0}+p_{3,1})(p_{4,0}+p_{4,1}) \\ \approx 0.001679. \tag{15}$$

Finally, the probability of DPLD $\partial\phi(3\to2)/\partial x_1(3\to2)$ being nonzero for $k=1$ agrees with the probability that state vector $(._1,\boldsymbol{x})$ has the form of $(.,r^{<3},r^{<3},r^{<3})$. Using the notation "$\le$", this can be calculated as follows:

$$\Pr\left\{\frac{\partial\phi(3\to2)}{\partial x_1(3\to2)}=1\right\}=\Pr\{(._1,\boldsymbol{x})\le(.,2,2,2)\} \\ =(p_{2,0}+p_{2,1}+p_{2,2})(p_{3,0}+p_{3,1}+p_{3,2})(p_{4,0}+p_{4,1}+p_{4,2}) \\ \approx 0.004943. \tag{16}$$

Equations (14) – (16) identify the probability that DPLDs of the form of $\partial\phi(j\to j-1)/\partial x_1(j\to j-1)$ takes nonzero value. Since this probability agrees with values of $\mathrm{BI}_{1,j}^{j\downarrow}$ and $\mathrm{BI}_{1,j}^{\ge j}$, it allows us to investigate importance of degradation of state j of pipeline 1 on state/availability level j of the oil supply system. Since all other BI measures investigating importance of a degradation of a component on system state/availability level are equal to 0 (formulae (13)), these values can be used to investigate importance of a given component state for the system, or total importance of pipeline 1 for a specific system state/availability level, or total importance of pipeline 1 for the entire system. All these numbers are presented in the upper left sub-table of Table 7. Clearly, the same results can be obtained if we investigate importance of component 2, since state probabilities for these components are same.

Table 7. Birnbaum's importance measures investigating pipelines 1 and 2 for $k = 1, 2, 3, 4$

$k = 1$	Component state 1	Component state 2	Component state 3	**Average**
System state 1	0.00005	0	0	0.00002
System state 2	0	0.0017	0	0.0006
System state 3	0	0	0.0049	0.0016
Sum	0.00004	0.0017	0.0049	**0.0022**

$k = 2$	Component state 1	Component state 2	Component state 3	**Average**
System state 1	0.0038	0	0	0.0013
System state 2	0	0.0377	0	0.0126
System state 3	0	0	0.0733	0.0244
Sum	0.0038	0.0377	0.0733	**0.0382**

$k = 3$	Component state 1	Component state 2	Component state 3	**Average**
System state 1	0.1023	0	0	0.0341
System state 2	0	0.2797	0	0.0932
System state 3	0	0	0.3563	0.1188
Sum	0.1023	0.2797	0.3563	**0.2461**

$k = 4$	Component state 1	Component state 2	Component state 3	**Average**
System state 1	0.8939	0	0	0.2980
System state 2	0	0.6809	0	0.2270
System state 3	0	0	0.5654	0.1885
Sum	0.8939	0.6809	0.5654	**0.7134**

Using the similar procedure as has been presented above, the BI measures of all system components can be obtained (upper left sub-tables of Table 7 and Table 8). Furthermore, if we repeat this procedure for other values of k, i.e. for $k = 2,3,4$, we can investigate importance of individual pipelines for all versions of the oil supply system (the remaining parts of Table 7 and Table 8). (Please note that we obtain the same results for components 1 and 2 and for components 3 and 4 since their state probabilities have the same values.) According to the data presented in Table 7 and Table 8, we can state that pipelines 1 and 2 have less influence on the activity of the oil supply system than pipelines 3 and 4 if $k = 1,2,3$ but greater if $k = 4$. Another interesting fact that can be noticed based on Table 7 and Table 8 is that all system components have greater influence on greater states of the system in case of $k \in \{1,2,3\}$, (e.g., if $k = 1$, then degradation of component 1 results in degradation of system state 3 with the probability 0.0016 while in degradation of system state 1 with the probability 0.00002. However, in case of $k = 4$, all pipelines are more important for lower states of the system (e.g., a degradation of pipeline 1 causes degradation of system state 1 with the probability 0.2980 and degradation of system state 3 with the probability 0.1885). The similar facts can be observed for the BI measures investigating the total importance of individual states of the pipelines for the oil supply system (the bottom rows in sub-tables of Table 7 and Table 8). All these results imply that importance of individual system components in case of *k*-out-of-*n* systems largely depends on mutual values of *k* and *n*.

Table 8. Birnbaum's importance measures investigating pipelines 3 and 4 for $k = 1, 2, 3, 4$

	$k = 1$				$k = 2$			
	Component state			**Average**	**Component state**			**Average**
System state	1	2	3		1	2	3	
1	0.00008	0	0	0.00003	0.0053	0	0	0.0018
2	0	0.0023	0	0.0008	0	0.0454	0	0.0151
3	0	0	0.0069	0.0023	0	0	0.0897	0.0299
Sum	0.00008	0.0023	0.0069	**0.0031**	0.0053	0.0454	0.0897	**0.0468**

	$k = 3$				$k = 4$			
	Component state			**Average**	**Component state**			**Average**
System state	1	2	3		1	2	3	
1	0.1192	0	0	0.0397	0.8755	0	0	0.2918
2	0	0.2999	0	0.1000	0	0.6524	0	0.2175
3	0	0	0.3788	0.1263	0	0	0.5246	0.1786
Sum	0.1192	0.2999	0.3788	**0.2660**	0.8754	0.6524	0.5246	**0.6841**

5 Conclusion

In this paper, importance analysis of a *k*-out-of-*n* MSS was considered. We summarized some results from the qualitative and quantitative analysis of MSSs and proposed the method for calculation of all range of the SI and BI measures. Furthermore, we showed that a *k*-out-of-*n* MSS is a special type of MSSs in which a minor degradation of any system component can result only in a minor degradation of system state. Because of that, it is not important to distinguish between IMs focusing on system state and IMs dealing with system availability level. Next, using logical differential calculus, we found closed-form expressions for calculation of the SI measures for a *k*-out-of-*n* MSS. All these results were used in the analysis of the oil supply system considered in [12] and [13]. Based on our approach, we identified topological importance of individual system components for different values of *k* and identified which components of the oil supply system were the most important if the state probabilities of individual system components were known. Finally, we would like to mention that the results presented in this paper could also be applied in the analysis of other types of systems such as medical and temporal database systems studied in [17] and [18].

Furthermore, they could also be used in other research fields, such as data mining, where they can be used to find key attributes in a dataset [19].

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