Anti-Modules

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Abstract. Modules of a logical theory are subtheories that are conservatively extended by the theory. There are, on the other hand, subtheories that are not contained in any proper module of the theory, which we refer to as the residue of the theory. In this paper, we characterize properties of residues and explore their role in ontology modularization. We discuss that there are ontological commitments that cannot be captured by modules of a theory and must be axiomatized by residues. We observe that sentences in the residue of a theory eliminate some possible combinations of models of modules of the theory. In that sense, ontological commitments that are captured by residues basically determine how arbitrary models of modules must be combined.

Keywords. Ontology Modularization, Residues, Ontology Composition, Ontology Decomposition

1. Introduction

The modularization of an ontology is a indispensable technique in the analysis of an ontology. Modules tell us about the organization of an ontology through the relationships between its logical subtheories. Modules also constrain how we can extend an ontology; one of the original motivations for modularity was the safety – how can we extend an ontology and still preserve the intended interpretations of the terminology in the original ontology? Nevertheless, modules do not tell us the whole story.

In general, an ontology is not equivalent to the union of its modules; there will be axioms that are not contained in any module of the ontology. Since almost all of the research within applied ontology has focused on the identification and extraction of the modules of an ontology, it has neglected to investigate such sentences which cannot be contained in any module.

In this paper, we formally define the notion of residue of a theory[1] as the set of sentences which are not contained in any module, and show how the residue gives us insight into ontology design and evaluation. After introducing the notion of residue in Section 3, and proving some basic properties, in Section 4 we explore the role that residues play in the decomposition of an ontology that is the result of ontology verification. In Section 5, we see the role of residues in understanding how ontologies can be composed as modules of larger ontologies.

[1] In this paper, we use the words "ontology" and "theory" interchangeably. We consider a theory to be a set of first-order sentences closed under logical entailment, and a subtheory to be a theory which is contained by another theory.

2. Relationships Among Ontologies

We first provide a short review of the relationships between ontologies that will be necessary for introducing the notion of the residue of a theory.[2]

Definition 1 *Let T_1, T_2 be two first-order theories such that $\Sigma(T_1) \subseteq \Sigma(T_2)$.*
T_2 is an extension of T_1 if and only if for any sentence $\Phi \in \mathscr{L}(T_1)$,

$$T_1 \models \Phi \;\Rightarrow\; T_2 \models \Phi.$$

T_2 is a conservative extension of T_1 if and only if for any sentence $\Phi \in \mathscr{L}(T_1)$,

$$T_1 \models \Phi \;\Leftrightarrow\; T_2 \models \Phi.$$

Non-conservative and conservative extensions are generalized to theories with different signatures through the notions of non-faithful and faithful interpretations. We adopt the definitions of these notions from [2]:

Definition 2 *An interpretation π of a signature Σ_1 into a theory T_2 is a function on the set of symbols in Σ_1 such that*

1. *π assigns to $\forall$ a formula $\pi_\forall$ of $\mathscr{L}(T_2)$ in which at most the variable v_1 occurs free, and*

$$T_2 \models (\exists v_1)\, \pi_\forall$$

2. *π assigns to each n-place predicate symbol P a formula π_P of $\mathscr{L}(T_2)$ in which at most n variables occur free.*
3. *π assigns to each n-place function symbol f a formula π_f of $\mathscr{L}(T_2)$ in which at most the variables $v_1, ..., v_n, v_{n+1}$ occur free, and*

$$T_2 \models (\forall v_1, ..., v_n)\, \pi_\forall(v_1) \wedge ... \wedge \pi_\forall(v_n) \supset$$
$$(\exists x)(\pi_\forall(x) \wedge ((\forall v_{n+1})(\pi_f(v_1, ..., v_{n+1}) \equiv (v_{n+1} = x))))$$

Let T_1 be a theory. An interpretation π of $\Sigma(T_1)$ into T_2 is an interpretation of T_1 in T_2 if and only if for all sentences $\Phi, \Phi_1, \Phi_2 \in \mathscr{L}(T_1)$,

- *if Φ is an atomic sentence with predicate symbol P, then $\pi(\Phi) = \pi_P$;*
- $\pi(\neg\Phi) = \neg\pi(\Phi)$;
- $\pi(\Phi_1 \wedge \Phi_2) = \pi(\Phi_1) \wedge \pi(\Phi_2)$;
- $\pi(\Phi_1 \vee \Phi_2) = \pi(\Phi_1) \vee \pi(\Phi_2)$;
- $\pi(\Phi_1 \supset \Phi_2) = \pi(\Phi_1) \supset \pi(\Phi_2)$;
- $\pi(\exists x \Phi) = (\exists x)\, \pi_\forall(x) \wedge \pi(\Phi)$;
- $\pi(\forall x \Phi) = (\forall x)\, \pi_\forall(x) \supset \pi(\Phi)$;

[2] For every theory T, $\Sigma(T)$ denotes the signature of T, which is the set of all the constant, function, and relation symbols used in T, $\mathscr{L}(T)$ denotes the language of T, which is the set of all first-order formulae generated by symbols in $\Sigma(T)$, and $Mod(T)$ denotes the class of all models of T.

- *For any sentence* $\Phi \in \mathscr{L}(T_1)$,

$$T_1 \models \Phi \;\Rightarrow\; T_2 \models \pi(\Phi).$$

An interpretation π *of a theory* T_1 *into a theory* T_2 *is faithful iff for any sentence* $\Phi \in \mathscr{L}(T_1)$,

$$T_1 \not\models \Phi \;\Rightarrow\; T_2 \not\models \pi(\Phi).$$

If an (faithful) interpretation from T_1 into T_2 exists, we say that T_1 is (faithfully) interpretable in T_2, or T_2 (faithfully) interprets T_1. Interpretation preserves theorems of the original theory, while faithful interpretation preserves decidability and satisfiability.

An interpretation of T_1 into T_2 defines symbols of T_1 in terms of the language of T_2. This indicates that relative interpretations are related to the notion of definitional extension:

Definition 3 (adopted from [7]) *Let* T *be a theory with signature* $\Sigma(T)$.
T' *is a definitional extension of* T *if and only if there exists a set of sentences* Δ *such that*

1. *for every predicate symbol* $P \in \Sigma(T') \setminus \Sigma(T)$, Δ *includes a sentence of the form*

$$(\forall \bar{x})\, P(\bar{x}) \equiv \Phi(\bar{x}),$$

where Φ *is a formula in* $\mathscr{L}(T)$*;*

2. $T \cup \Delta$ *is a conservative extension of* T*, and is logically equivalent with* T'.

Interpretations between theories can be axiomatized in terms of translation definitions between the theories:

Definition 4 Δ *is a set of translation definitions for* T_1 *into* T_2 *if and only if* $T_2 \cup \Delta$ *is a definitional extension of* T_2 *and*

$$T_2 \cup \Delta \models T_1.$$

More formally, the results in [5] show that T_1 is interpretable in T_2 iff there exists a set of translation definitions Δ for T_1 into T_2 such that $T_2 \cup \Delta$ entails T_1.

Finally, just as interpretations can be considered to be a generalization of extensions to theories with disjoint signatures, the notion of logical synonymy generalizes the notion of logical equivalence.

Definition 5 (adopted from [7]) *Two ontologies* T_1 *and* T_2 *are logically synonymous if and only if there exists an ontology* T_3 *with the signature* $\Sigma(T_1) \cup \Sigma(T_2)$ *that is a definitional extension of* T_1 *and* T_2.

If T_1 and T_2 are logically synonymous, then there exists a set of translation definitions Δ for T_1 into T_2, and a set of translation definitions Π for T_2 into T_1 such that $T_2 \cup \Delta$ is logically equivalent with $T_1 \cup \Pi$.

3. What is a Residue?

3.1. Modules and Residues

We consider a module of a theory T to be a subtheory of T, and adopt the following definition from [4].

Definition 6 *T_1 is a module of T_2 if and only if T_2 is a conservative extension of T_1.*

Note that this is more general than the notion of module used in work such as [11] and [8], in which a module is required to be a subset of the axioms in T. We believe that it is a natural generalization of the notion of modularity because it is more robust with respect to different possible logically equivalent axiomatizations of a theory.

We are not interested in finding a module of a theory, but rather in understanding how a theory is related to sets of its modules.

Definition 7 *A theory T is perfectly modularized into a set of proper modules $T_1,...,T_n$ iff*

$$T = T_1 \cup ... \cup T_n$$

Many theories, however, cannot be perfectly modularized, that is, there are sentences in the theory which are not contained in any proper module of the theory. We will refer to the set of all such sentences as the residue of the theory, since it is the set of sentences "leftover" after all of the modules have been determined:

Definition 8 *Let $T_1,...,T_n$ be all proper modules of a theory T.*
The residue R of T is the subtheory of T that is logically equivalent to

$$T \setminus (T_1 \cup ... \cup T_n).$$

Thus, a theory has a nontrivial residue iff it is not perfectly modularized.

Example: Let T be the theory[3] with axioms $\{(\exists x)\,A(x), (\exists x)\,B(x), (\forall x)\,A(x) \vee B(x)\}$.
T has two modules $\{(\exists x)\,A(x)\}$, $\{(\exists x)\,B(x)\}$.
The residue of T is $\{(\forall x)\,A(x) \vee B(x)\}$. □

Example: Consider the ontology $T_{dolce\_present}$[4] which is a subtheory of the DOLCE Ontology that captures intuitions about how objects can be present at different points in time. One module is equivalent to $T_{dolce\_time\_mereology}$[5] (which axiomatizes the mereology on temporal regions); the other module consists of the axioms:

$$(\forall x)\,(ED(x) \vee PD(x) \vee Q(x)) \supset (\exists t)\,PRE(x,t),$$

$$(\forall x,t)\,PRE(x,t) \supset T(t).$$

[3]Thank you to one of the reviewers, who suggested this example.
[4]`colore.oor.net/dolce_present/dolce_present.clif`
[5]`colore.oor.net/dolce_time_mereology/dolce_time_mereology.clif`

In addition, there are two axioms which are not contained in any module of $T_{dolce\_present}$:

$$(\forall x, t, t_1)\, PRE(x,t) \wedge P(t_1,t) \supset PRE(x,t_1),$$

$$(\forall x, t, t_1, t_2)\, PRE(x,t_1) \wedge PRE(x,t_2) \wedge SUM(t,t_1,t_2) \supset PRE(x,t).$$

and these axioms form the residue for $T_{dolce\_present}$. Note that the signature of this subset of axioms is the union of the signature of the two modules. □

Example: Consider the ontology T_{wog}[6] that plays a role in the verification of OWL-Time [3].
One module is equivalent to the theory $T_{partial\_bipartite}$[7] and the other module is equivalent to the theory $T_{betweenness}$[8]. The two sentences

$$(\forall x, y, z, w, l)\, in(x,l) \wedge in(y,l) \wedge in(z,l) \wedge in(w,l)$$
$$\wedge\, between(x,y,z) \wedge (y \neq z) \wedge between(y,z,w) \supset between(x,y,w),$$

$$(\forall x, y, z, w, l)\, in(x,l) \wedge in(y,l) \wedge in(z,l) \wedge in(w,l)$$
$$\wedge\, between(x,y,w) \wedge between(y,z,w) \supset between(x,y,z)$$

are not contained in any module, and hence form the residue for T_{wog}. □

Example: There exist theories which have no modules, and hence are equivalent to their residue.
The theories of partial orderings, semilinear orderings, and linear orderings are all of the signature **let**, and no subtheory of these theories is conservatively extended by the theory. □

It is easy to see that a theory is a non-conservative extension of any subtheory that contains its residue. Furthermore, if T has a residue R, then R is unique, up to logical equivalence, since the set of all modules of T is uniquely determined. The following theorem characterizes sentences in the residue of a theory:

Theorem 1 *If R be the residue of a theory T, then* $\Sigma(R) = \Sigma(T)$.

Proof: Let R be the residue of T. By definition, R is a subtheory of T, so $\Sigma(R) \subseteq \Sigma(T)$.

Suppose, for a contradiction, that $\Sigma(R) \neq \Sigma(T)$. Then $\Sigma(R)$ must be a proper subset of $\Sigma(T)$. Since $\Sigma(R) \subset \Sigma(T)$, there must be a module T' of T which contains sentences with the signature $\Sigma(R)$. Since T' is a module of T, T has to be a conservative extension of T'. This means that any sentence in $T \setminus T'$ extends the signature of T' (otherwise T would be a non-conservative extension of T'). Thus, R must be included in T', which is a contradiction. □

The above theorem shows that sentences in the residue of a theory specify how the modules of the theory are composed; we will explore this in more detail in Section 5.

[6] colore.oor.net/ordered_geometry/wog.clif
[7] colore.oor.net/bipartite_incidence/partial_bipartite
[8] colore.oor.net/betweenness/betweenness.clif

3.2. Residues and Strong Reducibility

The notion of reducibility presented in [5] uses the metalogical relationships among ontologies to modularize an ontology.

Definition 9 *A theory T is strongly reducible to a set of theories $S_1, ..., S_n$ iff*

1. *T faithfully interprets each theory S_i;*
2. *T is synonymous with $S_1 \cup ... \cup S_n$.*

If a theory T is strongly reducible to theories $S_1, ..., S_n$, then $S_1, ..., S_n$ is called a strong reduction of T.

Example: Consider the theory $T_{endpoints}$[9] of time endpoints which relates the notion of linear time points with the notion of time intervals by defining the functions *beginof*, *endof*, and *between*. $beginof(i)$, $endof(i)$ indicate the begin and the end point of an interval i respectively, while $between(p,q)$ denotes the interval between time points p and q. The theory includes a binary relation $before$ over time points which is transitive and irreflexive.
$T_{endpoints}$ is strongly reducible to the theory $T_{linear\_ordering}$[10] of linear ordering and the theory $T_{strict\_graphical}$[11] of strict graphical incidence structures.
A strict graphical incidence structure [6] is a tuple $\mathbb{G} = \langle X, Y, \mathbf{in} \rangle$ such that

1. X and Y are disjoint sets, and
$$\mathbf{in} \subseteq (X \times Y) \cup (Y \times X) \cup (X \times X) \cup (Y \times Y).$$
Two elements of $\mathbb{G}$ that are related by **in** are called incident;
2. all elements of Y are incident with exactly two elements of X, and for each pair $\mathbf{p}, \mathbf{q} \in X$ there exists a unique element in Y that is incident with both **p** and **q**.

The reductive modules of $T_{endpoints}$ are the subtheories which are logically synonymous with $T_{linear\_ordering}$ and $T_{strict\_graphical}$. □

Lemma 1 *If T has an empty residue, then T is strongly reducible.*

Proof: Let $T_1, ..., T_n$ be all modules of a theory T, and suppose T has empty residue.
Then T is logically equivalent to $T_1 \cup ... \cup T_n$, and consequently is synonymous with $T_1 \cup ... \cup T_n$. Moreover, T faithfully interpret each T_i, $1 \leq j \leq n$, since T is a conservative extension of T_i. Consequently, T is strongly reducible to $T_1, ..., T_n$. □

Thus, any theory which is not strongly reducible will have a nonempty residue; however, there do exist strongly reducible theories with nonempty residues. For example, although $T_{endpoints}$ is reducible, it has a nonempty residue. The following sentence, for example, is not entailed by any module of $T_{endpoints}$:

$$(\forall i)\, timeinterval(i) \supset before(beginof(i), endof(i)). \quad (1)$$

[9] `colore.oor.net/combined_time/endpoints.clif`
[10] `colore.oor.net/orderings/linear_ordering.clif`
[11] `colore.oor.net/bipartite_incidence/strict_graphical.clif`

This observation is rather counter-intuitive as it seems that if a strong reduction of a theory exists, then each theory in the reduction should correspond with a module of the theory. The following theorem shows that this is indeed the case. However, later on in Theorem 3, we will show that residues are not preserved by reductions. Therefore, having a strong reduction does not necessarily imply that a perfect modularization for a theory exists.

Theorem 2 (from [4]) *Let $S_1, ..., S_n$ be a strong reduction of a theory T.*

There exist theories $T_1, ..., T_n$ such that

1. *T_i is synonymous with S_i.*
2. *T_i is a module of T, for $1 \leq i \leq n$;*

If $S_1, ..., S_n$ is a strong reduction of T and T_i is a module of T which is synonymous with a theory in the reduction, then T_i is called a reductive module of T.

It is easy to see that not all modules of a theory are reductive modules. For example, neither $T_{dolce\_present}$ nor T_{wog} have reductive modules since neither of them is reducible. Moreover, the following theorem proves our earlier observation that residues are not preserved by strong reducibility:

Theorem 3 *Let $S_1, ..., S_n$ be a strong reduction of a theory T, and Δ be a set of translation definitions from T into $S_1 \cup ... \cup S_n$.*

Let R be the residue of T. Then $\Delta \models R$.

Proof: Suppose $S_1, ..., S_n$ is a strong reduction of T. Then T and $S_1 \cup ... \cup S_n$ are synonymous. There exist sets of translation definitions Π and Δ such that $T \cup \Pi$ and $S_1 \cup ... \cup S_n \cup \Delta$ are logically equivalent. Hence,

$$S_1 \cup ... \cup S_n \cup \Delta \models R.$$

On the other hand, by Theorem 2, T has a reductive modularization $T_1, ..., T_n$ such that T_i and S_i are synonymous, for $1 \leq i \leq n$. That is, for each T_i and S_i, $1 \leq i \leq n$, there exist translation definitions Π_i and Δ_i such that $T_i \cup \Pi_i$ and $S_i \cup \Delta_i$ are logically equivalent, and $\Pi_i \subset \Pi$ and $\Delta_i \subset \Delta$. Then we have ($\equiv$ denotes logical equivalence)

$$T_1 \cup \Pi_1 \cup ... \cup T_n \cup \Pi_n \equiv S_1 \cup \Delta_1 \cup ... \cup S_n \cup \Delta_n.$$

Since R is disjoint from $T_1, ..., T_n$, we can say that

$$S_1 \cup \Delta_1 \cup ... \cup S_n \cup \Delta_n \not\models R.$$

Thus, Δ must entail R. □

Consider again $T_{endpoints}$; using an automated theorem prover it can be verified that the following sentences are translation definitions from $T_{endpoints}$ to $T_{linear\_ordering} \cup T_{strict\_graphical}$:

$$\Delta: \quad (\forall x)\, timepoint(x) \equiv point(x).$$
$$(\forall x)\, timeinterval(x) \equiv line(x).$$
$$(\forall x,y)\, (beginof(y) = x) \equiv$$
$$(in(x,y) \wedge point(x) \wedge (x \neq y) \wedge (\forall z)\, point(z) \wedge in(z,y)) \supset leq(x,z).$$
$$(\forall x,y)\, (endof(y) = x) \equiv$$
$$(in(x,y) \wedge point(x) \wedge (x \neq y) \wedge (\forall z)\, point(z) \wedge in(z,y)) \supset leq(z,x).$$
$$(\forall x,y)\, before(x,y) \equiv leq(x,y).$$
$$(\forall x,y,z)\, (z = between(x,y)) \equiv (point(x) \wedge point(y) \wedge line(z) \wedge in(x,z) \wedge in(y,z)).$$

It can be verified that the residue of $T_{endpoints}$ (including sentence (1)) are entailed by Δ.

3.3. Residues and Weak Reducibility

Reducibility places strong conditions on the relationship between a theory and its modules. Yet, we have already seen several examples of theories which are not reducible and still can be decomposed into modules. In order to capture this larger class of theories, we introduce the following:

Definition 10 *A theory T is weakly reducible to the theories $S_1,...,S_n$ iff*

1. *T faithfully interprets each theory S_i, and*
2. *$S_1 \cup ... \cup S_n$ is synonymous with a subtheory T' of T which has the same signature as T (that is, $\Sigma(T') = \Sigma(T)$.*

It turns out that there are many theories which are weakly reducible.

Example: The ontology T_{period}, proposed by van Benthem in [10], specifies the weakest ontology that is satisfied by time periods. The signature of the ontology consists of two primitive relations, **precedence** and **inclusion**, and two defined relations, **glb** and **overlaps**. According to the ontology, **precedence** is a transitive and irreflexive relation which induces a strict partial ordering over elements of the domain, while **inclusion** relation is transitive, reflexive, and antisymmetric and induces a partial ordering over elements. The ontology further includes axioms which specify the interplay between the **precedence** and **inclusion** relations, and guarantee the existence of greatest lower bounds between overlapping intervals. It can be shown that $T_{periods}$ is weakly reducible to $T_{prod\_mereology}$ and $T_{partial\_ordering}$. □

As we saw with reducibility, the notion of weak reducibility also leads to the identification of a class of modules for a theory:

Theorem 4 *Let $S_1,...,S_n$ be a weak reduction of a theory T.*
There exist theories $T_1,...,T_n$ such that
1. *T_i is synonymous with S_i;*
2. *T_i is a module of T, for $1 \leq i \leq n$.*

If $S_1, ..., S_n$ is a weak reduction of T, and T_i is a module of T which is synonymous with a theory in the weak reduction, then T_i is called a weak reductive module of T.

If we look closely at the definition of weak reducibility, we can see that the sentences in $T \setminus T'$ are not synonymous with any sentences in any of the S_i theories. Thus, the sentences in $T \setminus T'$ are also in the residue of T. Further, we can show that the residue of T' is included in the residue of T.

Theorem 5 *Let $S_1, ..., S_n$ be a weak reduction of a theory T and T' be the subtheory of T with $\Sigma(T') = \Sigma(T)$ which is synonymous with $S_1 \cup ... \cup S_n$.* [12]

Let R' and R be residues of T' and T, respectively. Then $R' \subseteq R$.

Proof: By definition, $\Sigma(T') = \Sigma(T)$. Also, by Theorem 1, $\Sigma(T') = \Sigma(R')$ and $\Sigma(T) = \Sigma(R)$. Therefore, $\Sigma(R') = \Sigma(T)$ and $\Sigma(R') = \Sigma(R)$.

Now suppose $\Phi \in R'$. Then $\Sigma(\Phi) = \Sigma(T') = \Sigma(T)$ (otherwise T' would be a non-conservative extension of its module with the signature $\Sigma(\Phi)$, which is a contradiction with the definition of modules). Then Φ cannot be in any module of T since T is a non-conservative extension of any theory that contains Φ. Thus, Φ has to be in R. □

For another perspective of the role that residues play in weakly reducible theories, we can take a closer look at the relationship between models of the theory and the models of its weak reductive modules. Consider, for example, a model $\mathscr{M}$ of T_{period}. The reduct of $\mathscr{M}$ to the signature **inclusion** is a model of the weak reductive module that is synonymous with $T_{prod\_mereology}$, while the reduct of $\mathscr{M}$ to the signature **precedence** is a model of the weak reductive module that is synonymous with $T_{partial\_ordering}$. However, we cannot amalgamate arbitrary models of these modules to construct a model of T_{period}; any such amalgamation must also satisfy the axioms in the residue. We can therefore see that the residue eliminates some of the possible amalgamations of models of the weak reductive modules.

4. Residues and Ontology Decomposition

Given the relationship between reducibility, weak reducibility, and residues, an obvious question is how residues can be found through the reduction of a theory. This effectively provides a decomposition of a theory into the residue and a set of reductive and weak reductive modules. In this section, we use the reduction of a subtheory of the DOLCE Ontology, and the resulting modularization, as a case study to illustrate the role that residues play in the decomposition of an ontology.

The residue often determines whether or not a particular module is a reductive or a weak reductive module, and such a modularization can be strikingly different from other approaches to modularization. For example, The work in [9] used a modularization of the DOLCE Ontology to prove the consistency of the ontology. This modularization can be seen in Figure 1. On the other hand, the verification of the DOLCE Ontology presented in [1] led to a set of reductive modules (and no residue for the ontology as a whole), yet the reductive modules were organized around the ontological categories of endurants and perdurants.

[12] Note that by Definition 10, if T is weakly reducible, which is the case in this theorem, then T' exists.

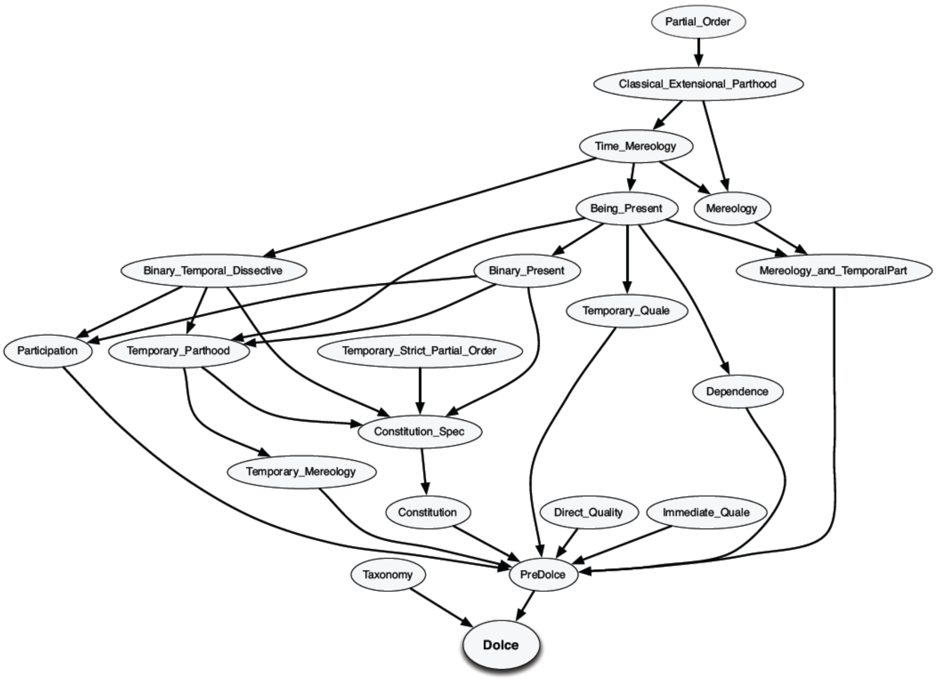

Figure 1. Modularization of DOLCE. Ovals denote the modules found in [9].

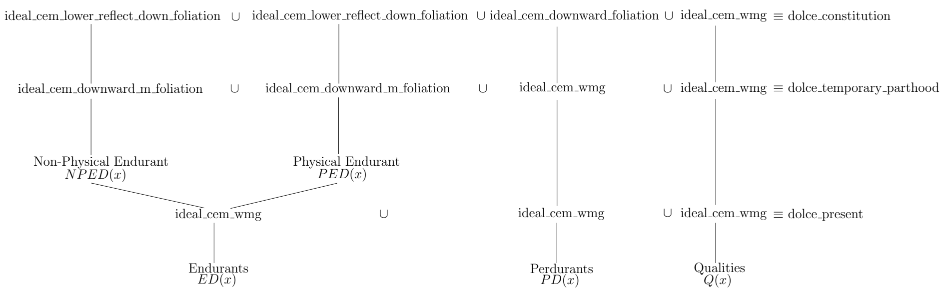

Figure 2. Modularization of DOLCE from [1]. Vertical sets of theories are reductive modules, with each theory in the set being a weak reductive module. Horizontal sets correspond to the modules found in [9].

Looking more closely at these reductive modules, we can see that they are themselves weakly reducible. Effectively, the residues of each of these subtheories prevented a further decomposition into smaller reductive modules. On the other hand, we can see that the combination of the weak reductive modules of the reductive modules does lead to an organization that is more aligned with that in Figure 1.

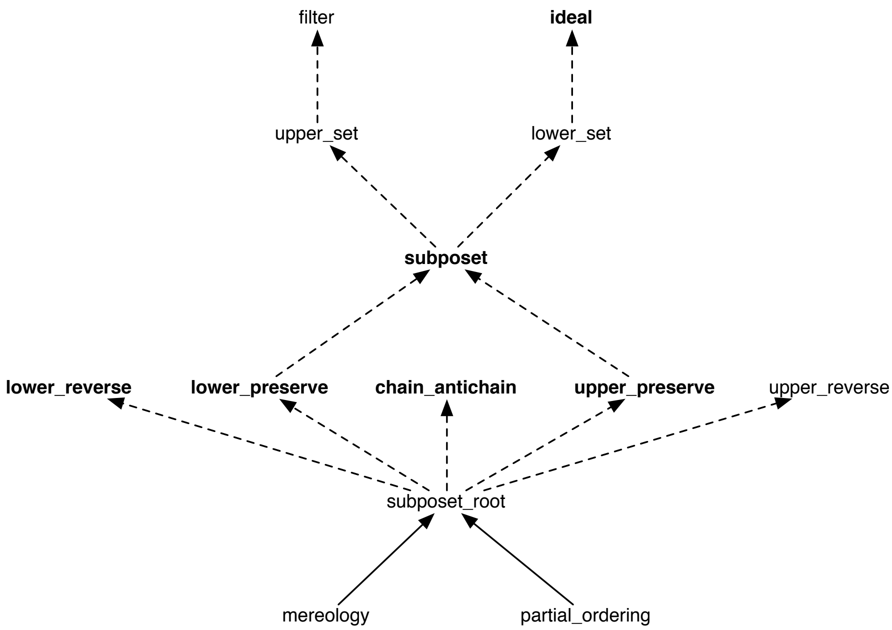

Figure 3. The hierarchy of ontologies obtained by composing mereologies and theories of partial orderings. Dashed lines denote non-conservative extension and solid lines denote conservative extension.

5. Residues and Ontology Composition

So far, we have considered the role of residues in ontology decomposition. In this section we explore the role of residues in ontology construction and composition.

A common practice in ontology design is to identify existing ontologies that satisfy parts of the ontological commitments and requirements, and reuse them as building modules of the new ontology. However, often there are ontological commitments that cannot be captured by the modules alone, and residues are needed to properly axiomatize such commitments.

To illustrate the role of residues in ontology composition, we use some of the ontologies in the subposet hierarchy. [13] The subposet hierarchy is a collection of mathematical ontologies which are developed for verification of spacial an temporal ontologies. The weakest ontology in this hierarchy is constructed by taking the weakest mereology and the weakest axiomatization of orderings. Other ontologies in the hierarchy are constructed by taking stronger mereologies or stronger theories of orderings. However, strengthening the building modules does not always provide the required ontology.

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[13] A hierarchy is a set of ontologies with the same signature [5]. More formally, a hierarchy $\mathbb{H} = \langle \mathscr{H}, \leq \rangle$ is a partially ordered, finite set of theories $\mathscr{H} = T_1, ..., T_n$ such that

1. $\Sigma(T_i) = \Sigma(T_j)$, for all i, j;
2. $T_1 \leq T_2$ iff T_2 is an extension of T_1;
3. $T_1 < T_2$ iff T_2 is a non-conservative extension of T_1.

For example, the work in [6] shows that verification of $T_{periods}$ requires two ontologies, $T_{lower\_preserve}$ and $T_{upper\_preserve}$, in the subposet hierarchy where $T_{lower\_preserve}$ is constructed by adding the following residue axiom to $T_{mereology} \cup T_{partial\_ordering}$

$$(\forall x,y,z)\, part(y,z) \wedge leq(x,y) \supset leq(x,z)$$

and $T_{upper\_preserve}$ is constructed by adding the following residue axiom to $T_{mereology} \cup T_{partial\_ordering}$

$$(\forall x,y,z)\, part(z,x) \wedge leq(x,y) \supset leq(z,y).$$

Notice that both of these two axioms specify the relationship between the parthood relation and the ordering relation. In other words, these axioms determine how models of modules of $T_{lower\_preserve}$ and $T_{upper\_preserve}$ must be combined.

Another example is the theory of subposets ($T_{subposet}$) itself, which is obtained by extending $T_{lower\_preserve} \cup T_{upper\_preserve}$ with the residue axiom

$$(\forall x,y)\, part(x,y) \supset leq(x,y).$$

In all of these three examples, the residue eliminates some possible combinations of models of the modules. All theories in Figure 3 are obtained by adding residues to $T_{mereology} \cup T_{partial\_ordering}$.

This observation can be generalized in terms of an ontology composition principle: Consider the root theory T of an ontology hierarchy so that T contains weak reductive modules $T_1, ..., T_n$ and residue R. Other ontologies in the hierarchy are obtained by strengthening at least one of $T_1, ..., T_n$, or strengthening R.

6. Summary

In this paper we have introduced the notion of the residue of a theory, which is the set of sentences not contained in any module of the theory. Given the original motivations for the study of modularity in the notion of ontology extension (safety) and ontology reuse, it is perhaps not surprising that there does not appear to have been any earlier work on the notion of residue. However, we have shown that residues play a critical role in the design and verification of ontologies. For verification, residues are used to distinguish between reductive modules and weak reductive modules of a theory. For design, the residue constrains how models of the theory can be constructed from models of its modules.

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