Heterotic Continuous Time Real-valued/Boolean-valued Networks

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1 Extended abstract

Heterotic models of computation were introduced in 2012 by Stepney et al. in [2011]. Heterotic models of computation *seamlessly* combine computational models such as classical/quantum, digital/analog, synchronous/asynchronous, imperative/functional/relational, etc. to obtain increased computational power, both practically and theoretically.

Although much greater generality is possible – we have previously reported on heterotic quantum/classical dynamical systems, [2014] – we here concentrate on heterotic dynamical systems that are given by continuous time real-valued/boolean-valued networks, which in the sequel we refer to as a heterotic Boolean network (HBN). A network of this kind is a finite directed graph with a reflexive edge relation where each vertex is of type **real** or of type **boolean**. Each vertex updates its value in continuous time according to a dynamics specified by a set of *autonomous* first-order differential equations

$$\{dx_i/dt \;=\; f_i(x_1,\ldots,x_n) \mid i=1,\ldots,n\} \tag{1}$$

where each variable x_i corresponds to a vertex in the network.

Nearly all continuous time dynamical systems can be expressed as a system of 1st-order differential equations provided a differential calculus satisfying a functorial chain rule is available for functions mapping between the spaces corresponding to the types of the variables involved in the system. The purpose of the proposed differential calculus in this application to HBN's is to allow for the uniform and seamless specification of an HBN via such systems of 1st-order differential equations. Since the calculus is mathematically rigorous it provides a formal semantics for such specifications and therefore a rigorous basis for verification and validation of an HBN with respect to those specifications. In the HBN applications we need to have a differential calculus that agrees with the familiar calculus on Euclidean spaces that extends to functions mapping reals to Booleans, and Booleans to Booleans. We use the apparatus of *convergence spaces*, [2016].

To conservatively extend the notion of differentiation to general convergence spaces, we note first that the set of continuous functions from X to Y which we denote here by Y^X has a convergence structure uniformly constructible from the convergence structures on X and Y such that the category of convergence spaces is Cartesian-closed; the details of the convergence structure on the exponential spaces Y^X need not concern us at present; it is enough that we have convergence structures available on the exponential spaces sufficient for obtaining a chain rule. Also for the present application of convergence spaces differences $x_0 - x$ are needed; they are constructible from an Abelian regular action on the convergence space, [1995], which renders the space as a module over a ring. Modules are nearly vector spaces where non-zero scalars are not guaranteed to have inverses.

Suppose convergence spaces Y and X are each equipped with regular actions and the sum and difference of pairs of points in X and Y have been determined. Then choose a subspace $\mathsf{Diff}(X,Y)$ of Y^X to serve as values of the derivative operation on these functions spaces. For example, we choose $\mathsf{Diff}(\mathbb{R}^m,\mathbb{R})$ to be the space of linear functionals on $\mathbb{R}^m$, and similarly for $\mathsf{Diff}(\mathbb{B}^m,\mathbb{B})$. Then $g \in \mathsf{Diff}(X,Y)$ is a *differential* of f at x_0 iff for every filter $F \downarrow_X x_0$ there is a filter $G \downarrow_{Y^X} g$ such for each $W \in G$ there is a $V \in F$ such that for each $x \in V$, there is $h \in W$ such that $h(x - x_0) = f(x) - f(x_0)$.

In this application we also need to choose $\mathsf{DIff}(\mathbb{R},\mathbb{B})$. We identify $\mathbb{B}$ with $\{0,1\}$ with the indiscrete convergence structure. We take $\mathsf{Diff}(\mathbb{R},\mathbb{B})$ to be the discrete space of the following four functions: (1) the constant function mapping all real number to 0, (2) the "step" function $f(x) = (x \le 0)\ ?\ 0 : 1$, (3) $f(x) = (x \ge 0)\ ?\ 0 : 1$ and (4) $f(x) = (x = 0)\ ?\ 0 : 1$. Under these definitions the system (1) is well-defined, but may or may not have a solution. After all, even with a rigorous denotational semantics, not every syntactically correct program in an ordinary programming language has a solution in the sense that it will produce a well-defined trajectory (i.e. no run-time errors.)

We conclude with a small example of an HBN:

$$dx/dt = ((b = 1)\ ?\ [-\sin x] : [\sin x]), \quad dy/dt = ((b = 1)\ ?\ [\cos x] : [-\cos x])$$
$$db/dt = (\mathsf{prime}(t)\ ?\ (\lambda t'.((t' \le t)\ ?\ 0 : 1) : \lambda t'.0)$$

The solutions to this equation have the boolean value of b changing whenever t is a prime nonnegative integer and have point given by cordinates (x, y) traversing a circle of radius 1 centered at the origin. The point reverses direction whenever t is prime.

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