J. Hlaváčová (Ed.): ITAT 2017 Proceedings, pp. 112–119
CEUR Workshop Proceedings Vol. 1885, ISSN 1613-0073, © 2017 M. Navara, J. Šindelář

The Role of Information in the Two Envelope Problem

Mirko Navara, Jiří Šindelář

Center for Machine Perception, Department of Cybernetics,
Faculty of Electrical Engineering, Czech Technical University in Prague,
Technická 2, 166 27 Prague, Czech Republic,
navara@cmp.felk.cvut.cz, sindeji3@fel.cvut.cz,
WWW home page: http://cmp.felk.cvut.cz/~navara/

Abstract: We offer a new view on the two envelope problem (also called the exchange paradox). We describe it as a zero-sum game of two players, having only partial information. We first explain a standard situation and show that the mean gain—when defined—is really zero. However, there are even more paradoxical situations in which the information obtained by the players supports the exchange of envelopes. We explain that this does not lead to a contradiction and we demonstrate it also by computer simulation. The reason for this paradox is that the mean gain does not exist and that the players have different information, supporting their contradictory decisions.

1 Formulation of the Problem

The *two envelope problem* (also called the *exchange paradox*) is a famous logical puzzle demonstrating a paradox in logic and probability. We adopt its formulation from [14], expressed here as a game of two players:

> There are two indistinguishable envelopes, each containing money, one contains twice as much as the other. Player A picks one envelope of his choice; player B receives the second envelope. They can keep the money contained in their envelopes or switch the envelopes (if both agree on it). Should they switch?

There is an easy answer:

Argument 1. *The situation is symmetric. Thus there is no reason for (or against) switching.*

However, there are other interpretations suggesting something else:

Argument 2. *The situation is symmetric. Thus the probability of having the envelope with the higher or lower amount is* $1/2$. *If the envelope of player* A *contains the amount* a, *then the other envelope contains* $2a$ *(and the exchange results in a gain of* a*) or* $a/2$ *(and the exchange results in a loss of* $a/2$*). In average, the mean gain is*

$$\frac{1}{2}a-\frac{1}{2}\frac{a}{2}=\frac{a}{4},$$

thus switching is always recommended.

Another point of view is the following:

Argument 3. *The smaller amount is* x, *the bigger is* $2x$. *They are assigned randomly (with probabilities* $1/2$*) to players* A *and* B. *The mean values for both players are*

$$\frac{1}{2}x-\frac{1}{2}2x=\frac{3}{2}x$$

and there is no reason for (or against) switching.

We presented several arguments; each of them seems correct, but their conclusions are contradictory.

Surprisingly, the debate about this paradox is still not finished (cf. [3]).

> "Currently, there is no consensus on a demonstration, since most people generally reject each other's demonstrations." [5]

One reason is that many authors merely defended their solution (mostly correct), cf. [6, 13]. However, to resolve the paradox, it is necessary to explain the errors in the contradicting arguments.[1] The topic was studied not only by mathematicians and logicians but also by philosophers (e.g., [4, 11]). For some of them, a sufficient explanation is that a in Argument 2 denotes different amounts; the smaller one in the first case and the bigger one in the second case [4, 13]. However, this is not forbidden, this is just what a random variable means. Thus a more advanced analysis is needed.

The paradox has more variants (cf. [11]). The method of choice of the amounts was not specified. (This is usual in such puzzles. They rarely start with a precise definition of a random experiment generating the data. Instead of that, it was said that two amounts are given, one of them twice greater than the other.) Here we assume that they were drawn as realizations of some random variable with a given distribution (known or unknown to players). Nevertheless, the formulation of the problem does not specify this at all, and some authors (e.g. [9]) consider this amount as given (without any randomness); such formulation excludes a probabilistic analysis. Thus we do not consider it here. Besides, it is not specified whether we first draw the (realization of) random variable X (the smaller amount) or the contents of the envelope given to player A, described

[1] We experienced this misunderstanding also during the reviewing process of this paper: "Argument 3 is correct, so there is no paradox." However, what is wrong on Argument 2?

by random variable A. It is natural to assume that the binary choice of envelopes is made with equal probabilities and independently of all other random events (or parameters) of the experiment. Here we apply the probabilistic approach to the problem in its original form: We first draw a positive amount x from some distribution. We put this amount into one envelope and $2x$ into the other envelope. Both envelopes have probability $1/2$ to be chosen by player A. The remaining envelope is given to player B.

It is also not specified whether the players know the amounts in their envelopes. This knowledge is useless if the distribution is unknown. On the other hand, knowing the distribution, the amounts bring useful information for the decision. We suppose that the players know the distribution from which x was drawn. We discuss this case in detail. Some of its consequences seem to determine the strategy also without looking inside the envelope, but—as we shall show—this need not correspond to conclusions made in the former case.

We present an explanation of the paradox (based on [12]) using results of probability and information theory. The standard explanation of the exchange paradox (following [7]) is presented in Section 2 and demonstrated by an example in Section 3. As a new contribution, we modify this example to two even more surprising and counterintuitive versions of the paradox, which we explain in detail in Sections 4 and 5. In Section 6, we verify the results by a computer simulation.

2 Exchange Paradox: First Level

We introduce a third member of the experiment, the banker C, who controls the game and puts (his) money in the envelopes. We assume that the smaller amount, x, was chosen by a realization of a random variable X. (The larger amount in the second envelope is $2x$.) The distribution of random variable X is known to the banker and also to the players. For simplicity, we assume that the distribution is discrete and the amounts are positive. Then we do an independent random experiment (e.g., tossing a coin) with two equally probable results, expressed by a random variable U, whose possible values are 0 and 1 and expectation $\mathrm{E}U = 1/2$. If $U = 0$, player A receives the smaller amount, x; if $U = 1$, player A receives the bigger amount, $2x$. He does not know x, only the contents of his envelope, specified by realization a of random variable A,

$$A = \begin{cases} x & \text{if } X = x \text{ and } U = 0\,, \\ 2x & \text{if } X = x \text{ and } U = 1\,, \end{cases}$$
$$A = (1+U)\,X\,.$$

Player B receives the other envelope and knows only its contents, specified by realization b of random variable B,

$$B = \begin{cases} x & \text{if } X = x \text{ and } U = 1\,, \\ 2x & \text{if } X = x \text{ and } U = 0\,, \end{cases}$$
$$B = (2-U)\,X\,.$$

If the players exchange the envelopes, the gain of A is $G = B - A$. For player B, G is the loss and $-G$ is the gain.

Random variables X and U are independent. If X has an expectation $\mathrm{E}X$, then

$$\mathrm{E}A = (1+\mathrm{E}U)\,\mathrm{E}X = \frac{3}{2}\,\mathrm{E}X\,,$$
$$\mathrm{E}B = (2-\mathrm{E}U)\,\mathrm{E}X = \frac{3}{2}\,\mathrm{E}X\,,$$
$$\mathrm{E}G = \mathrm{E}B - \mathrm{E}A = 0\,.$$

This is in accordance with Arguments 1 and 3. It remains to find an error in Argument 2.

As U, X are independent,

$$P(U=0|X=x) = P(U=0) = \frac{1}{2}\,,$$
$$P(U=1|X=x) = P(U=1) = \frac{1}{2}$$

for all x. However, this does not apply to conditional probabilities $P(U=0|A=a), P(U=1|A=a)$ because U,A are dependent:

$$\begin{aligned} P(U=0|A=a) &= \frac{P(U=0,A=a)}{P(A=a)} \\ &= \frac{P(U=0,X=a)}{P(A=a)} = \frac{P(X=a)}{2\,P(A=a)}\,, \\ P(U=1|A=a) &= \frac{P(U=1,A=a)}{P(A=a)} \\ &= \frac{P(U=1,X=\frac{a}{2})}{P(A=a)} = \frac{P(X=\frac{a}{2})}{2\,P(A=a)}\,, \end{aligned}$$

where

$$\begin{aligned} P(A=a) &= P(U=0,A=a) + P(U=1,A=a) \\ &= P(U=0,X=a) + P(U=1,X=\tfrac{a}{2}) \\ &= \frac{1}{2}\big(P(X=a) + P(X=\tfrac{a}{2})\big)\,. \end{aligned}$$

Notice that $P(U=0|A=a)$ is the conditional probability of gain and $P(U=1|A=a)$ is the conditional probability of loss given $A=a$. Their ratio is

$$\frac{P(X=a)}{P(X=\frac{a}{2})}\,.$$

As there is no uniform distribution on an infinite countable set (a fact ignored even in [10]), $P(X=a), P(X=\frac{a}{2})$ cannot be equal for all a. Typically, the conditional probability of gain, $P(U=0|A=a)$, is higher for “small” values of a and smaller for "high" values, although the notions "small" and "high" are relative. In any case, these probabilities converge to 0 when a goes to infinity, hence there must be values "sufficiently large" so that $P(X=a) < P(X=\frac{a}{2})$ and the conditional probability of gain $P(U=0|A=a) < \frac{1}{2}$. This can also lead to an effective strategy based on random switching [9, 10].

Given $A=a$, switching brings a gain with conditional probability distribution

$$P(G=g|A=a) = \frac{P(G=g, A=a)}{P(A=a)},$$

where

$$P(G=g, A=a) = \begin{cases} P(U=0, X=a) & \text{if } g=a, \\ P(U=1, X=\frac{a}{2}) & \text{if } g=-\frac{a}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \frac{1}{2}P(X=a) & \text{if } g=a, \\ \frac{1}{2}P(X=\frac{a}{2}) & \text{if } g=-\frac{a}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

$$P(G=g|A=a) = \begin{cases} \dfrac{P(X=a)}{P(X=a)+P(X=\frac{a}{2})} & \text{if } g=a, \\ \dfrac{P(X=\frac{a}{2})}{P(X=a)+P(X=\frac{a}{2})} & \text{if } g=-\frac{a}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

The conditional expectation of the gain is always defined and it is

$$\mathrm{E}(G|A=a) = \frac{aP(X=a) - \frac{a}{2}P(X=\frac{a}{2})}{P(X=a)+P(X=\frac{a}{2})}. \qquad (1)$$

These values may differ from 0.

The (unconditional) distribution of the gain is

$$P(G=g) = \begin{cases} \frac{1}{2}P(X=g) & \text{if } g>0, \\ \frac{1}{2}P(X=-g) & \text{if } g<0, \\ 0 & \text{otherwise} \end{cases}$$

and its expectation is

$$\mathrm{E}G = \sum_g g\,P(G=g) \qquad (2)$$

$$= \frac{1}{2}\left(\sum_{g>0} g\,P(X=g) + \sum_{g<0} g\,P(X=-g)\right) \qquad (3)$$

$$= \frac{1}{2}\left(\sum_{g>0} g\,P(X=g) - \sum_{h>0} h\,P(X=h)\right) = 0$$

(after substitution $g := -h$), provided that the sum (2) is absolutely convergent. In this case

$$\mathrm{E}G = \sum_a P(A=a)\,\mathrm{E}(G|A=a) \qquad (4)$$

$$= \sum_a \frac{1}{2}\left(aP(X=a) - \tfrac{a}{2}P(X=\tfrac{a}{2})\right)$$

$$= \frac{1}{2}\left(\sum_a aP(X=a) - \sum_b bP(X=b)\right) = 0$$

(after substitution $a := 2b$). This explains the error in Argument 2 provided that the expectation of G is defined.

Remark 1. There is another arrangement suggested in [8, 11]: First, the amount a in the envelope of player A is drawn from some distribution. Then the random variable U (as before) decides whether the second envelope will contain $2a$ or $\frac{a}{2}$. In this arrangement, random variables U and A are independent and Argument 2 is valid. Arguments 1 and 3 fail because of an intervention of the banker; it is him who puts additional money in the second envelope, so that the total amount may be $3a$ or $\frac{3}{2}a$. This is not a zero-sum game, and it is not symmetric.

3 Example of the First Level of Paradox

Let T be a random variable with geometrical distribution with quotient $q \in (0,1)$:

$$P(T=t) = \frac{q^t}{1-q}, \qquad t \in \{0,1,2,\dots\}.$$

Let $X = 2^T$, thus X attains values $1,2,4,8,\dots$ with probabilities

$$P(X=2^t) = \frac{q^t}{1-q}, \qquad t \in \{0,1,2,\dots\}.$$

In this arrangement, a player can deduce the contents of both envelopes only if he holds 1. For any other value, both cases are possible—switching may bring a gain or a loss.

Suppose that the expectation of X exists; this happens iff $q < \frac{1}{2}$. Then

$$\mathrm{E}X = \sum_{t=0}^{\infty} 2^t \frac{q^t}{1-q} = \frac{1}{(1-q)(1-2q)}.$$

The joint distribution of U and A is given (for $a=2^s$, $s \in \{0,1,2\dots\}$) by

$$P(U=0, A=a) = P(U=0, X=a) = \frac{q^s}{1-q},$$

$$P(U=1, A=a) = P(U=1, X=\tfrac{a}{2}) = \begin{cases} \dfrac{q^{s-1}}{1-q} & \text{if } s \geq 1, \\ 0 & \text{if } s=0. \end{cases}$$

For $q = 0.25$, it is shown in Fig. 1.

If player A has $a = 2^s$, the conditional probability of his gain by switching is

$$P(U=0|A=2^s) = \frac{P(X=a)}{P(X=a)+P(X=\frac{a}{2})} = \frac{q}{q+1} \quad (5)$$

for all $s \in \{1,2,\dots\}$ (and 1 for $s=0$). As $q < 1/2$, this probability is less than $1/3$. The conditional expectation

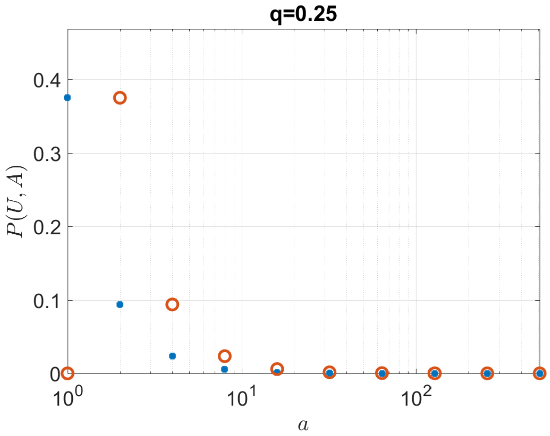

Figure 1: Joint distribution of U and A for $q = 0.25$. Values a of A are on the horizontal axis; blue dots denote $P(U = 0, A = a)$, orange circles $P(U = 1, A = a)$.

of his gain is

$$\mathrm{E}(G|A = 2^s) = \frac{2^s P(T = s) - 2^{s-1} P(T = s-1)}{P(T = s) + P(T = s-1)} = \begin{cases} 2^{s-1} \cdot \dfrac{2q-1}{q+1} & \text{if } s \in \{1,2,\dots\}, \\ 1 & \text{if } s = 0. \end{cases} \tag{6}$$

For $q = 0.25$, it is shown in Fig. 2. The contributions of conditional expectations $\mathrm{E}(G|A = a)$ to the unconditional expected gain $\mathrm{E}G$ are $\mathrm{E}(G|A = a) \cdot P(A = a)$, see Fig. 3. The conditional expectation is positive only for $s = 0$ (i.e., $a = 1$), negative otherwise. This determines the right strategy of switching: Switch only if you hold 1. The two players never both agree on switching the envelopes because at most one of them holds 1.

If player A does not know the contents of his envelope, he may use only its distribution

$$P(A = 2^s) = \begin{cases} \dfrac{1}{2}\left(\dfrac{q^s}{1-q} + \dfrac{q^{s-1}}{1-q}\right) & \text{if } s \in \{1,2,\dots\}, \\ \dfrac{1}{2}\,\dfrac{1}{1-q} & \text{if } s = 0, \\ 0 & \text{otherwise}, \end{cases} = \begin{cases} \dfrac{1}{2} q^{s-1} \dfrac{q+1}{1-q} & \text{if } s \in \{1,2,\dots\}, \\ \dfrac{1}{2}\,\dfrac{1}{1-q} & \text{if } s = 0, \\ 0 & \text{otherwise}. \end{cases}$$

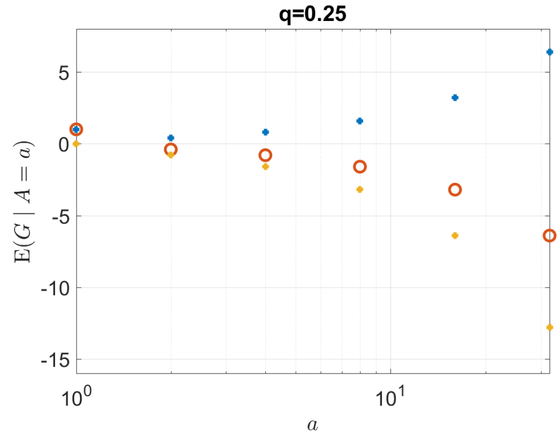

Figure 2: Conditional expectation of gain given $A = a$ for $q = 0.25$ (orange circles). It is a mixture (=convex combination) of the cases $U = 0$ (blue dots) and $U = 1$ (yellow dots).

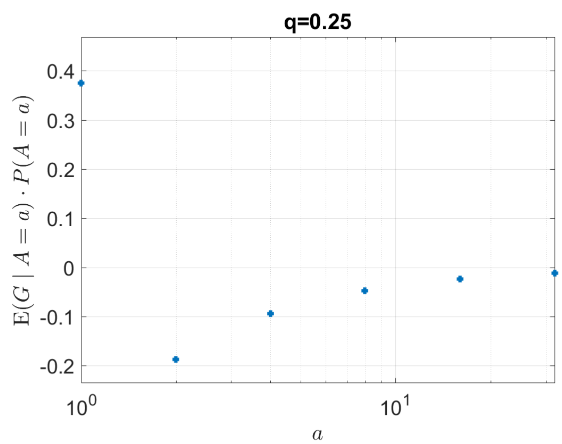

Figure 3: The contributions of conditional expectations of gain given $A = a$ to the unconditional expected gain for $q = 0.25$.

The unconditional expectation of the gain is

$$\mathrm{E}G = \sum_{s=0}^{\infty} P(A = 2^s)\,\mathrm{E}(G|A = 2^s) = \frac{1}{2}\left(\frac{1}{1-q} + \sum_{s=1}^{\infty} (2q)^{s-1} \frac{2q-1}{1-q}\right) = \frac{1}{2}\left(\frac{1}{1-q} + \frac{1}{1-2q} \cdot \frac{2q-1}{1-q}\right) = 0, \tag{7}$$

in accordance with our arguments.

4 Exchange Paradox: Second Level

In Section 2, we presented a standard explanation of the exchange paradox. It is based on the assumption that the amount in the envelopes has an expectation. This can fail even for some common distributions. (This fact is ignored, e.g., in [9].) We discovered that this leads to a more advanced paradox. We have found out that Nalebuff [8] proposed the same example, and similar ones can be found in [2]. However, Nalebuff only noticed that both players might be convinced that switching brings gain to them and that the above arguments are not applicable if the expectation does not exist. It seems that no detailed analysis was published since, and this is what we do here.

Let us consider the situation from Section 3 if $q > \frac{1}{2}$. Then the expectation $\mathrm{E}X$ does not exist. (The respective sum of a geometric series with quotient $2q > 1$ is $+\infty$.) For $q = 0.75$, the joint distribution of U and A is shown in Fig. 4.

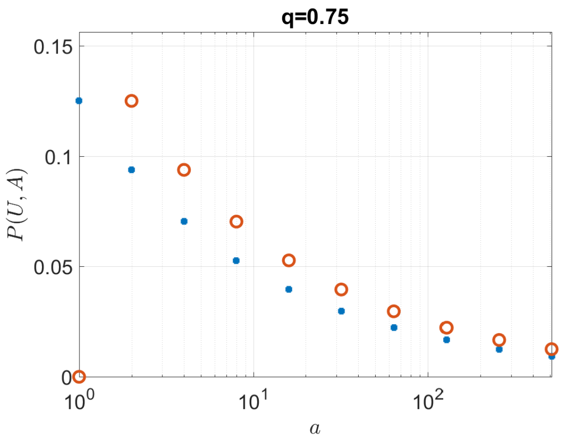

Figure 4: Joint distribution of U and A for $q = 0.75$. Values a of A are on the horizontal axis; blue dots denote $P(U = 0, A = a)$, orange circles $P(U = 1, A = a)$.

The expectation of the gain, $\mathrm{E}G$, does not exist because the sum (2) is not absolutely convergent; it is a difference of two infinite sums in (3).

Still formula (6) for the conditional expectation of the gain is valid,

$$\mathrm{E}(G|A = 2^s) = \begin{cases} 2^{s-1} \cdot \dfrac{2q-1}{q+1} & \text{if } s \in \{1, 2, \dots\}, \\ 1 & \text{if } s = 0. \end{cases}$$

Thus $\mathrm{E}(G|A = 2^s) > 0$ for all $s \in \{1, 2, \dots\}$. For $q = 0.75$, see Fig. 5. (Notice that formula (5) for the conditional probability of gain $P(U = 0|A = 2^s)$ for $s \neq 0$ still holds and gives a constant value from the interval $(\frac{1}{3}, \frac{1}{2})$.) Player A has a strong argument for switching the envelopes, independently of the amount in his envelope. (Thus he may "rationally" decide for switching without looking inside the envelope.) Such distributions are called *paradoxical* in [2].

The same argument applies to player B. Although he holds a different amount in his envelope, he also prefers switching. We have again a paradox, now supported by a

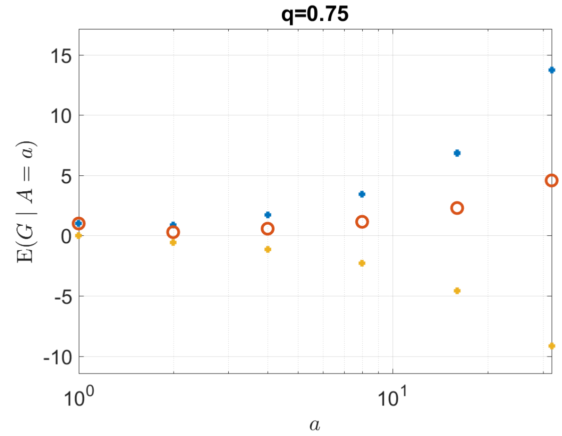

Figure 5: Conditional expectation of gain given $A = a$ for $q = 0.75$ (orange circles). It is a mixture of the cases $U = 0$ (blue dots) and $U = 1$ (yellow dots).

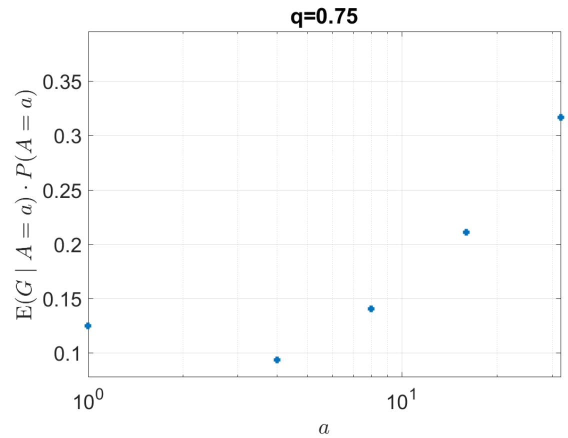

Figure 6: The contributions of conditional expectations of gain given $A = a$ to the unconditional expected gain for $q = 0.75$.

probabilistic analysis. The only thing which does not work as in Section 3 is formula (7) for unconditional gain; the sum is not absolutely convergent. However, the unconditional gain is not needed for decision if the conditional one is always positive. How can we now defend Argument 1?

First of all, we refuse the possibility (considered in [2, 4, 6]) that players A and B will change the envelopes there and back forever. After one exchange and looking inside, they would know the contents of both envelopes and decide deterministically with full information. One of the players has the larger amount (and he knows that), so he would not agree to switch again.

If the players know only the amount in the envelope they received first, they have *different information*, and this is the key difference.

Example 1. *Suppose that A has* 4 *and B has* 8. *Then*

A knows that B can have 2 *or* 8*, while B knows that A can have* 4 *or* 16*. Using Argument 2, A might expect that switching brings him a mean gain of* 1*. Using the model described in this section (formula (5) remains valid), he knows that his chance of gain is lower,*

$$\frac{q}{q+1} \in \left(\frac{1}{3}, \frac{1}{2}\right).$$

For $q = 0.75$*, this chance is* $3/7 \doteq 0.43$*, still sufficient to give a positive conditional mean gain of*

$$\frac{8q+2}{q+1} - 4 = \frac{4}{7}.$$

Player B may apply the same arguments, leading to twice higher estimates of his *gain.*

Example 2. *Suppose now that A has* 4 *and B has* 2*. Then A knows that B can have* 2 *or* 8*, while B knows that A can have* 1 *or* 4*. From the point of view of player A, the situation is the same as in Ex. 1. Player B may apply the same arguments, leading to twice lower estimates of* his *gain, still supporting the decision to switch.*

In Exs. 1 and 2, we saw that probabilistic analysis suggests switching to both players. This apparently brings a gain to only one of them, but their arguments overestimate their chances. This explains why they may have contradictory views on the effect of the switching of envelopes (both thinking that the other envelope is "better").

To understand this paradox better, imagine the reverse game: Suppose that the players see the contents of the other player's envelope (and not of their own). Then the same reasoning (based on the information received) would support keeping the envelopes (and no switching). This shows that it is the *different incomplete information* which supports their paradoxical behavior. (The role of incomplete information and other arrangements of the experiment are discussed in [11] for the "first level" of the paradox.)

This situation is not so counterintuitive. Imagine for instance a poker game where two players hold a poker in their hands. They both evaluate their chances of winning as very high, although it is clear that only one is in the winning position. In the reverse game, where they see the cards of the opponent (and not their own), each player would estimate the chances of his opponent as very high, and he would surrender.

The two envelope problem in this setting possesses the same feature: the partial information given to players is overly optimistic. Thus looking inside the envelopes is not so helpful as it seems. Therefore, if rational players do not look inside any of the envelopes, the latter argument makes their choice ambivalent, and they would accept Argument 1. Even if they look in the envelopes, they will not accept Argument 2, knowing (from the above analysis of the model) that its prediction is biased and too optimistic.

5 Exchange Paradox: Third Level

Another modification of the example of Section 3 with $q = 1/2$ is also of particular interest. The joint distribution of U and A is shown in Fig. 7. Formula (5) for the conditional

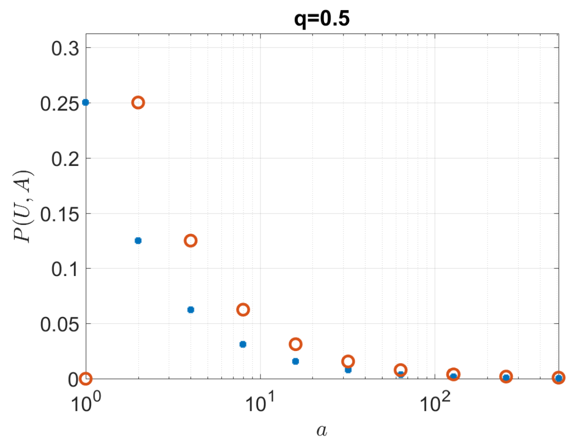

Figure 7: Joint distribution of U and A for $q = 0.5$. Values a of A are on the horizontal axis; blue dots denote $P(U = 0, A = a)$, orange circles $P(U = 1, A = a)$.

probability of gain gives $P(U = 0 | A = 2^s) = 1/3$ if $s \neq 0$. The loss is twice more probable but twice smaller. Thus the conditional expectation of the gain simplifies to

$$\mathrm{E}(G | A = 2^s) = \begin{cases} 0 & \text{if } s \in \{1, 2, \ldots\}, \\ 1 & \text{if } s = 0, \end{cases}$$

see Fig. 8 for the conditional expectations and Fig. 9 for their contributions to the unconditional expectation $\mathrm{E}G$.

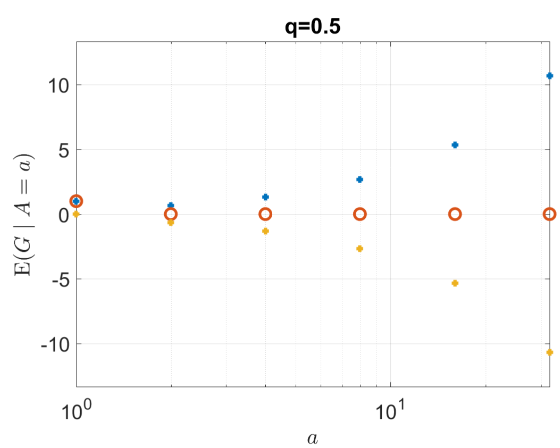

Figure 8: Conditional expectation of gain given $A = a$ for $q = 0.5$ (orange circles). It is a mixture of the cases $U = 0$ (blue dots) and $U = 1$ (yellow dots).

It seems that player A (as well as B) may only gain by

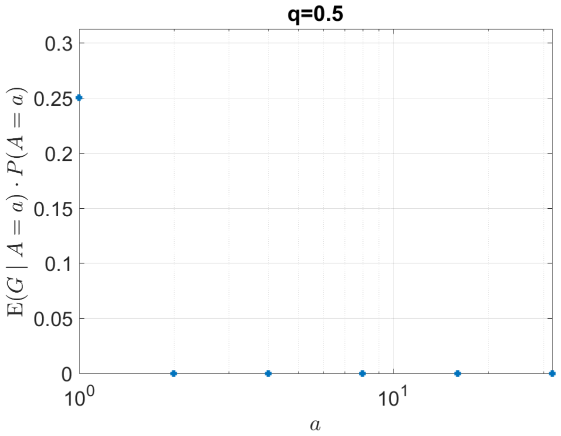

Figure 9: The contributions of conditional expectations of gain given $A = a$ to the unconditional expected gain for $q = 0.5$.

switching (if he holds 1), in all other cases the risk of loss is compensated by the same expected gain. In the sum in (7), only the first summand is nonzero, and it is positive. So the sum exists and evaluates to

$$\frac{1}{2}\,\frac{1}{1-q} > 0\,.$$

The arguments from Section 4 are applicable. Moreover, switching is supported by a computation which results in a positive unconditional gain (of a player not looking in his envelope). However, this argument is wrong. As in Section 4, the expectation of the gain, EG, does not exist because the sum (2), as a difference of two infinite sums in (3), is not absolutely convergent. Formula (7) uses only one possible arrangement of the summands, leading to an invalid conclusion. If player B applies the same argument, he uses another arrangement of the summands and gets a positive expected gain for himself, loss for A. Thus the sum (2) does not exist and the discussion from Section 4 fully applies, despite the seemingly trivial (wrong) sum (7).

6 Simulations

To verify the results, we also used computer simulations. We computed the average gain from 1000 samples and repeated this 5000 times. The results were displayed as histograms of the averages, see Figs. 10, 11, 12 for quotients $q = 0.25, 0.5, 0.75$, respectively. For $q = 0.75$, the linear scale could not be used for the horizontal axis. The semilogarithmic scale would not allow negative values. Therefore, we used the 31$^{\text{st}}$ root as a compromise which combines non-linearity similar to the logarithm and possibility of displaying negative values.

As expected, the histograms show relatively frequent occurrences of averages with high absolute value in cases of $q = 0.5, 0.75$, where the expectation does not exist. The values are distributed approximately symmetrical with respect to zero, verifying that no envelope appears "better" and Argument 2 does not apply in practice, as predicted by the theoretical analysis in previous sections.

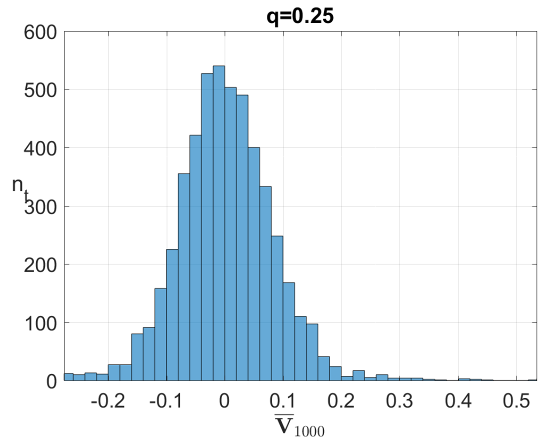

Figure 10: Histogram of average gains of 1000 samples for $q = 0.25$.

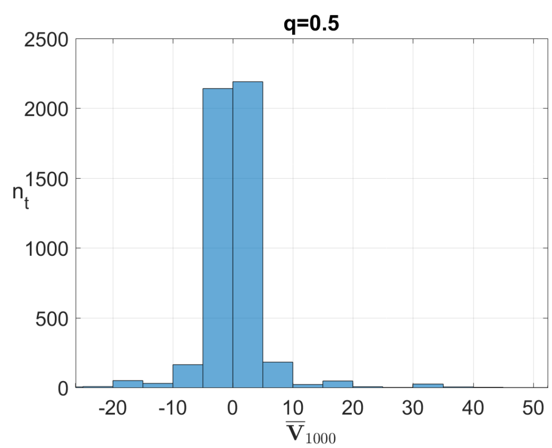

Figure 11: Histogram of average gains of 1000 samples for $q = 0.5$.

7 Conclusions

We explained the two envelope paradox in its classical form, as well as in two advanced instances in which the players find rather convincing (and still insufficient) probabilistic arguments for switching the envelopes. The latter is our novel contribution to the discussion of the paradox. We confirmed the following conclusion

> "a perfectly rational player would simply recognize that his subjective probabilities provide a

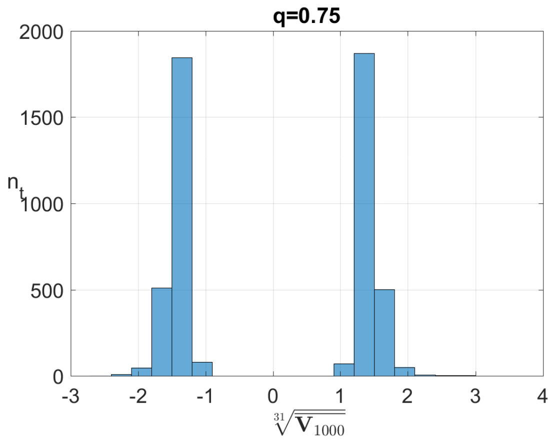

Figure 12: Histogram of average gains of 1000 samples for $q = 0.75$. The values on the horizontal axis are mapped by the 31$^{\text{st}}$ root.

> misleading account using Bayesian decision theory and would therefore ignore those results" [1]

also in the case of nonexisting expectation, which was not considered in the cited source.

This topic has consequences in psychology, but it is also important in economics because it explains behavior at a market which is seemingly well-motivated, but in fact wrong. Besides, this paradox can be a good example and motivation for the study of statistics and information theory.

Acknowledgments. The first author was supported by the Ministry of Education of the Czech Republic under Project RVO13000.

References

[1] Bliss, E.: A Concise Resolution to the Two Envelope Paradox. 2012, arXiv:1202.4669

[2] Broome, J.: The Two-envelope Paradox. *Analysis* 55 (1995): 6–11 doi:10.1093/analys/55.1.6

[3] Cover, T.M.: Pick the largest number. In: Cover, T, Gopinath, B., *Open Problems in Communication and Computation*. Springer-Verlag, 1987

[4] Falk, R.: The Unrelenting Exchange Paradox. *Teaching Statistics* 30 (2008), 86–88 doi:10.1111/j.1467-9639.2008.00318.x

[5] Gerville-Réache, L.: Why do we change whatever amount we found in the first envelope: the Wikipedia "two envelopes problem" commented. University of Bordeaux - IMB UMR 52-51, 2014, arXiv:1402.3311

[6] Green, L.: The Two Envelope Paradox. 2012, `http://www.aplusclick.com/pdf/LeslieGreenTwoEnvelopes.pdf`, 2017-06-03

[7] Jackson, F., Menzies, P., Oppy, G.: The two envelope "paradox", *Analysis* 54 (1994) 43-45.

[8] Nalebuff, B.: Puzzles: The Other Person's Envelope is Always Greener. *Journal of Economic Perspectives* 3 (1988): 171–81 doi:10.1257/jep.3.1.171

[9] Martinian, E.: The Two Envelope Problem. `https://web.archive.org/web/20071114230748/http://www.mit.edu/~emin/writings/envelopes.html`, 2017-06-03, archive version of an original from 2007-11-14

[10] McDonnell, M. D., Abbott, D.: Randomized switching in the two-envelope problem. *Proceedings of the Royal Society A* 465 (2111): 3309–3322 doi:10.1098/rspa.2009.0312

[11] Priest, G., Restall, G.: Envelopes and Indifference, In: *Dialogues, Logics and Other Strange Things*, College Publications, 2007, 135–140

[12] Šindelář, J.: Two Envelopes Problem. Preprint of Bachelor Thesis, CTU, Prague, 2017-05-04

[13] Schwitzgebe, E., Dever, J.: The Two Envelope Paradox and Using Variables Within the Expectation Formula, *Sorites* (2008) 135–140

[14] Two envelopes problem. Wikipedia, 2017-05-10, `https://en.wikipedia.org/wiki/Two_envelopes_problem`