Algorithmic Correspondence and Canonicity for Possibility Semantics (Abstract)

Zhiguang Zhao

Delft University of Technology, Netherlands

Unified Correspondence. Correspondence and completeness theory have a long history in modal logic, and they are referred to as the "three pillars of wisdom supporting the edifice of modal logic" [22, page 331] together with duality theory. Dating back to [20,21], the Sahlqvist theorem gives a syntactic definition of a class of modal formulas, the *Sahlqvist class*, each member of which defines an elementary (i.e. first-order definable) class of Kripke frames and is canonical. Since modal logic on the frame level is essentially second-order, computing the first-order correspondence of a modal formula is a kind of second-order quantifier elimination.

Recently, a uniform and modular theory which subsumes the above results and extends them to logics with a *non-classical* propositional base has emerged, and has been dubbed *unified correspondence* [5]. It is built on duality-theoretic insights [9] and uniformly exports the state-of-the-art in Sahlqvist theory from normal modal logic to a wide range of logics which include, among others, intuitionistic and distributive and general (non-distributive) lattice-based (modal) logics [6,8], non-normal (regular) modal logics based on distributive lattices of arbitrary modal signature [19], hybrid logics [12], many valued logics [16] and bi-intuitionistic and lattice-based modal mu-calculus [1,3,2]. Unified correspondence theory has two components: the first one is a very general syntactic definition of Sahlqvist and inductive formulas, which applies uniformly to each logical signature and is given purely in terms of the order-theoretic properties of the algebraic interpretations of the logical connectives; the second one is the Ackermann lemma based algorithm ALBA, which is a generalization of SQEMA based on order-theoretic and algebraic insights, which effectively computes first-order correspondents of input formulas/inequalities, and is guaranteed to succeed on the Sahlqvist and inductive classes of formulas/inequalities. The algorithm aims at eliminating all propositional variables, which are, on the relational semantics side, second-order variables, and rewrite the formula into a quasi-inequality which contains only nominals and co-nominals, which are, on the relational semantics side, essentially first-order. In this sense, unified correspondence theory is essentially second-order quantifier elimination on the algebraic side.

The breadth of this work has stimulated many and varied applications. Some are closely related to the core concerns of the theory itself, such as understanding the relationship between different methodologies for obtaining canonicity results [18,7], the phenomenon of pseudocorrespondence [10], and the investigation of

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the extent to which the Sahlqvist theory of classes of normal distributive lattice expansions can be reduced to the Sahlqvist theory of normal Boolean algebra expansions, by means of Gödel-type translations [11]. Other, possibly surprising applications include the dual characterizations of classes of finite lattices [13], the identification of the syntactic shape of axioms which can be translated into structural rules of a proper display calculus [14] and of internal Gentzen calculi for the logics of strict implication [17], and the epistemic interpretation of lattice-based modal logic in terms of categorization theory in management science [4]. These and other results (cf. [9]) form the body of a theory called unified correspondence [5], a framework within which correspondence results can be formulated and proved abstracting away from specific logical signatures, using only the order-theoretic properties of the algebraic interpretations of logical connectives.

Possibility Semantics. Possibility semantics for modal logic is a generalization of standard Kripke semantics. In this semantics, a possibility frame has a refinement relation which is a partial order between states, in addition to the accessibility relation for modalities. From an algebraic perspective, full possibility frames are dually equivalent to complete Boolean algebras with complete operators which are not necessarily atomic, while filter-descriptive possibility frames are dually equivalent to Boolean algebras with operators.

In recent years, the theoretic study of possibility semantics has received more attention. In [23], Yamamoto investigates the correspondence theory in possibility semantics in a frame-theoretic way and prove a Sahlqvist-type correspondence theorem over full possibility frames, which are the possibility semantic counterpart of Kripke frames, using insights from the algebraic understanding of possibility semantics. In [15, Theorem 7.20], it is shown that all inductive formulas are filter-canonical and hence every normal modal logic axiomatized by inductive formulas is sound and complete with respect to its canonical full possibility frame. However, the correspondence result for inductive formulas is still missing, as well as the correspondence result over filter-descriptive possibility frames (see [15, page 103]) and soundness and completeness with respect to the corresponding elementary class of full possibility frames. The present paper aims at giving a closer look at the aforementioned unsolved problems using the algebraic and order-theoretic insights from a current ongoing research project, namely *unified correspondence.*

Methodology. Our contribution is methodological: we analyze the correspondence phenomenon in possibility semantics using the dual algebraic structures, namely complete (not necessarily atomic) Boolean algebras with complete operators, where the atoms are not always available. For the correspondence over full possibility frames, our strategy is to identify two different Boolean algebras with operators as the dual algebraic structures of the possibility frame, namely the Boolean algebra of regular open subsets $\mathbb{B}_{\mathsf{RO}}$ (when viewing the possibility frame as a possibility frame itself) and the Boolean algebra of arbitrary subsets

$\mathbb{B}_{\mathsf{Full}}$ (when viewing the possibility frame as a bimodal Kripke frame), where a canonical order-embedding map $e : \mathbb{B}_{\mathsf{RO}} \to \mathbb{B}_{\mathsf{Full}}$ can be defined. The embedding e preserves arbitrary meets, therefore a left adjoint $c : \mathbb{B}_{\mathsf{Full}} \to \mathbb{B}_{\mathsf{RO}}$ of e can be defined, which sends a subset X of the domain W of possibilities to the smallest regular open subset containing X. This left adjoint c plays an important role in the dual characterization of the interpretations of the expanded language, which form the ground of the regular open translation, i.e. the counterpart of standard translation in possibility semantics. When it comes to canonicity, we use the fact that filter-canonicity is equivalent to constructive canonicity [15, Theorem 5.46, 7.20], and prove a topological Ackermann lemma, which justifies the soundness of propositional variable elimination rules and forms the basis of the correspondence result with respect to the class of filter-descriptive frames as well as the canonicity and completeness result with respect to the corresponding class of full possibility frames.

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