On proving and argumentation

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Abstract

In this paper, we demonstrate how argumentation theory can be used to explore certain aspects of the development of discovery proof-events in time. The concept of *proof-event* was introduced by Joseph Goguen, [10], who understood mathematical proof, not as a purely syntactic object, but as a social event, that takes place in specific place and time and involving agents or communities of agents. Proof-events are sufficiently general concepts that can be used to study besides the "traditional" formal proofs, other proving activities, such as incomplete proofs, purported proofs or attempts to verify a conjecture. Since argumentation is inseparable from the process of searching for a mathematical proof, we suggest a modified model of the *proof-events calculus*, initially suggested by Vandoulakis and Stefaneas [32], based on the versions of argumentation theories advanced by Pollock [26], Toulmin [28] and Kakas and Loizos [14]. We claim that the exchange of arguments and counterarguments set forward to clarify eventual gaps or implicit assumptions occurring in the course of a proof-event can be formalized by appealing to argumentation theories. Additionally, we highlight the connection between proving, human reasoning, cognitive processes and creativity.

Keywords: proof events, argumentation, proving

Introduction

The concept of mathematical proof has undergone significant changes in the 20th century. Proof, particularly formalized proof, was initially identified in philosophy of mathematics with truth. However, many mathematicians dealing with real proofs did not accept the paradigm of formalized proof. Goguen [10] suggested the broader concept of *proof-event*, which is actually a social event that takes place in specific place and time and involves public communication. The concept of proof-event is designed to embrace any proving activity, such as incomplete proofs or attempts to verify a conjecture. Vandoulakis and Stefaneas [31] described proof-events as activity of a *multi-agent system* incorporating their history, insofar as they form *sequences of proof-events* evolving in time. Thus, they modelled certain temporal aspects of proof-events, using the language of the calculus of events developed in Kowalski's and *Event Calculus* [19].

Our approach combines proof-events with logic-based argumentation to study in a more adequate way the categories of purported, faulty or incomplete proofs, setting forward the concept of *dialogue* between agents with the use of *arguments and counterarguments*. Hence, we extend the calculus of proof-events through argumentation theory by depicting the relevant stages of a discovery proof-event (incomplete or even false proofs, ideas, valid or invalid inference steps, comments, etc.) in a form of a dialogue of

agents that use arguments and counterarguments or counterexamples in their attempt to clarify the validity of a purported proof.

Many researchers claim that the role of argumentation is central in mathematics. Mathematicians do much more, than simply prove theorems. Most of their proving activity might be understood as kinds of argumentation [2]. Lakatos' *Proofs and Refutations* [20] is an enduring classic that highlights the role of dialogue between agents [a teacher and some students) by attempts at proofs and critiques of these attempts. The comparison between argumentation supporting an assumption or a purported proof and its proof is based on the case that proof can be regarded as a specific argumentation in mathematics [22].

A methodological tool that has been widely used to examine argumentation is Toulmin's model [28], in which argument is constituted by "claim", "data", and "warrant" that are considered the substantial elements of applied arguments. The procedure by which mathematicians evaluate reasoning resembles to argumentation, as various researchers, such as Alcolea [6], Aberdein [0], Pedemonte [22,23], Aberdein, Dove [4] tried to show by adapting Toulmin's model to mathematical examples. We propose to integrate argumentation theory into the calculus of proof-events. By doing so, we can depict all the relevant stages that we go through when proving, from the statement of a problem until its validation or rejection by the relevant community that uses arguments and counterarguments or counterexamples in checking the validity of a purported proof.

To combine the proof-event calculus with argumentation theory, we use the basic structure of Toulmin's model for the representation of an argument and Pollock's logic-based argumentation theory. We rely on Pollock's view of defeasible reasoning and we associate the procedure of human reasoning with proving. *Defeasible reasoning* has a non-monotonic character. Pollock presented in his work the concept of an argument in the form $<\Phi,\alpha>$, where Φ is a set of data and α is a claim [25]. He separated the rebutting and undercutting defeat and presented one of the first monotonic logics with concepts of argument and defeat, even though he did not explicitly distinguish between them [24].

In this paper, we proceed from a comparison of proof-events and argumentation. Then, we suggest a formalization of proof-events involving argumentation theory. We model the argument moves and the calculus of the temporal predicates. The aforementioned calculus is analyzed through the levels of argumentation. In the last section, Fermat's Last Theorem as a proof-event is investigated as a case study illustrating the concepts introduced.

1. Proof-events vs. arguments.

Comparison of the basic elements of proof-events and argumentation theory shows similarities in the structure, the sequence of events, the agents, the layers of communication, and the levels of argumentation.

Arguments and proof-events have three common fundamental components: a set of premises for a task or problem, a method of reasoning and a conclusion. Moreover, each proof-event has temporal extension and, thereby history: it has a starting point and a termination point and what is posed to be proved, emerges often out of the history of provings (sequences of proof-events or sequences of arguments and counterarguments) [31]. A sequence of a proof-event is complete when the community in-

volved in it concludes that they have understood the proof and agree that a proof has actually been given or that a proof is invalid, based on a suggested counterargument (or counterexample).

Proof-events presuppose the existence of at least two agents: a prover and an interpreter [31]. Similarly, argumentation involves agents or group of agents, enacting the roles of supporter and opponent of an argument [17]. The layers of communication, understanding, interpretation, and validation that agents use to disseminate their knowledge, are common in both approaches. An agent is a proactive and intelligent system that enacts a specific role. We may consider some software systems as agents, if they possess these characteristics. Argumentation has encouraged its adoption as a technology for multi-agent systems developments. The main concept advanced in agent-based approaches is that of *autonomy*: agents operate as independent individual entities trying often to collaborate and coordinate with others [13]. However, the steps that an individual agent wants to perform in order to accomplish a mathematical proof may interfere with the steps attempted or already performed by other agents. A number of significant questions appear from these processes of inter-agent debates, such as the systematization of atomic agent contributions as phases in a goal-directed plan (such as proving) and the review in the formalization of logic-based languages in terms of both syntactic and semantic aspects [16].

2. Argumentation models

Argumentation models generally contain the following main elements: an underlying logical language with the definition of the concepts of *argument*, *conflict* between arguments and counterarguments, and *status* of argument. We will outline formalizations of proof-events based on argumentation theory, using a list of structures, which represent arguments and counterarguments. Our approach suggests a multi-agent system, enacting the roles of provers and interpreters. An argument has premises, sentences, and conclusion. In the definition of an argument given in this section, the knowledge grounds and claims are formulae in classical logic and the method of inference by which a claim follows from a set of formulae is deductive inference and is denoted by $\vdash$. A proof-event e can be represented as a communicated argument $\langle \boldsymbol{\Phi}, \boldsymbol{c} \rangle$ [26] designated by the pair $e \langle \boldsymbol{\Phi}, \boldsymbol{c} \rangle$:

$$e \cap < communicate < \Phi, c >, w >,$$

where $\boldsymbol{\Phi}$ is the Data of the argument, **c** is the Claim that refers to a fixed problem (proposition), specified by certain conditions (predicates) and **w** are the inference rules (Warrant) which allow $\boldsymbol{\Phi}$ to be connected with **c**, so that:

- $\boldsymbol{\Phi} \nvdash \perp$
- $\boldsymbol{\Phi} \vdash \boldsymbol{c}$
- *There is no $\boldsymbol{\Phi}' \subseteq \boldsymbol{\Phi}$ such that $\boldsymbol{\Phi}' \vdash$ c.*

Counterarguments are represented by the corresponding pair $\boldsymbol{e} * (\boldsymbol{\Psi}, \boldsymbol{\beta})$, where $\boldsymbol{\Psi}$ is the data that the claim $\boldsymbol{\beta}$ (that refers to the same fixed problem (proposition), specified by the same conditions (predicates)) of the counterargument is based.

Argumentation may require chains (or trees) of reasoning, where claims are used in the assumptions for obtaining further claims [8], so that a proof-event could be an atomic argument or a sequence of arguments (fluent). *Fluents f* are sequences of proof-events (proving instances) evolving in time that refer to a fixed problem, speci-

fied by certain conditions [30]. Let R be a set of rules of inference. A fluent f is a formula of the form $e_1, e_2, e_3 \rightarrow e$, where $e_1\langle\boldsymbol{\Phi_1}, \boldsymbol{c_1}\rangle, e_2\langle\boldsymbol{\Phi_2}, \boldsymbol{c_2}\rangle, e_3\langle\boldsymbol{\Phi_3}, \boldsymbol{c_3}\rangle$ is a finite, possibly empty, sequence of arguments, such that the conclusion of proof-event e_i is the claim $\mathbf{c_i}$, i.e.: $concl(e_1) \equiv \boldsymbol{c_1}, concl(e_2) \equiv \boldsymbol{c_2}, concl(e_3) \equiv \boldsymbol{c_3}$ for some rule $\boldsymbol{c_1, c_2, c_3} \rightarrow \boldsymbol{c} \,\epsilon R$ [33]. Accordingly, the meaning of the three substantial components of the argument based on Toulmin's model [28], which abbreviated by corresponding prefixes, are defined as follows for the notion of fluent:

$$\boldsymbol{data}\colon prem(e) = prem(e_1) \cup prem(e_2) \cup prem(e_3) \equiv \Phi_1 \cup \Phi_2 \cup \Phi_3 \quad (1)$$
$$\mathbf{claim}\colon conc(e) \equiv c = c_1 \cap c_2 \cap c_3 \quad\quad\quad\quad (2)$$
$$\mathbf{warrant}\colon \mathrm{sent}(e) = \mathrm{sent}(e_1) \cup \mathrm{sent}(e_2) \cup \mathrm{sent}(e_2 3) \equiv w_1 \cup w_2 \cup w_3 \quad (3)$$

where*:*
c (Claim): the statement communicated by the speaker.
Φ (Data): facts as the ground of the claim.
w (Warrant): the inference rules, which allows data to be connected to the claim.
    The aforementioned elements are frequently used to define a consequence relation between the arguments and/or the counter-arguments.

2.1 Argument moves

        In the course of a proving procedure, we pass through various inference stages, such as attempts, impasses, confirmed or unconfirmed steps, false suggestions or implicit assumptions, intuitive ideas, intentions, etc. Arguments can then be specified as chains of reasoning leading to a conclusion with consideration of possible counterarguments at each step. With the explicit construction of the chain of reasoning (a chain $x_0, x_1, .. , x_n$ where the argument x_i attacks the argument x_{i-1} for $i > 0$), distinct concepts of defeat can be conceptualized. When an agent has gained control of an argument, he must select which argument-move to apply. Gordon [11] refers to *"argument moves"* as analogs of three roles for legal cases. This term was also previously used by Rissland [27], Asley and Aleven [7], Pease et al [21]. We reserve the term "argument moves" for specific, active tactics or strategy that a prover can choose to support his claim. We present four fundamental relations that indicate links and conflicts at the sequence of proof-events. The possible argument moves could provide support or attack the claim.
        Given a claim **c** and and an argument communicated during the proof-event e, possible argument moves, which provide support for **c** [12] include:
    Equivalent: an argument for a claim, which is equivalent to (or is) $\boldsymbol{c}$;
    Elaboration: an argument for an elaboration of **c**, and
        Argument moves, which oppose **c** [26] include:
    Rebutting: an argument for a claim which disagrees with **c**;
    Undercutting: an argument for a claim which disagrees with a premise of **c**.

2.1.1. Argument moves that support the claim:

$$\mathbf{Equivalent}(e, e')\colon e(\boldsymbol{\Phi}, \mathbf{c}) = e'(\boldsymbol{\Phi'}, \mathbf{c'}), \quad\quad\quad (4)$$

when Φ=Φ´, c=c´ (although it might be **w≠w′**),

where the proof-event e_1 is equivalent with the proof-event e_2, whenever it has the same data and the same conclusion (although different warrants). Thus, equivalent proof-events can have different ways of proving. For instance, the Pythagorean Theorem has been proved in numerous ways, such as by Euclid's geometrical proving or by James Abram Garfield algebraic proving.

$$\textbf{Elaboration}(e, \textbf{S}) \colon \textbf{sent}(e) \cap \textbf{sent}(\textbf{S}) \rightarrow \textbf{concl}(e), \quad (5)$$

where $e\langle \boldsymbol{\Phi}, \boldsymbol{c} \rangle$, is a proof-event and S is a set of sentences which elaborate or embellish upon e, iff $\boldsymbol{\Phi} \cup \boldsymbol{S} \vdash \boldsymbol{c}$.

These moves are used for backing our claim and supporting our proving, so that

$$\textbf{Support}(e, \textbf{t}) \rightarrow \textbf{Equivalent}(e, e') \cup \textbf{Elaboration}(e, \textbf{S}) \quad (6)$$

2.1.2. Counterargument moves that attack the claim:

$$\textbf{Rebutting}(e^*, e) \colon \textbf{rebut}(e^*, e) \rightarrow \neg \textbf{concl}(e) \quad (7)$$

where a counterargument communicated during the proof-event $e^*\langle \boldsymbol{\Psi}, \boldsymbol{\beta} \rangle$ attacks (*rebuts*) the *conclusion* of an argument communicated during the proof-event $e\langle \boldsymbol{\Phi}, \textbf{c} \rangle$, iff $\vdash \boldsymbol{\beta} \leftrightarrow \neg \textbf{c}$,

$$\textbf{Undercutting}(e^*, e) \colon \textbf{undercut}(e^*, e) \rightarrow \neg \textbf{prem}(e) \quad (8)$$

where a counterargument communicated during the proof-event $e^*\langle \boldsymbol{\Psi}, \boldsymbol{\beta} \rangle$ *undercuts* (attacks) *some of the premises* (defeasible inference) of the argument communicated during the proof-event $e\langle \boldsymbol{\Phi}, \boldsymbol{c} \rangle$, iff $\vdash \boldsymbol{\beta} \leftrightarrow \neg(\cap \Phi i)$, for some $\{\boldsymbol{\Phi}1, .., \boldsymbol{\Phi}n\} \subset \boldsymbol{\Phi}$

Given an argument communicated during the proof-event $e\langle \boldsymbol{\Phi}, \textbf{c} \rangle$, a *counterargument* communicated during the proof-event $e^*\langle \boldsymbol{\Psi}, \boldsymbol{\beta} \rangle$ **attacks** the argument communicated during the proof-event e, at time t, iff e^* rebuts e or e^* undercuts e. Therefore:

$$\textbf{attack}(e^*, t) \rightarrow \textbf{rebut}(e^*, e) \cup \textbf{undercut}(e^*, e) \quad (9)$$

2.2 Temporal predicates

Even though proof-events can be regarded as taking place instantaneously, the *Event Calculus* is actually neutral with respect to whether events have duration or are instantaneous [19]. RAC is concerned with the study of how fluents change when new information is acquired and how this view of the problem is affected by the observation of some events remaining active or terminated at a particular time. The language in RAC [16] uses causal propositions (*c-propositions*), of the form "A initiates F when C" or "A terminates F when C", which in this paper are represented in more detailed and specific form with the arguments' and the counterarguments' moves that initiate or terminate a fluent. In most cases, we will take into consideration only the starting point of a proof-event, with the exception of those proof-events that terminate, or when duration plays a significant role. In these cases, we mention both the starting and termination points.

We apply the abovementioned *operators* combined with the basic temporal predicates from [32]: *Happens(e,t),Initiates(e,f,t), Terminates(e,f,t), ActiveAt(f,t), Clipped(t_1,f,t_2)*

$$\textbf{Happens}(e, t)\textbf{,} \text{ which means that a proof-event } e \text{ occurs at time } t. \quad (10)$$

$$\textbf{Initiates}(e, f, t_1)\textbf{: happens}(e, t_1) \rightarrow \neg\textbf{attack}(e^*, t_1) \cup$$
$$\textbf{support}(e, t_1), \quad (11)$$
$$\text{at time } t_1,$$

which means that if a proof-event e occurs at time t, then there are no counter-arguments that attack the validity of the outcome of the proof-event and there is adequate support for our claim at the specific time t_1.

$$\textbf{Clipped}(t_1, et_1, t_2)\textbf{: } \exists e_1, e_1^*, t_1, t[\textbf{Happens}(e, t1) \cap (t1 \leq t < t2) \cap$$
$$\textbf{attack}(e_1^*, t)] \cap [\nexists e_2, t_2(\textbf{Happens}(e_2, t_2) \rightarrow \neg\textbf{attack}(e_1^*, t))], \quad (12)$$
$$\text{for } t_1 \leq t < t_2$$

which means that a proof-event clips when there is no proof-event e_2 that attacks the counter-argument e_1^* attacking the proof-event e_1 between t_1 and t_2.

$$\textbf{Terminates}(e, f, e_1)\textbf{: } \exists e, e^*, t_1([\textbf{attack}(e^*, t_1) \rightarrow \neg\textbf{conc}(e)$$
$$\cup \neg\textbf{prem}(e)] \cap [\nexists(\textbf{Happens}(e_2, t_2) \rightarrow \neg\textbf{attack}(e^*, t_1))], \quad (13)$$
$$\text{with } t_1 < t_2$$

which means that a fluent terminates when there is a counterargument attacking our sequence and there is <u>no</u> proof-event e_2 that Happens in time t_2, with $t_1 < t_2$, to defend our claim. The termination of a sequence of proof-events may be caused by the proof of the falsity of the problem (there are counter-arguments that attack the conclusion of the proof-event), or the undecidability of the problem (there is a lack of adequate warrants to prove the desideratum).

$$\textbf{ActiveAt}(e, f, t_{n+1})\textbf{: Happens}(te_{n+1}, t_{n+1}) \rightarrow \neg\textbf{attack}(e_n^*, t_n) \cup$$
$$\textbf{support}(e_n^*, t_n), \textbf{for every } n \epsilon \mathbb{N}, t_{n+1} > t_n \quad (14)$$

which means that a fluent is active, if there is an argument to support our claim for every counterargument attacking our claim,. This means that for every counterargument $e^* < \mathbf{\Psi i, \beta i} >$, $i = 1, \ldots, n$, $n \epsilon \mathbb{N}$, there is a proof-event $e_{n+1}(\mathbf{\Phi_{n+1}, c_{n+1}})$, which Happens$(e_{n+1}, t_{n+1})$ and defeats the attack of the counterargument $e_n < \mathbf{\Psi_n, \beta_n} >$, for $t_{n+1} > t_n$.

From the above-mentioned, we can conclude that:

$$\textbf{Happens}(e, t_1) \cap \textbf{Initiates}(e, f, t_1) \cap (t_1 < t_{12}) \cap \neg\textbf{attack}(e*, t_2)$$
$$\rightarrow \textbf{ActiveAt}(e, f, t_2) \quad (15)$$

which means that a fluent remains active at time t_2, if a proof-event e has taken place at time t_1, with $t_1 < t_2$ and has not been terminated at a time between t_1 and t_2.

$$\forall i \leq n[\textbf{ActiveAt}(e, f, t_i) \cap (t_i < t_n) \cap \neg\textbf{Terminates}(e, f, t_i)]$$
$$\rightarrow \textbf{Valid}(e, t_n), \textbf{at time} t_n, i = 1, ., n, n \epsilon \mathbb{N} \quad (16)$$

which means that a fluent could consider valid at time t_n, if it is active and there is no counter-arguments to terminate it at time t_i for every $i = 1, \ldots, n$, $n \epsilon \mathbb{N}$.

3. Levels of argumentation

In order to define the warranted premises that are justified by a set of arguments in the sequence, we need a mechanism, which by recursion could examine the representation of the arguments. Pollock introduces defeasible reasoning where arguments are chains of reasoning that may lead to a conclusion, whereas additional information may destroy the chain of reasoning. Kakkas and Moraitis [17] presented three levels of arguments: *Object level arguments*, which represent the possible decisions or actions in a specific domain. *First-level priority arguments*, which express justifications on the object-level arguments in order to resolve possible conflicts. Then, *higher-order priority arguments* are used to deal with potential conflicts between priority arguments of the previous level until all conflicts are resolved.

We can apply the same levels in mathematical proving, in order to understand the history of proof-events, starting from the statement of a problem until its validation or rejection, including attempts or failures [32]. The data and the claim of the initial proof-events constitute *the object-level arguments*. Proof-events constitute *the first-level priority arguments*, in which we have preferences and justifications in the object-level arguments. The proof-events that have fulfilled their purpose terminate, while the rest of them continues to the *higher-order priority arguments*. As proof-events continue from lower levels to higher, they constitute fluents. In the example below, we describe the possible steps and conflicts for the justification of a proof-event e through the levels of argumentation.

Object level arguments

$$\textbf{Happens}(e_i, t_i), i = 1, \ldots, m, m\epsilon\mathbb{N}, t_i \leq t_m < t \tag{17}$$

$$\forall e_i: [(\textbf{Happens}(e_i, t_i)) \rightarrow \neg\textbf{attack}(e_i^*, t_i) \cap (t_i \leq t_m)]$$
$$\rightarrow \textbf{Initiates}(e_i, f_0, t_m) \tag{18}$$

for $i=1,\ldots,m$, $m\epsilon\mathbb{N}$, $t_i{\leq}t_m{<}t$.

In the object level arguments, we have our claim and the initial representations of arguments. The proof-events that are not attacked constitute the fluent f_0 and continue to the first level priority arguments.

First-level priority arguments

$$\textbf{Initiates}(e_{m+1}, f_1, t_{m+1}),$$
$$\textbf{attacks}(e_{m+1}^*, f_1, t_{m+1}),$$
$$i = 1, \ldots, m_1, m_1\epsilon\mathbb{N}, t_{m+i} \leq t_{m+m_1} <t, \tag{19}$$

for every $i\epsilon\mathbb{N}$ that we have:

$$\exists e_{m+i}, e_{m+i}^*, t_{m+i}[\textit{attack}(e_{m+i}^*, t_{m+i})$$
$$\rightarrow \neg\textbf{conc}(e_{m+i}) \cup \neg\textbf{prem}(e_{m+i})] \cap$$
$$(t_{m+i} \leq t_{m+m_1} < t) \cap [\nexists e_{m+i+1}, t_{m+i+1}(\textbf{Happens}(e_{m+i+1}, t_{m+i+1}) \tag{20}$$
$$\rightarrow \neg\textbf{attack}(e_{m+i}^*, t_{m+i})]$$
$$\rightarrow \textbf{Terminates}(e_{m+i}, f_1, t_{m+m_1})]$$

so that the proof-events that have been attacked and could not resolve the conflict, terminate in this fluent. The rest of them remain active, so we have:

$$\textbf{ActiveAt}(e_{m+j}, f_1, t_{m+m_1}) \text{ for every } j \neq i, j \in \mathbb{N} \tag{21}$$

and continues to the second-level priority arguments.

The same pattern continues for n-level priority arguments and for n fluents f_n, that deal with potential conflicts between priority arguments of the previous level until all conflicts are resolved or our claim proved invalid. In the final level we have:

Higher-order priority arguments

If the proof-events fail to resolve all the conflicts, our claim cannot be proved and it clips:

$$\textbf{Clipped}(t_i, e, t_n) \text{ at the time } t_n = t_{m(n-1)+mn} \geq t_i \tag{22}$$

If the proof –events manage to deal with all the attacks and:

$$\exists j, j \in \mathbb{N}: \left[\textbf{ActiveAt}\left(e_{m(n-1)+j}, f_n, t_n\right) \cap \neg\textbf{Terminates}(e, f_n, t_n) \right] \rightarrow$$
$$\textbf{Valid}(e, t_n), \text{ at the time } t_n = t_{m(n-1)+mn} \geq t_i, \tag{23}$$

then our claim is proved valid.

4. An example using the Fermat's Last Theorem

We use the famous Fermat's Last Theorem to illustrate our approach. Fermat's Last Theorem was formulated in 1637 by Pierre de Fermat, who stated that there are no three distinct positive integers a, b, and c, other than zero, that can satisfy the equation $a^n + b^n = c^n$, whenever n is an integer greater than two ($n > 2$). The statement of the problem marks the starting-point of a proof-event. Even though Fermat claimed to have proved this theorem, it actually took 358 years and numerous attempts undertaken by many famous mathematicians and amateurs to prove it until its final proof by Andrew Wiles in 1995. Thus, Fermat's alleged proof cannot be included in the initial proof-event, since it was never communicated. Fermat communicates the Theorem only for the cases $n=3$ and $n=4$ in his letters and gives a solution for the latter case. The statement of the problem marks the beginning of a sequence of proof-events that evolved in time for 358 years. This sequence of proof-events was evolving in time, since many famous mathematicians and amateurs (agents) were involved in various distinct proof-events that took place in different places and times in their attempt to solve the problem posed.

We cannot expose here the whole sequence of such proof-events. We confine ourselves to select some of these historical attempts (proof-events) until the proof-event that includes the communication and validation of the final proof of the theorem and demonstrate how argumentation is involved in the process of search for proof.

The first attempts to prove the Theorem were proofs for specific exponents. The case $n = 3$ was first explored by Abu-Mahmud Khojandi (c. 940 - 1000), but his attempt has not survived (and thereby cannot be considered as a proof-event) and it is conjectured that it was incorrect. Leonhard Euler gave a proof for $n = 3$ in 1755 and for n=4 in 1747, but his proof of the former case contained a basic fallacy [9, p.39-40]. Many other mathematicians proved the theorem for $n=3$ using various methods. Gabriel Lamé (1795 –1870) proved it for the case $n = 7$. In 1847, he communicated a

proof of the Theorem, but it was flawed. Gabriel Lame's proof failed because it was claimed incorrectly that complex numbers could be factored into primes uniquely. This gap was indicated instantly by Joseph Liouville [9, p.76-77]. In 1984, Gerhard Frey pointed out a connection between the modularity theorem and Fermat's equation, but Fermat's Last Theorem remained to be a conjecture. The Taniyama-Shimura-Weil conjecture, which was proposed in 1955, was the method that led to a successful proof of Fermat's Last Theorem, when Andrew Wiles accomplished a partial proof of this conjecture in 1994 [29].

Wiles, after spending six years applying various methods that were proved unsuccessful, he approached the problem in a new way. He decided to present his work in June 1993 at the Isaac Newton Institute for Mathematical Sciences [29].

However, during the peer review, it became evident that there was an incorrect critical point in the proof. Wiles tried almost a year to resolve this point, firstly by himself and then in collaboration with Richard Taylor, but without success [18]. When Wiles was on the verge to quit his attempt, he experienced an insight that the Kolyvagin–Flach approach and Iwasawa theory were each insufficient on their own, but in combination they could be strong enough to overcome this final barrier. In 1994, Wiles submitted two papers that established the modularity theorem for the case of semistable elliptic curves, which was the last step in proving Fermat's Last Theorem [29].

This example illustrates the contribution of the agents in the process of proving. Firstly, the central aim of the proving itself is to convince the rest of the community about the justification and the validity of your approach. Moreover, the other agents also contribute significantly in the procedure. So many people had to participate in order to reach the initial goal, which was the proving of the Fermat's Last Theorem. This participation is depicted with two ways, either with the rejection of someone else's attempt by pointing out a fault and/or inaccuracy (ex. Liouville indicated Lame's gap concerning complex numbers) or with the dialogue between cooperators in order to detect and resolve weak or deficiently supported areas in the proving (ex. Wiles asked other colleagues' help, like Richard Taylor, whenever he found a dead-end or fault in his attempt). Argumentation is more efficient in more interactive contexts, as they let counterarguments to be addressed and stronger arguments to surface. An audience with mainly common beliefs will generate less differentiated counterarguments, making them easier to address. Thus, a mathematician is in a favored position if he wants to ask the assistance of a few colleagues in order to point out most of the possible counterarguments and resolve them in the final proof. By doing this, the proving could be more convincing not only to these few colleagues, but probably to the whole community.

The arguments and the counterarguments also play an essential role in the process of proving, contributed equally in the building and the justification of the proving. The warranted parts of the initial provings acted as a groundwork for the next provings, while the counterarguments that signalize the faults in those unsuccessful provings open the way for better justified provings and in some cases turn the interest of the mathematical community in new unexplored areas. Those incomplete provings may add more or less to the proof of Fermat's Last Theorem, but the methods that were created with them lead to major discoveries and creation of new fields in the era of Mathematics like the foundation of modern algebra. Discoveries that are

even more significant than the proving of the theorem itself and might have not been found if it weren't for the warranted proof-events and the counterarguments emerged from the previous attempts of proving.

In the next part we present a brief representation of this example through the levels of argumentation[1].

Object level arguments – Fermat's Conjecture

In the object level arguments, we have Fermat's conjecture as the initial proof-event (e_{Fermat}) and his claim that he has a proving for this conjecture, without any claim-counterargument (e_{Fermat}^*) clearly opposes this conjecture.

$$Happens(e_{Fermat}, t_{1637}) \cap \neg attack(e_{Fermat}^*, t_{1637}) \rightarrow Initiates(e_{Fermat}, f_0, t_{1637})$$

First-level priority arguments - Proofs for specific exponents

In the first-level priority arguments, we have proofs for specific exponent *n* of the Fermat's Last theorem from various mathematicians in different time points.

For the exponent n=3 ($e_{n=3}$), **Leonhard Euler (e_{Euler})** gave a proof in 1755, so we have: $Happens(e_{Euler}, t_{1755})$.

Many other well-known mathematicians followed with equivalent proofs that support the validity of the proof for n=3. Each prover used a different way (warrant) for proving the conclusion, so their provings are equivalent.

$$Support(e_{n=3}, t_i) \rightarrow Equivalent(e_{n=3}, e_i), \text{ for } i=1,\dots,14 \text{ with:}$$

$i=1$:(e_{Euler}, t_{1707}), $i=2$:($e_{Kausler}, t_{1802}$), $i=3$:($e_{Legendre}, t_{1823}$), $i=4$:($e_{Calzolari}, t_{1855}$), $i=5$:(e_{Lame}, t_{1865}), $i=6$:(e_{Tait}, t_{1872}), $i=7$:($e_{Gunther}, t_{1878}$).

$$Happens(e_{Euler}, t_{1755}) \cap Initiates(e_{n=3}, f_1, t_{1755}) \cap [\neg attack(e_{n=3}^*, t_i) \cup support(e_{n=3}, t_i)] \cap (t_{1755} < t_i) \rightarrow ActiveAt(e_{n=3}, f_1, t_i), \text{ for } t_{1755} < t_i$$

Fermat's Last Theorem was also proved for the exponents *n* = 5, 6,7, 10, and 14.

Second-level priority arguments – Lame's Proving

In 1847, Gabriel Lame's proving (e_{Lame}) failed because it claimed incorrectly that complex numbers can be factored into primes uniquely. This gap was indicated instantly by Joseph Liouville ($e_{Liouville}^*$).

$$\exists e_{Lame}, e_{Liouville}^*, t_{1847}[attack(e_{Liouville}^*, t_{1847}) \rightarrow \neg conc(e_{Lame})]$$
$$\cap (t_{1847} \leq t_1 < t_2)$$
$$\cap [\nexists e_{Lame}, t_2(Happens(e_{Lame}, t_2) \rightarrow \neg attack(e_{Liouville}^* t_{1847}))]$$
$$\rightarrow Terminates(e_{Lame}, f_3, t_2)$$

Third-level priority arguments – Andrew Wiles

Andrew Wiles presented his work in June 1993, but it became evident that there was an incorrect critical point (e_{Wiles}^*) in the proving. Wiles tried almost a year to

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[1] Names of provers and dates are taken from Wikipedia «Fermat's Last Theorem». Retrieved from: https://en.wikipedia.org/wiki/Fermat%27s_Last_Theorem]

resolve this point, firstly by himself and then with the contribution of Richard Taylor, but without success.

$$\boldsymbol{Clipped}(t_{1993}, e_{Wiles}, t_{1994}):$$
$$\exists e_{Wiles}, e^*_{Wiles}, t_1, t_2 [\boldsymbol{Happens}(e_{Wiles}, t_1) \cap (t_{1993} \leq t_1 <$$
$$t_{1994}) \cap \boldsymbol{attack}(e^*_{Wiles}, t_1)] \cap [\nexists e_2, t_2(\boldsymbol{Happens}(e_{Taylor}, t_2) \rightarrow$$
$$\neg \boldsymbol{attack}(e^*_{Wiles}, t_1))], for\ t_{1993} \leq t_2 < t_{1993}.$$

Finally, in 1994, Wiles submitted two papers that combined Kolyvagin–Flach approach and Iwasawa theory which was the last step in proving Fermat's Last Theorem.

$$\boldsymbol{ActiveAt}(e_{Wiles}, f_5, t_{1994}): \boldsymbol{Happens}(e_{Wiles}, t_{1994}) \rightarrow \neg \boldsymbol{attack}(e^*_{Wiles}, t_{1994})$$
$$\cap \boldsymbol{Elaboration}(e_{Wiles}, S_{Kolyvagin-Flach}) \cap \boldsymbol{Elaboration}(e_{Wiles}, S_{Iwasawa})$$

<u>Higher-order priority arguments-Fermat's Last Theorem</u>
The proof–event managed to deal with all the attacks and we have:

$$\boldsymbol{ActiveAt}(e_{Wiles}, f_n, t_{1994}) \cap \neg \boldsymbol{Terminates}(e_{Fermat}, f_n, t_{1994})] \rightarrow$$
$$\boldsymbol{Valid}(e_{Fermat}, t_{1994}),$$

at the time $t_{1994.}$

Thus, Fermat's Last Theorem is proved valid by Wiles, with the contribution of the other agents that opened the way before him in this ages-long sequence of proof-events.

5. Conclusion

We have developed a model of the proof-events calculus [32] based on Pollock's [26], Toulmin's [28] and Kakas' argumentation theories, extending the proof-events calculus with the integration of arguments and counterarguments. The combination of Vandoulakis's and Stefaneas's proof-events-based theory and logic-based argumentation has the advantage of highlighting weak areas in a proof. Proof-events are not considered as infallible facts before their ultimate validation, thus enabling the exploration of flawed approaches and proofs to be found and resolved. We focused on the connection between proving, human reasoning, cognitive processes and creativity. We presented a calculus for the proof-event argument, argument moves, and temporal predicates and we analyzed them in terms of levels of argumentation. The calculus adds an additional dimension and performs a significant role in making these connections sufficiently detailed and specific. In our future work, we can apply this model to express the proof-events sequence and history in specific examples of mathematical proving, as in the example of Fermat's Last

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