General and Fractional Hypertree Decompositions: Hard and Easy Cases (Extended Abstract)*

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Abstract. Hypertree decompositions, generalized hypertree decompositions, and fractional hypertree decompositions are hypergraph decomposition methods successfully used for answering conjunctive queries and for the solution of constraint satisfaction problems. In this work, we present new intractability and tractability results for the problem of recognizing if a given hypergraph has a generalized or fractional hypertree decomposition of low width.

1 Introduction

Answering conjunctive queries (CQs) and solving constraint satisfaction problems (CSPs) are fundamental tasks in Computer Science. Both problems are NP-complete [2]. Consequently, the search for tractable fragments of these problems has been an active research area in the Database and AI communities for several decades.

The most powerful methods known to date for defining tractable fragments are based on various decompositions of the hypergraph structure underlying a given CQ or CSP. The most important forms of hypergraph decompositions are *hypertree decompositions (HDs)* [5], *generalized hypertree decompositions (GHDs)* [5], and *fractional hypertree decompositions (FHDs)* [8]. These decomposition methods give rise to three notions of width of a hypergraph H: the *hypertree width* $hw(H)$, *generalized hypertree width* $ghw(H)$, and *fractional hypertree width* $fhw(H)$, where, $fhw(H) \leq ghw(H) \leq hw(H)$ holds for every hypergraph H. For definitions, see Section 2.

Both, answering CQs and solving CSPs, become tractable for classes of instances where the underlying hypergraphs have bounded hw, ghw, or fhw, and an appropriate decomposition is given. This gives rise to yet another crucial computational problem, namely the problem of recognizing if a given CQ or CSP (strictly speaking, its hypergraph) has hw, ghw, or, fhw bounded by some constant k. Formally, for *decomposition* $\in$ {HD, GHD, FHD} and $k > 0$, we consider the following family of problems:

CHECK(*decomposition*, k)
instance: hypergraph $H = (V, E)$;
question: does H have a *decomposition* of width $\leq k$?

Clearly, bounded fhw defines the largest tractable class while bounded hw defines the smallest one. On the other hand, only the problem CHECK(HD, k) is known to be feasible in polynomial time [5]. CHECK(GHD, k) has been shown to be NP-complete

* This is an extended abstract of [4].

for $k \geq 3$ [6]. The status of CHECK(FHD, k) has been open for over a decade [7]. The goal of this work is therefore as follows: first, we want to identify the complexity of the CHECK(FHD, k) problem. Ideally, we would also like to close the gap of CHECK(GHD, k) for $k = 2$. Note that this is an important special case since CQs tend to have low hw and, thus also low ghw [1]. Finally, we want to define meaningful classes of hypergraphs for which the CHECK(GHD, k) and CHECK(FHD, k) problems become tractable.

2 Basic Definitions

Formally both, CQs and CSPs, are first-order formulae using only $\{\exists, \wedge\}$ as connectives. but not $\{\forall, \vee, \neg\}$. Now consider an arbitrary CQ or CSP, i.e., an FO-formula ϕ with connectives $\{\exists, \wedge\}$. The *hypergraph corresponding to* ϕ is defined as hypergraph $H = (V(H), E(H))$, where the set of vertices $V(H)$ is defined as the set of variables in ϕ and the set of edges $E(H)$ is defined as $E(H) = \{e \mid \phi$ contains an atom A, s.t. e equals the set of variables occurring in $A\}$.

We consider here three notions of hypergraph decompositions with associated notions of width. To this end, we first introduce the notion of (fractional) edge covers. Let $H = (V(H), E(H))$ be a hypergraph and consider a function $\gamma\colon E(H) \to [0, 1]$. Then, we define the set $B(\gamma)$ of all vertices covered by γ and the weight of γ as

$$B(\gamma) = \left\{ v \in V(H) \mid \sum_{e \in E(H), v \in e} \gamma(e) \geq 1 \right\}, \; weight(\gamma) = \sum_{e \in E(H)} \gamma(e).$$

The special case of a function with values restricted to $\{0, 1\}$, will usually be denoted by λ, i.e., $\lambda\colon E(H) \to \{0, 1\}$. Note that, following [5], λ can also be seen as a set with $\lambda \subseteq E(H)$ (i.e., the set of edges e with $\lambda(e) = 1$) and the weight as the cardinality of such a set of edges.

We are now ready to introduce our three notions of hypergraph decompositions and their width measures.

Definition 1. *A* generalized hypertree decomposition *(GHD) of a hypergraph* $H = (V(H), E(H))$ *is a tuple* $\left\langle T, (B_u)_{u \in N(T)}, (\lambda)_{u \in N(T)} \right\rangle$*, such that* $T = \langle N(T), E(T) \rangle$ *is a rooted tree and the following conditions hold:*

(1) *for each* $e \in E(H)$*, there is a node* $u \in N(T)$ *with* $e \subseteq B_u$*;*
(2) *for each* $v \in V(H)$*, the set* $\{u \in N(T) \mid v \in B_u\}$ *is connected in* T*;*
(3) *for each* $u \in N(T)$*,* λ_u *is a function* $\lambda_u\colon E(H) \to \{0, 1\}$ *with* $B_u \subseteq B(\lambda_u)$*.*

We use the following notational conventions. To avoid confusion, we will consequently refer to the elements in $V(H)$ as *vertices* of the hypergraph and to the elements in $N(T)$ as the *nodes* of the decomposition. For a node u in T, we write T_u to denote the subtree of T rooted at u. By slight abuse of notation, we will write $u' \in T_u$ to denote that u' is a node in the subtree T_u of T. Finally, we define $V(T_u) := \bigcup_{u' \in T_u} B_{u'}$.

Definition 2. *A* hypertree decomposition *(HD) of a hypergraph* $H = (V(H), E(H))$ *is a GHD, which in addition also satisfies the following condition:*

(4) for each $u \in N(T)$, $V(T_u) \cap B(\lambda_u) \subseteq B_u$

Definition 3. *A* fractional hypertree decomposition *(FHD) [8] of a hypergraph $H = (V(H), E(H))$ is a tuple $\langle T, (B_u)_{u \in N(T)}, (\gamma)_{u \in N(T)} \rangle$, where conditions (1) and (2) of Definition 1 plus condition (3') hold:*

(3') for each $u \in N(T)$, γ_u is a function $\gamma_u : E(H) \rightarrow [0, 1]$ with $B_u \subseteq B(\gamma_u)$.

The width of a GHD, HD, or FHD is the maximum weight of the functions λ_u or γ_u, respectively, over all nodes u in T. The generalized hypertree width, hypertree width, and fractional hypertree width of H (denoted $ghw(H)$, $hw(H)$, $fhw(H)$) is the minimum width over all GHDs, HDs, and FHDs of H, respectively.

3 Main Results

On the negative side, the main result of our work is the following NP-completeness (in particular, the NP-hardness) result:

Theorem 1. *The* CHECK*(FHD, k) problem is* NP*-complete for $k = 2$.*

The NP-hardness proof is by reduction from the 3-SAT problem. It can be easily extended to arbitrary (possibly fractional) $k \geq 3$ (see [4] for details). For fractional k with $1 < k < 3$, the precise complexity remains an open question for future work.

The construction in the proof of Theorem 1 also gives us the following result:

Theorem 2. *The* CHECK*(GHD, k) problem is* NP*-complete for $k = 2$.*

On the positive side, we identify general, realistic, and non-trivial restrictions that make the CHECK(GHD, k) and CHECK(FHD, k) problems tractable. More precisely, we concentrate on the following three properties:

Definition 4. *We say that a class $\mathscr{C}$ of hypergraphs has the* bounded intersection property (BIP) *if there exists some constant i such that for every hypergraph H in $\mathscr{C}$, the cardinality of any intersection $e_1 \cap e_2$ of two distinct edges e_1 and e_2 of H is $\leq i$.*

We say that a class $\mathscr{C}$ of hypergraphs has the bounded multi-intersection property (BMIP) *if there exist constants c and i such that for every hypergraph H in $\mathscr{C}$, the cardinality of any intersection $e_1 \cap \cdots \cap e_c$ of c distinct edges $e_1, \ldots, e_c$ of H is $\leq i$.*

We say that a class $\mathscr{C}$ of hypergraphs has bounded degree, if there exists $d \geq 1$, such that for every hypergraph $H \in \mathscr{C}$, $|\{e \in E(H) \mid v \in E(H)\}| \leq d$ holds, i.e., every vertex occurs in at most d edges.

In an empirical analysis of several CQ and CSP instances we have verified that an overwhelming number of the instances enjoy these properties for low constants i, c, or d, respectively [4]. With these restrictions, we get the following tractability result.

Theorem 3. *Let $\mathscr{C}$ be a class of hypergraphs. If $\mathscr{C}$ has the BMIP, then the* CHECK*(GHD, k) problem is in* P *for arbitrary $k > 0$. Moreover, in case of a positive instance, a GHD of width $\leq k$ can be computed in polynomial time. Consequently, this tractability holds if $\mathscr{C}$ has bounded degree or the BIP (which each imply the BMIP).*

Unfortunately, the tractability proof for the BMIP in Theorem 3 does not directly carry over to FHDs. However, by combining two restrictions, we also manage to identify an interesting tractable fragment for the CHECK(FHD, k) problem.

Theorem 4. *Let $\mathscr{C}$ be a class of hypergraphs. If $\mathscr{C}$ has the BIP and bounded degree, then the* CHECK*(FHD, k) problem is in* P *for arbitrary $k > 0$. Moreover, in case of a positive instance, an FHD of width $\leq k$ can be computed in polynomial time.*

4 Conclusion

Our main results shown in this work are, on the one hand, the NP-completeness proof of the CHECK(*decomp*, k) problem for *decomp* $\in$ {GHD, FHD} and $k = 2$ and, on the other hand, the identification of tractable fragments of these problems. The tractability results for CHECK(FHD, k) are significantly weaker than for CHECK(GHD, k). Further results in [4] therefore deal with efficient approximations of the CHECK(FHD, k) problem. More precisely, we could show that polynomial-time approximation up to a logarithmic factor is possible for any class of hypergraphs with bounded Vapnik–Chervonenkis dimension. This is a much closer approximation than the cubic approximation for the general case shown in [9]. Moreover, in a follow-up work, we have meanwhile managed to show tractability of CHECK(FHD, k) in case of bounded degree [3]. An interesting open problem is, if the weaker restriction of the BMIP also suffices to guarantee tractability. If not, then the effect of the BIP alone on the CHECK(FHD, k) problem has to be investigated.

Acknowledgements. This work was supported by the Engineering and Physical Sciences Research Council (EPSRC), Programme Grant EP/M025268/ VADA: Value Added Data Systems — Principles and Architecture as well as by the Austrian Science Fund (FWF):P25518-N23 and P30930-N35.

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