

GeoGebra as Means of Improving the Quality of Education

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Abstract. The article substantiates the use of dynamic mathematics software as effective means of formation of the functional thinking of pupils, which directly impact on the quality of mathematical education. The constructive approaches to solving mathematical problems by GeoGebra reduce the weight of analytical calculations. Such approaches put forward the need for skills to construct the desired configuration, take into account the dependencies between its parameters, visualize positions of possible results, even "see" the desired function, for which you need to determine extreme.

The authors use GeoGebra in solving extreme problems using method based on constructing an empirical graph of the relations between the values and defining of extremum. Another method is based on the visualisation of spreadsheets of the values of the empirical function and their analysis.

The effectiveness of the proposed approach was tested during 2015-2017 and was experimentally confirmed in the work on the research topic "The use of information technology in education" through the organization of math group works for pupils in the Sumy region. We tracked the overall level of academic achievement and its dynamics. Since the scale had two positions (right/wrong) and the results of educational achievements were not dependent on each other, we used the sign test. The statistical check at the significance level of 0.05 confirmed the positive impact of the group works on the quality of mathematical preparation of pupils.

Keywords: dynamic mathematics software, GeoGebra, mathematical preparation, quality of mathematical education, constructive approaches, extremum problems

1 Introduction

The realities of modern society determine the technologies that are used in the training of the younger generation. The extension of portable devices (PDAs, smartphones, tablets) and implementation of mobile and blended learning technologies contribute to the strengthening of scientific and methodical searches in the course of the special software, among which dynamic mathematics software (DMS) are allo-

cated in the field of mathematics. Such software is characterized by the ability of dynamically handling of mathematical objects and getting information about their properties. Among this software we allocate The Geometer's SketchPad, GeoGebra, Cabri and similar.

Attraction of such software as means of improving the quality of mathematical education is mentioned in the findings of V. Dubrovskiy, M. Zhaldak, S. Pozniakova, S. Rakov, V. Rakuta, M. Hohenwarter, I. Khrapovytskyi, M. Shabanova, T. Shyrykova and others. We note works [1-5], where the problems of usage of this software at math lessons in secondary schools are considered. The authors offer solution examples of plane geometry, solid geometry, beginnings of the analysis and indicate the implementation of such software for automation of calculations, visualisation of results, investigation of properties of objects etc.

Analysis of these and other works suggests a typical usage of the software when the solution of the problem in the software duplicates the traditional solutions in the notebook, and non-traditional, when the solutions of problems are based on the discovery of mathematical facts through changes of a dynamic structure [5], on the "computer proof" of certain statements at the empirical level through the search of a large number of variants [4], on the construction of correspondences and the use of spreadsheets of values for some parameters for discovery of a certain fact. The last two approaches we have mentioned in the works [6-7] and dwell on them in this article to illustrate the use of DMS as effective means of formation of the functional thinking of the pupils, which directly impacts the quality of mathematical education.

Among the variety of DMS the authors selected software *GeoGebra* as one of the most powerful and free. Each new version of the software is enriched with services and expands its application not only in the field of school mathematics [8].

2 Constructive Approaches to the Solution of Extremum Problems

Geometric extremum problems often cause difficulties even among pupils, whose level of mathematical education is above average. Such problems are supposed to be difficult because of the unusual formulation of the conditions and search for the answer – you need to determine variables, to construct a function which will associate these values with unknown one, and then to explore this function on the presence of the extremum. Ordinary pupils don't understand such actions, because they additionally require already established geometric concepts and analytical skills.

Constructive approaches to the solution of such problems with the help of DMS tools, reduce the weight of analytical calculations and highlight the need for skills to construct desired configuration, to take into account relations between parameters, to visualize some positions of possible results, even to "see" the desired function for which you need to determine the extremum.

Example 1. A cone is inscribed in a sphere of radius 4. What should be the height of the cone with the largest volume? [9, p. 202]

The solution of this problem in *GeoGebra* is implemented through construction of a dynamic configuration and visual observation of the cone volume value, which will be interactively changed by the movement of the base point – the point *F*, which is the center of the cone base (Fig.1).

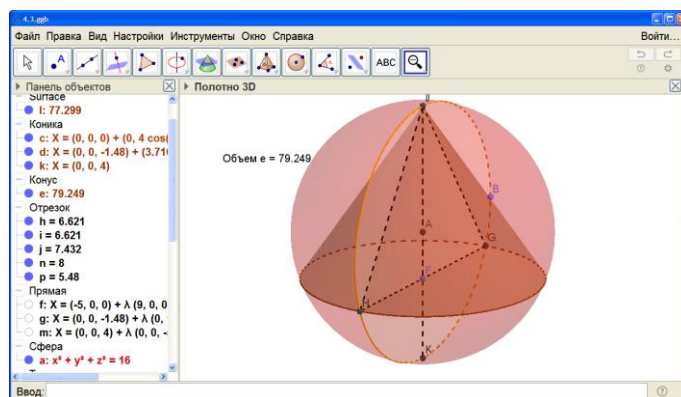


Fig.1. Visual observation of cone volume value

In Table 1 we propose the algorithm for the construction of the configuration of this problem by *GeoGebra*.

Table 1. Algorithm for the construction of geometric combination of the bodies

	Constructive actions of a pupil	Computer tool
1	To construct a sphere of radius 4.	<i>Sphere with Center and Radius</i>
2	To construct an arbitrary line passing through the centre of the sphere – the point <i>A</i> .	<i>Line</i>
3	To construct the plane α that is perpendicular to the given line and passes through the centre of the sphere.	<i>Perpendicular Plane</i>
4	To construct the curve of intersection of the given plane and the sphere – the big circle of the sphere.	<i>Intersect Two Surfaces</i>
5	To construct an arbitrary point <i>U</i> on the circle and the line <i>UA</i> , which passes through it and the centre of the sphere– the axis of the cone.	<i>Point, Line</i>
6	To construct another point of intersection of this line and the circle – the point <i>K</i> .	<i>Intersect</i>
7	To construct the segment <i>UK</i> which connects these two points of intersection.	<i>Segment</i>
8	To construct an arbitrary point <i>F</i> on the segment <i>UK</i> .	<i>Point</i>
9	To construct in the plane α the line, which is perpendicular to the cone axis and passes through the point <i>F</i> .	<i>Perpendicular Line</i>
10	To construct the point of intersection of this line and the big	<i>Intersect</i>

	Constructive actions of a pupil	Computer tool
	circle of the sphere – the points H and G.	
11	To construct the triangle UHG. It is inscribed in the big circle of the sphere and is the axial cross-section of the cone inscribed in the sphere.	<i>Segment</i>
12	To construct the circle which passes through the point H. The axis of the cone is its axis. Built circle is the base of the cone.	<i>Circle with Axis through Point</i>
13	To build the cone.	<i>Extrude to Pyramid or Cone</i>
14	To calculate the height and the volume of the cone.	<i>Distance or Length, Volume</i>

Let consider other methods of solving this problem.

1. Method is based on the construction of the empirical graph of the relation of the cone height and its volume by the *Trace* tool.

The use of the *Trace* tool expects the construction of a curve, points of which have the particular property. If you use this tool, than during dynamic changes of the initial construction the selected point will leave a trace, which will be the locus with the desired property.

If you do steps 1-14 (Table 1), than additionally constructed point *L* may leave such a trace. Its abscissa is equal to the value of the height of the cone, and its ordinate is equal to the value of the cone volume. In the parametres of point *L* you need to order the service *Trace On*. The point *F* moves with the movement of the point *A*. The built trace of the point *L* is the empirical graph of the function of the cone volume value that we are interested in (Fig.2).

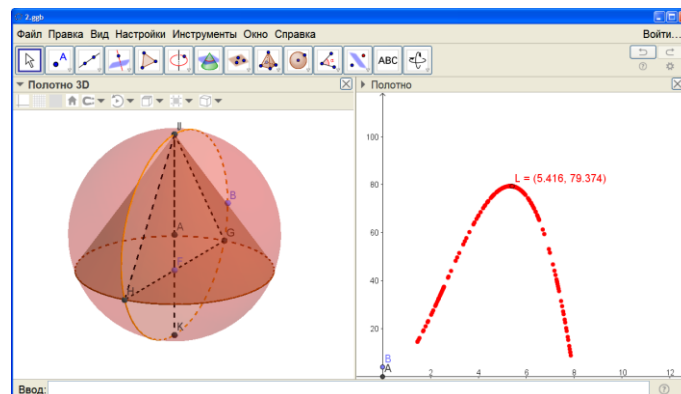


Fig.2 Construction of the empirical graph of the function of the cone volume value by the *Trace* tool

The extremum of this function is obvious. There is no doubt that the maximum for the cones volumes exists and it is unique.

Note that you can also use the *Locus* tool, which will automatically build the empirical function of volume value (Fig. 3). The result of this tool is similar to the result of the *Trace* tool, and the difference is in the output format of the result: after the *Locus* tool the graph is continuous curve, and after the *Trace* tool the graph is bitmap.

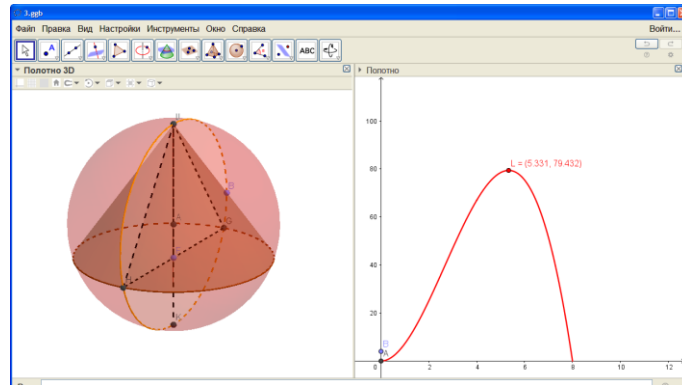


Fig.3 Construction of the empirical graph of the function of the cone volume value by the *Locus* tool

2. Method is based on the displaying of the spreadsheet of values of the empirical functions and their analysis (Fig. 4).

After constructing the main configuration the spreadsheet, that displays the values of the height of the cone and the cone volume, is created. During the change of the position of the base point this spreadsheet is filled with the appropriate value sets. The analysis of these values allows identifying the functional relation between the cone height and the cone volume, to see extreme volume value and make a conclusion about the corresponding value of the cone height.

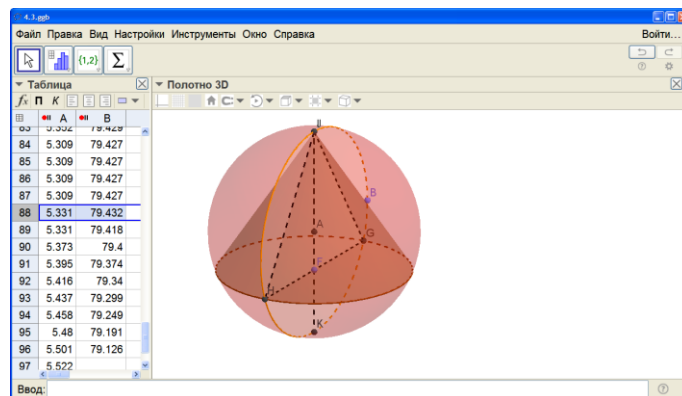


Fig.4 The spreadsheet with values of the empirical function of the cone volume

For the application of the described method pupils are acquired to master the tools and the awareness of a configuration to search for answer: to see the relation between the input and the result without the use of derivative, to know only the definition of the cone and its volume.

The algorithms for solving the problem based on the described methods are shown in Table 2.

Table 2. The algorithm for solving the extremum problems based on the constructive method

Method	Description of possible solving algorithm
Using the <i>Trace</i> tool (Fig. 2)	1-14. The steps are similar to the previous method of solution. 15. To construct the point <i>L</i> with the following coordinates: the x coordinate is the value of the cone height, the y coordinate is the value of the cone volume. 16. To order the service <i>Trace On</i> in the point properties. 17. To define the maximum 79,374 of the empirical function with the help of the trajectory of the point <i>L</i> .
Using the <i>Locus</i> tool (Fig. 3)	1-15. The steps are similar to the previous method of solution. 16. To construct locus using the <i>Locus</i> tool, choosing the point <i>L</i> as a point creating locus and the point <i>F</i> as a driver point. 17. To determine the maximum 79,374 for the continuous graph of the locus (of the point <i>L</i>).
Using the Spread-sheet of values (Fig.4)	1-14. The steps are similar to the previous method of solution. 15. To add <i>Spreadsheet View</i> . To order the service <i>Tracing to Spreadsheet</i> for values of the cone height (in Fig.4 it is the value <i>p</i>) and the cone volume using the context menu. 16. To observe appearance of the numerical value of the cone height and the cone volume value changing the position of the base point <i>F</i> . 17. To analyze the dynamics of changes in the volume value – value is growing to a certain point, and then falling. Critical value 79,432 is achieved when the height of the cone is equal to 5,331.

The analytical method for solving this problem requires to write the formula of the cone volume function and differentiate it further. According to the task:

$$V = \frac{1}{3} \pi (8h^2 - h^3), V' = \frac{1}{3} \pi (16h - 3h^2) = 0, h = \frac{16}{3} \approx 5,33, V_{max} \approx 79,43.$$

The results of the analytical solution coincide with the results obtained by the constructive method (Fig.5).

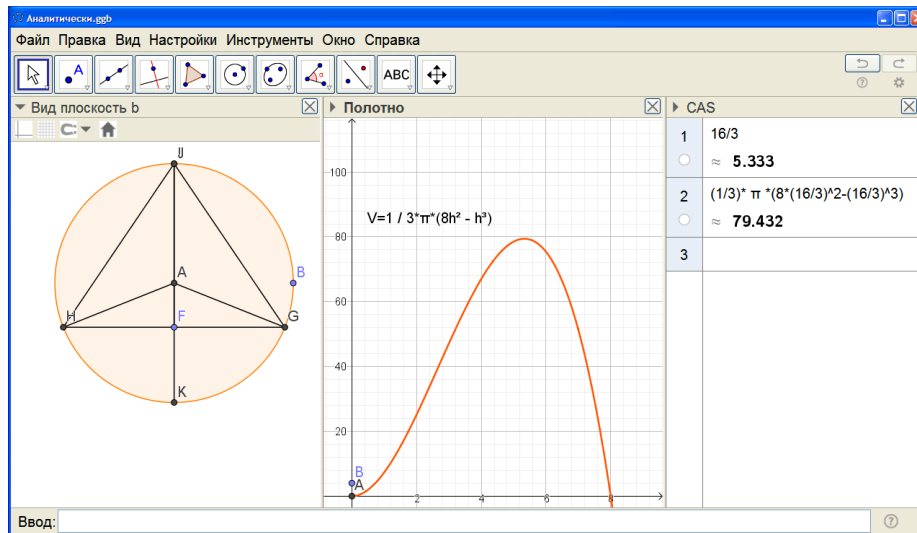


Fig.5 Graph of the cone volume function built analytically

Remark.

1. The empirical function of volume built by the *Locus* tool is not perceived by the software as an independent object, so using the *Extremum* tool or the *Function Inspector* tool to determine the maximum quickly is impossible. Extreme values of built function need to be determined visually.

2. Numerical results can often be "unattractive" or approximate, as are calculated in numeric format with early prescribed accuracy. This causes additional need either in formulaic expressions of desired function and analytical finding of its extreme values or at least in the check of coincidence of graphs of the empirical function and one that is found analytically.

Example 2. The sum of the lengths of the cone base radius and its height is constant and equal to 10. At what ratio of the radius and the height volume of the cone will be the biggest?

Let's describe the possible solution algorithms.

Method 1 (traditional, Fig.6).

1. To construct points $O(0,0)$, $A(10,0)$ and the segment OA .
2. To construct the point B on the segment OA (segment OB determines the cone base radius).
3. To construct the point $D(0,BA)$ (the segment OD determines the cone height) and add *3D Graphics View*. As *Graphics View* and *3D Graphics View* are interactively linked, then the points O , A , B , D will appear on *3D Graphics View*.
4. To construct the circle with the centre in the point O and the point A on the circle – the base of the cone.
2. To construct the cone with the height OD using the *Extrude to Pyramid or Cone* tool.
3. To calculate the volume of the cone using the *Volume* tool.

4. To calculate (via the *Input Field*) the ratio of the lengths of the cone base radius and the height of the cone – number h .

Changing the position of the point B we are observing the value of the cone volume. The biggest value of the cone volume 155,1 is achieved when the ratio of radius and height is equal to 2.

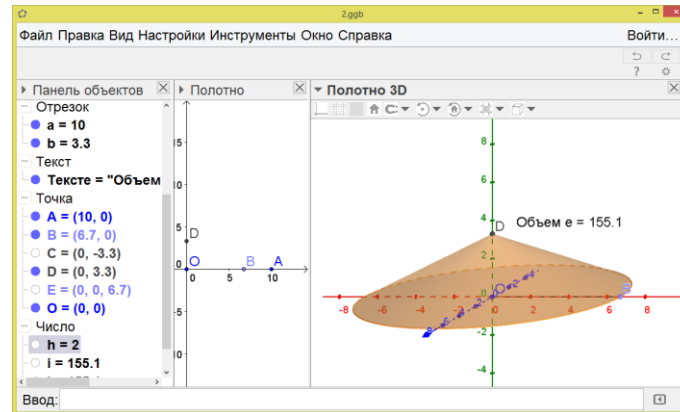


Fig.6 Visual observation of the cone volume value.

Method 2 (using the *Trace* tool).

1. To construct a slider for the parameter a in the range $[0,10]$.
2. To construct points $O(0,0)$ and $A(a,0)$ (the segment OA determines the cone base radius).
3. To construct the point $B(0,10-a)$ (the segment OB determines the cone height).
4. To construct the cone and calculate its volume as in the previous method of solution by adding the *3D Graphics View*.
5. To construct the point $C(a/(10-a), volume b)$ and order the service *Trace On*.
6. To determine the maximum 155,1 of empirical function using the trajectory of the point C . It is achieved if the cone base radius is twice its height.

Method 3 (using the *Locus* tool, Fig.7).

- 1-4. Steps are similar to Method 2.
5. To construct the point $C(a/(10-a), volume b)$ and the locus, using the *Locus* tool choosing the parameter a as a "driver" point and the point C as a point creating locus.
6. To determine the maximum 155,1 of the empirical function using the continuous graph of the locus (of the point C). It is reached at the ratio which is equal to 2.

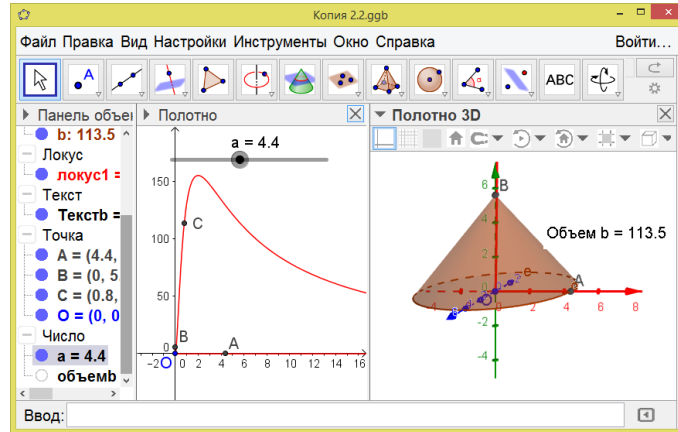


Рис.7 The empirical graph of the function of cone volume built by the *Locus* tool.

Method 4 (using the spreadsheet of values of the empirical functions, Fig.8)

1-4. Steps are similar to Method 2.

5. To construct the point $C(a/(10-a), \text{volume}b)$.

6. To add the Spreadsheet and order *Tracing to Spreadsheet* for values of the ratio of the length of the cone base radius to its height (value f) and volume.

7 To observe the appearance of numeric values, relations and volume value changing the position of the base point A.

8. To analyze the dynamics of changes in the volume value – value is growing up to the certain point, and then falling. Critical value 155,1 is achieved at the ratio of 2.

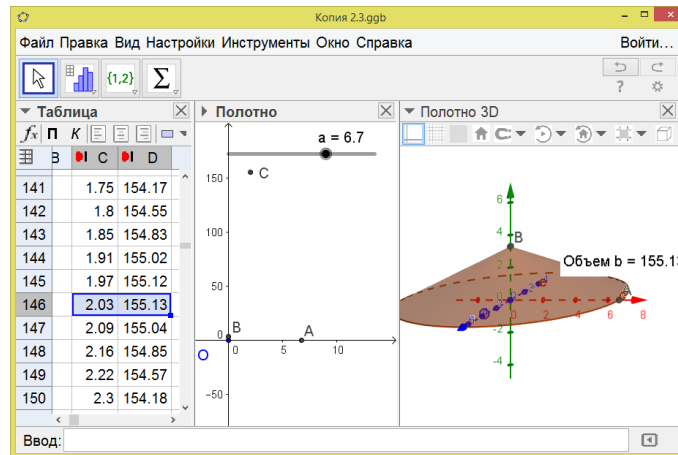


Fig.8 The spreadsheet with values of the empirical function of the cone volume value

An analytical method of solving the problem by writing the cone volume function and its differentiation gives the following results:

$$V = \frac{1}{3}\pi(10r^2 - r^3), V' = \frac{1}{3}\pi(20r - r^2), r = \frac{20}{3}, h = \frac{10}{3}, \frac{r}{h} = 2, V \approx 155,1$$

The construction of the graph of this function shows an absolute coincidence with the empirical graph (Fig. 9).

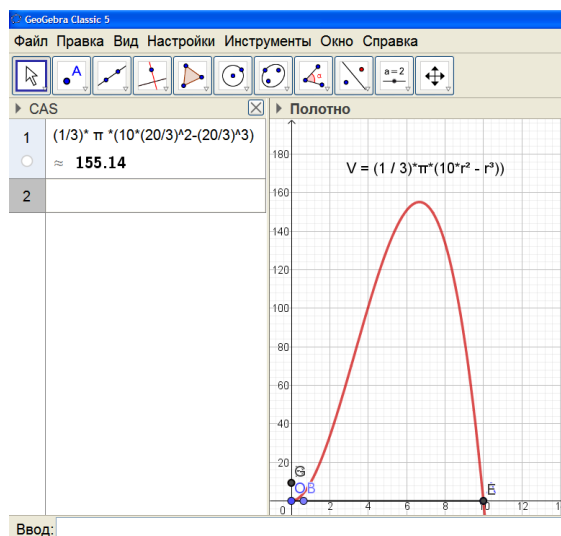


Fig.9 Graph of the cone volume function built analytically

Remark. By analyzing the spreadsheet it becomes evident that the value of the volume and the value of the ratio of radius and height are given with a certain approximation (it is additionally possible to demonstrate changing the format of output values: one digit after the decimal point, two, etc.). So you need to find the results in additional searches of more accurate solution by analytical methods for the confirmation of the empirical fact.

It is very difficult for students to aware an analytic formula of the volume function. The students are not always able to express the relation between the values correctly according to the condition of the task, to determine which variable is independent, and which one is dependent. Mistakes are often in finding the derivative of the function. Such problems are fixed by the subsequent construction of the graph of the analytical function and the coincisidness of the last one with the empirical graph.

In General, the solution of geometric extremum problems with the help of described methods is subordinated to the following algorithm of actions (Table 3). This algorithm contributes to the development of functional thinking of pupils due to the possibility of dynamic representation and processing of a variety of graphical, numerical or algebraic data: we can measure parameters of geometric figures, verify on the basis of these measurements quantitative ratios (between the lengths, areas, angles,

etc.), dynamically changing the shape of the original object. Functional relations can be obtained in the form of a spreadsheet of values or graphically.

Table 3. Algorithm for solving the extremum problems based on the constructive method

№	Action description	Remark
1	Constructing geometric figures or its part using the given parameters	A figure should be built in such a way that the analyzed value depends on only one variable, the other parameters must be fixed
2	Output of functional dependencies (graph or spreadsheet of empirical functions values)	As the dependence is described by one variable, the independent point must provide dynamic changing of only one parameter. As geometrical quantities are nonnegative, then for constructing the empirical functions we can use the first quadrant
3	Interpretation of the empirical dependence	Dependence will be empirical, because the analytical expression of the function is unknown
4	Analytical confirmation (without a computer)	The output of the dependence formula in a notebook or on the board remains an obligatory element of the solution and requires the ability to operate the relevant mathematical apparatus
5	Verification of the matching between analytical and empirical dependences	The verification has more applied and formal character ("analytic" graph should coincide with "empirical")
6	Analysis of the answer (if there are critical points, how many of them are exist, what are the conditions of their existence or absence, etc.)	

3 Improving the Quality of Mathematical Education During Math Group Works

Below we will describe the peculiarities of organization and conducting of pedagogical experiment involving the use of dynamic mathematics software as means of improving the quality of mathematical education.

The effectiveness of the described approach was approved (2015-2017) and experimentally confirmed by the results of pedagogical research within the research topic "Use of information technologies in education", which was performed by faculty of the Departments of Informatics and Mathematics of Makarenko Sumy State Pedagogical University in the experimental classes of secondary schools of the Sumy region.

Described approaches to the solution of mathematical problems were implemented in the frame of math group works for pupils. Members of the group works had the opportunity to attend classes once a week. The class work was conducted one hour and provided solution of several problems of varying complexity from separate topics

of school mathematics course. Also it was expected to solve similar problems at home.

Below there are the examples of problem conditions with the research content, which were offered to pupils on group works "The extremum of the function" (task level is different, so you should previously check the level of educational achievements of those to whom they will be offered).

1. Taking a segment of length 20 as a flexible wire, construct a rectangle from it. Changing the lengths of the sides, study its area. When will it be the biggest?

2. Construct rectangles: a) with different area and different perimeter; b) with different area but the same perimeter; c) with different perimeter but the same area; d) with the same perimeter and the same area.

Methodological comment. Problems are simple at first glance, but experience of their solutions with pupils shows that tasks a) and b) are solved faster than tasks c) and d). Common mistake is constructing of a rectangle that is a rectangle only visually, and it is transformed into a quadrilateral by changing the position of one of the vertices. We advise pupils firstly to construct a rectangle (!), then a rectangle in which the perimeter is fixed (for example, 20 as in problem 1). It requires from pupils to know the basic constructions of figures using compasses and ruler, which were studied in the 7th class. After that it is better to change the value of perimeter and see, whether the constructed figure remains a rectangle. If actions are correct, then you can study the area value with a fixed perimeter (using the *Trace* tool, the *Locus* tool or the spreadsheet).

3. There is a rectangle No. 1, the form of which can be changed by movement of the point A (the lower right vertex moves along Ox). The perimeter of the rectangle is always equal to 20. Construct the dependence graph between its length and width.

4. There is a rectangle No. 2, the form of which can be changed by movement of the point A (the lower right vertex moves along Ox). The area of the rectangle is always equal to 15. Construct the dependence graph between the length and width of rectangles with equal areas.

The answer is: the hyperbola $x*y=15$ (constructed, for example, as a trace of the upper right vertex).

5. The rectangle No. 1 retains its perimeter, and the rectangle No. 2 retains its area. Find the rectangle that has these perimeter and area simultaneously. Does the problem always have a solution?

Answer: the intersection points of the traces constructed in problems 3 and 4 define the upper right vertices of the desired rectangles.

The following problems were proposed for independent research and solutions.

6. Study the area value of a net of paper packaging (in the form of a parallelepiped), which has constant volume of 200 cm^3 . Is it possible to save material?

7a. Study whether the area of an isosceles triangle, in which only the length of the leg is known, has a maximum.

7b. Study whether the volume of a cone, in which only the length of the generatrix is known, has a maximum.

8a. Among inscribed in an isosceles triangle rectangles one has a maximum perimeter and the other has a maximum area. Do these extremums preserve when you change the base of the triangle?

8b. Among the inscribed cylinders one has a maximum volume and the other has a maximum area. Do these extremums preserve when you change shapes of the cone?

At the beginning and at the end of the group works pupils were asked to give answers to the test questions, which were compiled based on questions of External independent testing (the first and the second part of the test of 2008-2010, which, we believe, were more difficult than recent years tests) and which were positively evaluated by experts in the field of learning math as a test which can verify the level of mathematical preparation of a high school pupil.

The maximum number of test scores is 25.

We tracked the general level of educational achievements and its dynamics. As the nominal scale had two positions – right/wrong, the results of each member of the sample were dependent, but the results among members in the sample were mutually independent, we have applied the sign test for the processing the general results.

The experiment involved 72 people.

The null hypothesis is that the group works does not impact on the quality of mathematical preparation of pupils. The alternative hypothesis is that the quality of mathematical preparation is changed.

At a significance level of 0.05 critical statistics value $G_{crit}=28$.

The Table 5 shows the results of the tests (green – the result increased, yellow – the result does not change, red – the result decreased).

Table 5. The test results

№ resp.	change type	№ resp.	change type	№ resp.	change type
1	+5	25	0	49	-2
2	+3	26	+2	50	-2
3	+2	27	+4	51	-1
4	+6	28	-1	52	+6
5	-4	29	-3	53	+2
6	-1	30	0	54	+5
7	0	31	-2	55	+6
8	-2	32	0	56	+3
9	-1	33	-1	57	-2
10	+6	34	+2	58	-3
11	+2	35	+3	59	-1
12	+3	36	+2	60	-1
13	-1	37	0	61	0
14	-2	38	+4	62	0

15	-4	39	+2	63	+2
16	-2	40	+3	64	-2
17	+3	41	+1	65	0
18	+3	42	+1	66	-3
19	0	43	+1	67	-1
20	+2	44	-1	68	+2
21	0	45	-2	69	+2
22	+3	46	+2	70	-1
23	+3	47	+2	71	-1
24	0	48	+2	72	+2

According to the rules of decision making we have that $G_{\text{empir}}=26$. As the empirical value is less than critical, the alternative hypothesis about the impact of the group works on the quality of mathematical preparation of pupils is accepted, and such impact is positive, as the number of positive shifts (35 respondents have increased their rates) exceeds the number of negative ones (26 respondents showed a decrease in the general test result).

4 Conclusion

1. The purpose of mathematics education is not only learning of the concept of function, but the readiness to analyze the obtained values and their relations. So the tools that accelerate, simplify and visualize the calculations, construction and provide the ability to dynamically vary the variables for awareness essential relations between them, should be involved in the training process. Such tools in the teaching of mathematics are dynamic mathematics software. Their usage allows not only to organize the heuristic search, but also to free up time for additional independent studies to demonstrate the output of mathematics to the practical level of children's experiences and the need for its study.

Note that the involvement of DMS to solving mathematical problems does not provide the knowledge of math formulas, concepts, or functional dependencies, but it is a tool that contributes to developing pupils' research abilities, mathematical thinking and critical view on any statement.

2. Nowadays math teacher needs to demonstrate all the possible ways of solving math problems. It concerns not only analytical or geometric approaches, but also the use of specialized software. The wider the list of ways of finding the answer, the greater the probability of a correct solution of the problem is (at least through the possibility of answer verification).

3. As our experience shows, the involvement of DMS to the solution of problems not only demonstrates an additional way of using information devices (tablets, smartphones, computers, etc.) to an average pupil, but also forms and helps to im-

prove the quality of mathematical preparation, which is confirmed experimentally on a significance level of 0.05 on the sign test.

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